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MODULI OF STABLE OBJECTS IN A TRIANGULATED CATEGORY

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Abstract. We introduce the concept of strict ample sequence in a fibered triangulated category and define the stability of the objects in a triangulated category. Then we construct the moduli space of (semi) stable objects by GIT construction.

1. Introduction

Let $X \to S$ be a projective and flat morphism of noetherian schemes. We consider the functor $\mathrm{Splcpx}_{X/S} : (\mathrm{Sch}/S) \to (\mathrm{Sets})$ defined by

$$\mathrm{Splcpx}_{X/S}(T) = \left\{ E \in D^b(\mathrm{Coh}(X \times_S T)) \;\middle|\; \begin{array}{l} \text{for any geometric point } t \text{ of } T, \\ E(t) := E \otimes^L k(t) \text{ is a bounded complex and} \\ \mathrm{Ext}^i(E(t), E(t)) \cong \left\{ \begin{array}{ll} k(t) & \text{if } i = 0 \\ 0 & \text{if } i = -1 \end{array} \right. \end{array} \right\} \Big/ \sim,$$

where $E \sim E'$ if there is a line bundle L on T such that $E \cong E' \otimes L$ in $D^b(\mathrm{Coh}(X \times_S T))$. We denote the étale sheafification of $\mathrm{Splcpx}_{X/S}$ by $\mathrm{Splcpx}^{\text{ét}}_{X/S}$. Then the result of [4] is that $\mathrm{Splcpx}^{\text{ét}}_{X/S}$ is an algebraic space over S. M. Lieblich extends this result in [7] to the case when $X \to S$ is a proper flat morphism of algebraic spaces. So the problem on the construction of the moduli space of objects in a derived category is solved in some sense. However, the moduli space $\mathrm{Splcpx}^{\text{ét}}_{X/S}$ is not separated and it is not a good space in geometric sense. So we want to construct a projective moduli space (or quasi-projective moduli space with a good compactification) as a Zariski open set of $\mathrm{Splcpx}^{\text{ét}}_{X/S}$ such as the moduli space of stable sheaves.

This problem is also motivated by Fourier-Mukai transform. Let X, Y be projective varieties over an algebraically closed field k and $\mathcal{P}$ be an object of $D^b(\mathrm{Coh}(X \times Y))$. The functor

$$\Phi : D^b(\mathrm{Coh}(X)) \longrightarrow D^b(\mathrm{Coh}(Y))$$
$$E \mapsto \mathrm{R}(p_Y)_*(p_X^*(E) \otimes^L \mathcal{P})$$

is called a Fourier-Mukai transform if it is an equivalence of categories. Here $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ are the projections. Fourier-Mukai transform induces the isomorphisms on moduli spaces and for example the image $\Phi(M_X^P)$ of a moduli space of stable sheaves M_X^P on X by Φ sometimes becomes a moduli space of stable sheaves on Y. The problem on the preservation of stability under Fourier-Mukai transform is investigated by many people and this problem is clearly pointed out by K. Yoshioka in [11]. However, the image $\Phi(M_X^P)$ of the moduli space of stable sheaves by the Fourier-Mukai transform may not be contained in the category of coherent sheaves on Y in general and so we must consider certain moduli space of stable objects in the derived category $D^b(\mathrm{Coh}(Y))$.

In this paper we introduce the concept "strict ample sequence" in a triangulated category. "Strict ample sequence" satisfies the condition of ample sequence defined by A. Bondal and D. Orlov in [2], but it also satisfies many other conditions because we expect that a "polarization" is determined by strict ample sequence. Indeed we can define stable objects determined by a strict ample sequence and construct the moduli space of stable objects (resp. S-equivalences classes of semistable objects) as a quasi-projective scheme (resp. projective scheme). This is the main result of this paper (Theorem 4.4 and Theorem 4.8). If $\Phi : D^b(\mathrm{Coh}(X)) \to D^b(\mathrm{Coh}(Y))$ is a Fourier-Mukai transform and M_X^P is a moduli space of stable sheaves on X, then the image $\Phi(M_X^P)$ of M_X^P by Φ becomes a moduli space of stable objects in $D^b(\mathrm{Coh}(Y))$ whose stability is determined by some strict ample sequence on $D^b(\mathrm{Coh}(Y))$. So Fourier-Mukai transform always preserves certain stability in our sense (Example 5.3).

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T. Bridgeland defined in [1] the concept of stability condition on a triangulated category. So we are interested in the relation between the stability condition of Bridgeland and the deifinition of stability determined by a strict ample sequence. However, it seems rather impossible to expect the construction of a strict ample sequence from the stability condition defined by Bridgeland without any other condition. How to treat the relation between strict ample sequence and stability condition of Brigeland is a problem still unsolved.

2. Definition of fibered triangulated category

Let S be a noetherian scheme. We denote the category of noetherian schemes over S by (Sch/S) and the derived category of bounded complexes of coherent sheaves on U by $D^b_c(U)$ for $U \in (\mathrm{Sch}/S)$. We denote the derived category of lower bounded complexes of coherent sheaves on U by $D^+_c(U)$ for $U \in (\mathrm{Sch}/S)$. For a noetherian scheme X over S, we denote the full subcategory of $D^b_c(X)$ consisting of the objects of finite Tor-dimension over S by $D^b(\mathrm{Coh}(X/S))$. Then $D^b(\mathrm{Coh}(X/S))$ becomes a triangulated category. For a triangulated category $\mathcal{T}$ and for objects $E, F \in \mathcal{T}$, we write $\mathrm{Ext}^i(E,F) := \mathrm{Hom}_{\mathcal{T}}(E, F[i])$.

Definition 2.1. $p : \mathcal{D} \to (\mathrm{Sch}/S)$ is called a fibered triangulated category if

(1) $\mathcal{D}$ is a category, p is a covariant functor,
(2) for any $U \in (\mathrm{Sch}/S)$, the full subcategory $\mathcal{D}_U := p^{-1}(U)$ of $\mathcal{D}$ is a triangulated category,
(3) for any object $E \in \mathcal{D}_U$ and for any morphism $f : V \to U = p(E)$ in (Sch/S), there exist an object $F \in \mathcal{D}_V$ and a morphism $u : F \to E$ satisfying the condition: For any object $G \in \mathcal{D}_V$ and a morphism $v : G \to E$ with $p(v) = f$, there exists a unique morphism $w : G \to F$ satisfyfing $p(w) = \mathrm{id}_V$ and $u \circ w = v$, (we denote F by $f^*(E)$ or E_V and we call such morphism u a Cartesian morphism),
(4) any composition of Cartesian morphisms is Cartesian,
(5) for any morphism $V \to U$ in (Sch/S), $\mathcal{D}_U \ni E \mapsto E_V \in \mathcal{D}_V$ is an "exact functor", that is, for any distinguished triangle $E \to F \to G$ in $\mathcal{D}_U$, $E_V \to F_V \to G_V$ is a distinguished triangle in $\mathcal{D}_V$ and for any $E \in \mathcal{D}_U$ and any $i \in \mathbf{Z}$, there is an isomorphism $(E[i])_V \cong E_V[i]$ functorial in E.

Definition 2.2. A fibered triangulated category $p : \mathcal{D} \to (\mathrm{Sch}/S)$ has base change property if

(1) for each $U \in (\mathrm{Sch}/S)$, there is a bi-exact bi-functor $\otimes : \mathcal{D}_U \times D^b(\mathrm{Coh}(U/U)) \to \mathcal{D}_U$ such that there is a functorial isomorphism $E[i] \otimes P[j] \cong (E \otimes P)[i+j]$ for $E \in \mathcal{D}_U$, $P \in D^b(\mathrm{Coh}(U/U))$,
(2) for a morphism $\varphi : U \to V$ in (Sch/S), the diagram

$$\begin{array}{ccc} \mathcal{D}_V \times D^b(\mathrm{Coh}(V/V)) & \xrightarrow{\otimes} & \mathcal{D}_V \\ {\scriptstyle \varphi^* \times L\varphi^*}\downarrow & & \downarrow{\scriptstyle \varphi^*} \\ \mathcal{D}_U \times D^b(\mathrm{Coh}(U/U)) & \xrightarrow{\otimes} & \mathcal{D}_U \end{array}$$

is "commutative", precisely, there exists a functorial isomorphism $\varphi^* \circ \otimes \xrightarrow{\sim} \otimes \circ (\varphi^* \times L\varphi^*)$,
(3) for $U \in (\mathrm{Sch}/S)$, there is a bi-exact bi-functor

$$\mathbf{R}\,\mathrm{Hom}_p : \mathcal{D}_U \times \mathcal{D}_U \longrightarrow D^+_c(U)$$

such that for $E_1, E_2 \in \mathcal{D}_U$ and for intgers i, j, there is an isomorphism $\mathbf{R}\,\mathrm{Hom}_p(E_1[i], E_2[j]) \cong \mathbf{R}\,\mathrm{Hom}_p(E,F)[j-i]$ functorial in E_1 and E_2 and also for $E_1, E_2 \in \mathcal{D}_U$ there is an isomorphism $\mathrm{Hom}_{D(U)}(\mathcal{O}_U, \mathbf{R}\,\mathrm{Hom}_p(E_1,E_2)) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}_U}(E_1,E_2)$ functorial in E_1 and E_2,
(4) for any $U \in (\mathrm{Sch}/S)$ and for any objects $E_1, E_2 \in \mathcal{D}_U$, there exist a lower bounded complex $P^\bullet$ of locally free sheaves of finite rank on U and an isomorphism

$$P^\bullet \otimes \mathcal{O}_V \xrightarrow{\sim} \mathbf{R}\,\mathrm{Hom}_p((E_1)_V, (E_2)_V)$$

in $D^+_c(V)$ for any morphism $V \to U$ in (Sch/S), such that the diagram

$$\begin{array}{ccccc} H^0(\Gamma((U,P^\bullet)) & \longrightarrow & \mathrm{Hom}_{D(U)}(\mathcal{O}_U, \mathbf{R}\,\mathrm{Hom}_p(E_1,E_2)) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathcal{D}_U}(E_1,E_2) \\ \downarrow & & & & \downarrow \\ H^0(\Gamma((V,P^\bullet)) & \longrightarrow & \mathrm{Hom}_{D(U)}(\mathcal{O}_V, \mathbf{R}\,\mathrm{Hom}_p((E_1)_V,(E_2)_V)) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathcal{D}_V}((E_1)_V,(E_2)_V) \end{array}$$

is commutative,

(5) for $U \in (\mathrm{Sch}/S)$, $E_1, E_2 \in \mathcal{D}_U$ and $F_1, F_2 \in D^b(\mathrm{Coh}(U/U))$, there is a functorial isomorphism $\mathbf{R}\operatorname{Hom}_p(E_1 \otimes F_1, E_2 \otimes_U F_2) \cong \mathbf{R}\operatorname{Hom}_p(E_1, E_2) \otimes^L_{\mathcal{O}_U} \mathbf{R}\mathcal{H}om(F_1, F_2)$ such that for any morphism $\varphi : V \to U$ in (Sch/S), the diagram

$$\begin{array}{ccc} \mathbf{R}\operatorname{Hom}_p(E_1 \otimes F_1, E_2 \otimes F_2) & \xrightarrow{\sim} & \mathbf{R}\operatorname{Hom}_p(E_1, E_2) \otimes^L_{\mathcal{O}_U} \mathbf{R}\mathcal{H}om(F_1, F_2) \\ \downarrow & & \downarrow \\ \mathbf{R}\varphi_*(\mathbf{R}\operatorname{Hom}_p((E_1 \otimes F_1)_V, (E_2 \otimes F_2)_V)) & \xrightarrow{\sim} & \mathbf{R}\varphi_*(\mathbf{R}\operatorname{Hom}_p((E_1)_V, (E_2)_V) \otimes^L_{\mathcal{O}_V} \mathbf{R}\mathcal{H}om((F_1)_V, (F_2)_V)) \end{array}$$

is commutative.

