

Coarse Geometry of Metric Measure Spaces

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This talk is based on a joint work with

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What is coarse geometry?

Coarse geometry studies the large-scale structure of metric spaces.
For example, $\mathbb{R} \simeq_{\text{ce}} \mathbb{Z}$.

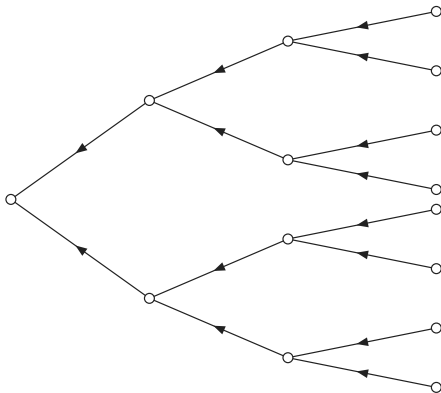
Motivation of considering measures in coarse geometry

- The volume growth is an important invariant in coarse geometry.
- Non-amenability $\iff$ isoperimetric inequality
 $\iff$ existence of a Ponzi scheme (ネズミ構)

A Ponzi scheme is described by transportation.

- A binary tree is non-amenable.
- $\mathbb{Z}^n$ is amenable.

A Ponzi scheme on the rooted binary tree



Our results

- We define the **measured coarse equivalence** between metric measure spaces.
- For a locally doubling metric measure space X , we define the **weighted ℓ^∞ homology $H_n^{(\infty)}(X)$** , which is a generalization of uniformly finite homology of Block–Weinberger.
- We prove the invariance of $H_n^{(\infty)}(X)$ up to measured coarse equivalence.
- Under the locally doubling condition for X , we prove
$$H_0^{(\infty)}(X) = 0 \iff X \text{ is weighted non-amenable.}$$

Remark

- Our results are generalizations of Block–Weinberger's work.
- Our proof relies on the optimal transport theory and disintegration techniques.

Coarse Equivalence

Def (Coarse equivalence; Gromov, Roe)

Let X and Y be metric spaces and $S \subset X \times Y$ a subset.

S is **bi-bornologous** if $\forall r > 0, \exists R = R(r) > 0$ s.t.

$\forall (x, y), (x', y') \in S,$

$$\textcircled{1} \quad d_X(x, x') \leq r \implies d_Y(y, y') \leq R,$$

$$\textcircled{2} \quad d_Y(y, y') \leq r \implies d_X(x, x') \leq R.$$

S is **boundedly dense** if $B_r(p_X(S)) = X$ and $B_r(p_Y(S)) = Y, \exists r > 0$, where $p_X : X \times Y \rightarrow X, p_Y : X \times Y \rightarrow Y$ are the projections.

S is a **coarse correspondence** between X and Y if it is bi-bornologous and boundedly dense.

$X \simeq_{\text{ce}} Y \stackrel{\text{def}}{\iff} \exists \text{ a coarse correspondence between } X \text{ and } Y$

Example

$$\mathbb{R} \simeq_{\text{ce}} \mathbb{Z}.$$

Measured Coarse Equivalence

In this talk, we assume a metric measure space (called mm-space) $X = (X, d_X, \mu_X)$ satisfies

- (X, d_X) is a complete separable metric space;
- μ_X is a boundedly finite Borel measure on X with full support, i.e., $\mu_X(A)$ is finite for any bounded Borel $A \subset X$.

How to define measured coarse equivalence?

A naive definition of measured coarse equivalence between two mm-spaces X and Y is the existence of a Borel isomorphism $f : X \rightarrow Y$ preserving the measures.

However, this is too strict, because we want to treat both Riemannian manifolds and discrete spaces.

The same problem happens for the Monge transport problem in the optimal transport theory.

We use Kantorovich's idea: coupling.

Def (Coupling; Kantorovich)

A **coupling** between μ_X and μ_Y is a Borel measure σ on $X \times Y$ s.t.
 $(p_X)_\bullet \sigma = \mu_X$ and $(p_Y)_\bullet \sigma = \mu_Y$,
where $(p_X)_\bullet$ is the push-forward by p_X , i.e.,
 $(p_X)_\bullet \sigma(A) := \sigma(p_X^{-1}(A)) = \sigma(A \times Y)$ for $A \subset X$ Borel.

