

Symmetries of Projective Varieties and Their Applications from the Viewpoint of Algebraic Dynamics

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Basic objects

A self-map $f : X \dashrightarrow X$ of a variety X (mainly X is a smooth complex projective variety and f is a birational or biregular automorphism of X in this talk) and its iterates

$$f, f^2, f^3, \dots$$

- Algebraic geometry: automorphism groups, birational transformation groups, cones, and fibration structures.
- Complex/algebraic/arithmetic dynamics: entropy and dynamical degrees of self-maps, behavior of point orbits, and primitivity.
- Applications in this talk: mainly K3 surfaces, Calabi–Yau varieties in a broad sense, rational varieties, and real forms.

Topological entropy $h_{\text{top}}(f)$

Let $X = (X, d)$ be a compact metric space and $f : X \rightarrow X$ a continuous surjective map. Set

$$d_{f,n}(x, y) := \text{Max}_{0 \leq j \leq n-1} d(f^j(x), f^j(y)).$$

Let $N_d(f, n, \epsilon)$ be the maximal cardinality of a set whose distinct points have $d_{f,n}$ -distance at least ϵ . The topological entropy of f is

$$h_{\text{top}}(f) := \lim_{\epsilon \rightarrow +0} \limsup_{n \rightarrow \infty} \frac{\log N_d(f, n, \epsilon)}{n} \geq 0.$$

- $h_{\text{top}}(f)$ measures the exponential divergence rate of typical orbits.
- It is independent of the choice of d . If $h_{\text{top}}(f) > 0$, then f has infinite order.
- Infinite-order maps can nevertheless have entropy 0 (for example, translations on complex tori).

The Gromov–Yomdin theorem

Let $M = (M, \eta)$ be a compact Kähler manifold (for instance, a smooth projective variety) and $f \in \text{Aut}(M)$.

Dynamical degrees [Dinh–Sibony]

$$d_p(f) := \lim_{n \rightarrow \infty} \left(\int_M (f^n)^*(\eta^p) \wedge \eta^{\dim M - p} \right)^{1/n}.$$

This is independent of the choice of the Kähler form η .

Theorem (Gromov–Yomdin)

$$h_{\text{top}}(f) = \log \max_{0 \leq p \leq \dim M} d_p(f) = \log r(f),$$

where $r(f)$ is the spectral radius of f^* on the total cohomology group.

Thus $h_{\text{top}}(f)$ can be computed from intersection numbers and the induced action on cohomology. In particular, if $f \in \text{Aut}^0(X)$, then $h_{\text{top}}(f) = 0$. Notice that

$$h_{\text{top}}(f) = 0 \iff d_1(f) = 1.$$

Extension to birational self-maps

For a birational map, iteration and pullback need not commute:

$$(f^n)^* \neq (f^*)^n.$$

For each n , resolve the indeterminacy of f^n :

$$X \xleftarrow{p_n} X_n \xrightarrow{q_n} X, \quad f^n = q_n \circ p_n^{-1},$$

and define $(f^n)^* := (p_n)_*(q_n)^*$. The dynamical degrees $d_p(f)$ are then defined by the same formula as above. Unlike $h_{\text{top}}(f)$, they enjoy the following very nice algebro-geometric property:

Theorem (Dinh–Sibony, Truong)

The dynamical degrees of a birational self-map of a compact Kähler manifold are birational invariants.

- Only countably many real numbers occur as dynamical degrees of birational self-maps of projective varieties (Urech). Transcendental values also occur (Bell–Diller–Jonsson–Krieger, Sugimoto).
- For general compact Kähler manifolds, countability remains open.

The Zariski-dense orbit conjecture

Let $V \subset \mathbb{P}^n$ be a projective variety over an algebraically closed field k , let $f \in \text{Bir}(V)$, and let $I(f)$ be the indeterminacy locus of f . Put

$$V_f := \{x \in V \mid f^n(x) \notin I(f) \ (\forall n \geq 0)\}.$$

Points with Zariski-dense orbit

$$Z_f := \left\{ x \in V_f \mid \overline{\{f^n(x)\}_{n \geq 0}} = V \right\}.$$

An obvious obstruction to $Z_f \neq \emptyset$ is the existence of a nonconstant f -invariant rational function:

$$k(V)^{f^*} \neq k \iff \exists \phi : V \dashrightarrow \mathbb{P}^1 \text{ dominant with } f \in \text{Bir}(V/\mathbb{P}^1) \implies Z_f = \emptyset.$$

Conjecture (Medvedev–Scanlon)

$$k(V)^{f^*} = k \implies Z_f \neq \emptyset.$$

The theorem of Amerik–Campana

Theorem (Amerik–Campana; Bell–Ghioca–Reichstein)

For $f \in \text{Bir}(V)$, there exists a dominant rational map

$$\phi : V \dashrightarrow W, \quad \phi \circ f = \phi \quad (= \text{id}_W \circ \phi),$$

whose fibers are the orbit closures of very general points.

