The Ordinary Poincaré Polynomial

The Poincaré polynomial of a complex algebraic variety $X$ is given by

$$P_X(q) = \sum_{i=0}^\ell \dim C^i(X)q^i$$

where $H^i(X)$ is the singular homology of $X$, viewed in its analytic topology. If $X_w$ is a Schubert variety, then its Poincaré polynomial can be described combinatorially by the formula

$$P_w(q) = \sum_{i \leq \ell} q^{i(u)}$$

where the sum is over all elements $x \leq w$ in the Bruhat-Chevalley order on $W$.

The Intersection Cohomology Poincaré Polynomial

The Poincaré polynomial for the full intersection cohomology $I_X(q)$ of $X$ is defined to be

$$I_X(q) = \sum_{i \leq \ell} \dim C^i(X)q^i$$

As was the case for the ordinary Poincaré polynomial $P_X(q)$, when $X_w$ is a Schubert variety the intersection cohomology Poincaré polynomial has a combinatorial description, described as follows. For $w \in W$ and $x \leq w$ in the Bruhat-Chevalley order on $W$, let $P_w(x)$ denote the Kazhdan-Lusztig polynomial indexed by $x$ and $w$ (see [5]). The Poincaré polynomial for the full intersection cohomology $I_{w}(q)$ is then given by

$$I_{w}(q) = \sum_{x \leq w} P_w(x)q^{i(u)}.$$ 

It has been shown that $X$ is rationally smooth if and only if the ordinary cohomology groups $H^i(X)$ coincide with the intersection cohomology groups $IH^i(X)$ [4]. In other words, we have $I_{w} = P_{w}$ if and only if $X_w$ is a rationally smooth Schubert variety.

Pattern Containment

An element $w \in S_n$ contains the pattern $v \in S_k$ if $w$ contains a subword of length $k$ whose entries are in the same relative order as the entries of $v$.

Ex. 541623 contains 3412 and 4231. Billey and Braden have extended the notion of pattern containment to general Weyl groups [2]. Many geometric properties of a Schubert variety $X_w$ are equivalent to combinatorial statements about patterns.

1. For $w \in W$, $P_w(x)$ is rationally smooth if and only if the ordinary cohomology groups $\dim_{k}H^i(X_w)$ coincide with the intersection cohomology groups $\dim_{k}IH^i(X_w)$ [4].

2. See [1] for the lists of patterns which are avoided precisely when $X_w$ is smooth/rational smooth for Weyl groups of types $B$, $D$ and $E$.

A Factorization of $P_w(q)$

For an element $w \in S_n$, a value $r \in [n]$ is a record position of $w$ if $w(r) > w(r-1)$ or $r = 1$. For $i \in [n]$, let $r_i$ be the record positions of $w$ such that $r_i \leq i < r$ and there are no other record positions of $w$ such that $r < i < r'$. Define $e_i = \# \{ j : r_i < j < i, \; w(j) > w(i) \}$ and $f_i = \# \{ k : r' \leq k \leq n, \; w(k) < w(i) \}$.

If $w$ avoids the patterns 3412 and 4231, then the ordinary Poincaré polynomial for $w$, $P_w(q)$, is given by

$$P_w(q) = \prod_{i=1}^{n} e_i + f_i$$

where $[a+1]_q := q^{a} + q^{a-1} + \ldots + q + 1$.

The Inversion Polynomial

Main Goal: to combinatorially define a new polynomial which will coincide with $I_w(q)$.

Let $N(w)$ denote the collection of all positive roots sent negative by $w$. Say a set $S \subset N(w)$ is $N$-closed if whenever $\alpha, \beta \in S$, we have $\alpha + \beta \in S$. Define $N(w)$ to be the collection of all sets $S$ such that both $S$ and $N(w) \setminus S$ are $N$-closed. The inversion polynomial for $w$ is then defined to be

$$N_w(q) = \sum_{S \subset N(w)} q^{i(S)}.$$