Def (Measured coarse equivalence; OS)

$X \simeq_{\text{mce}} Y \stackrel{\text{def}}{\iff} \exists$ a coupling σ between μ_X and μ_Y
s.t. $\text{supp } \sigma$ is a coarse correspondence between X and Y .
Such a σ is called a **measured coarse correspondence**.

Example

$\mathbb{R} \simeq_{\text{mce}} \mathbb{Z}$ for the Lebesgue measure on $\mathbb{R}$ and counting measure on $\mathbb{Z}$.

Weighted ℓ^∞ Homology

Def (BGR discrete mm-space; OS)

We say that a discrete mm-space Γ is **BGR** if

$$(\text{BG}) \quad \sup_{x \in \Gamma} \#B_r(x) < \infty, \quad \forall r > 0,$$

$$(\text{BR}) \quad \forall r > 0, \exists \lambda_r \geq 1 \text{ s.t.}$$

$$d_\Gamma(x, y) \leq r \implies \mu_\Gamma(y) \leq \lambda_r \mu_\Gamma(x).$$

Let Γ be a BGR discrete mm-space. Define $C_n^{(\infty)}(\Gamma)$, $n \geq 0$, to be the set of boundedly finite signed measures ν on Γ^{n+1} s.t.

$$(\Delta) \quad \Delta_\nu := \sup_{(x_0, \dots, x_n) \in \text{supp } \nu} \max_{i,j} d_\Gamma(x_i, x_j) < \infty;$$

$$(\ell^\infty) \quad \frac{d\nu}{d\mu_{\Gamma^{n+1}}} \in \ell^\infty(\Gamma^{n+1}),$$

$$\mu_{\Gamma^{n+1}} := \sum_{(x_0, \dots, x_n) \in \Gamma^{n+1}} (\mu_\Gamma(x_0) \cdots \mu_\Gamma(x_n))^{\frac{1}{n+1}} \delta_{(x_0, \dots, x_n)}.$$

$$C_n^{(\infty)}(\Gamma) := 0 \text{ for } n \leq -1.$$

The **boundary operator** $\partial_n : C_n^{(\infty)}(\Gamma) \rightarrow C_{n-1}^{(\infty)}(\Gamma)$ is defined by

$$\partial_n v := \sum_{i=0}^n (-1)^i (p_{\hat{i}})_\bullet v,$$

where $p_{\hat{i}} : \Gamma^{n+1} \rightarrow \Gamma^n$ is the projection defined by

$$p_{\hat{i}}(x_0, \dots, x_n) := (x_0, \dots, \hat{x}_i, \dots, x_n).$$

Then, $\{(C_n^{(\infty)}(\Gamma), \partial_n)\}_{n \in \mathbb{Z}}$ is a chain complex.

Def (Weighted ℓ^∞ homology; OS)

We define the **weighted ℓ^∞ homology** by

$$H_n^{(\infty)}(\Gamma) := \text{Ker } \partial_n / \text{Im } \partial_{n+1}.$$

This is a generalization of uniformly finite homology of Block–Weinberger and controlled coarse homology of Nowak–Špakura.

Thm A (OS)

If $\Gamma \simeq_{\text{mce}} \Lambda$ for two BGR discrete mm-spaces Γ and Λ , then

$$H_n^{(\infty)}(\Gamma) \cong H_n^{(\infty)}(\Lambda).$$

Outline of proof.

An idea of the proof is to introduce the **transport operator**

$\sigma_{\#} : C_n^{(\infty)}(\Gamma) \rightarrow C_n^{(\infty)}(\Lambda)$ for a measured coarse corresp. σ by

$$\sigma_{\#} \nu := \int_{\Gamma^{n+1}} \sigma_{\bar{x}}^{\otimes(n+1)} d\nu(\bar{x}),$$

where $\sigma_{\bar{x}}^{\otimes(n+1)}$ is the disintegration of $\sigma^{\otimes(n+1)}$ w.r.t. the projection $p_{\Gamma^{n+1}} : \Gamma^{n+1} \times \Lambda^{n+1} \rightarrow \Gamma^{n+1}$, i.e.,

$$\sigma_{\bar{x}}^{\otimes(n+1)}(\bar{y}) := \frac{\sigma^{\otimes(n+1)}(\bar{x}, \bar{y})}{\mu_{\Gamma}^{\otimes(n+1)}(\bar{x})} \quad (\bar{x} \in \Gamma^{n+1}, \bar{y} \in \Lambda^{n+1}).$$

