\[ \{ \text{HEAD, TAIL} \}^N = \text{LENGTH - N SEQUENCES} = \text{N-DIM CUBE} \]

\[ f := \frac{1}{N} \text{ # "HEADS" IN THE SEQUENCE} \]

THEN \[ f = \frac{1}{2} \pm \frac{1}{\sqrt{N}} \]
\[
S^{n-1} \subset \mathbb{R}^n
\]

UNIT SPHERE

\[
f = \text{PROJECTION ON AN AXIS}
\]

THEN \[
f = 0 \pm \frac{1}{\sqrt{n}}
\]
$\mathbb{R}^N$ with Gaussian measure

$$e^{-\frac{||x||^2}{2\nu^2}}$$

$$\left(\sqrt{2\pi\nu}\right)^N$$

$$E\left[||x||^2\right] = 1$$
$X$ an $n$-dim. Riem. manifold with Ricci $\geq$ Ricci ($CS^N$)

More generally, $\exists m \in \mathbb{R}$, $f = m \pm \frac{1}{\sqrt{n}}$

More precisely,

$\mu(x, |f(x) - m| > tD) \leq e^{-\frac{t^2}{2}}$

with $D = \frac{1}{\sqrt{n}}$
SIGN OF CURVATURE

→ NEGATIVE CURVATURE
≈ HYPERBOLIC PLANE $H^n$
NEGATIVE SECTIONAL CURVATURE

DISCRETE VERSIONS:
 $\delta$-HYPERBOLIC SPACES,
CAT(0), CAT(-1), ...

→ POSITIVE CURVATURE
≈ SPHERE $S^n$
POSITIVE RICCI CURVATURE

DISCRETE VERSION: ?
SECTIONAL CURVATURE

\[ d(x', y') = d(x, y) \times (1 - \frac{\delta^2}{2} K(\nu, \omega)) \]

\[ K(\nu, \omega) = \text{SECTIONAL CURVATURE IN PLANE } \nu, \omega \]
\[ \text{Ric} (\omega) := N \int K(\nu, w) \]

\( \nu \in \text{TANGENT UNIT SPHERE} \)

"Balls are closer than their centers are" \( \iff \text{Ric} > 0 \)

\[ d(x', y') = d(x, y)_x \left( 1 - \frac{\delta^2}{2N} \text{Ric}(\nu) \right) \]

On average
$\text{Vol}(F_t) = \text{Vol}(F_0) \left(1 - \frac{t^2}{2} \text{Ric}(u) + O(t^3)\right)$

**Ricci curvature**

$= \text{contraction of volume by the geodesic flow}$
Positive Ricci Curvature

Let $M$ be an $n$-dim Riemannian manifold with

$\text{Ric}(M) \geq k \text{Ric}(S^n)$

$(k > 0)$

Then:

- $\text{diam}(M) \leq \frac{\pi}{\sqrt{k}}$

  (Bonnet-Myers)

- $\lambda_1(\Delta) \geq k^n$

  (Lichnerowicz)

$\Delta = \text{Laplace-Beltrami operator on } M$
A \ f : M \rightarrow \mathbb{R} \text{ 1-LIPSCHITZ}

\exists m \in \mathbb{R}

\mu(\{x \in M, |f(x) - m| > tD\}) < 2 \exp\left(-\frac{t^2}{2}\right)

\text{with } D = \frac{1}{\sqrt{NK}}

\text{(LEVY-GROMOV)}

\text{(CONSEQUENCE OF ISOPERIMETRIC INEQUALITY)}
POSITIVE RICCI CURVATURE 3

Let $P_t, t > 0$ be the heat semi-group on $M$.

- $\sup_t |\nabla P_t f| \leq e^{-tK(N-1)} \sup_t |\nabla f|$
- $\sup_t P_t f(\cdot) \leq e^{-tK(N-1)} P_t f(\cdot)$
- $\text{Ent}_t f \leq \frac{1}{2K(N-1)} \int_M \frac{|\nabla f|^2}{f}$

(LSI) (Bakry-Emery)

Where $\text{Ent}_f f := \int f \log \frac{f}{\int f}$
Ricci > 0

↔

Balls are closer than their centers are
DEF

$(X, d)$ POLISH METRIC SPACE. A SYSTEM OF BALLS IS A FAMILY $(m_x)_{x \in X}$ WHERE EACH $m_x$ IS A PROBABILITY MEASURE ON $X$.