Remark 2.3. For $E, F \in \mathcal{D}_U$, we denote the i-th cohomology $H^i(\mathbf{R}\operatorname{Hom}_p(E, F))$ by $R^i \operatorname{Hom}_p(E, F)$. We notice that for three objects $E, F, G \in \mathcal{D}_U$, there is a canonical morphism

$$R^0 \operatorname{Hom}_p(E, F) \times R^0 \operatorname{Hom}_p(F, G) \to R^0 \operatorname{Hom}_p(E, G).$$

Example 2.4. Let $X \to S$ be a flat projective morphism. Then $\{D^b(\mathrm{Coh}(X_U/U))\}_{U \in (\mathrm{Sch}/S)}$ becomes a fibered triangulated category over S which has base change property.

Example 2.5. Let X be a projective scheme over $\mathbf{C}$ and G a finite group acting on X. For a scheme $U \in (\mathrm{Sch}/\mathbf{C})$, let $D^G(\mathrm{Coh}(X_U/U))$ be the derived category of bounded complexes of G-equivariant coherent sheaves on X_U of finite Tor-dimension over U. Then $\{D^G(\mathrm{Coh}(X_U/U))\}_{U \in (\mathrm{Sch}/\mathbf{C})}$ becomes a fibered triangulated category over $\mathbf{C}$ which has base change property.

3. Strict ample sequence and stability

Definition 3.1. Let $p : \mathcal{D} \to (\mathrm{Sch}/S)$ be a fibered triangulated category with base change property. A sequence $\mathcal{L} = \{L_n\}_{n \geq 0}$ of objects of $\mathcal{D}_S$ is said to be a strict ample sequence if it satisfies the following conditions:

(1) $\operatorname{Ext}^i((L_N)_s, (L_n)_s) = 0$ for any $i \neq 0$, $N > n$ and $s \in S$.

(2) There exist isomorphisms

$$\theta_k : R^0 \operatorname{Hom}_p(L_n, L_m) \xrightarrow{\sim} R^0 \operatorname{Hom}_p(L_{n+k}, L_{m+k})$$

for non-negative integers k, m, n with $n \geq m$ such that $\theta_k \circ \theta_l = \theta_{k+l}$ for any k, l and the diagram

$$\begin{array}{ccc} R^0 \operatorname{Hom}_p(L_n, L_m) \otimes R^0 \operatorname{Hom}_p(L_m, L_l) & \xrightarrow{\theta_k \otimes \theta_k} & R^0 \operatorname{Hom}_p(L_{n+k}, L_{m+k}) \otimes R^0 \operatorname{Hom}_p(L_{m+k}, L_{l+k}) \\ \downarrow & & \downarrow \\ R^0 \operatorname{Hom}_p(L_n, L_l) & \xrightarrow{\theta_k} & R^0 \operatorname{Hom}_p(L_{n+k}, L_{l+k}) \end{array}$$

is commutative for non-negative integers k, l, m, n with $n \geq m \geq l$.

(3) There exists a subbundle $V_1 \subset R^0 \operatorname{Hom}_p(L_1, L_0)$ such that the diagram

$$\begin{array}{ccc} V_1 \times R^0 \operatorname{Hom}_p(L_n, L_0) & \xrightarrow{\theta_n \times 1} & R^0 \operatorname{Hom}_p(L_{n+1}, L_n) \times R^0 \operatorname{Hom}_p(L_n, L_0) \\ {\scriptstyle 1 \times \theta_1}\downarrow & & \downarrow \\ V_1 \times R^0 \operatorname{Hom}_p(L_{n+1}, L_1) & \longrightarrow & R^0 \operatorname{Hom}_p(L_{n+1}, L_0), \end{array}$$

is commutative for $n \geq 0$, where the right vertical arrow and the bottom horizontal arrow are the canonical composition maps and there exists an integer n_0 such that for any $n \geq n_0$,

$$R^0 \operatorname{Hom}_p(L_n, L_1) \otimes V_1 \longrightarrow R^0 \operatorname{Hom}_p(L_n, L_0)$$

is surjective for any $n \geq n_0$.

(4) For any object $E \in \mathcal{D}_U$ and for any non-negative integer m, there exists a bounded complex $P^\bullet$ of locally free sheaves of finite rank on U such that $\mathbf{R}\operatorname{Hom}_p((L_m)_V, E_V) \cong P^\bullet \otimes \mathcal{O}_V$ for any $V \to U$. Moreover, there exists an integer n_0 such that for any $n \geq n_0$, exists an integer N_0 such that for any integers i, N with $N \geq N_0$ and for any $s \in U$,

$$\operatorname{Hom}((L_N)_s, (L_n)_s) \otimes \operatorname{Ext}^i((L_n)_s, E_s) \to \operatorname{Ext}^i((L_N)_s, E_s)$$

is surjective.

(5) If there exist integers i, n_0 and an object $E \in \mathcal{D}_U$ satisfying $\mathrm{Ext}^i((L_n)_s, E_s) = 0$ for any $n \geq n_0$ and for any $s \in U$, then there exist an object $F \in \mathcal{D}_U$ and a morphism $u : E \to F$ such that for any $j > i$, $R^j \mathrm{Hom}_p((L_n)_U, E) \to R^j \mathrm{Hom}_p((L_n)_U, F)$ are isomorphic for $n \gg 0$, and for any $j \leq i$, $R^j \mathrm{Hom}_p((L_n)_U, F) = 0$ for $n \gg 0$.

(6) Take two objects $E, F \in \mathcal{D}_U$ such that for any $i \geq 0$, $R^i \mathrm{Hom}_p((L_n)_U, E) = 0$ for $n \gg 0$ and that for any $i < 0$, $R^i \mathrm{Hom}_p((L_n)_U, F) = 0$ for $n \gg 0$. Then we have $\mathrm{Hom}_{\mathcal{D}_U}(E, F) = 0$.

Proposition 3.2. *Take $E \in \mathcal{D}_U$ such that for any i, $R^i \mathrm{Hom}_p((L_n)_U, E) = 0$ for $n \gg 0$. Then we have $E = 0$.*

Proof. Applying the condition (6) of Definition 3.1, we have $\mathrm{Hom}(E, E) = 0$. In particular $\mathrm{id}_E = 0$. So, for any object $F \in \mathcal{D}_U$ and for any morphism $f \in \mathrm{Hom}(F, E)$ (resp. $g \in \mathrm{Hom}(E, F)$), $f = \mathrm{id}_E \circ f = 0$ (resp. $g = g \circ \mathrm{id}_E = 0$). Thus $E = 0$. □

Remark 3.3. By the condition in Definition 3.1 (2), we can see that $\theta_0 = \mathrm{id}$ and $\theta_k(\mathrm{id}) = \mathrm{id}$. We put $A := \bigoplus_{n \geq 0} R^0 \mathrm{Hom}_p(L_n, L_0)$ and define a multiplication

$$\alpha : R^0 \mathrm{Hom}_p(L_n, L_0) \times R^0 \mathrm{Hom}_p(L_m, L_0) \longrightarrow R^0 \mathrm{Hom}_p(L_{n+m}, L_0)$$

by $\alpha = (\text{composition}) \circ (\theta_m \times \mathrm{id})$. Then A becomes an associative graded ring which is a finitely generated module over $S^*(V_1)$, where $S^*(V_1)$ is the symmetric algebra of V_1 over $\mathcal{O}_S$.

Proposition 3.4. *Let E_1, E_2 be objects of $\mathcal{D}_U$ and $u : E_1 \to E_2$ be a morphism such that for any integer i the induced morphism $R^i \mathrm{Hom}_p((L_n)_U, E_1) \to R^i \mathrm{Hom}_p((L_n)_U, E_2)$ is isomorphic for $n \gg 0$. Then u is an isomorphism.*

Proof. For any i, there is an exact sequence

$$\begin{aligned} R^i \mathrm{Hom}_p((L_n)_U, E_1) &\xrightarrow{\sim} R^i \mathrm{Hom}_p((L_n)_U, E_2) \longrightarrow R^i \mathrm{Hom}_p((L_n)_U, \mathrm{Cone}(u)) \\ &\longrightarrow R^{i+1} \mathrm{Hom}_p((L_n)_U, E_1) \xrightarrow{\sim} R^{i+1} \mathrm{Hom}_p((L_n)_U, E_2) \end{aligned}$$

for $n \gg 0$. Thus we have $R^i \mathrm{Hom}_p((L_n)_U, \mathrm{Cone}(u)) = 0$ for $n \gg 0$. By Proposition 3.2 we have $\mathrm{Cone}(u) = 0$, which means that u is an isomorphism. □

Proposition 3.5. *For an integer i and an object $E \in \mathcal{D}_U$ such that for $n \gg 0$, $\mathrm{Ext}^i((L_n)_s, E_s) = 0$ for $s \in U$, the object F given in Definition 3.1 (5) is unique up to an isomorphism.*

Proof. Let $F' \in \mathcal{D}_U$ be another object with a morphism $u' : E \to F'$ having the same property as F. Consider the composite

$$v : \mathrm{Cone}(u)[-1] \longrightarrow E \xrightarrow{u'} F'.$$

Since there is a long exact sequence

$$\begin{aligned} \cdots \longrightarrow R^j \mathrm{Hom}_p((L_n)_U, E) &\longrightarrow R^j \mathrm{Hom}_p((L_n)_U, F) \longrightarrow R^j \mathrm{Hom}_p((L_n)_U, \mathrm{Cone}(u)) \\ &\longrightarrow R^{j+1} \mathrm{Hom}_p((L_n)_U, E) \longrightarrow R^{j+1} \mathrm{Hom}_p((L_n)_U, F) \longrightarrow \cdots, \end{aligned}$$

we have, for any $j \geq i$, $R^j \mathrm{Hom}_p((L_n)_U, \mathrm{Cone}(u)) = 0$ for $n \gg 0$. Note that for any $j \leq i$, we have $R^j \mathrm{Hom}_p((L_n)_U, F') = 0$ for $n \gg 0$. Then we have $\mathrm{Hom}_{\mathcal{D}_U}(\mathrm{Cone}(u), F') = 0$ and $\mathrm{Hom}_{\mathcal{D}_U}(\mathrm{Cone}(u)[-1], F') = 0$ by condition (6) of Definition 3.1. So we have $v = 0$ and there is a unique morphism $\varphi : F \to F'$ which makes the diagram

$$\begin{array}{ccc} E & \xrightarrow{u} & F \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \varphi} \\ E & \xrightarrow{u'} & F' \end{array}$$

commute. We can see that for any integer j, the morphism $R^j \mathrm{Hom}_p((L_n)_U, F) \to R^j \mathrm{Hom}_p((L_n)_U, F')$ induced by φ is isomorphic for $n \gg 0$. Hence φ is an isomorphism by Proposition 3.4 □

Remark 3.6. In the situation of Definition 3.1 (5), for $n \gg 0$, the induced morphism

$$\mathrm{Ext}^j((L_n)_s, E_s) \to \mathrm{Ext}^j((L_n)_s, F_s)$$

is isomorphic for any $j > i$ and for any $s \in U$, and we have, for $n \gg 0$, $\mathrm{Ext}^j((L_n)_s, F_s) = 0$ for any $j \le i$ and for any $s \in U$.

Indeed consider the distinguished triangle $E \xrightarrow{u} F \to \mathrm{Cone}(u)$. Note that there is a long exact sequence

$$R^j \mathrm{Hom}_p((L_n)_U, E) \longrightarrow R^j \mathrm{Hom}_p((L_n)_U, F) \longrightarrow R^j \mathrm{Hom}_p((L_n)_U, \mathrm{Cone}(u)) \\ \longrightarrow R^{j+1} \mathrm{Hom}_p((L_n)_U, E) \longrightarrow R^{j+1} \mathrm{Hom}_p((L_n)_U, F).$$

Since $R^i \mathrm{Hom}_p((L_n)_U, F) = 0$ for $n \gg 0$, and for any $j > i$, $R^j \mathrm{Hom}_p((L_n)_U, E) \to R^j \mathrm{Hom}_p((L_n)_U, F)$ are isomorphic for $n \gg 0$, we have, for any $j \ge i$, $R^j \mathrm{Hom}_p((L_n)_U, \mathrm{Cone}(u)) = 0$ for $n \gg 0$.

