An Iwasawa Conjecture for a Hyperbolic
Threefold of a Finite Volume

Ken-ichi SUGIYAMA
Department of Mathematics and Informatics
Faculty of Science
Chiba University

November 24, 2006
1 The motivation and a brief overview of our results

The Riemann’s zeta function and its cousins have two main natures:

1. The geometric nature (the distribution of zeros);
   - The Riemann hypothesis.

2. The arithmetic nature (special values);
   - The class number formula (for Riemann or Dedekind zeta functions).
   - The Birch and Swinnerton-Dyer conjecture (for an L-function of an elliptic curve).

In the number theory, it has been considered two models of the zeta function.

1. The Hasse-Weil’s congruent zeta function for a smooth projective variety over a finite field.
   - The geometric nature ⇒ The Weil conjecture.
   - The arithmetic nature ⇒ The Artin-Tate conjecture.

A Point of the theory

The Euler product

\[ \prod \text{The Grothendieck-Lefschetz trace formula} \]

An alternating product of the characteristic polynomials of the Frobenius on étale cohomologies

2. The $p$-adic zeta function.
   - The geometric nature ⇒ ?.
   - The arithmetic nature ⇒ The Iwasawa conjecture.
A Point of the theory

The $p$-adic zeta function

† The Euler system

The Iwasawa polynomial

But the model over a finite field seems to be too special a little because we do not have the Frobenius on a number field!.

So we want to propose an another model following Ruelle and Selberg. We will investigate the Ruelle L-function for a local system of rank one over a hyperbolic threefold of finite volume. Although there is no apparent “Frobenius”, we will show it enjoys:

1. an analogue of the Riemann hypothesis.
2. an analogue of the Iwasawa conjecture.

A Point of our theory

The Ruelle L-function

† The Selberg trace formula

The Alexander invariant
2 The Iwasawa power series of a cyclotomic $\mathbb{Z}_p$-extension

We will fix an odd prime $p$ and will use the following notations.

**Notations 2.1.**

1. $K_n = \mathbb{Q}(\zeta_{p^n})$, $\zeta_{p^n} = \exp\left(\frac{2\pi i}{p^n}\right)$

2. $A_n = \text{Cl}(K_n)\{p\}$: the $p$-primary part of the ideal class group of $K_n$,

3. $\Gamma = \text{Gal}(K_\infty/K_1)$, where $K_\infty = \lim_{n \to \infty} K_n$.

Here are some remarks.

1. The cyclotomic character $\chi_{\text{cyc}}$ yields an isomorphism:

$$\Gamma \xrightarrow{\chi_{\text{cyc}}} \mathbb{Z}_p.$$ 

2. By the action of $\text{Gal}(K_1/\mathbb{Q}) \cong \mathbb{F}_p^\times$, $A_n$ is decomposed as

$$A_n = \bigoplus_{i=0}^{p-2} A_{\omega_i}^n$$

where

$$A_{\omega_i}^n = \{ \alpha \in A_n | \gamma \alpha = \omega(\gamma)^i \alpha \text{ for } \gamma \in \text{Gal}(K_1/\mathbb{Q}) \}.$$ 

**Definition 2.1.** For $0 \leq i \leq p - 2$, the Iwasawa module $X_i$ is defined to be

$$X_i = \lim_{\leftarrow} A_{\omega_i}^n.$$ 

Here the limit is taken for the norm map.

We set

$$\Lambda_p = \mathbb{Z}_p[[\Gamma]],$$

which is isomorphic to a formal power series ring $\mathbb{Z}_p[[s]]$ in a non-canonical way. Then Iwasawa has shown:

$X_i$ is a torsion $\Lambda_p$-module.

Let $L_{p,\text{alg},i} \in \Lambda_p$ be a generator of its characteristic ideal $\text{Char}_{\Lambda_p}X_i$. It will be referred as the **Iwawasa power series**.
3 The Alexander invariant

Let $X$ be a topological threefold which has a surjective homomorphism:

$$\pi_1(X) \xrightarrow{\pi} \mathbb{Z},$$

and

$$\pi_1(X) \xrightarrow{\rho} U(m)$$

a unitary representation. Here are some notations.

Notations 3.1.  
1. $X_\infty$ is the infinite cyclic covering of $X$ which corresponds to $\text{Ker} \pi$.
2. $\Lambda = \mathbb{C}[\mathbb{Z}]$, which is isomorphic to $\mathbb{C}[t^{-1}, t]$ in a non-canonical way.
3. $\Lambda_\infty = \mathbb{C}[[s]]$, where $s = t - 1$.