\(\quad\) MARKOV CHAIN ON $X$:

\[ x_{t+1} \sim_{\text{law}} m_{x_t} \]
**DEF**

Let $x, y \in X$. The **coarse Ricci curvature** in direction $(x, y)$ is $k(x, y)$ defined by

$$C(m_x, m_y) = d(x, y)(1 - k(x, y))$$

where $C(m_x, m_y) =$ **Wasserstein transportation distance**

$$C(\mu, \nu) = \inf \int d(x, x') \, d\pi(x, x')$$

$\pi$ coupling between $\mu$ and $\nu$. 
SOME HISTORY

DOBUSHIN 1970: USE OF "VASERSHTEIN" DISTANCE FOR CONVERGENCE OF MARKOV FIELDS. LATER:
DOBUSHIN-SHLOSMAN 1984
DOBUSHIN 1994, CHEN-WANG 1997
BUBLEY-DYER 1997,
DJELLOUT-GUILLIN-WU 2004

BAKRY-ÉMERY 1984: RICCI CURVATURE FOR DIFFUSIONS. MANY WORKS INCLUDING
RENESSE-STURM 2005
STURM/LOTT-VILLANI/ÖHTA 2007

CONVERGENCE OF THESE TWO LINES

CF. ALSO JOULIN (2007), OLIVEIRA
OVERVIEW OF THEOREMS

POSITIVE $x$ IMPLIES

- Diameter bounds (Bonnet-Myers)
- Spectral gap bound (Lichnerowicz)
- Concentration of measure (Levy-Gromov)
- Gradient norm contraction
- Modified LSI (Bakry-Emery)
- Convergence of empirical mean / MCMC (New?)
  (Joulin + Yo)

With essentially sharp constants,
+ Gromov-Hausdorff CV
+ Non-negative $x \Rightarrow$
  Exponential concentration
  (with some assumptions)
CHOICE OF $m_x$

**EXAMPLE**

$(X, d, \mu)$ METRIC MEASURE SPACE.

TAKE

$\mu$-L $B(x, a)$

$\mu(B(x, a))$

$\mu_x := \frac{\mu(\mu, a)}{\mu(B(x, a))}$

$\sim$ RICCI CURVATURE "AT SCALE $\alpha$"

MANIFOLD: $\alpha \to 0$

GRAPH: $\alpha = 1$
$$\mathcal{C}(m_x, m_y) = d(x, y)$$

$$\Rightarrow \mathbf{x} = \mathbf{0}$$

By translation.

Same for $\mathbb{R}^n$ with any norm.
DISCRETE CUBE

\[ C(m_x, m_y) = \frac{N-1}{N+1} \times 1 + \frac{2}{N+1} \times 0 \]

\[ = 1 - \frac{2}{N+1} \]

\[ \kappa = \frac{2}{N+1} \]
HOW TO CHECK $k > 0$?

**Exercise**

If $(X, d)$ is $\epsilon$-geodesic, then $x(x, y) > k_0$ for $d(x, y) \leq \epsilon$ implies $x(x, y) > k_0$ for all $x, y$.

