Convex geometric Demazure operators

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Convex polytopes in algebraic geometry and in representation theory

0. Toric geometry
*Newton (or moment) polytopes*

1. Representation theory
*Gelfand–Zetlin polytopes* and *string polytopes* (Berenstein–Zelevinsky, Littelmann, 1998)

2. Algebraic geometry
*Newton–Okounkov convex bodies* (Kaveh–Khovanskii, Lazarsfeld–Mustata, 2009)

1 & 2. Schubert calculus
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Flag varieties

Definition
The flag variety $X$ is the variety of complete flags in $\mathbb{C}^n$:

$$X = \{\{0\} = V^0 \subset V^1 \subset \ldots \subset V^{n-1} \subset V^n = \mathbb{C}^n | \dim V^i = i\}$$

Remark
Alternatively, $X = GL_n(\mathbb{C})/B$, where $B$ denotes the group of upper-triangular matrices (Borel subgroup). In this form, the definition can be extended to arbitrary connected reductive groups.

Dimension

$$\dim X = \frac{n(n-1)}{2}$$
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Flag varieties and representation theory

Definition
A collection of integers $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ is a strictly dominant weight of the group $GL_n(\mathbb{C})$ if $\lambda_i < \lambda_{i+1}$ for all $i = 1, \ldots, n - 1$.

Fact
very ample line bundles on $X$ $\longleftrightarrow$ irreducible representations of $GL_n(\mathbb{C})$ with strictly dominant weights.

Construction

- $V_\lambda$ — the irreducible $GL_n$-module with the highest weight $\lambda$ $\implies$ $X \hookrightarrow \mathbb{P}(V_\lambda)$, $g \mapsto gv_\lambda$ — embedding;
- $\mathcal{L}$ — very ample line bundle $\implies H^0(X, \mathcal{L})^* = V_\lambda$
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Gelfand–Zetlin polytope
For each strictly dominant weight $\lambda$, define a convex polytope $P_\lambda \subset \mathbb{R}^d$ (where $d = n(n - 1)/2$) with integer vertices.

Origins
Gelfand and Zetlin constructed a natural basis in $V_\lambda$. The basis elements are parameterized by the integer points inside and at the boundary of $P_\lambda$.

Dimension
$$\dim P_\lambda = d = \dim X$$
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Gelfand–Zetlin polytopes

The Gelfand–Zetlin polytope $P_\lambda$ is defined by inequalities:

$$
\begin{array}{ccccccc}
\lambda_1 & \lambda_2 & \lambda_3 & \cdots & \lambda_n \\
 x_1^1 & x_2^1 & \cdots & \cdots & x_{n-1}^1 \\
 x_1^2 & \cdots & \cdots & \cdots & x_{n-2}^2 \\
 \vdots & \cdots & \cdots & \cdots & \ddots \\
 x_1^{n-2} & \cdots & \cdots & \cdots & x_2^{n-2} \\
 x_1^{n-1} & & & & & x_1^1 \\
\end{array}
$$

where $(x_1^1, \ldots, x_{n-1}^1; \ldots; x_1^{n-1})$ are coordinates in $\mathbb{R}^d$, and the notation

$$
\begin{array}{ccc}
a & b \\
& c \\
\end{array}
$$

means $a \leq c \leq b$. 
Gelfand–Zetlin polytopes

A Gelfand–Zetlin polytope for $GL_3$:

$$
\begin{bmatrix}
-1 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z \\
\end{bmatrix}
$$
Flag varieties and Gelfand–Zetlin polytopes

Goal

Use combinatorics of $P_\lambda$ to study geometry of $X$.

Results

• Relation between Schubert varieties and preimages of rc-faces of $P_\lambda$ under the Guillemin–Sternberg moment map $X \to P_\lambda$ (Kogan, 2000)

• Degenerations of Schubert varieties to (reducible) toric varieties given by (unions of) faces of $P_\lambda$ (Kogan–Miller, Knutson–Miller, 2003)

• Description of $H^*(X, \mathbb{Z})$ using volume polynomial of $P_\lambda$ (Kaveh, 2003)

• Schubert calculus: intersection product of Schubert cycles in $H^*(X, \mathbb{Z}) = \text{intersection of faces in } P_\lambda$ (K.—Smirnov–Timorin, 2011)
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Schubert calculus and Gelfand–Zetlin polytopes

**Definition**
For each permutation $w \in S_n$, the *Schubert variety* $X_w \subset X$ is

$$X_w = \overline{B w B},$$

where $w$ acts on the standard basis vectors $e_i$ by the formula $e_i \mapsto e_{w(i)}$.