Corollary

If $k = \mathbb{C}$ (hence uncountable), then $k(V)^{f^*} = k \implies Z_f \neq \emptyset$.

Therefore, over $\mathbb{C}$, one has $k(V)^{f^*} = k \iff Z_f \neq \emptyset$ and in this case Z_f is a dense subset of V .

Widely open over $\mathbb{Q}$.

Definition (D.-Q. Zhang)

A map $f \in \text{Bir}(V)$ is *imprimitive* if there exist a dominant rational map $\phi : V \dashrightarrow W$ and a rational self-map $f_W : W \dashrightarrow W$ such that

$$0 < \dim W < \dim V, \quad \phi \circ f = f_W \circ \phi.$$

Interpretation

Primitivity is a birational-algebraic-geometric irreducibility condition: the dynamical system is not obtained by lifting a lower-dimensional one.

$$k = \mathbb{C}, \quad f \text{ primitive} \implies k(X)^{f^*} = k \implies Z_f \neq \emptyset.$$

Definition

A *Salem number* is a real algebraic integer $\alpha > 1$ whose Galois conjugates consist of $1/\alpha$ and complex numbers on the unit circle.

There are only finitely many Salem numbers of bounded degree and bounded size. The smallest known Salem number is Lehmer's number, of degree 10.

Theorem (Cantat, McMullen)

Let S be a smooth compact Kähler surface and $f \in \text{Aut}(S)$.

1. (Cantat) If $d_1(f) > 1$, then S is a rational surface, a K3 surface, an Enriques surface, a two-dimensional complex torus, or a blowup of one of these.
2. (McMullen) $d_1(f)$ is either 1 or a Salem number (at least Lehmer's number).

Key ingredients: the classification of compact Kähler surfaces and the Hodge index theorem; the intersection form on $H^{1,1}(S, \mathbb{R})$ has signature $(1, h^{1,1} - 1)$.

Definition

A K3 surface is a compact complex surface S admitting a nowhere-vanishing holomorphic 2-form and no nonzero holomorphic 1-forms. Thus

$$H^0(S, \Omega_S^2) = \mathbb{C}\omega_S.$$

- $b_2(S) = 22$, and the Picard number satisfies

$$0 \leq \rho(S) := \text{rank NS}(S) \leq 20, \quad \text{NS}(S) := \text{Im}(c_1 : \text{Pic}(S) \rightarrow H^2(S, \mathbb{Z})).$$

- All K3 surfaces are deformation equivalent (Kodaira), hence simply connected; moreover they are Kähler (Siu).
- A K3 surface is uniquely recovered from its period.
- A very general K3 surface has $\rho(S) = 0$ and is non-projective. More precisely, K3 surfaces with Picard number at least ρ form a countable union of subvarieties of dimension $20 - \rho$ in the period domain. In every nontrivial small deformation, projective members are dense in the Euclidean topology (Kodaira).

Examples of K3 surfaces

- Smooth quartic surfaces $(4) \subset \mathbb{P}^3$, for example the Fermat quartic

$$(X_0^4 + X_1^4 + X_2^4 + X_3^4 = 0) \subset \mathbb{P}^3.$$

- Kummer surfaces $\text{Km}(A)$, where A is a two-dimensional complex torus. In fact, the Fermat quartic is isomorphic to

$$\text{Km}(E_{\sqrt{-1}} \times E_{2\sqrt{-1}}), \quad E_\tau = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau)$$

(Shioda).

- A very general $(2, 2, 2) \subset (\mathbb{P}^1)^3$ (a Wehler K3 surface).

A very general smooth quartic surface in $\mathbb{P}^3$ has only the identity automorphism, but many other K3 surfaces have rich automorphism groups.

A Wehler K3 surface has three degree-two projections $p_{ij} : S \rightarrow (\mathbb{P}^1)^2$ ($1 \leq i < j \leq 3$), yielding involutions $\iota_{ij} \in \text{Bir}(S) = \text{Aut}(S)$. For very general S ,

$$\text{Aut}(S) = \langle \iota_{12}, \iota_{23}, \iota_{13} \rangle \simeq {}^3(\mathbb{Z}/2),$$

and $\iota_{12} \circ \iota_{23} \circ \iota_{13}$ has positive entropy.