In the following, we always assume:

Assumption:

$$\dim H_*(X_{\infty}, \mathbb{C}), \ \dim H_*(X_{\infty}, \rho) < \infty.$$ 

Remark 3.1. Under the assumption Milnor has shown:

1. $$H^i(X_{\infty}, \mathbb{C}) = H^i(X_{\infty}, \rho) = 0, \ i \geq 3$$

and

$$H^2(X_{\infty}, \mathbb{C}) = \mathbb{C}.$$

2. (Milnor duality) For each $0 \leq i \leq 2$, the dimension of $H^i(X_{\infty}, \rho)$ is finite and there is a perfect pairing

$$H^i(X_{\infty}, \mathbb{C}) \times H^{2-i}(X_{\infty}, \mathbb{C}) \to H^2(X_{\infty}, \mathbb{C}) = \mathbb{C}$$

and

$$H^i(X_{\infty}, \rho) \times H^{2-i}(X_{\infty}, \rho) \to H^2(X_{\infty}, \mathbb{C}) = \mathbb{C}.$$
Thus the assumption implies

“$H^i(X_\infty, \rho)$ is a torsion $\Lambda$-module.”

Let $\tau^*$ be the action of $t$ on $H^i(X_\infty, \rho)$. We define the twisted Alexander polynomial $A_{\rho,i}$ to be the characteristic polynomial of $\tau^*$:

$$A_{\rho,i}(t) = \det[ t - \tau^* | H^i(X_\infty, \rho) ].$$

**Remark 3.2.** The characteristic ideal of $H^i(X_\infty, \rho)$ is generated by $A_{\rho,i}$:

$$\text{Char}_\Lambda(H^i(X_\infty, \rho)) = (A_{\rho,i}).$$

The Alexander invariant $A_\rho$ is defined to be

$$A_\rho = \frac{A_{\rho,0} \cdot A_{\rho,2}}{A_{\rho,1}}.$$

**Example 3.1.** (Milnor) Let $S^1 \xrightarrow{\kappa} S^3$ be a knot and $X$ its complement. Then

$$H_1(X, \mathbb{Z}) \simeq \mathbb{Z},$$

and we have an infinite cyclic covering

$$X_\infty \xrightarrow{\pi} X.$$

Moreover the dimension of $H_*(X_\infty, \mathbb{C})$ are finite.
4 The Iwasawa main conjecture

The cyclotomic character $\chi_{cyc}$ induces a ring homomorphism:

$$\Lambda_p = \mathbb{Z}_p[[\Gamma]] \xrightarrow{\chi_{cyc}} \mathbb{Z}_p,$$

by

$$\chi_{cyc}(\sum a_\gamma \gamma) = \sum a_\gamma \chi_{cyc}(\gamma).$$

For an integer $0 < i < p - 1$, Kubota-Leopoldt, Iwasawa and Coleman have independently constructed an element $L_{p,ana,i}$ (referred as the $p$-adic $\zeta$-function) of $\Lambda_p$ which satisfies

$$\chi_r(L_{p,ana,i}) = (1 - p^r)\zeta(-r)$$

for any

$$r \in \mathbb{N}, \quad r \equiv i \mod p - 1$$

Remark 4.1. Special values of the Riemann zeta function at non-positive integers are given by

$$\zeta(1 - n) = -\frac{B_n}{n}, \quad n = 1, 2, 3, \ldots,$$

where $B_n$ is the $n$-th Bernoulli number. In particular they are all rational numbers.

Now the Iwasawa main conjecture is

Theorem 4.1. (Mazur-Wiles) For an odd $i$ such that $0 < i < p - 1$, we have

$$(L_{p,ana,i}) = (L_{p,alg,i}),$$

as an ideal of $\Lambda_p$. 
5 An Iwasawa conjecture for a compact hyperbolic threefold

Let $X$ be a compact hyperbolic threefold which admits an infinite cyclic covering $X_\infty$. Thus $X$ is a quotient of $\mathbb{H}^3$ by a cocompact discrete subgroup $\Gamma_g$ of $PSL_2(\mathbb{C})$.

**Definition 5.1.** For a complex number $z$, the Ruelle L-function is defined to be

$$R_\rho(z) = \prod_{\gamma} \det[1 - \rho(\gamma)e^{-zl(\gamma)}]^{-1}.$$ 

Here we have used the following conventions:

1. Closed geodesics are identified with the hyperbolic conjugacy classes of $\Gamma_g$.
2. The index $\gamma$ runs through prime closed geodesics.
3. $l(\gamma)$ is the length of $\gamma$.