By a straight forward calculation, $\sigma_{\#}$ is a chain map and induces a homomorphism

$$\sigma_* : H_n^{(\infty)}(\Gamma) \rightarrow H_n^{(\infty)}(\Lambda).$$

Let σ^{-1} be the pushforward of σ by $(x, y) \mapsto (y, x)$,
let $\sigma \circ \sigma^{-1}$ be the gluing of σ and σ^{-1}

and let I_{Γ} be the pushforward of μ_{Γ} by $x \mapsto (x, x)$.

Then

$$(\sigma^{-1})_{\#} \circ \sigma_{\#} = (\sigma^{-1} \circ \sigma)_{\#} \stackrel{\sim}{\sim} (I_{\Gamma})_{\#} = \text{id}_{C_n^{(\infty)}(\Gamma)}.$$

Hence

$$(\sigma^{-1})_* \circ \sigma_* = \text{id}_{H_n^{(\infty)}(\Gamma)}.$$

In the same way,

$$\sigma_* \circ (\sigma^{-1})_* = \text{id}_{H_n^{(\infty)}(\Lambda)}.$$

This proves Thm A. $\square$

Def (Locally doubling condition)

An mm-space X is **locally doubling** if for $\forall r > 0, \exists C_r > 0$ s.t.

$$\mu_X(B_{2r}(x)) \leq C_r \mu_X(B_r(x))$$

for any $x \in X$.

- Complete Riemannian manifold with Ricci curvature bounded below are locally doubling.
- Finitely generated groups with word metric and counting measure are locally doubling.

Lem–Def

For a locally doubling mm-space X , $\exists$ a BGR discretization Γ s.t. $\Gamma \simeq_{\text{mce}} X$. We define

$$H_n^{(\infty)}(X) := H_n^{(\infty)}(\Gamma).$$

Thm A' (OS)

If $X \simeq_{\text{mce}} Y$ for two locally doubling mm-spaces X and Y , then

$$H_n^{(\infty)}(X) \cong H_n^{(\infty)}(Y).$$

Weighted Amenability and Weighted ℓ^∞ Homology

Def (Weighted amenable)

An mm-space X is **weighted non-amenable** if $\exists C, r > 0$ s.t.

$$\mu_X(A) \leq C\mu_X(\partial_r A)$$

for any bounded Borel set $A \subset X$, where $\partial_r A := B_r(A) \cap B_r(A^c)$.

Remark

Let Γ be a BG discrete mm-space with the counting measure.

Γ is weighted amenable $\iff \Gamma$ is amenable as a metric space.

Thm B (OS)

For a locally doubling mm-space X , the following are equivalent.

- ① $H_0^{(\infty)}(X) = 0$.
- ② X is weighted non-amenable.

Thm B (OS)

For a locally doubling mm-space, the following are equivalent.

- 1 $H_0^{(\infty)}(X) = 0$.
- 2 X is weighted non-amenable.

Outline of proof.

Prop (OS)

Under $X \simeq_{\text{mce}} Y$,

$X : \text{weighted non-amenable} \iff Y : \text{weighted non-amenable}$.

Take a BGR discretization Γ of X s.t. $\Gamma \simeq_{\text{mce}} X$.

Since $H_n^{(\infty)}(X) = H_n^{(\infty)}(\Gamma)$, it suffices to prove the thm. for Γ .

‘(1) $\implies$ (2)’ for Γ is easy.

For ‘(2) $\implies$ (1)’ for Γ , our strategy is

$$\begin{aligned} \Gamma : \text{weighted non-amenable} &\stackrel{(a)}{\implies} [\mu_\Gamma] = 0 \text{ in } H_0^{(\infty)}(\Gamma) \\ &\stackrel{(b)}{\implies} H_0^{(\infty)}(\Gamma) = 0. \end{aligned}$$

(a) is proved in the same way as Block–Weinberger.