**Dimension**
$$\dim X_w = \ell(w)$$

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The *Schubert cycle* $[X_w]$ is the class of $X_w$ in $H^*(X, \mathbb{Z})$. 
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\[
\begin{align*}
[X_{s_1 s_2 s_1}] & = \\
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\end{align*}
\]
Schubert calculus and Gelfand–Zetlin polytopes

\[
[J_{s_1 s_2 s_1}] = \\
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\]

\[
= +
\]
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\[ [X_{s_1 s_2 s_1}] = \]

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\[ = + \]

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\[ = + \]

\[ = + \]
Schubert calculus and Gelfand–Zetlin polytopes

\[
[X_{s_1}] = \cdots = [X_{s_2}] = \cdots \\
= [X_{s_1}] + [X_{s_2}]
\]
Schubert calculus and Gelfand–Zetlin polytopes

\[ [X_{s_1}] = \quad = \quad [X_{s_2}] = \quad = \quad \]

\[ [X_{s_2 s_1}]^2 = \quad \cdot \quad (\quad + \quad ) = \quad = \quad [X_{s_1}] \]
Schubert calculus and Gelfand–Zetlin polytopes

$GL_n$

Any two Schubert cycles $[X_w]$ and $[X_{w'}]$ can be represented as sums of faces so that every face appearing in the decomposition of $[X_w]$ is transverse to every face appearing in the decomposition of $[X_{w'}]$\(^1\).

**Corollary**

Intersection of any two Schubert cycles can be represented by linear combinations of faces with nonnegative coefficients.

**Question**

Why intersecting faces is better than multiplying Schubert polynomials?

\(^1\)see ArXiv:1101.0278v1 [Math.AG] for precise formulas
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Answer
Faces are more “positive” than monomials: all computations with faces are cancelation free.

Example for $GL_3$
Compute $[X_{s_1}] \cdot [X_{s_2 s_1}]$ in two ways: via Schubert polynomials and via faces.

\[ x_1 x_2 (x_1 + x_2) = x_1^2 x_2 - x_1 x_2^2 = 1 - 1 = 0 \text{ (cancelation)} \]
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Arbitrary reductive groups
How to relate Schubert cycles to (unions of) faces of polytopes?

Main tool
A convex geometric incarnation of divided difference operators.
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How to relate Schubert cycles to (unions of) faces of polytopes?

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Tool: divided difference operators

Definition (for $GL_n$)

Divided difference operator $\delta_i$ (for $i = 1, \ldots, n - 1$) acts on $\mathbb{Z}[x_1, \ldots, x_n]$ as follows:

$$
\delta_i : f \mapsto \frac{f - s_i(f)}{x_i - x_{i+1}}.
$$

Example

$$
\delta_1(x_1^2) = \frac{x_1^2 - x_2^2}{x_1 - x_2} = x_1 + x_2.
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$$\delta_1(x_1^2) = \frac{x_1^2 - x_2^2}{x_1 - x_2} = x_1 + x_2$$
Theorem (Bernstein–Gelfand–Gelfand, Demazure, 1971) 

Let \( w = s_{i_1} \ldots s_{i_\ell} \) be a reduced representation. In the Borel presentation, 

\[
[X_w] = \delta_{i_\ell} \ldots \delta_{i_1} [X_{id}],
\]

where \([X_{id}]\) is the class of a point.

Remark

For \( GL_n \),

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[X_{id}] = x_1^{n-1} x_2^{n-2} \ldots x_{n-1}. 
\]

For other reductive groups, there is sometimes no denominator-free formula.
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Topological meaning of divided difference operators

Gysin morphism

Let $P_i$ be the minimal parabolic subgroup, and $p_i : G/B \to G/P_i$ the natural projection. Then the action of $\delta_i$ on $H^*(G/B, \mathbb{Z})$ coincides with the action of $p_i^* \circ p_{i*}$:

$$\delta_i : H^*(G/B, \mathbb{Z}) \xrightarrow{p_i^*} H^*(G/P_i, \mathbb{Z}) \xrightarrow{p_i^*} H^*(G/B, \mathbb{Z}).$$

Example

If $G = GL_n$, then $G/P_i$ is obtained by forgetting the $i$-th space in a flag.
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Generalizations of divided difference operators