Jacobian fibrations and Mordell–Weil groups

A K3 surface S often admits an elliptic fibration with a global section,

$$\phi : S \rightarrow \mathbb{P}^1.$$

The set of global sections forms the Mordell–Weil group $\text{MW}(\phi) \subset \text{Bir}(S) = \text{Aut}(S)$. It is abelian, and

$$f \in \text{MW}(\phi) \implies d_1(f) = 1,$$

so these are dynamically mild, zero-entropy automorphisms.

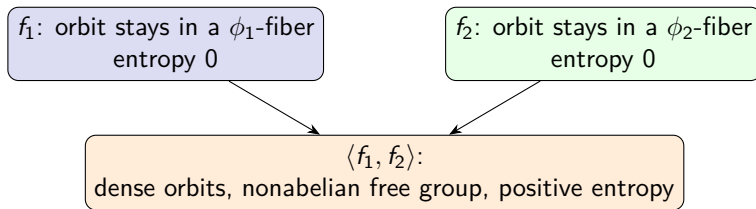
Proposition (Cantat, Oguiso)

Assume that S has two distinct Jacobian fibrations ϕ_i ($i = 1, 2$), each of positive Mordell–Weil rank. Choose infinite-order elements $f_i \in \text{MW}(\phi_i)$.

- (Cantat) For a very general point P , the orbit $\langle f_1, f_2 \rangle \cdot P \subset S$ is dense in the Euclidean topology.
- (Oguiso) For all sufficiently large n , $\langle f_1^n, f_2^n \rangle \simeq \mathbb{Z} * \mathbb{Z}$, and this group contains an element of positive entropy.

This proposition implies that $\mathbb{Z} * \mathbb{Z} \subset \text{Aut}(S)$ for every complex K3 surface S with maximal $\rho(S) = 20$.

From mild dynamics to complicated dynamics



Two different fibration structures can turn mild dynamical systems into a complicated one. For a projective K3 surface S and an automorphism f with $d_1(f) > 1$, it is unknown whether there exists a point P such that $\langle f \rangle \cdot P$ is dense in S in the Euclidean topology: neither an example nor a counterexample is known. McMullen constructed counterexamples on rational surfaces; without projectivity, counterexamples also exist for K3 surfaces.

McMullen's K3 surfaces

Theorem (McMullen (Crelle, 2002))

There exist a K3 surface S with $\rho(S) = 0$ and an automorphism $f \in \text{Aut}(S)$ such that

- $d_1(f)$ is a Salem number of degree 22; and
- f admits a two-dimensional Siegel disk.

In particular, $\langle f \rangle \cdot P \subset S$ ($P \in S$) is never dense in the Euclidean topology.

Siegel disk

An F -invariant unit bidisk $\Delta_{x,y}^2$ on which

$$F(x, y) = (ax, by), \quad |a| = |b| = 1,$$

with a and b multiplicatively independent, is called a (two-dimensional) Siegel disk of F .

The characteristic polynomial of $f^*|H^2(S, \mathbb{Z})$ is irreducible. Hence every small deformation of the pair (S, f) is trivial: for a projective deformation S_t , the Néron–Severi group $\text{NS}(S_t) \subset H^2(S_t, \mathbb{Z})$ would give a nontrivial f_t -invariant subspace. Moreover, $f^*\omega_S = ab\omega_S$, where $|ab| = 1$ but ab is not a root of unity (whereas in the projective case it always is). This also implies the rigidity of the pair (S, f) .

Beauville–Bogomolov decomposition

After a finite étale cover, a compact Kähler manifold V with vanishing first Chern class in $H^2(V, \mathbb{Q})$ decomposes as a product of manifolds of the following three types.

- Complex tori (abelian varieties in the projective case).
- Calabi–Yau manifolds in the strict sense: simply connected compact Kähler manifolds of dimension d with

$$H^0(X, \Omega_X^k) = 0 \quad (1 \leq k < d), \quad H^0(X, \Omega_X^d) = \mathbb{C}\omega_X,$$

where ω_X is nowhere vanishing. In particular, $K_X \simeq \mathcal{O}_X$.

- Irreducible holomorphic symplectic manifolds: simply connected compact Kähler manifolds with

$$H^0(X, \Omega_X^2) = \mathbb{C}\eta_X,$$

where the alternating two form η_X on TX is everywhere nondegenerate. In particular, the dimension is even and $K_X \simeq \mathcal{O}_X$.