By Definition 3.1 (4), there are integers n_0 and N_0 with $N_0 > n_0$ such that

$$\mathrm{Hom}((L_N)_s, (L_{n_0})_s) \otimes \mathrm{Ext}^j((L_{n_0})_s, \mathrm{Cone}(u)_s) \longrightarrow \mathrm{Ext}^j((L_N)_s, \mathrm{Cone}(u)_s)$$

is suriective for any j, any $N \ge N_0$ and any $s \in U$. By Definition 3.1 (4), there are integers j_0, j_1 such that for $j < j_0$ and $j > j_1$, $\mathrm{Ext}^j((L_{n_0})_s, \mathrm{Cone}(u)_s) = 0$ for any $s \in U$. Then for any $N \ge N_0$, we have $\mathrm{Ext}^j((L_N)_s, \mathrm{Cone}(u)_s) = 0$ for any $j > j_1$ and $s \in U$. For each j with $i \le j \le j_1$, there exists an integer $N(j)$ such that for any $N \ge N(j)$, we have $R^j \mathrm{Hom}_p((L_N)_U, \mathrm{Cone}(u)) = 0$. Put

$$\tilde{N} := \max\{N(i), N(i+1), \dots, N(j_1), N_0\}.$$

By Definition 2.2 (4), we have $\mathrm{Ext}^j((L_N)_s, \mathrm{Cone}(u)_s) = 0$ for any $N \ge \tilde{N}$ and for each j with $i \le j \le j_1$ and for any $s \in U$, because $\mathrm{Ext}^{j_1+1}((L_N)_s, \mathrm{Cone}(u)_s) = 0$ for any $s \in U$ and $R^j \mathrm{Hom}_p((L_N)_U, \mathrm{Cone}(u)) = 0$ for $i \le j \le j_1$. Thus we have $\mathrm{Ext}^j((L_N)_s, \mathrm{Cone}(u)_s) = 0$ for any $N \ge \tilde{N}$, $j \ge i$ and $s \in U$.

Note that there are integers k_0, k_1 and a positive integer M_0 such that for any $M \ge M_0$ and for any $s \in U$, $\mathrm{Ext}^j((L_M)_s, F_s) = 0$ for $j < k_0$ and $j > k_1$. We may also assume that for any $M \ge M_0$ and for any $s \in U$, $\mathrm{Ext}^i((L_M)_s, E_s) = 0$. From the exact sequence

$$0 = \mathrm{Ext}^i((L_M)_s, E_s) \longrightarrow \mathrm{Ext}^i((L_M)_s, F_s) \longrightarrow \mathrm{Ext}^i((L_M)_s, \mathrm{Cone}(u)_s) = 0,$$

we have $\mathrm{Ext}^i((L_M)_s, F_s) = 0$ for $s \in U$ and $M \ge \max\{M_0, \tilde{N}\}$. By assumption, for each j with $k_0 \le j \le i$, there exists an integer $M(j)$ such that $R^j \mathrm{Hom}_p((L_M)_U, F) = 0$ for $M \ge M(j)$. Put

$$\tilde{M} := \max\{\tilde{N}, M_0, M(k_0), M(k_0+1), \dots, M(i)\}.$$

Then we have $\mathrm{Ext}^j((L_M)_s, F_s) = 0$ for $j \le i$, $s \in U$ and $M \ge \tilde{M}$ by using Definition 2.2 (4), because $\mathrm{Ext}^i((L_M)_s, F_s) = 0$ and $R^j \mathrm{Hom}_p((L_M)_U, F) = 0$ for $k_0 \le j \le i$. From the exact sequence

$$\mathrm{Ext}^{j-1}((L_M)_s, \mathrm{Cone}(u)_s) \longrightarrow \mathrm{Ext}^j((L_M)_s, E_s) \longrightarrow \mathrm{Ext}^j((L_M)_s, F_s) \longrightarrow \mathrm{Ext}^j((L_M)_s, \mathrm{Cone}(u)_s),$$

we have an isomorphism $\mathrm{Ext}^j((L_M)_s, E_s) \xrightarrow{\sim} \mathrm{Ext}^j((L_M)_s, F_s)$ for $j > i$, $s \in U$ and $M \ge \tilde{M}$.

Lemma 3.7. *If $E \in \mathcal{D}_U$ satisfies $\mathrm{Ext}^i((L_n)_s, E_s) = 0$ for $n \gg 0$, $i \ne 0$ and $s \in U$, then there exist locally free $\mathcal{O}_U$-modules W_0, W_1, W_2, positive integers $n_0 < n_1 < n_2$ and morphisms*

$$(L_{n_2})_U \otimes W_2 \xrightarrow{d^2} (L_{n_1})_U \otimes W_1 \xrightarrow{d^1} (L_{n_0})_U \otimes W_0 \xrightarrow{f} E$$

such that the induced sequence

$$\mathrm{Hom}((L_N)_s, (L_{n_2})_s) \otimes W_2 \longrightarrow \mathrm{Hom}((L_N)_s, (L_{n_1})_s) \otimes W_1 \longrightarrow \\ \mathrm{Hom}((L_N)_s, (L_{n_0})_s) \otimes W_0 \longrightarrow \mathrm{Hom}((L_N)_s, E_s) \longrightarrow 0$$

is exact for $N \gg 0$ and $s \in U$.

Proof. By Definition 3.1 (4), there exist integers n_0, N_0 with $N_0 > n_0$ such that for any $s \in U$,

$$\mathrm{Hom}((L_N)_s, (L_{n_0})_s) \otimes \mathrm{Hom}((L_{n_0})_s, E_s) \to \mathrm{Hom}((L_N)_s, E_s)$$

is surjective for $N \ge N_0$ and $\mathrm{Ext}^i((L_n)_s, E_s) = 0$ for $n \ge n_0$, $i \ne 0$ and $s \in U$. There is a canonical morphism

$$f : (L_{n_0})_U \otimes R^0 \mathrm{Hom}_p((L_{n_0})_U, E) \longrightarrow E$$

and we put $F^1 := \mathrm{Cone}(f)[-1]$. Then we can see that $\mathrm{Ext}^i((L_N)_s, (F^1)_s) = 0$ for $N \geq N_0$, $i \neq 0$ and $s \in U$. We can find integers n_1, N_1 with $N_1 > n_1$ such that for any $s \in U$,

$$\mathrm{Hom}((L_N)_s, (L_{n_1})_s) \otimes \mathrm{Hom}((L_{n_1})_s, (F^1)_s) \longrightarrow \mathrm{Hom}((L_N)_s, (F^1)_s)$$

is surjective for $N \geq N_1$ and $\mathrm{Ext}^i((L_n)_s, (F^1)_s) = 0$ for $n \geq n_1$, $i \neq 0$ and $s \in U$. Consider the canonical morphism

$$g : (L_{n_1})_U \otimes R^0 \mathrm{Hom}_p((L_{n_1})_U, F^1) \longrightarrow F^1$$

and put $F^2 := \mathrm{Cone}(g)[-1]$. We can find again integers n_2, N_2 with $N_2 > n_2$ such that for any $s \in U$,

$$\mathrm{Hom}((L_N)_s, (L_{n_2})_s) \otimes \mathrm{Hom}((L_{n_2})_s, (F^2)_s) \longrightarrow \mathrm{Hom}((L_N)_s, (F^2)_s)$$

is surjective for $N \geq N_2$ and $\mathrm{Ext}^i((L_n)_s, (F^2)_s) = 0$ for $n \geq n_2$, $i \neq 0$ and $s \in U$. There is a canonical morphism

$$h : (L_{n_2})_U \otimes R^0 \mathrm{Hom}_p((L_{n_2})_U, F^2) \longrightarrow F^2$$

and we obtain a sequence of morphisms

$$\begin{aligned} (L_{n_2})_U \otimes R^0 \mathrm{Hom}_p((L_{n_2})_U, F^2) &\longrightarrow (L_{n_1})_U \otimes R^0 \mathrm{Hom}_p((L_{n_1})_U, F^1) \\ &\longrightarrow (L_{n_0})_U \otimes R^0 \mathrm{Hom}_p((L_{n_0})_U, E) \longrightarrow E \end{aligned}$$

such that for $N \geq \max\{N_0, N_1, N_2\}$, the induced sequence

$$\begin{aligned} \mathrm{Hom}((L_N)_s, (L_{n_2})_s) \otimes R^0 \mathrm{Hom}_p((L_{n_2})_U, F^2) &\longrightarrow \mathrm{Hom}((L_N)_s, (L_{n_1})_s) \otimes R^0 \mathrm{Hom}_p((L_{n_1})_U, F^1) \\ \longrightarrow \mathrm{Hom}((L_N)_s, (L_{n_0})_s) \otimes R^0 \mathrm{Hom}_p((L_{n_0})_U, E) &\longrightarrow \mathrm{Hom}((L_N)_s, E_s) \longrightarrow 0 \end{aligned}$$

is exact for any $s \in U$. If we put $W_0 = R^0 \mathrm{Hom}_p((L_{n_0})_U, E)$ and $W_i = R^0 \mathrm{Hom}_p((L_{n_i})_U, F^i)$ for $i = 1, 2$, then we can see by Definition 2.2 (4) that W_i are locally free $\mathcal{O}_U$-modules and have the desired property. □

Proposition 3.8. *Let E_1, E_2 be objects of $\mathcal{D}_U$ such that $\mathrm{Ext}^i((L_n)_s, (E_j)_s) = 0$ for $j = 1, 2$, $n \gg 0$, $i \neq 0$ and $s \in U$. If $f : E_1 \to E_2$ is a morphism in $\mathcal{D}_U$ such that the induced morphisms $R^0 \mathrm{Hom}_p((L_n)_U, E_1) \to R^0 \mathrm{Hom}_p((L_n)_U, E_2)$ are zero for $n \gg 0$, then $f = 0$.*

Proof. By assumption, there is an integer N_0 such that for any $N \geq N_0$, the morphism

$$R^0 \mathrm{Hom}_p((L_N)_U, E_1) \to R^0 \mathrm{Hom}_p((L_N)_U, E_2)$$

induced by f is zero and $\mathrm{Ext}^i((L_N)_s, (E_j)_s) = 0$ for $j = 1, 2$, $i \neq 0$ and $s \in U$. By Lemma 3.7, there are locally free sheaves W_0, W_1, W_2, integers $n_0 < n_1 < n_2$ and morphisms

$$(L_{n_2})_U \otimes W_2 \longrightarrow (L_{n_1)_U} \otimes W_1 \longrightarrow (L_{n_0)_U} \otimes W_0 \xrightarrow{\varphi} E_1$$

such that the induced sequence

$$\begin{aligned} \mathrm{Hom}((L_N)_s, (L_{n_2})_s) \otimes W_2 &\longrightarrow \mathrm{Hom}((L_N)_s, (L_{n_1})_s) \otimes W_1 \longrightarrow \\ \mathrm{Hom}((L_N)_s, (L_{n_0})_s) \otimes W_0 &\longrightarrow \mathrm{Hom}((L_N)_s, (E_1)_s) \longrightarrow 0 \end{aligned}$$

is exact for $N \gg 0$ and $s \in U$. We can take n_0 so that $n_0 \geq N_0$. Consider the distinguished triangle

$$(L_{n_0})_U \otimes W_0 \longrightarrow E_1 \longrightarrow \mathrm{Cone}(\varphi).$$

We can see that $\mathrm{Ext}^i((L_n)_s, \mathrm{Cone}(\varphi)_s) = 0$ for $n \gg 0$, $i \neq -1$ and $s \in U$. So we have $\mathrm{Hom}_{\mathcal{D}_U}(\mathrm{Cone}(\varphi), E_2) = 0$ by (6) of Definition 3.1 and the homomorphism

$$(\dagger) \quad \mathrm{Hom}_{\mathcal{D}_U}(E_1, E_2) \to \mathrm{Hom}_{\mathcal{D}_U}((L_{n_0})_U \otimes W_0, E_2)$$

induced by φ is injective. On the other hand, the homomorphism

$$R^0 \mathrm{Hom}_p((L_{n_0})_U \otimes W_0, E_1) \longrightarrow R^0 \mathrm{Hom}_p((L_{n_0})_U \otimes W_0, E_2)$$

induced by f is zero. So we have $f \circ \varphi = 0$. By the injectivity of $(\dagger)$, we have $f = 0$. □

Since $A = \bigoplus_{n \geq 0} R^0 \mathrm{Hom}_p(L_n, L_0)$ becomes a finite algebra over $S^*(V_1)$, the associated sheaf $\mathcal{A} := \tilde{A}$ becomes a coherent sheaf of algebras on $\mathbf{P}(V_1)$. For each object $E \in \mathcal{D}_U$ satisfying $\mathrm{Ext}^i((L_n)_s, E_s) = 0$ for $n \gg 0$, $i \neq 0$ and $s \in U$, the associated sheaf $\left(\bigoplus_{n \geq 0} R^0 \mathrm{Hom}_p((L_n)_U, E)\right)^{\sim}$ on $\mathbf{P}(V_1)_U = \mathbf{P}(V_1) \times_S U$ becomes a coherent $\mathcal{A}_U$-module flat over U.