Generalized cohomology theories

Let $A^*$ be a generalized oriented cohomology theory. Define the \emph{generalized divided difference operator} $\delta_i^A$ as the composition

$$
\delta_i^A : A^*(G/B, \mathbb{Z}) \xrightarrow{p_i^A} A^*(G/P_i, \mathbb{Z}) \xrightarrow{p_i^{*A}} A^*(G/B, \mathbb{Z}).
$$

Examples

- classical cohomology $H^*$ or Chow ring $CH^*$
- $K$-theory $K^*_0$
- complex cobordism $MU^*$ or algebraic cobordism $\Omega^*$
Generalizations of divided difference operators

Generalized cohomology theories
Let $A^*$ be a generalized oriented cohomology theory. Define the \textit{generalized divided difference operator} $\delta_i^A$ as the composition

$$
\delta_i^A : A^*(G/B, \mathbb{Z}) \xrightarrow{p_i^A} A^*(G/P_i, \mathbb{Z}) \xrightarrow{p_i^{*,A}} A^*(G/B, \mathbb{Z}).
$$

Examples

- classical cohomology $H^*$ or Chow ring $CH^*$
- $K$-theory $K_0^*$
- complex cobordism $MU^*$ or algebraic cobordism $\Omega^*$
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Question
Is there an algebraic formula for $\delta_i^A$?

Formal group law
There exists a formal power series $F_A(x, y) = x + y + \ldots$ with coefficients in $A^0$ such that

$$F(c_1^A(L), c_1^A(M)) = c_1^A(L \otimes M)$$

in $A^*(X)$ for any pair of line bundles $L$ and $M$ on a variety $X$.

Examples

$CH^*$ $F(x, y) = x + y$

$K_0^*$ $F(x, y) = x + y - xy$

$\Omega^*$ $F(x, y) = x + y - [\mathbb{P}^1]xy + ([\mathbb{P}^1]^2 - [\mathbb{P}^2])x^2y + \ldots$

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\[ \delta_i^A = (1 + s_i) \frac{1}{x_i - A x_{i+1}} \]

Applications

Formulas for “Schubert cycles” in the “Borel presentation” for \( A^*(G/B) \). Algorithms for multiplying “Schubert cycles”.

- \( H^* \) Bernstein–Gelfand–Gelfand, Demazure, 1973
- \( K_0^* \) Demazure, 1974
- \( MU^* \) Bressler–Evens, 1992
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Demazure operators

Notation
Let $G$ be a connected reductive group of semisimple rank $r$, $\Lambda_G$ the weight lattice of $G$, and $\mathbb{Z}[\Lambda_G]$ the group ring. Simple roots of $G$ are denoted by $\alpha_1, \ldots, \alpha_r$.

Remark
Elements of $\Lambda_G$ are written in the form

$$\sum_{\mu \in \Lambda_G} m(\mu)e^\mu.$$

Definition
Demazure operator $D_i$ (for $i = 1, \ldots, n$) acts on $\mathbb{Z}[\Lambda_G]$ as follows:

$$D_i : f \mapsto \frac{f - e^{\alpha_i}s_i(f)}{1 - e^{\alpha_i}}.$$
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Example for $GL_n$

$$D_1(e^{\alpha_2}) = \frac{e^{\alpha_2} - e^{\alpha_1}e^{\alpha_1+\alpha_2}}{1 - e^{\alpha_1}} = e^{\alpha_2} + e^{\alpha_1+\alpha_2}$$

Exercise

Define $(\lambda, \alpha_i)$ by the identity $s_i(\lambda) = \lambda - (\lambda, \alpha_i)\alpha_i$.

$$\begin{cases} 
D_i(e^\lambda) = e^\lambda(1 + e^{\alpha_i} + \ldots + e^{-(\lambda,\alpha_i)\alpha_i}), & (\lambda, \alpha_i) \leq 0 \\
D_i(e^\lambda) = 0, & (\lambda, \alpha_i) = 1 \\
D_i(e^\lambda) = -e^\lambda(1 + e^{-\alpha_i} + \ldots + e^{-(\lambda,\alpha_i)-2)\alpha_i}), & (\lambda, \alpha_i) > 1 
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Remark
For $G = GL_n$, put $x_i := 1 + e^{\chi_i}$ where the character $\chi_i$ is given by the $i$-th entry of the diagonal torus. Then

$$D_i = -\delta_i^K,$$

that is, the Demazure operator is equal up to a sign to the $K$-theory divided difference operator (= isobaric divided difference operator).
Demazure characters

Definition
Demazure $B$-module $V_{\lambda,w} := H^0(X_w, \mathcal{L}_\lambda|_{X_w})^*$ is the dual space to the space of global sections of the line bundle $\mathcal{L}_\lambda$ on $G/B$ (corresponding to $V_\lambda$) restricted to $X$.