A K3 surface is both a two-dimensional Calabi–Yau manifold in the strict sense and a two-dimensional irreducible holomorphic symplectic manifold.

Application 1: the Kodaira problem

Kodaira problem

Can every compact Kähler manifold be deformed, by a small deformation, to a projective variety?

Voisin constructed a stronger counterexample (in dimension ≥ 6) using complex tori: a compact Kähler manifold whose rational cohomology ring is not isomorphic to the rational cohomology ring of any projective variety.

McMullen's K3 surface (S, f) gives a simply connected counterexample (in dimension 4). Consider $S \times S$, the diagonal $\Delta \subset S \times S$, and the graph $\Gamma_f \subset S \times S$ of f , and perform the following blowups:

- blow up $\Delta \cap \Gamma_f$;
- then blow up the strict transforms of Δ and Γ_f .

The resulting manifold M is a simply connected, non-projective compact Kähler manifold. The rigidity of (S, f) implies that M is rigid. Thus M gives a simply connected counterexample to the Kodaira problem (Oguiso, 2008).

Later, Linsheng Wang (2021) showed that M is also a counterexample in the stronger sense above.

Application 2: birational groups of non-projective hyperkähler manifolds

Theorem (Oguiso (2008))

For a non-projective hyperkähler manifold M , the birational automorphism group $\text{Bir}(M)$ is virtually abelian of finite rank. In particular, it is finitely generated.

Theorem (Oguiso (2007))

Let M be a projective hyperkähler manifold and $G \subset \text{Bir}(M)$ a subgroup such that $d_1(g) = 1$ for every $g \in G$. Then G is virtually abelian of rank at most $\rho(M) - 2$.

The Beauville–Bogomolov–Fujiki form and Salem numbers play an essential role in the proofs. Later, finite generation of $\text{Bir}(M)$ for projective hyperkähler manifolds was also established (Boissière–Sarti, 2012).

Application 3: Non-lifting automorphisms of K3 surfaces in positive characteristic

Theorem (Esnault–Oguiso (2015); Esnault–Oguiso–Yu (2016))

For every prime p with $p = 11$ or $p \geq 17$, there exists a K3 surface over an algebraically closed field of characteristic p carrying an automorphism that cannot be geometrically lifted to characteristic 0.

Later, Brandhorst and Xun Yu extended this to every odd characteristic (2018) and Xun Yu received the JMSJ Outstanding Paper Prize for this extension.

- Every K3 surface in positive characteristic lifts to characteristic 0 (Deligne).
- The issue is whether the automorphism can be lifted simultaneously.

Idea behind non-liftability

By Shioda, for $p = 11$ or $p \geq 17$, the supersingular K3 surface $S(p)$ of Artin invariant 1 has

- a Jacobian fibration of Mordell–Weil rank $20 = 22 - 2 = \rho(S(p)) - 1$;
- a Jacobian fibration of Mordell–Weil rank 4, arising from a Kummer structure.

Using these two Jacobian fibrations, one constructs an automorphism f for which $d_1(f)$ has degree 22, via the following theorem.

Theorem (EOY (2016))

Let L be a hyperbolic lattice of rank $2d$ and $G < \mathrm{SO}^+(L)$. If $L \otimes_{\mathbb{Z}} \mathbb{R}$ has no nontrivial G -invariant subspace, then G contains an element g whose characteristic polynomial is a Salem polynomial of degree $2d$.

$$\deg d_1(f) = 22 > 20 \geq \rho(\text{any K3 surface in characteristic 0}).$$

This discrepancy with the injectivity of the specialization map yields non-liftability of f .

Theorem (D.-Q. Zhang)

Let V be a smooth complex projective variety admitting a primitive birational self-map. Then one of the following holds:

1. $\kappa(V) = -\infty$ and $q(V) = 0$;
2. $\kappa(V) = 0$ and $q(V) = 0$;
3. $\kappa(V) = 0$ and $q(V) = \dim V$.

V in case 3 is birational to an abelian variety (Kawamata).

The proof uses the fact that the pluricanonical map, Albanese map are preserved by birational self-maps. Assuming the MMP and a weak abundance conjecture, case 1 reduces to rationally connected varieties, while case 2 is birational to a minimal Calabi–Yau variety: a projective variety with only $\mathbb{Q}$ -factorial terminal singularities, numerically trivial canonical divisor, and irregularity zero.

Proposition (Oguiso)

Let A be an abelian variety and $f \in \text{Aut}(A) = \text{Bir}(A)$.