Proposition 3.9. *The correspondence* $E \mapsto \left(\bigoplus_{n\geq 0} R^0 \operatorname{Hom}_p((L_n)_U, E)\right)^{\sim}$ *gives an equivalence of categories between the full subcategory of* $\mathcal{D}_U$ *consisting of the objects* E *of* $\mathcal{D}_U$ *satisfying* $\operatorname{Ext}^i((L_n)_s, E_s) = 0$ *for* $n \gg 0$*,* $i \neq 0$ *and* $s \in U$ *and the category of coherent* $\mathcal{A}_U$*-modules flat over* U*.*

Proof. First we will prove that the functor

$$\psi : E \mapsto \left(\bigoplus_{n\geq 0} R^0 \operatorname{Hom}_p((L_n)_U, E)\right)^{\sim}$$

is fully faithful. Take any objects E, F of $\mathcal{D}_U$ which satisfy $\operatorname{Ext}^i((L_n)_s, E_s) = 0$, $\operatorname{Ext}^i((L_n)_s, F_s) = 0$ for $n \gg 0$, $i \neq 0$ and $s \in U$. By Proposition 3.8,

$$(\dagger) \quad \operatorname{Hom}(E, F) \longrightarrow \operatorname{Hom}(\psi(E), \psi(F))$$

is injective. Take any homomorphism $f \in \operatorname{Hom}(\psi(E), \psi(F))$. There exists an integer n_0 such that for any $n \geq n_0$, $\operatorname{Ext}^i((L_n)_s, E_s) = 0$, $\operatorname{Ext}^i((L_n)_s, F_s) = 0$ for $i \neq 0$ and $s \in U$ and the homomorphisms

$$\operatorname{Hom}((L_N)_s, (L_{n_0})_s) \otimes \operatorname{Hom}((L_{n_0})_s, E_s) \longrightarrow \operatorname{Hom}((L_N)_s, E_s)$$
$$\operatorname{Hom}((L_N)_s, (L_{n_0})_s) \otimes \operatorname{Hom}((L_{n_0})_s, F_s) \longrightarrow \operatorname{Hom}((L_N)_s, F_s)$$

are surjective for $N \gg 0$ and $s \in U$. For a coherent $\mathcal{A}_U$-module $\mathcal{E}$, we denote $\mathcal{E} \otimes \mathcal{O}_{\mathbf{P}(V_1)_U}(n)$ simply by $\mathcal{E}(n)$. We denote the structure morphism $\mathbf{P}(V_1)_U \to U$ by π. Then we may assume that $R^i\pi_*(\psi(E)(n_0)) = 0$, $R^i\pi_*(\psi(F)(n_0)) = 0$ for $i > 0$ and that the homomorphisms

$$\pi_*(\psi(E)(n_0)) \otimes \mathcal{A}(-n_0) \longrightarrow \psi(E)$$
$$\pi_*(\psi(F)(n_0)) \otimes \mathcal{A}(-n_0) \longrightarrow \psi(F)$$

are surjective. We may also assume that

$$R^0 \operatorname{Hom}((L_{n_0})_U, E) \longrightarrow \pi_*(\psi(E)(n_0))$$
$$R^0 \operatorname{Hom}((L_{n_0})_U, F) \longrightarrow \pi_*(\psi(F)(n_0))$$

are isomorphic. Consider the distinguished triangles

$$\operatorname{Cone}(v)[-1] \xrightarrow{\iota_1} (L_{n_0})_U \otimes R^0 \operatorname{Hom}_p((L_{n_0})_U, E) \xrightarrow{v} E$$
$$\operatorname{Cone}(w)[-1] \xrightarrow{\iota_2} (L_{n_0})_U \otimes R^0 \operatorname{Hom}_p((L_{n_0})_U, F) \xrightarrow{w} F.$$

Then we can see that $\operatorname{Ext}^i((L_N)_s, \operatorname{Cone}(v)_s) = 0$, $\operatorname{Ext}^i((L_N)_s, \operatorname{Cone}(w)_s) = 0$ for $N \gg 0$, $i \neq 0$ and $s \in U$. The homomorphism $f : \psi(E) \to \psi(F)$ induces a homomorphism

$$f(n_0) : R^0 \operatorname{Hom}_p((L_{n_0})_U, E) \cong \pi_*(\psi(E)(n_0)) \longrightarrow \pi_*(\psi(F)(n_0)) \cong R^0 \operatorname{Hom}((L_{n_0})_U, F).$$

Then $f(n_0)$ induces a homomorphism

$$\tilde{f} : (L_{n_0})_U \otimes R^0 \operatorname{Hom}_p((L_{n_0})_U, E) \longrightarrow (L_{n_0})_U \otimes R^0 \operatorname{Hom}_p((L_{n_0})_U, F).$$

Consider the composite

$$w \circ \tilde{f} \circ \iota_1 : \operatorname{Cone}(v)[-1] \xrightarrow{\iota_1} (L_{n_0})_U \otimes \operatorname{Hom}((L_{n_0})_U, E) \xrightarrow{\tilde{f}} (L_{n_0})_U \otimes \operatorname{Hom}((L_{n_0})_U, F) \xrightarrow{w} F.$$

Then we have $\psi(w \circ \tilde{f} \circ \iota_1) = \psi(w) \circ \psi(\tilde{f}) \circ \psi(\iota_1) = f \circ \psi(v) \circ \psi(\iota_1) = f \circ \psi(v \circ \iota_1) = 0$. Since

$$\operatorname{Hom}(\operatorname{Cone}(v)[-1], F) \longrightarrow \operatorname{Hom}(\psi(\operatorname{Cone}(v)[-1]), \psi(F))$$

is injective, we have $w \circ \tilde{f} \circ \iota_1 = 0$. So there is a morphism $f' : E \to F$, which makes the diagram

$$\begin{array}{ccccc}
\operatorname{Cone}(v)[-1] & \xrightarrow{\iota_1} & (L_{n_0})_U \otimes R^0 \operatorname{Hom}_p((L_{n_0})_U, E) & \xrightarrow{v} & E \\
 & & \Big\downarrow{\scriptstyle \tilde{f}} & & \Big\downarrow{\scriptstyle f'} \\
\operatorname{Cone}(w)[-1] & \xrightarrow{\iota_2} & (L_{n_0})_U \otimes R^0 \operatorname{Hom}_p((L_{n_0})_U, F) & \xrightarrow{w} & F
\end{array}$$

commute. This commutative diagram induces a commutative diagram

$$\begin{array}{ccc} \pi_*(\psi(E)(n_0)) \otimes \mathcal{A}(-n_0) & \xrightarrow{\psi(v)} & \psi(E) \\ \downarrow & & \downarrow{\scriptstyle \psi(f')} \\ \pi_*(\psi(F)(n_0)) \otimes \mathcal{A}(-n_0) & \longrightarrow & \psi(F). \end{array}$$

Since $\psi(v) \circ (\psi(f') - f) = \psi(v) \circ \psi(f') - \psi(v) \circ f = \psi(w) \circ \psi(\tilde{f}) - \psi(w) \circ \psi(\tilde{f}) = 0$, we have $\psi(f') - f = 0$ because $\psi(v)$ is surjective. So we have $\psi(f') = f$. Thus (†) is surjective and ψ becomes a fully faithful functor.

Take any coherent $\mathcal{A}_U$-module $\mathcal{E}$ flat over U. There is an exact sequence of coherent $\mathcal{A}_U$-modules

$$W_2 \otimes \mathcal{A}(-n_2) \xrightarrow{\delta^2} W_1 \otimes \mathcal{A}(-n_1) \xrightarrow{\delta^1} W_0 \otimes \mathcal{A}(-n_0) \longrightarrow \mathcal{E} \longrightarrow 0,$$

where W_0, W_1, W_2 are locally free sheaves on U and $n_2 \gg n_1 \gg n_0 \gg 0$. The above sequence induces a sequence of morphisms

$$(L_{n_2})_U \otimes W_2 \xrightarrow{d^2} (L_{n_1})_U \otimes W_1 \xrightarrow{d^1} (L_{n_0})_U \otimes W_0.$$

By construction we have $d^1 \circ d^2 = 0$. So there is a morphism $u : \mathrm{Cone}(d^2) \to (L_{n_0})_U \otimes W_0$ such that the diagram

$$\begin{array}{ccccc} (L_{n_1})_U \otimes W_1 & & \xrightarrow{d^1} & & (L_{n_0})_U \otimes W_0 \\ & \searrow & & \nearrow_u & \\ & & \mathrm{Cone}(d^2) & & \end{array}$$

is commutative. Note that $\mathrm{Ext}^i((L_N)_s, \mathrm{Cone}(d^2)_s) = 0$ for $N \gg 0$, $i \neq -1, 0$ and $s \in U$. So we have $\mathrm{Ext}^i((L_N)_s, \mathrm{Cone}(u)_s) = 0$ for $N \gg 0$, $i \neq -2, -1, 0$ and $s \in U$. Since $\mathcal{E}$ is flat over U, the sequence

$$W_2 \otimes \mathcal{A}(-n_2) \otimes k(s) \longrightarrow W_1 \otimes \mathcal{A}(-n_1) \otimes k(s) \longrightarrow W_0 \otimes \mathcal{A}(-n_0) \otimes k(s) \longrightarrow \mathcal{E} \otimes k(s) \longrightarrow 0$$

is exact for any $s \in U$. So we obtain the exact commutative diagram

$$\begin{array}{ccccc} H^0(W_2 \otimes \mathcal{A}(N-n_2) \otimes k(s)) & \longrightarrow & H^0(W_1 \otimes \mathcal{A}(N-n_1) \otimes k(s)) & \longrightarrow & H^0(W_0 \otimes \mathcal{A}(N-n_0) \otimes k(s)) \\ \cong\downarrow & & \cong\downarrow & & \cong\downarrow \\ \mathrm{Hom}((L_N)_s, (L_{n_2})_s \otimes (W_2)_s) & \longrightarrow & \mathrm{Hom}((L_N)_s, (L_{n_1})_s \otimes (W_1)_s) & \longrightarrow & \mathrm{Hom}((L_N)_s, (L_{n_0})_s \otimes (W_0)_s) \end{array}$$

for $N \gg 0$ and $s \in U$. Here we denote $W_i \otimes k(s)$ by $(W_i)_s$ for $i = 0, 1, 2$. We have a factorization

$$\begin{array}{ccccc} \mathrm{Hom}((L_N)_s, (L_{n_1})_s \otimes (W_1)_s) & & \longrightarrow & & \mathrm{Hom}((L_N)_s, (L_{n_0})_s \otimes (W_0)_s) \\ & \searrow & & \nearrow & \\ & & \mathrm{Hom}((L_N)_s, \mathrm{Cone}(d^2)_s) & & \end{array}$$

for $N \gg 0$ and $s \in U$, and the homomorphism $\mathrm{Hom}((L_N)_s, (L_{n_1})_s \otimes (W_1)_s) \longrightarrow \mathrm{Hom}((L_N)_s, \mathrm{Cone}(d^2)_s)$ is surjective for $N \gg 0$ and $s \in U$, because $\mathrm{Ext}^1((L_N)_s, (L_{n_2})_s \otimes (W_2)_s) = 0$ for $N \gg 0$ and $s \in U$. So we can see that the homomorphism

$$\mathrm{Hom}((L_N)_s, \mathrm{Cone}(d^2)_s) \longrightarrow \mathrm{Hom}((L_N)_s, (L_{n_0})_s \otimes (W_0)_s)$$

