1 Coxeter groups and Poincaré polynomials

Let \( W \) be a Coxeter group with finite reflection set \( S \). By definition, \( W \) is the group generated by \( S \) subject to the relations \( s^2 = e \) where \( m_{ij} \in \{1, 2, 3, \ldots, \infty\} \) and \( m_{ij} = 1 \) if \( i = j \).

If \( t \) and \( s \) denote the length function and Bruhat order on \( W \), then for any \( w \in W \) we define the Poincaré polynomial

\[
P_w(q) = \sum_{w \leq t} q^{l(w)}.
\]

If \( W \) is crystallographic (i.e. \( m_{ij} \in \{1, 2, 3, 4, 6, \infty\} \)), then \( W \) is the Weyl group of some Kac-Moody group \( G \). Each element \( w \in W \) indexes a Schubert variety \( X_w \subseteq G/B \). Topologically we have

\[
P_w(q) = \dim H^*(X_w, \mathbb{C}) q^l.
\]

Question 1. When is \( P_w(q) \) a palindromic polynomial?

A degree \( l \) polynomial \( \sum_{i=0}^{l} a_i q^i \) is palindromic if \( a_i = a_{l-i} \) for all \( i \). If \( P_w(q) \) is palindromic, then \( w \) is maximal in \( W \). If \( W \) is a simple Weyl group, then \( P_w(q) \) is palindromic by the Bruhat order.

For finite Weyl groups, the answer to Question 1 is well understood. In particular, palindromic functions can be characterized using permutation pattern-avoidance in classical types and using root system avoidance in all types \([1, 3, 5]\). The characterization using permutation pattern avoidance has been extended to the affine type \( A \) case as well \([2]\).

Combinatorial answers to Question 1 for general Coxeter groups are unknown. We introduce a family of Coxeter groups (mostly) outside the finite and affine cases for which it is possible to determine if \( P_w(q) \) is palindromic by calculating a few of its coefficients.

2 Triangle group avoidance

A triangle group is a Coxeter group with \( |S| = 3 \). If \( S = \{r, s, t\} \), then a triangle group is completely characterized by the triple \((m_r, m_s, m_t)\). We will denote a triangle group by its corresponding triple.

Definition 1. A Coxeter group \( W \) contains the triangle \((a, b, c)\) if there exists a subset \( \{r, s, t\} \subseteq S \) such that \((a, b, c) = (m_r, m_s, m_t)\).

If \( S \) contains no such subset, then \( W \) avoids the triangle \((a, b, c)\).

Definition 2. A polynomial \( \sum_{i=0}^{l} a_i q^i \) is \( k \)-palindromic if \( a_i = a_{l-k} \) for all \( i \).

Consider the set of triangle groups: \( HQ := \{(2, b, c) \mid b, c \geq 3 \text{ and } b < c\} \).

Theorem 1. (R-Slofstra) Suppose \( W \) avoids all triangle groups in \( HQ \). Let \( w \in W \) be \( 2 \)-palindromic and fix a parabolic factorization \( w = w_1w_2 \) such that \( [S \cap [e, u]] = [S \cap [e, u]] + 1 \).

Then \( w \) is a \( BP \)-factorization where \( |S \cap [e, u]| \leq 3 \).

Moreover, if \( S \cap [e, u] = \{r, s, t\} \), then \( 3 \leq m_u \leq \infty \) and \( m_u < \infty \) with one of the following:

1. \( v = trd^{|u|} \) where \( \langle r, s, t \rangle \) generates the triangle \((3, m_r, m_u)\).
2. \( v = strd^{|u|} \) where \( \langle r, s, t \rangle \) generates the triangle \((3, 3, m_u)\).
3. \( v = strst \cdots \) where \( \langle r, s, t \rangle \) generates the triangle \((3, 3, 3)\).

If \( w \) is \( 2 \)-palindromic, \( w \) is also \( 2 \)-palindromic. Hence Theorem 2 can be applied inductively to factor \( P_w(q) \). Define the \( q \)-integer

\[
[k]_q := 1 + q + \cdots + q^k.
\]

Corollary 1. Suppose \( w = w_1w_2 \in W \) satisfies the conditions in Theorem 2. Then \( P_w(q) \) equals one of the following polynomials.

1. \( [k]_q + [k]_q^q \).
2. \( [k]_q + [k]_q^q + q^2[k]_q(q^2 - 3) \).
3. \( [k]_q + [k]_q^q + q^2[k]_q(q^2 - 3) + q^2[k]_q(q^2 - 6) \).
4. \( \sum_{i=0}^{k} q^i[k]_q - 4k + 1 \) with \( k = \frac{[k]}{\binom{q}{2}} \).

Observe that all the polynomials listed are palindromic except the third which is 3-palindromic but not 4-palindromic. This third polynomial corresponds to part 2 of Theorem 2 which proves Theorem 1.

Example 1. Consider the Coxeter group \( W \) with \( S = \{s_1, s_2, s_3\} \) defined by \( m_{s_1} = 3 \) \( \forall s \in S \). Let \( w = s_1s_2s_3s_1s_2s_3s_1s_2s_3s_1 \). Then \( w \) is 2-palindromic with factorization:

\[
w = (s_1)(s_2s_3)(s_1s_2s_3s_1s_2s_3s_1s_2s_3s_1).
\]

The corresponding parabolic polynomial factorization is

\[
P_w(q) = [2]_q^2[3]_q^2[2]_q^2 = (1 + q)(1 + q + q^2)(1 + q + q^2 + q^4 + q^5)(1 + q).
\]

4 Enumeration results

One consequence of Theorem 2 is that the number of elements with palindromic Poincaré polynomials is finite if \( W \) avoids triangles in \( HQ \) and \( (1, 3, 3) \).

We can explicitly enumerate the number of palindromic elements in uniform Coxeter groups. For any positive integers \( m, n \), let \( W(m, n) \) denote the Coxeter group with \( |S| = n \) and \( m_{s_1} = m, s \in S \). Define the generating series

\[
\Phi_m(q, t) = \sum_{n>0} P_{m_{s_1}}(q, t)^n
\]

where \( P_{m_{s_1}} \) denotes the number of palindromic \( w \in W(m, n) \) of length \( k \).

Corollary 2. For any \( m \geq 4 \), the series \( \Phi_m(q, t) = \frac{\exp(t)}{(1 + 4q + 6q^2 + 3q^3)} \).

Example 2. The expansion of \( \Phi_4(q, t) = 1 + 4q + 6q^2 + 3q^3 \) is

\[
(1 + 4q + 6q^2 + 3q^3)(1 + 2q + 2q^2 + 3q^3 + 3q^4 + 3q^5 + 2q^6 + q^7).
\]

The following table lists the number of palindromic elements in \( W(m, n) \).

| \( m \times n \) | 1 2 3 4 5 6 7 |
|----------------|-------|-------|-------|-------|-------|-------|
| 2              | 2     | 8     | 67    | 893   | 15596 | 330082 | 8165963 |
| 3              | 10    | 115   | 2057  | 47356 | 1314229 | 42584795 |
| 4              | 12    | 175   | 3893  | 110436 | 3769882 | 150113447 |
| 5              | 14    | 247   | 6545  | 219956 | 884312 | 418725119 |
| 6              | 16    | 331   | 10157 | 399316 | 18351562 | 997538291 |

References