1. If A is simple, then the following are equivalent: f is primitive; f has a Zariski-dense orbit; f is of infinite order.
2. For a translation t_p of A , t_p is primitive if and only if A is simple and t_p has infinite order. In this case $d_p(t_p) = 1$ for all p .

Theorem (Lo Bianco)

A birational automorphism f of a projective hyperkähler manifold satisfying $d_1(f) > 1$ is primitive. In particular, it has a Zariski-dense orbit.

Thus dynamically mild translations and positive-entropy automorphisms provide primitivity in strikingly different ways.

Calabi–Yau varieties in the strict sense

Theorem (Oguiso (in preparation))

Let V be a minimal Calabi–Yau variety with $\dim V \geq 3$ and Picard number $\rho(V) \geq 2$. Then $f \in \operatorname{Bir}(V)$ is primitive under the following assumptions.

1. Every effective movable divisor is semiample on a suitable minimal model.
2. The characteristic polynomial of $f^*|N^1(V)_{\mathbb{Q}}$ is irreducible over $\mathbb{Q}$, or is the product of an irreducible polynomial and a linear factor $(t + a)$ with $a > 0$.

Condition 1 follows from the log MMP and abundance conjecture. Condition 2 is invariant under replacing V by another minimal model, as any birational maps between minimal models are isomorphisms in codimension one. Our formulation is motivated by the idea that primitivity resembles irreducibility. The proof also uses the following theorem.

Theorem (Lo Bianco)

Let X, Y be smooth projective varieties and $f \in \operatorname{Bir}(X)$. If there exists an f -equivariant dominant rational map $\phi : X \dashrightarrow Y$ whose general fiber is of general type of positive dimension, then f has no Zariski-dense orbit.

This can be viewed as a relative version of the finiteness of the birational automorphism group of a variety of general type.

Example 1: Wehler-type Calabi–Yau varieties

A general hypersurface V_d of multidegree $(2, 2, \dots, 2)$ in $(\mathbb{P}^1)^{d+1}$, $d \geq 3$, is a d -dimensional Calabi–Yau variety in the strict sense with $\rho(V_d) = d + 1$; call it a Wehler-type Calabi–Yau variety. For a Wehler K3 surface S , one has $\text{Aut}(S) \simeq {}^3(\mathbb{Z}/2)$, whereas for V_d with $d \geq 3$ the biregular automorphism group is trivial (Cantat–Oguiso). On the other hand:

Application 1

Every Wehler-type Calabi–Yau variety V_d ($d \geq 3$) admits a primitive birational automorphism.

The proof uses the facts that $\text{Bir}(V_d)$ is the universal Coxeter group ${}^{d+1}(\mathbb{Z}/2)$ and that the movable cone conjecture holds for V_d (Cantat–Oguiso), allowing one to verify the two conditions in the preceding theorem.

Example 2: Calabi–Yau varieties of Picard number 2

For a Calabi–Yau variety X in the strict sense with $\rho(X) = 1$, one has $\text{Bir}(X) = \text{Aut}(X)$, and $\text{Aut}(X)$ is finite.

Application 2 (Oguiso, JAG 2014 revisited)

1. If X is an odd dimensional Calabi–Yau variety in the strict sense with $\rho(X) = 2$, then $\text{Aut}(X)$ is finite.
2. A general

$$X = (2, 2) \cap (1, 1) \cap (1, 1) \subset (\mathbb{P}^3)^2$$

is a three-dimensional Calabi–Yau variety in the strict sense with $\rho(X) = 2$. Moreover,

$$\text{Bir}(X) \simeq \mathbb{Z}/2 * \mathbb{Z}/2,$$

and every infinite-order element of $\text{Bir}(X)$ is primitive.

Lesieutre (JAG 2022) used the variety in (2) and its birational transformations to show that several notions of numerical Kodaira dimension for divisors can differ in general, a result that was quite surprising to specialists.

Biregular automorphisms of strict Calabi–Yau manifolds

Proposition (Oguiso)

Let S be a general Enriques surface. The universal cover X_n of the Hilbert scheme $\text{Hilb}^n(S)$ is a $2n$ -dimensional Calabi–Yau manifold in the strict sense (Oguiso–Schröer), and X_n admits an automorphism f such that $d_1(f) > 1$ and f is primitive.