is injective for $N \gg 0$ and $s \in U$. Since there is an exact sequence

$$0 = \mathrm{Ext}^{-1}((L_N)_s, (L_{n_0})_s \otimes (W_0)_s) \longrightarrow \mathrm{Ext}^{-1}((L_N)_s, \mathrm{Cone}(u)_s)$$
$$\xrightarrow{0} \mathrm{Hom}((L_N)_s, \mathrm{Cone}(d^2)_s) \longrightarrow \mathrm{Hom}((L_N)_s, (L_{n_0})_s \otimes (W_0)_s)$$

for $N \gg 0$ and $s \in U$, we have $\mathrm{Ext}^{-1}((L_N)_s, \mathrm{Cone}(u)_s) = 0$ for $N \gg 0$ and $s \in U$. By Definition 3.1 (5) and Remark 3.6, there is an object $E \in \mathcal{D}_U$ and a morphism $\alpha : \mathrm{Cone}(u) \to E$ such that $R^0 \mathrm{Hom}_p((L_N)_U, \mathrm{Cone}(u)) \to R^0 \mathrm{Hom}_p((L_N)_U, E)$ is isomorphic for $N \gg 0$ and that $\mathrm{Ext}^j((L_N)_s, E_s) = 0$ for $N \gg 0$, $j \neq 0$ and $s \in U$. We can see that the sequence

$$R^0 \mathrm{Hom}_p((L_N)_U, (L_{n_2})_U \otimes W_2) \longrightarrow R^0 \mathrm{Hom}_p((L_N)_U, (L_{n_1})_U \otimes W_1) \longrightarrow$$
$$R^0 \mathrm{Hom}_p((L_N)_U, (L_{n_0})_U \otimes W_0) \longrightarrow R^0 \mathrm{Hom}_p((L_N)_U, \mathrm{Cone}(u)) \longrightarrow 0$$

is exact. Since $R^0 \operatorname{Hom}_p((L_N)_U, \operatorname{Cone}(u)) \cong R^0 \operatorname{Hom}_p((L_N)_U, E)$ for $N \gg 0$, there is an integer N_0 such that for any $N \geq N_0$, there is a unique isomorphism $R^0 \operatorname{Hom}_p((L_N)_U, E) \xrightarrow{\sim} \pi_*(\mathcal{E}(N))$ which makes the diagram

$$\begin{array}{ccccc} R^0 \operatorname{Hom}_p((L_N)_U, (L_{n_1})_U \otimes W_1) & \longrightarrow & R^0 \operatorname{Hom}_p((L_N)_U, (L_{n_0})_U \otimes W_0) & \longrightarrow & R^0 \operatorname{Hom}_p((L_N)_U, E) \longrightarrow 0 \\ \cong\downarrow & & \cong\downarrow & & \cong\downarrow \\ \pi_*(\mathcal{A}(N-n_1) \otimes W_1) & \longrightarrow & \pi_*(\mathcal{A}(N-n_0) \otimes W_0) & \longrightarrow & \pi_*(\mathcal{E}(N)) \longrightarrow 0 \end{array}$$

commute. Note that there is a canonical commutative diagram

$$\begin{array}{ccc} R^0 \operatorname{Hom}_p((L_{N+m})_U, (L_N)_U) \otimes R^0 \operatorname{Hom}_p((L_N)_U, E) & \longrightarrow & R^0 \operatorname{Hom}_p((L_{N+m})_U, E) \\ \downarrow & & \downarrow \\ \pi_*(\mathcal{A}(m)) \otimes \pi_*(\mathcal{E}(N)) & \longrightarrow & \pi_*(\mathcal{E}(N+m)) \end{array}$$

for $N \geq N_0$ and a non-negative integer m. Then we have an isomorphism

$$\bigoplus_{n \geq N_0} R^0 \operatorname{Hom}_p((L_n)_U, E) \xrightarrow{\sim} \bigoplus_{n \geq N_0} \pi_*(\mathcal{E}(n))$$

of graded A_U-modules. So we obtain an isomorphism

$$\psi(E) = \left(\bigoplus_{n \geq N_0} R^0 \operatorname{Hom}_p((L_n)_U, E) \right)^{\sim} \xrightarrow{\sim} \left(\bigoplus_{n \geq N_0} \pi_*(\mathcal{E}(n)) \right)^{\sim} \cong \mathcal{E}.$$

Thus ψ becomes an equivalence of categories. □

Definition 3.10. For a geometric point $\operatorname{Spec} k \to S$, an object $E \in \mathcal{D}_k$ is said to be $\mathcal{L}$-stable (resp. $\mathcal{L}$-semistable) if $\operatorname{Ext}^i((L_n)_k, E) = 0$ for $n \gg 0$ and $i \neq 0$ and the inequality

$$\frac{\dim \operatorname{Hom}((L_m)_k, F)}{\dim \operatorname{Hom}((L_n)_k, F)} < \frac{\dim \operatorname{Hom}((L_m)_k, E)}{\dim \operatorname{Hom}((L_n)_k, E)} \quad \left(\text{resp. } \frac{\dim \operatorname{Hom}((L_m)_k, F)}{\dim \operatorname{Hom}((L_n)_k, F)} \leq \frac{\dim \operatorname{Hom}((L_m)_k, E)}{\dim \operatorname{Hom}((L_n)_k, E)} \right)$$

holds for $n \gg m \gg 0$ and for any non-zero object $F \in \mathcal{D}_k$ satisfying $\operatorname{Ext}^i((L_N)_k, F) = 0$ for $N \gg 0$ and $i \neq 0$ with a morphism $\iota : F \to E$ such that ι is not isomorphic and $\operatorname{Hom}((L_n)_k, F) \to \operatorname{Hom}((L_n)_k, E)$ is injective for $n \gg 0$.

Remark 3.11. Let $\operatorname{Spec} k \to S$ be a geometric point and E an object of $\mathcal{D}_k$ satisfying $\operatorname{Ext}^i((L_n)_k, E) = 0$ for $i \neq 0$ and $n \gg 0$. Let $\mathcal{E}$ be the coherent $\mathcal{A}_k$-module corresponding to E as in Proposition 3.9. Then E is $\mathcal{L}$-stable (resp. $\mathcal{L}$-semistable) if and only if for any coherent $\mathcal{A}_k$-submodule $\mathcal{F}$ of $\mathcal{E}$ with $0 \neq \mathcal{F} \subsetneq \mathcal{E}$, the inequality

$$\frac{\chi(\mathcal{F}(m))}{\chi(\mathcal{F}(n))} < \frac{\chi(\mathcal{E}(m))}{\chi(\mathcal{E}(n))} \quad \left(\text{resp. } \frac{\chi(\mathcal{F}(m))}{\chi(\mathcal{F}(n))} \leq \frac{\chi(\mathcal{E}(m))}{\chi(\mathcal{E}(n))} \right) \tag{1}$$

holds for $n \gg m \gg 0$. We say a coherent $\mathcal{A}_k$-module $\mathcal{E}$ stable (resp. semistable) if the corresponding object E of $\mathcal{D}_k$ is $\mathcal{L}$-stable (resp. $\mathcal{L}$-semistable).

Remark 3.12. For a field K with a morphism $\operatorname{Spec} K \to S$ and an object $E \in \mathcal{D}_K$, we say that E is $\mathcal{L}$-stable (resp. $\mathcal{L}$-semistable) if $E_{\bar{K}}$ is $\mathcal{L}$-stable (resp. $\mathcal{L}$-semistable), where $\bar{K}$ is the algebraic closure of K.

4. Existence of the moduli space of stable objects

Definition 4.1. Let $p : \mathcal{D} \to (\mathrm{Sch}/S)$ be a fibered triangulated category with base change property and $\mathcal{L} = \{L_n\}_{n \geq 0}$ be a strict ample sequence. For a numerical polynomial $P(t) \in \mathbf{Q}[t]$, we define a moduli functor $\mathcal{M}_{\mathcal{D}}^{P,\mathcal{L}} : (\mathrm{Sch}/S) \to (\mathrm{Sets})$ by

$$\mathcal{M}_{\mathcal{D}}^{P,\mathcal{L}}(T) := \left\{ E \in \mathcal{D}_T \;\middle|\; \begin{array}{l} \text{for any geometric point } s \text{ of } T, \text{ for } n \gg 0, \operatorname{Ext}^i((L_n)_s, E_s) = 0 \\ \text{for } i \neq 0 \text{ and } \operatorname{Hom}((L_n)_s, E_s) = P(n) \text{ and } E_s \text{ is } \mathcal{L}\text{-stable} \end{array} \right\} \Big/ \sim,$$

where $E \sim E'$ if there exists a line bundle L on T and an isomorphism $E \xrightarrow{\sim} E' \otimes L$.

We also define a moduli functor $\overline{\mathcal{M}_{\mathcal{D}}^{P,\mathcal{L}}} : (\mathrm{Sch}/S) \to (\mathrm{Sets})$ by

$$\overline{\mathcal{M}_{\mathcal{D}}^{P,\mathcal{L}}}(T) := \left\{ E \in \mathcal{D}_T \;\middle|\; \begin{array}{l} \text{for any geometric point } s \text{ of } T, \text{ for } n \gg 0,\ \mathrm{Ext}^i((L_n)_s, E_s) = 0 \\ \text{for } i \neq 0 \text{ and } \mathrm{Hom}((L_n)_s, E_s) = P(n) \text{ and } E_s \text{ is } \mathcal{L}\text{-semistable} \end{array} \right\} \Big/ \sim,$$

where $E \sim E'$ if there exists a line bundle L on T such that $E \cong E' \otimes L$ or there exist sequences $0 = E_0 \to E_1 \to \cdots \to E_\alpha = E$ and $0 = E'_0 \to E'_1 \to \cdots \to E'_\alpha = E'$ such that $\mathrm{Ext}^i((L_n)_s, (E_j)_s) = \mathrm{Ext}^i((L_n)_s, (E'_j)_s) = 0$ for $n \gg 0$, $i \neq 0$ and $s \in T$, $\mathrm{Hom}((L_n)_s, (E_j)_s) \to \mathrm{Hom}((L_n)_s, (E_{j+1})_s)$ and $\mathrm{Hom}((L_n)_s, (E'_j)_s) \to \mathrm{Hom}((L_n)_s, (E'_{j+1})_s)$ are injective for $n \gg 0$ and $s \in T$ and $\bigoplus_{j=1}^{\alpha} F_j \cong \bigoplus_{j=1}^{\alpha} F'_j \otimes L$, where $F_j = \mathrm{Cone}(E_{j-1} \to E_j)$, $F'_j = \mathrm{Cone}(E'_{j-1} \to E'_j)$ and for any geometric point s of T, $(F_j)_s$ and $(F'_j)_s$ are $\mathcal{L}$-stable such that

$$\frac{\dim \mathrm{Hom}((L_m)_s, (F_j)_s)}{\dim \mathrm{Hom}((L_n)_s, (F_j)_s)} = \frac{P(m)}{P(n)} = \frac{\dim \mathrm{Hom}((L_m)_s, (F'_j)_s)}{\dim \mathrm{Hom}((L_n)_s, (F'_j)_s)}$$

for $n \gg m \gg 0$ and for $j = 1, 2, \ldots, \alpha$.

Proposition 4.2. *For any numerical polynomial $P(t) \in \mathbf{Q}[t]$, the family*

$$\left\{ E \;\middle|\; \begin{array}{l} E \in \mathcal{D}_k \text{ for some geometric point } \mathrm{Spec}\, k \to S, \\ E \text{ is } \mathcal{L}\text{-semistable and } \mathrm{Hom}((L_n)_k, E) = P(n) \text{ for } n \gg 0 \end{array} \right\}$$

is bounded.