**Remark 5.1.** The definition is still valid for a noncompact hyperbolic threefold of a finite volume.

$R_\rho(z)$ is absolutely convergent for $\text{Re} \, z >> 0$ and we can show that it is meromorphically continued on the whole plane.

In the following we assume:

**Assumption**

$$H^0(X_\infty, \rho) = 0.$$ 

**Remark 5.2.** By the Milnor duality, this implies

$$H^2(X_\infty, \rho) = 0.$$ 

We set

\[
\mathcal{L}_\rho(z) = A_\rho(z + 1)^{-1} \\
\quad ( = \det[(z + 1) - \tau^* | H^1(X_\infty, \rho)])).
\]
\[ h^1(\rho) = \dim H^1(X, \rho). \]

The next result is (a weak version of) a geometric analogue of the Iwasawa Main Conjecture.

**Theorem 5.1.** We have
\[ \text{ord}_{z=0} R_\rho(z) = 2h^1(\rho) \leq 2\text{ord}_{z=0} L_\rho(z), \]
and if the action of \( \tau^* \) on \( H^1(X_\infty, \rho) \) is semisimple, the identity holds. In particular if \( \tau^* \) is semisimple,
\[ (R_\rho(z)) = (L_\rho(z))^2 \]
as an ideal of \( \Lambda_\infty = \mathbb{C}[[z]] \).

Next we will compare their leading terms.

**Theorem 5.2.** Suppose \( H^i(X, \rho) = 0 \)
for each \( i \). Then we have
\[ |R_\rho(0)| = \delta_\rho |L_\rho(0)|^2. \]
Here \( \delta_\rho \) is a certain positive constant which can be computed explicitly.

If \( H^1(X, \rho) \) does not vanish, we need an additional structure on \( X \).

Suppose that \( X \) is homeomorphic to a mapping torus whose fiber is a compact Riemannian surface \( \Sigma \):
\[ X \xrightarrow{f} S^1, \quad f^{-1}(s) = \Sigma, \]
and that the surjection
\[ \pi_1(X) \xrightarrow{\pi} \mathbb{Z} \]
is induced from \( f \).

**Remark 5.3.** In this case, \( X_\infty \) is a product of \( \Sigma \) and the real axis.
Theorem 5.3. (A limit formula) Suppose that $H^0(\Sigma, \rho)$ vanishes and that the action of $\tau^*$ on $H^1(X, \rho)$ is semisimple. Then

$$\text{ord}_{z=0} R_\rho(z) = 2\text{ord}_{z=0} L_\rho(z) = 2h^1(\rho),$$

and

$$\lim_{z \to 0} |z^{-h^1(\rho)} L_\rho(z)|^2 = \lim_{z \to 0} |z^{-2h^1(\rho)} R_\rho(z)|.$$

Remark 5.4. 1. Without semisimplicity of $\tau^*$, we only have

$$2h^1(\rho) = \text{ord}_{z=0} R_\rho(z) \leq 2\text{ord}_{z=0} L_\rho(z).$$

2. There is an example of a compact hyperbolic threefold which is a mapping torus. (due to W. Thurston.)
An Iwasawa conjecture for a hyperbolic threefold of a finite volume

Let $X = \Gamma \backslash \mathbb{H}^3$ be a complete hyperbolic threefold of finite volume which has only one cusp. As before we assume that it admits an infinite cyclic covering $X_\infty$.

Let $\rho$ be a unitary character of $\Gamma$. We will treat our problem according to its behavior at the cusp.

Let $\Gamma_\infty$ be the fundamental group at the cusp and $\rho|_{\Gamma_\infty}$ the restriction.

Theorem 6.1. Let us put $h^i(\rho) = \dim H^i(X, \rho)$.

1. Suppose $\rho|_{\Gamma_\infty}$ is trivial. Then
   \[ \text{ord}_{z=0} R_\rho(z) = -2(2h^0(\rho) - h^1(\rho) + 1). \]

2. Suppose $\rho|_{\Gamma_\infty}$ is nontrivial. Then
   \[ \text{ord}_{z=0} R_\rho(z) = 2h^1(\rho). \]

Suppose there is a surjective homomorphism from $\Gamma$ to $\mathbb{Z}$ and let $X_\infty$ be the corresponding infinite covering of $X$. Moreover suppose that all of the dimensions of $H_*(X_\infty, \mathbb{C})$ and $H_*(X_\infty, \rho)$ are finite. Let $g$ be a generator of the infinite cyclic group.