Proposition (Oguiso–Truong, Oguiso)

Let X_3 be a crepant resolution of

$$E_\omega^3 / \langle \omega I_3 \rangle,$$

and let X_7 be a crepant resolution of

$$\text{Jac}(XY^3 + YZ^3 + ZX^3 = 0) / \langle \text{diag}(\zeta_7, \zeta_7^2, \zeta_7^4) \rangle.$$

Then both X_3 and X_7 are Calabi–Yau manifolds in the strict sense and admit primitive biregular automorphisms.

Assuming two conjectures on nef cones of strict Calabi–Yau threefolds, one can show that these are the only strict Calabi–Yau threefolds admitting primitive biregular automorphisms.

What is a real form?

A *real form* of a complex quasi-projective variety $V \rightarrow \operatorname{Spec} \mathbb{C}$ is a real algebraic variety $W \rightarrow \operatorname{Spec} \mathbb{R}$ such that

$$W \times_{\operatorname{Spec} \mathbb{R}} \operatorname{Spec} \mathbb{C} \simeq V \quad (\text{isomorphism over } \operatorname{Spec} \mathbb{C}).$$

Equivalent description

The set $\operatorname{RF}(V)$ of isomorphism classes of real forms corresponds bijectively to the set $\operatorname{RS}(V)$ of $\operatorname{Aut}(V) = \operatorname{Aut}(V/\mathbb{C})$ -conjugacy classes of real structures on V , i.e. anti-holomorphic involutions $\iota : V \rightarrow V$ covering complex conjugation $c : \operatorname{Spec} \mathbb{C} \rightarrow \operatorname{Spec} \mathbb{C}$.

If V has at least one real form W with real structure ι_W , then

$$c \longmapsto (f \longmapsto \iota_W \circ f \circ \iota_W)$$

defines an action of $\operatorname{Gal}(\mathbb{C}/\mathbb{R})$ on $\operatorname{Aut}(V)$. By Borel–Serre, there are natural bijections

$$\operatorname{RF}(V) \simeq \operatorname{RS}(V) \simeq H^1(\operatorname{Gal}(\mathbb{C}/\mathbb{R}), \operatorname{Aut}(V)).$$

In particular, if $\operatorname{Aut}(V)$ is finite, then V has only finitely many real forms. If the $\operatorname{Gal}(\mathbb{C}/\mathbb{R})$ -action on $\operatorname{Aut}(V)$ is trivial, then this H^1 is the set of $\operatorname{Aut}(V)$ -conjugacy classes of the identity and involutions of V .

Example: real forms of $\mathbb{P}_{\mathbb{C}}^n$

The projective space $\mathbb{P}_{\mathbb{C}}^n$ has the standard real form $\mathbb{P}_{\mathbb{R}}^n$. With respect to it,

$$\begin{aligned} H^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), \mathrm{Aut}(\mathbb{P}_{\mathbb{C}}^n)) &\simeq H^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), \mathrm{PGL}_{n+1}(\mathbb{C})) \\ &\simeq H^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), \mathrm{Aut}_{\mathbb{C}\text{-alg}}(M_{n+1}(\mathbb{C}))) \\ &\simeq \begin{cases} \{M_{n+1}(\mathbb{R})\}, & n \text{ even,} \\ \{M_{n+1}(\mathbb{R}), M_{(n+1)/2}(\mathbb{H})\}, & n \text{ odd.} \end{cases} \end{aligned}$$

Hence $\mathbb{P}_{\mathbb{C}}^n$ has one real form when n is even and two when n is odd. In the odd-dimensional case, one is $\mathbb{P}_{\mathbb{R}}^n$, which has real points, and the other has no real points. The latter can be realized as the Hilbert scheme $\mathrm{Hilb}^n(C)$ of the smooth real conic

$$C = \mathrm{Proj} \mathbb{R}[X, Y, Z]/(X^2 + Y^2 + Z^2),$$

which has no real points.

Real forms of Fermat hypersurfaces, except for the Fermat quartic surface, are also explicitly described (Y. Sasaki).

Basic questions

1. (Folklore) Does every smooth complex projective variety have only finitely many real forms?
2. (Kharlamov) Does every smooth complex projective rational surface have only finitely many real forms?

Lesieutre constructed a six-dimensional (uniruled but not rationally connected) example with infinitely many real forms, giving the first negative answer to (1) (Invent. Math. 2018). Probably motivated in part by (2), his construction started from rational surfaces.

Subsequent work has led to a fairly broad picture of finiteness and infiniteness for smooth projective varieties.