$\epsilon$-geodesic: for all $x, x'$

$\exists x = x_0, x_1, \ldots, x_k = x'$ with

$d(x_i, x_{i+1}) \leq \epsilon$ and

$d(x, x') = \Sigma d(x_i, x_{i+1})$

Graph: $1$-geodesic

Manifold: $\epsilon$-geodesic $\forall \epsilon > 0$
FURTHER EXAMPLES: TREES

\[ C(m_x, m_y) > d(x, y) \]
\[ \Rightarrow k < 0 \]

SAME FOR $\delta$-HYPERBOLIC SPACES
INFLUENCE OF METRIC

LENGTH-N CYCLE WITH
GRAPH METRIC

LENGTH-N CYCLE WITH
INDUCED $\mathbb{R}^2$ METRIC

$K = 0$

$K > 0$

$\Rightarrow$ EXTRINSIC CURVATURE
INFLUENCE OF m\_x

SIMPLE RANDOM WALK

\[ K < 0 \]

\sim \text{TREE}

TWO-STEP RANDOM WALK

\[ X = 0 \]
INFLUENCE OF MEASURE

GAUSSIAN MEASURE ON $\mathbb{R}^n$

= INVARIANT MEASURE OF THE ORNSTEIN-UHLENBECK PROCESS

$$dY_t = -\frac{1}{2} Y_t \, dt + dB_t$$

FOR $m_T$: TAKE A RANDOM WALK WHICH APPROXIMATES THIS SDE (AS IN A NUMERICAL SIMULATION)
\[ C(m_x, m_y) < d(x, y) \]

\[ \Rightarrow k > 0 \]
SDE's on Manifolds

Consider the SDE
\[ dX_t = F dt + dB_t \]
with generator
\[ L = F \cdot D + \frac{1}{2} \Delta \]
on a manifold.

\( \mathbb{m}_x \): Discrete approx. of SDE

\( \sim \) Bakry-Émery Tensor
FURTHER EXAMPLE: ISING MODEL

G F INITE G R A P H
C O N F IG U R A T I O N S P A C E $X = \{ +, - \}^G$

E N E R G Y $E(\sigma) = -\beta \sum_{g \sim g'} \sigma_g \sigma_{g'}$

G I B B S M E A S U R E $\mu(\sigma) \propto e^{-E(\sigma)}$


$\sigma \rightarrow \begin{cases} 
\sigma_{x^+} \text{ with proba } \frac{\mu(\sigma_{x^+})}{\mu(\sigma_{x^+}) + \mu(\sigma_{x^-})} \\
\sigma_{x^-} \text{ with proba } \frac{\mu(\sigma_{x^-})}{\mu(\sigma_{x^+}) + \mu(\sigma_{x^-})}
\end{cases}$

F O R $x \in G$ C H O S E N A T R A N D O M.
\[\sum_{\mathclap{\mathbb{G}}}^{1} \left( 1 - \deg_{\mathbb{G}} \theta (2\beta) \right) \]

**Prop**

In particular, \( k > 0 \) for \( \beta < \ldots \)

\( \rightarrow \) easily recover known results about Ising model and its variants.
MARKOV CHAINS

$m_x \sim \text{MARKOV \ CHAIN \ WITH}$

TRANSITION KERNEL AT TIME $t$

$m^t_x(dy) = \int m^{t-1}_x(dz) m^z_x(dy)$

WITH $m^1_x := m_x$

NOTATION: $\mu$ MEASURE ON $X$

$\mu * m(dy) := \int \mu(dz) m^z_x(dy)$

"START WITH $\mu$ AND MAKE ONE STEP"
PROO: \[ \text{IF } x(x, y) > x \ \forall x, y \ \text{ THEN} \]

\[ C(\mu_1 \ast m, \mu_2 \ast m) \leq (1 - \kappa) C(\mu_1, \mu_2) \]

COR: \[ \exists ! \text{ INARIANT MEASURE } \]

COR: \[ C(\mu_x \ast t, t) \leq (1 - \kappa)^t \]

\[ \text{Diam } X \]

\[ \Rightarrow \text{ MIXING TIME ESTIMATES FOR EX. } N \log(N) \text{ FOR THE CUBE} \]
**DEF:** AVERAGING OPERATOR

\[(M_f)(x) := \int f(y) \, m_x \, (dy)\]

**PROP:** \[\lambda (x, y) \geq x \quad \forall (x, y)\]

IFF

\[\|Mf\|_{\text{Lip}} \leq (1-x) \|f\|_{\text{Lip}}\]