Theorem 6.2. 1. Suppose that $\rho|_{\Gamma_\infty}$ is trivial and that $H^0(X, \rho) = 0$. Then
   \[ \text{ord}_{z=0} R_\rho(z) \leq 2(1 + \text{ord}_{z=0} L_\rho(z)). \]

2. Suppose $\rho|_{\Gamma_\infty}$ is nontrivial. Then
   \[ \text{ord}_{z=0} R_\rho(z) \leq 2\text{ord}_{z=0} L_\rho(z). \]

Moreover if the action of $g$ on $H^1(X_\infty, \rho)$ is semisimple, they are equal.

Theorem 6.3. Suppose that $\rho|_{\Gamma_\infty}$ is nontrivial and that $h^1(\rho)$ vanishes. Then
   \[ R_\rho(0) = \tau_X(\rho)^2, \]
where $\tau_X(\rho)$ is the Reidemeister torsion of $X$ and $\rho$. In particular this implies
   \[ |R_\rho(0)| = \delta_\rho |L_\rho(0)|^2, \]
where $\delta_\rho$ is the positive constant in Theorem 4.2.
7 A philosophy of the proof

The proof of the theorems may be compared to one of the Grothendieck’s approach to the Weil conjecture.

Let $X$ be a proper smooth variety of dimension $d$ over a finite field $\mathbb{F}_q$ and $\mathfrak{M}$ the set of its closed points. The Hasse-Weil congruent zeta function of $X$ is defined to be

$$\zeta_X(t) = \prod_{P \in \mathfrak{M}} (1 - t^{\deg(P)})^{-1}.$$  

Here $\deg(P)$ is the extension degree of the residue filed $k_P$ of $P$ over $\mathbb{F}_q$. Its logarithmic derivative is given as

$$\frac{d}{dt} \log \zeta_X(t) = \frac{1}{\zeta_X(t)} \frac{d}{dt} \zeta_X(t) = \sum_{n=1}^{\infty} |X(\mathbb{F}_q^n)| t^{n-1},$$

where $|X(\mathbb{F}_q^n)|$ is the number of $\mathbb{F}_q^n$ points of $X$.

**Theorem 7.1. (The Grothendieck-Lefschetz Trace Formula)**

Let $\phi$ be the action of $q$-th power Frobenius map on the cohomology group. Then

$$|X(\mathbb{F}_q^n)| = \sum_{i=0}^{2d} (-1)^i \text{Tr}[(\phi^n | H^i_{et}(\overline{X}, \mathbb{Q}_l)],$$

where $\overline{X}$ is the base extension of $X$ to the algebraic closed field $\overline{\mathbb{F}}_q$.

By a simple computation,

$$\frac{d}{dt} \log \prod_i \det(1 - \phi t | H^i_{et}(\overline{X}, \mathbb{Q}_l))^{(-1)i+1} = \sum_{n=0}^{\infty} t^{n-1} \sum_{i=0}^{2d} (-1)^i \text{Tr}[(\phi^n | H^i_{et}(\overline{X}, \mathbb{Q}_l)].$$

Thus we have

$$\zeta_X(t) = \prod_i \det(1 - \phi t | H^i_{et}(\overline{X}, \mathbb{Q}_l))^{(-1)i+1}.$$  

We have changed this argument as
\[
\zeta_X(t) \Rightarrow \text{The Ruelle L-function}
\]

\[
\text{Frobenius} \Rightarrow \text{the heat operator}
\]

\[
\prod_i \det(1 - \phi t \mid H^i_{et}(X, \mathbb{Q}_l))^{(-1)^{i+1}}
\Rightarrow \text{the polynomial part of} \quad \prod_i \det(1 - e^{-t\Delta})^{(-1)^{i+1}}
\div \quad \text{The Alexander invariant}
\]

The Grothendieck-Lefschetz Trace Formula ⇔ The Selberg Trace Formula.

Thus our theorem may be summerized by the following diagram:

The Ruelle L-function \quad The Selberg Trace formula ⇐⇒ \quad The Alexander invariant,

which is quite similar to the solution of the Weil conjecture:

The H-W congruent zeta-function \quad The G-L Trace formula ⇐⇒ \quad A rational function.

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8 An outline of the proof of the theorems

For simplicity we will assume the nontriviality of $\rho|_{\Gamma_{\infty}}$.