Definition
Demazure character $\chi_w(\lambda)$ of $V_{\lambda,w}$ is the sum over all basis weight vectors of the exponentials of the corresponding weights:

$$
\chi_w(\lambda) := \sum_{\mu \in \Lambda} m_{\lambda,w}(\mu) e^{\mu}
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Demazure character formula [Andersen, 1985, ...]
Let $w = s_{i_1} \ldots s_{i_\ell}$ be a reduced representation. Then

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- \( V_{\lambda, id} = \mathbb{C}_\lambda, \chi_{id}(\lambda) = e^\lambda \)
- \( V_{\lambda, w_0} = V_\lambda, \chi_{w_0}(\lambda) — \text{Weyl character} \)

Remark

For \( GL_n \), the definition of Gelfand–Zetlin polytopes implies that

\[
\chi_{w_0}(\lambda) = \sum_{x \in P_\lambda \cap \mathbb{Z}^d} e^{p(x)},
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where \( p(x) := (\sum_{j=1}^{n-1} x_j^1)\alpha_1 + (\sum_{j=1}^{n-2} x_j^1)\alpha_2 + \ldots + x_1^{n-1}\alpha_{n-1} \) is the weight of \( x \).
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Demazure characters and Gelfand–Zetlin polytopes

A Gelfand–Zetlin polytope for $GL_3$:

$$
\begin{pmatrix}
-1 & 0 & 1 \\
 0 & x & y \\
 0 & 0 & z
\end{pmatrix}
$$

$$
p(x, y, z) = (x + y, z)
$$
Demazure characters and Gelfand–Zetlin polytopes

The weight polytope (\(=\)image of \(P_\lambda\) under the projection \(p\)): 
Convex geometric Demazure operators

Goal

Define operators $D_1, \ldots, D_r$ on convex polytopes in $\mathbb{R}^d$ and a weight map $\mathbb{R}^d \to \mathbb{R}^r$ such that for any reduced decomposition $w = s_{i_1} \ldots s_{i_\ell}$ the sequence of polytopes

$$pt(\lambda) \xrightarrow{D_{i_1}} P_1(\lambda) \xrightarrow{D_{i_2}} P_2(\lambda) \xrightarrow{D_{i_3}} \ldots \xrightarrow{D_{i_\ell}} P_\ell(\lambda)$$

yields the sequence of the Demazure characters

$$e^\lambda \xrightarrow{D_{i_1}} \chi_{s_{i_1}}(\lambda) \xrightarrow{D_{i_2}} \chi_{s_{i_1}s_{i_2}}(\lambda) \xrightarrow{D_{i_3}} \ldots \xrightarrow{D_{i_\ell}} \chi_w(\lambda)$$

that is,

$$\chi_w(\lambda) = \sum_{x \in P_\ell(\lambda) \cap \mathbb{Z}^d} e^{p(x)}.$$
Definition
A *root space* of rank $r$ is a real vector space $\mathbb{R}^d$ together with a direct sum decomposition

$$\mathbb{R}^d = \mathbb{R}^{d_1} \oplus \ldots \oplus \mathbb{R}^{d_r}$$

and a collection of linear functions $l_1, \ldots, l_r \in (\mathbb{R}^d)^{\ast}$ such that $l_i$ vanishes on $\mathbb{R}^{d_i}$.

Definition
A convex polytope $P \subset \mathbb{R}^d$ is called a *parapolytope* if for all $i = 1, \ldots, r$, and any vector $c \in \mathbb{R}^d$ the intersection of $P$ with the parallel translate $c + \mathbb{R}^{d_i}$ of $\mathbb{R}^{d_i}$ is a coordinate parallelepiped.
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Convex geometric Demazure operators

Notation
Coordinates in $\mathbb{R}^d$: $(x_1^1, \ldots, x_{d_1}^1; \ldots; x_1^n, \ldots, x_{d_n}^n)$.