**COR:** LET \[\Delta := M - \text{Id}\].

ASSUME THE RANDOM WALK DEFINED BY \[(m_x)\] IS REVERSIBLE

THEN

\[\lambda_1(\Delta) \geq x\]

(cf. Lichnerowicz)
**CONCENTRATION THEOREM**

**THM** (007)

**SUPPOSE** \( \kappa(x,y) \geq \kappa > 0 \).

**THEN** \( \forall f: X \to \mathbb{R} \) 1-LIPSCHITZ

\[
\nu(x, |f(x) - \mathbb{E}_x f| > t) \leq \exp - \frac{t^2}{2D}
\]

**FOR** \( 0 \leq t \leq t_{\text{max}} \).

**\( \nu \)**: INARIANT MEASURE

**\( t_{\text{max}} \)**: EXPLICIT, DEPENDS ON \( m \).

(USUALLY LARGE)

\[
D^2 = \mathbb{E}_x D_x^2
\]

**WITH**

\[
D_x = \frac{\sigma_x}{\sqrt{m_x \kappa}}
\]

**LOCAL DIMENSION**
\[ \sigma_x = \sqrt{\frac{1}{2} \int d(y, y')^2 \, m_x(dy) m_x(dy')} \]

~ AVERAGE SQUARE DISTANCE BETWEEN POINTS OF \( m_x \)

**Examples**

- \( X = \text{MANIFOLD}, \quad m_x = \epsilon - \text{BALL} \)
  \[ \Rightarrow \quad \sigma_x \leq \epsilon \]

- \( X = \text{GRAPH}, \quad m_x = \text{SIMPLE RANDOM WALK} \)
  \[ \Rightarrow \quad \sigma_x \leq 1 \]
COARSE DIMENSION

\((X, d, \mu)\) METRIC MEASURE SPACE.

DEFINE

\[
\text{CoarseDim} (X, d, \mu) = \frac{\int \int d(y, y')^2 \, d\mu(dy) \, d\mu(dy')} \sup \int \int (f(y) - f(y'))^2 \, d\mu(dy) \, d\mu(dy')
\]

\(f : X \to \mathbb{R}\) 1-LIPSCHITZ

LOCAL DIMENSION

\(n_x = \text{CoarseDim} (X, d, m_x)\)

EXAMPLES

- N-DIM MANIFOLD: \(n_x \approx N\)
- GRAPH: \(n_x \approx 1\)
BACK TO CONCENTRATION

- **N-DIM MANIFOLD WITH**
  \[ \text{Ric} \gg K \text{Ric (Sphere)} \]
  \[ \Rightarrow x = K \varepsilon^2 \]
  \[ \sigma_x = \varepsilon \]
  \[ m_x = N \]
  \[ D = \frac{\sigma_x}{\sqrt{m_x x}} = \frac{1}{\sqrt{NK}} \]
  AND \[ t_{\text{max}} \to \infty \]

- **N-DIM CUBE (WITH DIAM = 1)**
  \[ \Rightarrow x = \frac{1}{N} \]
  \[ \sigma_x = \frac{1}{N} \]
  \[ m_x = 1 \]
  \[ D = \frac{\sigma_x}{\sqrt{m_x x}} = \frac{1}{\sqrt{N}} \]
  AND \[ t_{\text{max}} \gg 1 \]

- **BINOMIAL/POISSON: OK, CORRECT VARIANCE AND** \[ t_{\text{max}} \]

- \[ t \gg t_{\text{max}}: \text{EXPONENTIAL CONCENTRATION} \]
LOG-SOB INEQUALITY

NEED A COARSE GRADIENT.

CHOOSE $\lambda > 0$. ("INVERSE SCALE")

DEFINE

$$
\nabla_{\lambda} f(x) = \sup_{y, y'} \frac{f(y) - f(y')}{d(y, y')} \cdot C(x, y, y')
$$

WITH

$$
C(x, y, y') = e^{-\lambda d(x, y)} - e^{-\lambda d(y, y')}
$$

- $\lambda = 0$: LIPSCHITZ NORM
- $\lambda \to \infty$: NORM OF GRADIENT

DESIGNED SO THAT THE HERBST ARGUMENT FOR $L^1 \Rightarrow$ CONCENTRATION STOPS AT $\lambda$. 
THM (007)

Suppose \( \kappa(x, y) \geq \kappa > 0 \).