Theorem (Dinh–Oguiso (2019); Dinh–Oguiso–Yu (2023); Dinh–Gachet–Lin–Oguiso–Wang–Yu (2025))

A smooth complex projective surface V has only finitely many real forms, except possibly in the following three cases:

1. rational surfaces;
2. non-minimal K3 surfaces;
3. non-minimal Enriques surfaces.

Conversely, in each of these three exceptional classes there exist examples with infinitely many real forms.

The examples in (1) give a negative answer to Kharlamov's question. All these examples are constructed from K3 surfaces, their blowups, and quotients. A major advantage of a K3 surface S compared with a rational surface is that for a composition of blowups $\nu : S' \rightarrow S$, conjugation by ν gives naturally

$$\mathrm{Aut}(S') \subset \mathrm{Bir}(S') = \mathrm{Bir}(S) = \mathrm{Aut}(S).$$

We consider higher dimensional cases. Let d be an integer such that $d \geq 3$.

Theorem (DO; Dinh–Oguiso–Yu (2022); DGLOWY)

If a smooth complex projective variety V of dimension d has infinitely many real forms, then

$$\kappa(V) \leq d - 2.$$

Conversely, for every

$$\kappa \in \{-\infty, 0, 1, \dots, d - 2\},$$

there exists a smooth projective variety V with $\dim V = d$, $\kappa(V) = \kappa$, and infinitely many real forms. Moreover, for every $d \geq 3$ there exists a smooth rational d -fold with infinitely many real forms.

Theorem (DGLOWY)

Let V be a smooth complex projective variety. If every $f \in \text{Aut}(V)$ has zero entropy, then

$\text{RF}(V)$ is finite.

Thus finiteness of real forms, an arithmetic phenomenon, can be guaranteed by entropy, a quantity from algebraic dynamics. In particular, every smooth projective curve has only finitely many real forms.

Theorem (Cattaneo–Fu (2019); DGLOWY)

Assume that there exists a rational polyhedral cone $\Sigma \subset \text{Nef}^+(V)$ such that

$$\text{Aut}^*(V) \cdot \Sigma$$

covers the ample cone. Then $\text{RF}(V)$ is finite.

- Finiteness follows when the nef cone is rational polyhedral, for example for Fano varieties.
- It also follows when the Kawamata–Morrison nef cone conjecture holds.
- Thus a cone conjecture arising from mirror symmetry appears naturally in the finiteness problem for real forms.

In particular, again smooth projective curves have finitely many real forms. Since the nef cone conjecture is expected for Calabi–Yau varieties in a broad sense, one expects their real forms to be finite as well.

Sketch of the proof in the surface case

Divide according to Kodaira dimension.

- $\kappa = 2$: V is of general type, so $\text{Aut}(V)$ is finite.
- $\kappa = 1$: automorphisms preserve the canonical genus-one fibration, hence have zero entropy.
- $\kappa = -\infty$ and $q(V) > 0$: automorphisms preserve the Albanese map, hence have zero entropy.
- $\kappa = 0$ and $q(V) = 1$: again, automorphisms preserve the Albanese map, hence have zero entropy.
- $\kappa = 0$ and V minimal: use the cone conjecture.

Thus infinitely many real forms can occur only for rational surfaces, non-minimal K3 surfaces, and non-minimal Enriques surfaces. The finiteness for non-minimal abelian surfaces is proved separately.

A non-minimal K3 surface with infinitely many real forms

Take the Kummer surface

$$S = \text{Km}(E \times F),$$

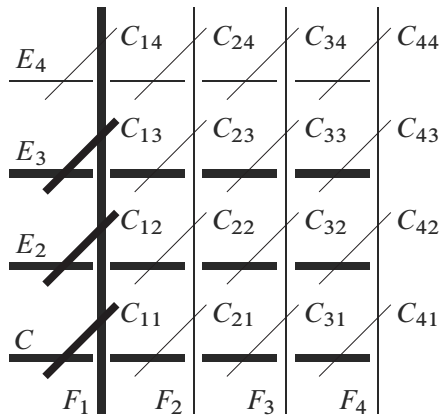
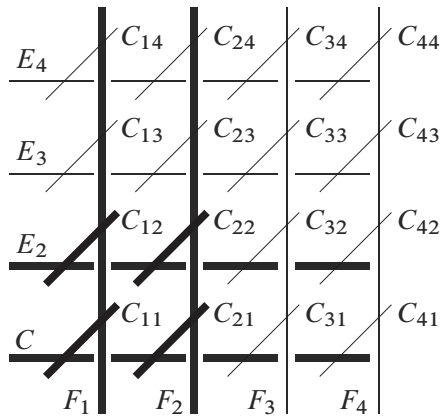
where E and F are non-isogenous elliptic curves defined over $\mathbb{R}$. The construction gives S a natural real form W , and the corresponding real structure ι_W commutes with $\text{Aut}(S)$.