Proof. It suffices to show that the corresponding family of coherent $\mathcal{A}$-modules on the fibers of $\mathbf{P}(V_1)$ over S is bounded. For a coherent sheaf $\mathcal{G}$ on $\mathbf{P}(V_1)$, we can write

$$\chi(\mathcal{G}(n)) = \sum_{i=0}^{d} a_i(\mathcal{G}) \binom{n+d-i}{d-i}$$

with $a_i(\mathcal{G})$ integers and we write $\mu(G) = a_1(\mathcal{G})/a_0(\mathcal{G})$. Let $\mathcal{E}$ be a coherent $\mathcal{A}_k$-module such that $\chi(\mathcal{E}(n)) = P(n)$ and the corresponding object of $\mathcal{D}_k$ is $\mathcal{L}$-semistable. Note that $\mathcal{E}$ is of pure dimension. We can take the slope maximal destabilizer $\mathcal{F}$ of $\mathcal{E}$ as a sheaf on $\mathbf{P}(V_1)$. Let $\tilde{\mathcal{F}}$ be the image of $\mathcal{F} \otimes \mathcal{A} \to \mathcal{E}$. Note that there exists a locally free sheaf W of finite rank on S, positive integer N and a surjection

$$W \otimes \mathcal{O}(-N) \longrightarrow \mathcal{A}$$

Then we obtain a surjection

$$W \otimes \mathcal{F}(-N) \longrightarrow \mathcal{F} \otimes \mathcal{A} \longrightarrow \tilde{\mathcal{F}}.$$

Since $W \otimes \mathcal{F}(-N)$ is slope semistable, we have

$$\mu(\mathcal{F}) - N = \mu(W \otimes \mathcal{F}(-N)) \leq \mu(\tilde{\mathcal{F}}) \leq \mu(\mathcal{E}).$$

So the maximal slope $\mu(\mathcal{F})$ is bounded by $N + \mu(\mathcal{E})$. Then we obtain the boundedness by [[6], Theorem 4.2]. □

Proposition 4.3. *Assume that $U \in (\mathrm{Sch}/S)$ and $E \in \mathcal{D}_U$ are given. Then the subsets*

$$U^s = \{x \in U \,|\, E_x \text{ is } \mathcal{L}\text{-stable}\}$$
$$U^{ss} = \{x \in U \,|\, E_x \text{ is } \mathcal{L}\text{-semistable}\}$$

of U are open.

Proof. First we will show that

$$U' = \left\{ x \in U \,\middle|\, \mathrm{Ext}^i((L_n)_x, E_x) = 0 \text{ for } n \gg 0 \text{ and } i \neq 0 \right\}$$

is open in U. By Definition 3.1 (4), there exists a positive integer n_0 such that for any $n \geq n_0$, exists an integer N_n with $N_n > n$ such that for any $N \geq N_n$,

$$\mathrm{Hom}((L_N)_s, (L_n)_s) \otimes \mathrm{Ext}^i((L_n)_s, E_s) \longrightarrow \mathrm{Ext}^i((L_N)_s, E_s)$$

is surjective for any i and $s \in U$. By Definition 3.1 (4), there are integers k_1, k_2 with $k_1 < k_2$ such that $\mathrm{Ext}^i((L_{n_0})_s, E_s) = 0$ for any $s \in U$ except for $k_1 \leq i \leq k_2$. Then we have $\mathrm{Ext}^i((L_N)_s, E_s) = 0$ for $N \geq N_{n_0}$

and $s \in U$, except for $k_1 \le i \le k_2$. Now take any point $x \in U'$. For each $i \ne 0$ with $k_1 \le i \le k_2$, there is an integer m_i with $m_i \ge n_0$ such that $\operatorname{Ext}^i((L_{m_i})_x, E_x) = 0$. For any $N \ge N_{m_i}$,

$$\operatorname{Hom}((L_N)_s, (L_{m_i})_s) \otimes \operatorname{Ext}^i((L_{m_i})_s, E_s) \longrightarrow \operatorname{Hom}((L_N)_s, E_s)$$

is surjective for any $s \in U$. By using Definition 2.2 (4), we can see that there exists an open neighborhood U_i of x such that $\operatorname{Ext}^i((L_{m_i})_y, E_y) = 0$ for any $y \in U_i$. Then we have $\operatorname{Ext}^i((L_N)_y, E_y) = 0$ for $N \ge N_{m_i}$. If we put

$$V := \bigcap_{k_1 \le i \le k_2, i \ne 0} U_i$$

then V is an open neighborhood of x. Put

$$\tilde{N} := \max(\{N_{m_i} | k_1 \le i \le k_2, i \ne 0\} \cup \{N_{n_0}\}).$$

Then we have $\operatorname{Ext}^i((L_N)_y, E_y) = 0$ for any $y \in V$, $i \ne 0$ and $N \ge \tilde{N}$, which means $V \subset U'$. Thus U' is an open subset of U.

By Proposition 3.9, $E_{U'}$ corresponds to a coherent $\mathcal{A}_{U'}$-module $\mathcal{E}$ flat over U'. We can see that U^s coincides with

$$\{x \in U' \,|\mathcal{E} \otimes k(x) \text{ is a stable } \mathcal{A}_x\text{-module}\} .$$

We can see by the argument similar to that of [[3], Proposition 2.3.1] that this subset is open in U'. By the same argument we can also see the openness of U^{ss}. □

Theorem 4.4. *There exists a coarse moduli scheme $\overline{M_{\mathcal{D}}^{P,\mathcal{L}}}$ of $\overline{\mathcal{M}_{\mathcal{D}}^{P,\mathcal{L}}}$ and an open subscheme $M_{\mathcal{D}}^{P,\mathcal{L}}$ of $\overline{M_{\mathcal{D}}^{P,\mathcal{L}}}$ which is a coarse moduli scheme of $\mathcal{M}_{\mathcal{D}}^{P,\mathcal{L}}$.*

Before constructing the moduli space, we first note the following lemma:

Lemma 4.5. *Let $P(x)$ be a numerical polynomial. Then there exists an integer m_0 such that for any $m \ge m_0$, any geometric point s of S, any semi-stable $\mathcal{A}_s$-module $\mathcal{E}$ with $\chi(\mathcal{E}(n)) = P(n)$,*

1. *$\mathcal{E}(m)$ is generated by global sections and $H^i(\mathcal{E}(m)) = 0$ for $i > 0$,*
2. *for any nonzero coherent $\mathcal{A}_s$-submodule $\mathcal{F} \subset \mathcal{E}$, the inequality*

$$\dim H^0(\mathcal{F}(m)) \le \frac{a_0(\mathcal{F})}{a_0(\mathcal{E})} \dim H^0(\mathcal{E}(m))$$

holds, where

$$\chi(\mathcal{E}(n)) = \sum_{i=0}^{d} a_i(\mathcal{E}) \binom{n+d-i}{d-i}, \quad \chi(\mathcal{F}(n)) = \sum_{i=0}^{d} a_i(\mathcal{F}) \binom{n+d-i}{d-i}.$$

Moreover the equality holds if and only if $\chi(\mathcal{E}(n))/a_0(\mathcal{E}) = \chi(\mathcal{F}(n))/a_0(\mathcal{F})$ as polynomials in n.

Proof. Proof is essentially the same as [[8], Proposition 4.10.] □

Take m_0 as in Lemma 4.5. Replacing S by its connected component, we may assume that S is connected. Replacing m_0 if necessary, we may assume by Proposition 4.2 that for any geometric point $E \in \overline{\mathcal{M}_{\mathcal{D}}^{P,\mathcal{L}}}(k)$ and for any $m \ge m_0$, $\operatorname{Ext}^i((L_m)_k, E) = 0$ for $i \ne 0$ and

$$\operatorname{Hom}((L_n)_k, (L_m)_k) \otimes \operatorname{Hom}((L_m)_k, E) \longrightarrow \operatorname{Hom}((L_n)_k, E)$$

is surjective for $n \gg 0$. For a geometric point $E \in \overline{\mathcal{M}_{\mathcal{D}}^{P,\mathcal{L}}}(k)$, we consider the canonical morphism

$$u : (L_{m_0})_k \otimes \operatorname{Hom}((L_{m_0})_k, E) \longrightarrow E$$

and put $E_1 := \operatorname{Cone}(u)[-1]$. We can take $m_1 \gg m_0$ such that for any such E and for any $m \ge m_1$, $\operatorname{Ext}^i((L_m)_k, E_1) = 0$ for $i \ne 0$ and

$$\operatorname{Hom}((L_n)_k, (L_m)_k) \otimes \operatorname{Hom}((L_m)_k, E_1) \longrightarrow \operatorname{Hom}((L_n)_k, E_1)$$

is surjective for $n \gg 0$. We consider the canonical morphism

$$v : (L_{m_1})_k \otimes \operatorname{Hom}((L_{m_1})_k, E_1) \longrightarrow E_1$$

and put $E_2 := \mathrm{Cone}(v)[-1]$. We can take $m_2 \gg 0$ such that for any E and for any $m \geq m_2$, $\mathrm{Ext}^i((L_m)_k, E_2) = 0$ for $i \neq 0$ and

$$\mathrm{Hom}((L_n)_k, (L_m)_k) \otimes \mathrm{Hom}((L_m)_k, E_2) \longrightarrow \mathrm{Hom}((L_n)_k, E_2)$$

is surjective for $n \gg 0$. We put

$$r_0 := \dim_k \mathrm{Hom}((L_{m_0})_k, E), \quad r_1 := \dim_k((L_{m_1})_k, E_1), \quad r_2 := \dim_k((L_{m_2})_k, E_2)$$

and

$$W_0 := \mathcal{O}_S^{\oplus r_0}, \quad W_1 := \mathcal{O}_S^{\oplus r_1}, \quad W_2 := \mathcal{O}_S^{\oplus r_2}.$$

Note that r_0, r_1, r_2 are independent of the choice of E and only depend on P and $\mathcal{L}$. We set

$$Z := \mathbf{V}\left(R^0 \mathrm{Hom}_p(L_{m_2}, L_{m_1})^\vee \otimes W_2 \otimes W_1^\vee\right) \times \mathbf{V}\left(R^0 \mathrm{Hom}_p(L_{m_1}, L_{m_0})^\vee \otimes W_1 \otimes W_0^\vee\right).$$

Let

$$(L_{m_2})_Z \otimes W_2 \xrightarrow{\tilde{v}} (L_{m_1})_Z \otimes W_1 \xrightarrow{\tilde{u}} (L_{m_0})_Z \otimes W_0$$

be the universal family. There exists a closed subscheme $Y \subset Z$ such that

$$Y(T) = \{g \in Z(T) \,|\, g^*(\tilde{u} \circ \tilde{v}) = 0\}$$

for any $T \in (\mathrm{Sch}/S)$. Since the sequence

$$\mathrm{Hom}(\mathrm{Cone}(\tilde{v}_Y), (L_{m_0})_Y \otimes W_0) \xrightarrow{\beta} \mathrm{Hom}((L_{m_1})_Y \otimes W_1, (L_{m_0})_Y \otimes W_0)$$
$$\xrightarrow{\tilde{v}^*} \mathrm{Hom}((L_{m_2})_Y \otimes W_2, (L_{m_0})_Y \otimes W_0)$$

is exact and $\tilde{v}^*(\tilde{u}_Y) = \tilde{u}_Y \circ \tilde{v}_Y = 0$, there exists a morphism $\tilde{w} : \mathrm{Cone}(\tilde{v}_Y) \to (L_{m_0})_Y \otimes W_0$ such that $\beta(\tilde{w}) = \tilde{u}_Y$. We put $\tilde{B} := \mathrm{Cone}(\tilde{w})$ and set

$$Y' := \left\{ x \in Y \,\middle|\, \mathrm{Ext}^{-1}((L_n)_x, \tilde{B}_x) = 0 \text{ for } n \gg 0 \right\}$$

