Universal mixed elliptic motive and derivation algebra of the fundamental group of one-punctured elliptic curve

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Let $G_{\mathbb{Q}}$ denote the absolute Galois group over $\mathbb{Q}$. Let $X$ be a three-point punctured projective line over $\mathbb{Q}$, and choose a tangential base point $\vec{01}$. Then we have a Galois representation on the $\mathbb{Q}_\ell$-prounipotent fundamental group

$$G_{\mathbb{Q}} \to \operatorname{Aut} \pi_1^{\mathrm{un}}(\overline{X}, \vec{01})/\mathbb{Q}_\ell.$$

We enumerate properties of (the Lie-algebrization of) this representation: (1) it has the weight filtration, (2) its weight graded quotients are Tate, (3) unramified outside $\ell$ and crystalline at $\ell$. Conversely, if we consider the category of finite dimensional $\mathbb{Q}_\ell$-linear Galois representation with these properties, then we have a Tannakian category whose Tannakian fundamental group is an extension of $G_m$ by a free prounipotent group generated by Soulé's elements. This coincides with that of the category of mixed Tate motives by Deligne-Goncharov, after extension of scaler to $\mathbb{Q}_\ell$.

One motivation of our research is to know what happens if we replace the family $X \to \operatorname{Spec} \mathbb{Q} = \mathcal{M}_{0,3}$ with the universal family of elliptic curves $\mathcal{E} \to \mathcal{M}_{1,1}$. We enumerate several properties of the corresponding monodromy representation

$$\pi_1(\mathcal{M}_{1,1}) \to \operatorname{Aut} \pi_1^{\mathrm{un}}(E_{\vec{01}}, \vec{01})/\mathbb{Q}_\ell.$$

Then the Tannakian fundamental group of the category of the representations $\pi_1(\mathcal{M}_{1,1})$ with these properties has a generating set consisting of Soulé's elements together with geometric generators corresponding to the Eisenstein series, and possible relations arising from the cusp forms. These possible relations turn out to be actual relations in the derivation algebra in the right hand side (due to Aaron Pollack's result).