Take \( \lambda < \lambda_{\text{max}} \).

Then \( \forall f: X \to \mathbb{R} \)

\[
\nabla_{\lambda} M f(x) \leq (1 - \frac{\kappa}{2}) M \nabla_{\lambda} f(x)
\]

Where \( M = \text{averaging operator} \).

Moreover for \( f > 0 \)

\[
\text{Ent } f \leq 4(\sup \frac{\sigma^2}{\mu x \kappa}) \int \frac{(\nabla_{\lambda} f)^2}{f}
\]

W.R.T. invariant measure

- **Manifold**: \( \lambda_{\text{max}} \to \infty \)
- **Cube**: \( \lambda_{\text{max}} = \frac{1}{\text{edge length}} \)
MCMC/CONVERGENCE OF EMPIRICAL MEANS

GOAL: ESTIMATE $E_{\mathcal{D}} f$ WITHOUT SAMPLING FROM $\mathcal{D}$.

EMPIRICAL MEAN
$$\hat{f}_T(x_0) = \frac{1}{T} \sum_{i=0}^{T-1} f(x_i)$$

WHERE $x_{i+1} \sim m x_i$

ERROR $\sim$ BIAS + VARIANCE

TO REDUCE BIAS, REPLACE WITH
$$\hat{f}_{T_0, T} = \frac{1}{T} \sum_{i=T_0}^{T-1} f(x_i)$$
BIAS ESTIMATION

Propose: Assume $x > 0$. Then

$$\left| E_{x_0, t} \hat{\theta}_{t_0, t} - E_{x_0} \hat{\theta} \right| \leq \frac{(1-x)^{t_0}}{T_x} E(x_0)$$

where $E$ is the eccentricity

$$E(x_0) := C(\delta_{x_0}, \nu)$$

- $E(x_0) \leq \text{Diam}$
- $E(x_0) \leq E(\theta) + d(\theta, x_0)$
- $E(x_0) \leq \frac{1}{\chi} \int d(x_0, x_1) m_{x_0}(dX_1)$
VARIANCE

**THM** (Joulin - 008). Suppose $\kappa > 0$.

\[
\text{Var} \begin{array}{c}
\hat{f}_{T_0, T} \\
\sup \frac{\sigma_x^2}{m_x \kappa}
\end{array} \leq \frac{8}{\kappa T} + \frac{\sigma_x^2}{m_x \kappa}
\]

+ GAUSSIAN/EXPONENTIAL CONCENTRATION

FOR ANY 1-LIPSCHITZ FUNCTION $f$.

REMEMBER, $\frac{\sigma_x^2}{m_x \kappa} \rightarrow$ VARIANCE OF THE INVARIANT DISTRIBUTION.
COROLLARY

LET $X_t$ BE THE BROWNIAN MOTION ON A RIEMANNIAN MANIFOLD WITH $\text{Ric} \geq K$.
LET $f$ BE A 1-LIPSCHITZ FUNCTION AND LET

$$\hat{f}_T(x_0) := \frac{1}{T} \int_0^T f(x_t) \, dt$$

THEN

$$\Pr_n \left( \hat{f}_T(x_0) - \mathbb{E} \hat{f}_T(x_0) \geq n \right) \leq \exp \left( - \frac{K^2 T n^2}{32} \right)$$

CF. GUILLIN-LEONARD, WU-YAO
OPEN PROBLEMS

- Link with Sturm-Lott-Villami?
- Alexandrov spaces $\Rightarrow x > 0$?
- Nilpotent groups?
- Spectral gap in non-reversible case?
- This is a discrete $\text{CD}(k, \infty)$.
- Define a discrete $\text{CD}(k, N)$.
- Discrete Ricci flow?
- Discrete sectional/scalar curvature?
- Expanders $\Rightarrow x < 0$?
- Explore the Finsler case.

...