Then we can see that Y' is an open subset of Y. Note that for any $x \in Y'$, $\mathrm{Ext}^i((L_n)_x, \tilde{B}_x) = 0$ for $n \gg 0$ except for $i = -2, 0$. By Definition 3.1 (5), there exist an object $\tilde{E} \in \mathcal{D}_{Y'}$ and a morphism $\tilde{B}_{Y'} \to \tilde{E}$ such that $\mathrm{Ext}^i((L_n)_x, \tilde{E}_x) = 0$ for $n \gg 0$, $x \in Y'$ and $i \neq 0$ and $\mathrm{Hom}((L_n)_x, \tilde{B}_x) \to \mathrm{Hom}((L_n)_x, \tilde{E}_x)$ is isomorphic for $n \gg 0$ and $x \in Y'$. If we set

$$\tilde{E}_1 := \mathrm{Cone}((L_{m_0})_{Y'} \otimes W_0 \to \tilde{E})[-1],$$

$\mathrm{Cone}(\tilde{v})_{Y'} \to (L_{m_0})_{Y'} \otimes W_0$ factors through $\tilde{E}_1$. Moreover, for any $x \in Y'$, $\mathrm{Ext}^i((L_n)_x, (\tilde{E}_1)_x) = 0$ for $i \neq 0$ and $\mathrm{Hom}((L_n)_x, \mathrm{Cone}(\tilde{v})_x) \to \mathrm{Hom}((L_n)_x, (\tilde{E}_1)_x)$ is isomorphic for $n \gg 0$. If we set

$$\tilde{E}_2 := \mathrm{Cone}((L_{m_1})_{Y'} \otimes W_1 \to \tilde{E}_1)[-1],$$

then $\tilde{v}_{Y'}$ factors through $\tilde{E}_2$. Now we put

$$Y^{ss} := \left\{ x \in Y' \,\middle|\, \begin{array}{l} W_0 \otimes k(x) \to \mathrm{Hom}((L_{m_0})_x, \tilde{E}_x) \text{ is isomorphic,} \\ W_j \otimes k(x) \to \mathrm{Hom}((L_{m_j})_x, (\tilde{E}_j)_x) \text{ are isomorphic for } j = 1, 2, \\ \mathrm{Hom}((L_n)_x, \tilde{E}_x) = P(n) \text{ for } n \gg 0 \text{ and } \tilde{E}_x \text{ is } \mathcal{L}\text{-semistable} \end{array} \right\}$$

and

$$Y^s := \left\{ x \in Y^{ss} \,\middle|\, \tilde{E}_x \text{ is } \mathcal{L}\text{-stable} \right\}.$$

Then we can check that Y^s, Y^{ss} are open subsets of Y'. If we put

$$G := GL(W_0) \times GL(W_1) \times GL(W_2),$$

then there is a canonical action of G on Z and Y, Y', Y^{ss}, Y^s are preserved by this action. For a sufficiently large integer N, we put

$$\alpha_0 := \mathrm{rank}\, W_2 + N \,\mathrm{rank}\, W_1$$
$$\alpha_1 := -N \,\mathrm{rank}\, W_0$$
$$\alpha_2 := -\mathrm{rank}\, W_0$$

and consider the character

$$\chi : G \longrightarrow \mathbf{G}_m; \quad (g_0, g_1, g_2) \mapsto \det(g_0)^{\alpha_0} \det(g_1)^{\alpha_1} \det(g_2)^{\alpha_2}.$$

Let us consider the quiver consisting of three vertices v_2, v_1, v_0 and $\operatorname{rank}_{\mathcal{O}_S} R^0 \operatorname{Hom}_p(L_{m_2}, L_{m_1})$-arrows from v_2 to v_1 and $\operatorname{rank}_{\mathcal{O}_S} R^0 \operatorname{Hom}_p(L_{m_1}, L_{m_0})$-arrows from v_1 to v_0. Then the points of Z correspond to the representations of this quiver (see [5] for the definition of quiver and its representation).

Lemma 4.6. *If we take $N \gg m_2 \gg m_1 \gg m_0 \gg 0$, Y^{ss} is contained in the set $Z^{ss}(\chi)$ of χ-semistable points of Z in the sense of* [5]. *Moreover, Y^s is contained in the set $Z^s(\chi)$ of χ-stable points of Z.*

Proof. Take any geometric point x of Y^{ss} and vector subspaces $W'_i \subset (W_i)_x$ $(0 \le i \le 2)$ which induce commutative diagrams

$$\begin{array}{ccccccc} W'_2 & \longrightarrow & W'_1 \otimes R^0 \operatorname{Hom}_p(L_{m_2}, L_{m_1})_x & & W'_1 & \longrightarrow & W'_0 \otimes R^0 \operatorname{Hom}_p(L_{m_1}, L_{m_0})_x \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ (W_2)_x & \longrightarrow & (W_1)_x \otimes R^0 \operatorname{Hom}_p(L_{m_2}, L_{m_1})_x & & (W_1)_x & \longrightarrow & (W_0)_x \otimes R^0 \operatorname{Hom}_p(L_{m_1}, L_{m_0})_x. \end{array}$$

From [5], we should say that

$$\alpha_0 \dim W'_0 + \alpha_1 \dim W'_1 + \alpha_2 \dim W'_2 \ge 0.$$

Let $\mathcal{E}$ be the Y^{ss}-flat $\mathcal{A}_{Y^{ss}}$-module corresponding to $\tilde{E}_{Y^{ss}}$ by Proposition 3.9. Then a morphism $\mathcal{A}(-m_0) \otimes W'_0 \to \mathcal{E}_x$ is induced and we denote its image by $\mathcal{E}(W'_0)$. Note that $\mathcal{E}_x$ is of pure dimension and so $\mathcal{E}(W'_0)$ is also of pure dimension. Since the family

$$\{\mathcal{E}(W'_0) \,|W'_0 \subset (W_0)_x, \ x \text{ is a geometric point of } Y^{ss}\}$$

is bounded, we can find an integer $m_1 \gg m_0$ such that for $K'_1 := \ker(W'_0 \otimes \mathcal{A}(-m_0) \to \mathcal{E}(W'_0))$, $K'_1(m_1)$ is generated by global sections and $H^i(K'_1(m_1)) = 0$, $H^i(\mathcal{A}_x(m_1 - m_0)) = 0$ for $i > 0$. Moreover we can find an integer $m_2 \gg m_1$ such that for $K'_2 := \ker(H^0(K'_1(m_1)) \otimes \mathcal{A}(-m_1) \to K'_1)$, $K'_2(m_2)$ is generated by global sections and $H^i(K'_2(m_2)) = 0$, $H^i(\mathcal{A}_x(m_2 - m_1)) = 0$, $H^i(\mathcal{A}_x(m_2 - m_0)) = 0$ and $H^i(K'_1(m_2)) = 0$ for $i > 0$. If we put $\tilde{W}'_1 := H^0(K'_1(m_1))$ and $\tilde{W}'_2 := H^0(K'_2(m_2))$, then we have

$$\dim H^0(\mathcal{E}(W'_0)(m_1)) = \dim H^0(\mathcal{A}_x(m_1 - m_0)) \dim W'_0 - \dim \tilde{W}'_1$$
$$\dim H^0(\mathcal{E}(W'_0)(m_2)) = \dim H^0(\mathcal{A}_x(m_2 - m_0)) \dim W'_0 - \dim H^0(\mathcal{A}_x(m_2 - m_1)) \dim \tilde{W}'_1 + \dim \tilde{W}'_2.$$

Since the family $\{\mathcal{E}(W'_0)\}$ is bounded, we can take by using Lemma 4.5 a positive integer $m_0 \gg 0$ and a positive number $\epsilon > 0$ such that

$$\frac{h^0(\mathcal{E}(W'_0)(m_0))}{P(m_0)} < \frac{a_0(\mathcal{E}(W'_0))}{a_0(P)} - \epsilon$$

for any W'_0 such that

$$\frac{\chi(\mathcal{E}(W'_0)(m))}{\chi(\mathcal{E}(W'_0)(n))} < \frac{P(m)}{P(n)}$$

for $n \gg m \gg 0$. Here we write

$$\chi(\mathcal{E}(W'_0)(n)) = \sum_{i=0}^{d} a_i(\mathcal{E}(W'_0)) \binom{n+d-i}{d-i}, \quad P(n) = \sum_{i=0}^{d} a_i(P) \binom{n+d-i}{d-i}$$

with $a_i(\mathcal{E}(W'_0))$ and $a_i(P)$ integers. Since

$$\lim_{m_1 \to \infty} \frac{h^0(\mathcal{E}(W'_0)(m_1))}{P(m_1)} = \frac{a_0(\mathcal{E}(W'_0))}{a_0(P)},$$

we can take $m_1 \gg m_0$ such that

$$\frac{h^0(\mathcal{E}(W'_0)(m_1))}{P(m_1)} > \frac{a_0(\mathcal{E}(W'_0))}{a_0(P)} - \frac{\epsilon}{2}.$$

Since

$$\lim_{N \to \infty} \frac{(h^0(\mathcal{A}_x(m_2 - m_1)) + N) h^0(\mathcal{E}(W'_0)(m_1)) - h^0(\mathcal{E}(W'_0)(m_2))}{(h^0(\mathcal{A}_x(m_2 - m_1)) + N) P(m_1) - P(m_2)} = \frac{h^0(\mathcal{E}(W'_0)(m_1))}{P(m_1)},$$

we can take $N \gg m_2$ such that

$$\frac{(h^0(\mathcal{A}_x(m_2 - m_1)) + N) h^0(\mathcal{E}(W'_0)(m_1)) - h^0(\mathcal{E}(W'_0)(m_2))}{(h^0(\mathcal{A}_x(m_2 - m_1)) + N) P(m_1) - P(m_2)} > \frac{h^0(\mathcal{E}(W'_0)(m_1))}{P(m_1)} - \frac{\epsilon}{2}.$$

Then we have

$$\begin{aligned}\frac{h^0(\mathcal{E}(W_0')(m_0))}{P(m_0)} &< \frac{a_0(\mathcal{E}(W_0'))}{a_0(P)} - \epsilon \\ &< \frac{h^0(\mathcal{E}(W_0')(m_1))}{P(m_1)} + \frac{\epsilon}{2} - \epsilon \\ &< \frac{(h^0(\mathcal{A}_x(m_2-m_1))+N)h^0(\mathcal{E}(W_0')(m_1)) - h^0(\mathcal{E}(W_0')(m_2))}{(h^0(\mathcal{A}_x(m_2-m_1))+N)P(m_1) - P(m_2)} + \frac{\epsilon}{2} + \frac{\epsilon}{2} - \epsilon \\ &= \frac{(h^0(\mathcal{A}_x(m_2-m_1))+N)h^0(\mathcal{E}(W_0')(m_1)) - h^0(\mathcal{E}(W_0')(m_2))}{(h^0(\mathcal{A}_x(m_2-m_1))+N)P(m_1) - P(m_2)}\end{aligned}$$

for any W_0' such that

$$\frac{\chi(\mathcal{E}(W_0')(m))}{\chi(\mathcal{E}(W_0')(n))} < \frac{P(m)}{P(n)}$$

for $n \gg m \gg 0$. Take W_0' such that

$$\frac{\chi(\mathcal{E}(W_0')(m))}{\chi(\mathcal{E}(W_0')(n)} = \frac{P(m)}{P(n)}$$

for $n \gg m \gg 0$. Then we can see by Lemma 4.5 that

$$\frac{h^0(\mathcal{E}(W_0')(m_0))}{P(m_0)} = \frac{a_0(\mathcal{E}(W_0'))}{a_0(P)} = \frac{(h^0(\mathcal{A}_x(m_2-m_1))+N)h^0(\mathcal{E}(W_0')(m_1)) - h^0(\mathcal{E}(W_0')(m_2))}{(h^0(\mathcal{A}_x(m_2-m_1))+N)P(m_1) - P(m_2)}.$$

Hence we have the inequality

$$h^0(\mathcal{E}(W_0')(m_0)) \leq \frac{(h^0(\mathcal{A}_x(m_2-m_1))+N)h^0(\mathcal{E}(W_0')(m_1)) - h^0(\mathcal{E}(W_0')(m_2)}{(h^0(\mathcal{A}_x(m_2-m_1))+N)P(m_1) - P(m_2)} P(m_0) \tag{2}$$

for any $\mathcal{E}(W_0')$. Moreover, the equality holds in (2) if and only if $\chi(\mathcal{E}(W_0')(n))/a_0(\mathcal{E}(W_0')) = P(n)/a_0(P)$ as polynomials in n. From the inequality (2), we obtain the inequality

$$(r_2 + Nr_1)\dim W_0' - Nr_0 \dim \tilde{W}_1' - r_0 \dim \tilde{W}_2' \geq 0$$

by using $\dim W_0' \leq h^0(\mathcal{E}(W_0')(m_0))$. Since $\dim W_1' \leq \dim \tilde{W}_1'$ and $\dim W_2' \leq \dim \tilde{W}_2'$, we have

$$\alpha_0 \dim W_0' + \alpha_1 \dim W_1' + \alpha_2 \dim W_2' \geq 0. \tag{3}$$

Thus x becomes a geometric point of $Z^{ss}(\chi)$.