Notation
1. $\Omega^j_X(\rho)$: a vector bundle of $j$-forms on $X$ twisted by $\rho$,
2. $L^2(X, \Omega^j_X(\rho))$: the space of its square integrable sections,
3. $\Delta$: the positive Hodge Laplacian on $L^2(X, \Omega^j_X(\rho))$.

Here are some remarks.
1. The Hodge star operator induces an isomorphism of Hilbert spaces:
   \[ L^2(X, \Omega^j_X(\rho)) \cong L^2(X, \Omega^{3-j}_X(\rho)), \quad j = 0, 1, \quad (1) \]
   which commutes with $\Delta$.
2. Since $\rho|_{\Gamma_{\infty}}$ is nontrivial we know that the spectrum of $\Delta$ consists of only eigenvalues.

Selberg trace formula

\[ \text{Tr}[e^{-t\Delta} | L^2(X, \Omega^0_X(\rho))] = \mathcal{I}_0(t) + \mathcal{H}_0(t) + \mathcal{U}_0(t), \]

where $\mathcal{I}_0(t)$, $\mathcal{H}_0(t)$ and $\mathcal{U}_0(t)$ are the identity, the hyperbolic and the unipotent term, respectively.

Notation

1.
\[
\begin{align*}
\delta_0(t) &= \text{Trace}[e^{-t\Delta} | L^2(X, \Omega^0_X(\rho))] \\
\delta_1(t) &= \text{Trace}[e^{-t\Delta} | L^2(X, \Omega^1_X(\rho))] - \delta_0(t).
\end{align*}
\]

2.
\[
\begin{align*}
H_0(t) &= \mathcal{H}_0(t), \quad H_1(t) = \mathcal{H}_1(t) - H_0(t), \\
I_0(t) &= \mathcal{I}_0(t), \quad I_1(t) = \mathcal{I}_1(t) - I_0(t), \\
U_0(t) &= \mathcal{U}_0(t), \quad U_1(t) = \mathcal{U}_1(t) - U_0(t).
\end{align*}
\]
In particular
\[
\delta_0(t) = H_0(t) + I_0(t) + U_0(t), \quad \delta_1(t) = H_1(t) + I_1(t) + U_1(t).
\]

We define the derivative of the Laplace transform of a function \( f \) on \( \mathbb{R} \) to be
\[
L'(f)(z) = 2z \int_{0}^{\infty} e^{-tz} f(t) dt,
\]
if the RHS is defined.

1. **Step 1.** We will compute:
   \[
   \frac{d}{dz} \log R_\rho(z) = L'(H_1)(z) - L'(e^tH_0)(z - 1) - L'(e^tH_0)(z + 1).
   \]

2. **Step 2.** We will show:
   \[
   L'(I_1)(z) - L'(e^tI_0)(z - 1) - L'(e^tI_0)(z + 1) = 0
   \]
   and
   \[
   L'(U_1)(z) - L'(e^tU_0)(z - 1) - L'(e^tU_0)(z + 1)
   \]
   is a polynomial.

3. **Step 3.** We will show
   \[
   L'(\delta_1)(z) - L'(e^t\delta_0)(z - 1) - L'(e^t\delta_0)(z + 1)
   \]
   is a meromorphic function on the whole plane with only simple poles whose residues are all integers. Moreover the Selberg trace formula, **Step 1** and **Step 2** imply
   \[
   \text{Res}_{z=0}\left\{ \frac{d}{dz} \log R_\rho(z) \right\} = \text{ord}_{z=0}[L'(\delta_1)(z) - L'(e^t\delta_0)(z - 1) - L'(e^t\delta_0)(z + 1)].
   \]
   Thus \( R_\rho(z) \) is meromorphically continued on the whole plane.
4. **Step 4.** Using the Hodge theory, we obtain

\[
\text{Res}_{z=0}[L'(\delta_1)(z) - L'(e^t\delta_0)(z - 1) - L'(e^t\delta_0)(z + 1)] = 2h^1(\rho),
\]

which implies

\[
\text{ord}_{z=0} R_\rho(z) = 2h^1(\rho).
\]

In the course of the proof, we will also obtain

**Theorem 8.1. (The Riemann hypothesis)** The zeros and poles of \( R_\rho(z) \) is, except for finitely many of them, are located on

\[
\{s \in \mathbb{C} | \text{Re } s = -1, 0, 1\}.
\]

**Remark 8.1.** If \( \rho|_{\Gamma_\infty} \) is trivial, there are another poles or zeros which derive from the scattering term. They are corresponding to the trivial zeros of the Riemann’s zeta function.