Definition
A coordinate parallelepiped in $\mathbb{R}^{d_i}$ is

$$\Pi(\mu, \nu) = \{(x_1^i, \ldots, x_{d_i}^i) \in \mathbb{R}^{d_i} | \mu_j \leq x_j^i \leq \nu_j, j = 1, \ldots, d_i\}.$$
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Example

\[ d = \frac{n(n-1)}{2}, \quad r = (n-1) \]

\[ \mathbb{R}^d = \mathbb{R}^{n-1} \oplus \mathbb{R}^{n-2} \oplus \ldots \oplus \mathbb{R}^1 \]

Exercise

The Gelfand–Zetlin polytope \( P_\lambda \) is a parapolytope.

\[
\begin{array}{cccccc}
\lambda_1 & \lambda_2 & \lambda_3 & \ldots & \lambda_{n-1} & \lambda_n \\
x_1^1 & x_2^1 & x_3^1 & \ldots & x_{n-1}^1 \\
x_1^2 & x_2^2 & \ldots & \ldots & x_{n-1}^2 \\
& \ldots & \ldots & \ldots & \ldots \\
x_1^{n-2} & x_2^{n-2} & \ldots & \ldots & \ldots \\
x_1^{n-1} & \end{array}
\]
Convex geometric Demazure operators

$$\mathbb{R}^3 = \mathbb{R}^2 \oplus \mathbb{R}$$

The slices \( \{ z = \frac{1}{2} \} \) and \( \{ z = -\frac{1}{2} \} \) are coordinate rectangles.
Convex geometric Demazure operators

Definition

1. $P = \prod(\mu, \nu) \subset (c + \mathbb{R}^{d_i})$ — coordinate parallelepiped
   Choose the smallest $j = 1, \ldots, d_i$ such that $\mu_j = \nu_j$. Put
   $$D_i(P) := \prod(\mu, \nu'),$$
   where $\nu'_k = \nu_k$ for all $k \neq j$ and $\nu'_j$ is defined by the equality
   $$\sum_{k=1}^{d_i} (\mu_k + \nu'_k) = l_i(c).$$

2. $P$ — any parapolytope
   $$D_i(P) = \bigcup_{c \in \mathbb{R}^d} \{D_i(P \cap (c + \mathbb{R}^{d_i}))\}$$
Convex geometric Demazure operators

Definition

1. \( P = \Pi(\mu, \nu) \subset (c + \mathbb{R}^{d_i}) \) — coordinate parallelepiped
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Examples

\[ \mathbb{R}^3 = \mathbb{R}^2 \oplus \mathbb{R}, \quad l_1(x, y, z) = z, \quad l_2(x, y, z) = x + y \]

\[ P = \{(a, b, c)\} \quad \text{— a point} \]

\[ D_1(P) = [(a, b, c), (a', b, c)] \]

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$GL_n$ root space

$$\mathbb{R}^d = \mathbb{R}^{n-1} \oplus \mathbb{R}^{n-2} \oplus \ldots \oplus \mathbb{R}^1$$

Functions $l_i$

$$l_i(x) = \sigma_{i-1}(x) + \sigma_{i+1}(x),$$

where $\sigma_i(x) = \sum_{j=1}^{d_i} x_j^i$ (=sum of coordinates in the $i$-th row) for $i = 1, \ldots, n-1$ and $\sigma_0 = \sigma_n = 0$
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Proposition
The Gelfand–Zetlin polytope $P_\lambda$ coincides with

$$[(D_1 \ldots D_{n-1})(D_1 \ldots D_{n-2}) \ldots (D_1)](p),$$

where $p \in \mathbb{R}^d$ is the point $(\lambda_1, \ldots, \lambda_{n-1}; \lambda_1, \ldots, \lambda_{n-2}; \ldots; \lambda_1)$.
Convex geometric Demazure operators

$\overline{w}_0$
Fix a reduced decomposition $w_0 = s_{i_1} \ldots s_{i_d}$ of the longest element in the Weyl group of $G$.

$(G, \overline{w}_0)$ root space

$$R^d = R^{d_1} \oplus \ldots \oplus R^{d_r},$$
where $d_i$ is the number of $s_{i_j}$ in $\overline{w}_0$ such that $i_j = i$.

$$l_i(x) = \sum_{k \neq i} (\alpha_k, \alpha_i) \sigma_k(x).$$

Example
For $G = GL_n$ and $w_0 = (s_1 \ldots s_{n-1})(s_1 \ldots s_{n-2}) \ldots (s_1)$, we get $GL_n$ root space.
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Theorem
For each dominant weight $\lambda$ of $G$, there exists a point $p_\lambda \in \mathbb{R}^d$ such that the polytope

$$P := D_{i_1} \ldots D_{i_d}(p_\lambda)$$

yields the Weyl character $\chi(V_\lambda)$ of the irreducible $G$-module $V_\lambda$, namely,

$$\chi(V_\lambda) = \sum_{x \in P \cap \mathbb{Z}^d} e^{p(x)}.$$
Geometric mitosis

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