- S has many Jacobian fibration structures (Oguiso, master's thesis).
- Blow up a very general real point P on C_{11} (next page) to obtain S_P . It inherits a natural real structure from ι_W and $\text{Aut}(S_P) = \text{Ine}(S, C_{11})$ (up to index 2).
- One constructs infinitely many involutions ι_n of S that are conjugate in $\text{Aut}(S)$, lift to involutions of S_P , but split into infinitely many distinct conjugacy classes in the smaller group $\text{Aut}(S_P)$, which gives the result.

Concretely, the involutions ι_n are obtained as

$$\iota_n := f^{-n} \circ \iota \circ f^n, \quad n \in \mathbb{Z},$$

where ι is the inversion of the right Jacobian fibration with zero section C_{31} and f is the translation of the left Jacobian fibration with $f(C_{31}) = C_{41}$ in the next page. Note that ι_n acts on C_{11} as identity, hence all lift to S_P . If $\iota_n = h^{-1} \circ \iota_m \circ h$ with $\exists h \in \text{Ine}(S, C_{11})$, then $f^m \circ h \circ f^{-n}$ commutes with ι , and thus preserves S^ι (which is big). Hence $f^m \circ h \circ f^{-n}$ is of finite order. From this and $h(C) = C$ (by the speciality of S), we deduce that n is finite for each m by considering the action on C of finite order.



Automorphism groups of K3 surfaces and hyperbolic groups

A finitely generated group G is called *hyperbolic* if its Cayley graph is a hyperbolic metric space in the sense of Gromov.

- Virtually free groups are elementary hyperbolic groups.
- Virtually abelian groups of rank at least 2 are not hyperbolic.
- Non-elementary hyperbolic groups, such as fundamental groups of curves of genus at least 2, are central objects in geometric group theory.

Virtually free groups of rank at most 1, i.e., virtually $\mathbb{Z}$ or finite groups are called elementary.

Question (Koji Fujiwara)

To what extent can full automorphism groups of K3 surfaces provide non-elementary hyperbolic groups?

Theorem (Fujiwara–Oguiso–Yu (Crelle, the special bicentennial issue 2026))

Let S be a projective K3 surface with $\rho(S) \geq 5$. The following are equivalent.

1. $\text{Aut}(S)$ is a non-elementary hyperbolic group.
2. $\mathbb{Z} * \mathbb{Z} \subset \text{Aut}(S)$ and $\mathbb{Z}^{\oplus 2} \not\subset \text{Aut}(S)$.
3. $\text{Aut}(S)$ contains a positive-entropy element, and every elliptic fibration on S has Mordell–Weil rank at most 1.

The last condition says that the algebraic dynamics and elliptic-fibration structures of a K3 surface determine hyperbolicity of its automorphism group.

The equivalence of (1) and (2) uses essentially the geometric finiteness of the full automorphism group of a K3 surface (Kohei Kikuta, Daiki Takatsu). For the equivalence of (2) and (3), besides the existence of elliptic fibrations when $\rho(S) \geq 5$, one uses the fact that every subgroup $\mathbb{Z}^r \subset \text{Aut}(S)$ with $r \geq 2$ is virtually realized inside the Mordell–Weil group of an elliptic fibration, not necessarily one with a section (Fujiwara–Oguiso–Yu).

Non-elementary hyperbolicity is unchanged after passing to a finite-index subgroup. By the global Torelli theorem, whether $\text{Aut}(S)$ is non-elementary hyperbolic depends only on the Néron–Severi lattice $\text{NS}(S)$.

Theorem (FOY)

Let S be a projective K3 surface with $\rho(S) \geq 6$. There are only finitely many isomorphism classes of lattices $\text{NS}(S)$ for which every elliptic fibration has Mordell–Weil rank at most 1. In particular, there are only finitely many isomorphism classes of $\text{NS}(S)$ for which $\text{Aut}(S)$ is non-elementary hyperbolic.

The finiteness proof can be made quite effective without computer assistance. A computer calculation further shows that there are exactly 166 lattices satisfying the condition, and determines them explicitly. By contrast, when $\rho(S) = 5$, there are infinitely many isomorphism classes of $\text{NS}(S)$ for which $\text{Aut}(S)$ is non-elementary hyperbolic by the results of Nikulin (in 80s) and Yu (JAG 2025).

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