(b) From $[\mu_\Gamma] = 0$, $\exists$ a **nonnegative** 1-chain ψ s.t.

$$(p_1)_\bullet \psi - (p_0)_\bullet \psi = \partial \psi = \mu_\Gamma.$$

ψ is a coupling and each point $x \in \Gamma$ gains a benefit of $\mu_\Gamma(x)$, meaning that ψ is a Ponzi scheme.

By the disintegration technique, we construct a 1-chain t_x for $x \in \Gamma$ s.t. $\partial t_x = \mu_\Gamma(x) \delta_x$. Then we have

$$\partial \sum_{x \in \Gamma} \frac{c(x)}{\mu_\Gamma(x)} t_x = c \quad \text{for any 0-chain } c,$$

which implies $H_0^{(\infty)}(\Gamma) = 0$. $\square$

Def (Ponzi scheme)

A **Ponzi scheme** on Γ is a nonnegative 1-chain ψ s.t. $\partial \psi \geq c \mu_\Gamma$ for some constant $c > 0$.

Cor (OS)

Γ has a Ponzi scheme $\iff \Gamma$ is weighted non-amenable.

The following is useful to prove weighted amenability.

Prop (OS)

If an mm-space X has **subexponential measure growth**, i.e.,

$$\lim_{r \rightarrow \infty} \frac{\ln \mu_X(B_r(x))}{r} = 0,$$

then X is weighted amenable.

Examples

- (1) By the prop., $(\mathbb{Z}, \mu_c)$ is weighted amenable for the counting measure μ_c and $H_0^{(\infty)}(\mathbb{Z}, \mu_c) \neq 0$.

Consider the measure $\mu := \sum_{n \in \mathbb{Z}} 2^n \delta_n$ on $\mathbb{Z}$.

Let

$$\mu_{\mathbb{Z}^2} := \sum_{(x_0, x_1) \in \mathbb{Z}^2} (\mu(x_0)\mu(x_1))^{1/2} \delta_{(x_0, x_1)}.$$

$\psi := \sum_{n \in \mathbb{Z}} 2^{n+1} \delta_{(n+1, n)}$ satisfies

$$\begin{aligned} \frac{d\psi}{d\mu_{\mathbb{Z}^2}}(n+1, n) &= \frac{2^{n+1}}{\sqrt{2^n 2^{n+1}}} = \sqrt{2}, \\ (p_1)_\bullet \psi - (p_0)_\bullet \psi &= \mu, \end{aligned}$$

so ψ is a Ponzi scheme. Therefore, $(\mathbb{Z}, \mu)$ is weighted non-amenable and $H_0^{(\infty)}(\mathbb{Z}, \mu) = 0$.

- (2) $(\mathbb{R}, dx)$ is weighted amenable and $H_0^{(\infty)}(\mathbb{R}, dx) \neq 0$,
 since $(\mathbb{R}, dx) \simeq_{\text{mce}} (\mathbb{Z}, \mu_c)$.
 $(\mathbb{R}, e^x dx)$ is weighted non-amenable and $H_0^{(\infty)}(\mathbb{R}, e^x dx) = 0$,
 since $(\mathbb{R}, e^x dx) \simeq_{\text{mce}} (\mathbb{Z}, \mu)$ (upto a constant multiple).
- (3) Let H^n be an n -dimensional complete simply connected
 hyperbolic space. By Yau's isoperimetric inequality,
 H^n is weighted non-amenable and $H_0^{(\infty)}(H^n) = 0$.
 Let $\nu := e^{-(n-1)d(\cdot, p)} \text{vol}$ for a fixed point $p \in H^n$.
 Then, (H^n, ν) is weighted amenable and $H_0^{(\infty)}(H^n, \nu) \neq 0$.

Future Problems

- 1 Relation to weighted scalar curvature (on going).
- 2 Define weighted ℓ^∞ homology directly without discretization.
- 3 Define weighted version of coarse cohomology and study it.
- 4 Study weighted version of abstract coarse spaces (previously defined by Miyauchi).

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Thank you for your attention!