In the inequality (3), the equality holds if and only if $\dim \tilde{W}_1' = \dim W_1'$, $\dim \tilde{W}_2' = \dim W_2'$, $h^0(\mathcal{E}(W_0')) = \dim W_0'$ and $\chi(\mathcal{E}(W_0')(n))/a_0(\mathcal{E}(W_0')) = P(n)/a_0(P)$ as polynomials in n. So, if x is a geometric point of Y^s, we have

$$(r_2 + Nr_1)\dim W_0' - Nr_0 \dim W_1' - r_0 \dim W_2' > 0.$$

for any (W_0', W_1', W_2') with $(0,0,0) \neq (W_0', W_1', W_2') \subsetneq ((W_0)_x, (W_1)_x, (W_2)_x)$, which means that x becomes a geometric point of $Z^s(\chi)$. □

By [5] and [9], there exists a GIT quotient $\phi : Y \cap Z^{ss}(\chi) \to (Y \cap Z^{ss}(\chi))//G$.

Lemma 4.7. $\phi^{-1}(\phi(Y^{ss})) = Y^{ss}$.

Proof. It is sufficient to show that $\phi^{-1}(\phi(Y^{ss})) \subset Y^{ss}$. Take any k-valued geometric point x of $\phi^{-1}(\phi(Y^{ss}))$. Let s be the induced k-valued goemetric point of S. Since $\phi(x)$ is a geometric point of $\phi(Y^{ss})$, there exists a k-valued geometric point y of Y^{ss} such that $\phi(x) = \phi(y)$.

Let $\mathcal{E}$ be the Y^{ss}-flat $\mathcal{A}_{Y^{ss}}$-module corresponding to $\tilde{E}_{Y^{ss}}$ as in the proof of Lemma 4.6. Then there is a Jordan-Hölder filtration

$$0 = F^{(0)} \subset F^{(1)} \subset \cdots \subset F^{(l)} = \mathcal{E} \otimes k(y)$$

of $\mathcal{E} \otimes k(y)$. For each i with $1 \leq i \leq l$, we define $K_1^{(i)}$, $K_2^{(i)}$ by exact sequences

$$0 \longrightarrow K_1^{(i)} \longrightarrow H^0(F^{(i)}(m_0)) \otimes \mathcal{A}(-m_0) \longrightarrow F^{(i)} \longrightarrow 0$$

$$0 \longrightarrow K_2^{(i)} \longrightarrow H^0(K_1^{(i)}(m_1)) \otimes \mathcal{A}(-m_1) \longrightarrow K_1^{(i)} \longrightarrow 0.$$

Then y corresponds to the representation of quiver given by

$$H^0(K_2^{(l)}(m_2)) \longrightarrow H^0(K_1^{(l)}(m_1)) \otimes H^0(\mathcal{A}_s(m_2-m_1))$$
$$H^0(K_1^{(l)}(m_1)) \longrightarrow H^0(F^{(l)}(m_0)) \otimes H^0(\mathcal{A}_s(m_1-m_0))$$

and the Jordan-Hölder filtration of $\mathcal{E} \otimes k(y)$ corresponds to the filtration of the quiver representation given by

$$0 \subset H^0(K_2^{(1)}(m_2)) \subset \cdots \subset H^0(K_2^{(l)}(m_2))$$
$$0 \subset H^0(K_1^{(1)}(m_1)) \subset \cdots \subset H^0(K_1^{(l)}(m_1))$$
$$0 \subset H^0(F^{(1)}(m_0)) \subset \cdots \subset H^0(F^{(l)}(m_0)).$$

We put $E^{(i)} := F^{(i)}/F^{(i-1)}$ and $\overline{\mathcal{E}} := \bigoplus_{i=1}^{l} E^{(i)}$. For $i = 1, \ldots, l$, we define $\bar{K}_1^{(i)}$, $\bar{K}_2^{(i)}$ by the exact sequences

$$0 \longrightarrow \bar{K}_1^{(i)} \longrightarrow H^0(E^{(i)}(m_0)) \otimes \mathcal{A}(-m_0) \longrightarrow E^{(i)} \longrightarrow 0$$
$$0 \longrightarrow \bar{K}_2^{(i)} \longrightarrow H^0(\bar{K}_1^{(i)}(m_1)) \otimes \mathcal{A}(-m_1) \longrightarrow \bar{K}_1^{(i)} \longrightarrow 0.$$

We can see from the proof of Lemma 4.6 that the quiver representation y_i given by

$$H^0(\bar{K}_2^{(i)}(m_2)) \longrightarrow H^0(\bar{K}_1^{(i)}(m_1)) \otimes H^0(\mathcal{A}_s(m_2-m_1))$$
$$H^0(\bar{K}_1^{(i)}(m_1)) \longrightarrow H^0(E^{(i)}(m_0)) \otimes H^0(\mathcal{A}_s(m_1-m_0))$$

is stable with respect to the weight $(\alpha_0, \alpha_1, \alpha_2)$. The direct sum $y_1 \oplus \cdots \oplus y_l$ corresponds to a point y' of Y_s^{ss} given by the exact sequence

$$H^0\left(\bigoplus_{i=1}^{l} \bar{K}_2^{(i)}(m_2)\right) \otimes \mathcal{A}(-m_2) \longrightarrow H^0\left(\bigoplus_{i=1}^{l} \bar{K}_1^{(i)}(m_1)\right) \otimes \mathcal{A}(-m_1)$$
$$\longrightarrow H^0\left(\bigoplus_{i=1}^{l} E^{(i)}(m_0)\right) \otimes \mathcal{A}(-m_0) \longrightarrow \bigoplus_{i=1}^{l} E^{(i)} \longrightarrow 0.$$

Then we can see that the quiver representations determined by y and y' are S-equivalent. So we have $\phi(x) = \phi(y) = \phi(y')$. Note that $G_s y'$ is a closed orbit in $(Y \cap Z^{ss}(\chi))_s$ by [[5], Proposition 3.2]. Thus the closure of the G_s-orbit of x must contain y'. Then, by Proposition 4.3, x becomes a geometric point of Y_s^{ss}. □

Proof of Theorem 4.4. If we put

$$\overline{M_{\mathcal{D}}^{P,\mathcal{L}}} := \phi(Y^{ss}),$$

then we can see by Lemma 4.7 that $\overline{M_{\mathcal{D}}^{P,\mathcal{L}}}$ is an open subset of $(Y \cap Z^{ss}(\chi))//G$. We can see by a similar argument to that of [[8], Proposition 7.3] that there is a canonical morphism $\Phi : \overline{\mathcal{M}_{\mathcal{D}}^{P,\mathcal{L}}} \to \overline{M_{\mathcal{D}}^{P,\mathcal{L}}}$. For two geometric points $x_1, x_2 \in Y^{ss}$ over a geometric point s of S, $\phi(x_1) = \phi(x_2)$ if and only if the corresponding representations of quiver are S-equivalent ([5]), that is, the corresponding objects of $\mathcal{D}_s$ are S-equivalent. Thus for any algebraically closed field k over S, $\Phi(k) : \overline{\mathcal{M}_{\mathcal{D}}^{P,\mathcal{L}}}(k) \to \overline{M_{\mathcal{D}}^{P,\mathcal{L}}}(k)$ is bijective. We can see by a standard argument that $\overline{M_{\mathcal{D}}^{P,\mathcal{L}}}$ has the universal property of the coarse moduli scheme. If we put $M_{\mathcal{D}}^{P,\mathcal{L}} := Y^s/G$, then $M_{\mathcal{D}}^{P,\mathcal{L}}$ becomes an open subset of $\overline{M_{\mathcal{D}}^{P,\mathcal{L}}}$ and we can easily see that $M_{\mathcal{D}}^{P,\mathcal{L}}$ is a coarse moduli scheme of $\mathcal{M}_{\mathcal{D}}^{P,\mathcal{L}}$. So we have proved Theorem 4.4. □

Theorem 4.8. *Assume that S is of finite type over a universally Japanese ring Ξ. Then the moduli scheme $\overline{M_{\mathcal{D}}^{P,\mathcal{L}}}$ is projective over S.*

For the proof of Theorem 4.8, the following lemma is essential.

Lemma 4.9. *Let R be a discrete valuation ring over S with quotient field K and residue field k. Assume that E is an object of $\mathcal{D}_K$ which is $\mathcal{L}$-semistable. Then there is an object $\tilde{E} \in \mathcal{D}_R$ such that $\tilde{E}_K \cong E$ and $\tilde{E}_k$ is $\mathcal{L}$-semistable.*

Proof. The above E corresponds to a coherent $\mathcal{A}_K$-module $\mathcal{E}$ and it suffices to show that there exists an R-flat coherent $\mathcal{A}_R$-module $\tilde{\mathcal{E}}$ such that $\tilde{\mathcal{E}} \otimes_R K \cong \mathcal{E}$ and $\tilde{\mathcal{E}} \otimes k$ satisfies the semistability condition given by the inequality in Remark 3.11. For a sufficiently large integer N, we have $H^i(\mathcal{E}(N)) = 0$ for $i > 0$ and $\mathcal{E}(N)$ is generated by global sections. Then there is a surjection $\mathcal{A}_K(-N)^{\oplus r} \to \mathcal{E}$ which determies a K-valued point η of the Quot-scheme $\mathrm{Quot}^P_{\mathcal{A}(-N)^{\oplus r}}$ for some numerical polynomial P, where $r = \dim H^0(\mathcal{E}(N))$. Let $\mathcal{F} \subset \mathcal{A}(-N)^{\oplus r}$ be the universal subsheaf and Y be the maximal closed subscheme of $\mathrm{Quot}^P_{\mathcal{A}(-N)^{\oplus r}}$ such that $\mathcal{A} \otimes \mathcal{F}_Y \to \mathcal{A}(-N)^{\oplus r}_Y$ factors through $\mathcal{F}_Y$. Then η is a K-valued point of Y and extends to an R-valued point ξ of Y because Y is proper over S. ξ corresponds to an R-flat quotient coherent $\mathcal{A}_R$-module $\mathcal{E}'$ of $\mathcal{A}(-N)^{\oplus r}_R$ and we have $\mathcal{E}' \otimes_R K \cong \mathcal{E}$. From the proof similar to that of Langton's theorem ([3], Theorem 2.B.1), we can obtain an R-flat coherent $\mathcal{A}_R$-module $\tilde{\mathcal{E}}$ by taking succesive elementary transforms of $\mathcal{E}'$ along $\mathbf{P}(V_1) \times \mathrm{Spec}\, k$ such that $\tilde{\mathcal{E}} \otimes_R K \cong \mathcal{E}' \otimes_R K \cong \mathcal{E}$ and $\tilde{\mathcal{E}} \otimes k$ is semistable as $\mathcal{A} \otimes k$-module. □

Now we prove Theorem 4.8. By construction, the moduli scheme $\overline{M^{P,\mathcal{L}}_{\mathcal{D}}}$ is quasi-projective over S. So it is sufficient to show that $\overline{M^{P,\mathcal{L}}_{\mathcal{D}}}$ is proper over S. Let R be a discrete valuation ring over S with quotient field K and let $\varphi : \mathrm{Spec}\, K \to \overline{M^{P,\mathcal{L}}_{\mathcal{D}}}$ be a morphism over S. Then there is a finite extension field K' of K such that the composite $\psi : \mathrm{Spec}\, K' \to \mathrm{Spec}\, K \xrightarrow{\varphi} \overline{M^{P,\mathcal{L}}_{\mathcal{D}}}$ is given by an $\mathcal{L}$-semistable object E'. We can take a discrete valuation ring R' with quotient field K' such that $K \cap R' = R$. Let k' be the residue field of R'. By Lemma 4.9, there exists an object E of $\mathcal{D}_{R'}$ such that $E_{K'} \cong E'$ and $E_{k'}$ is $\mathcal{L}$-semistable. Then E gives a morphism $\overline{\psi} : \mathrm{Spec}\, R' \to \overline{M^{P,\mathcal{L}}_{\mathcal{D}}}$ which is an extension of ψ. We can easily see that $\overline{\psi}$ factors through $\mathrm{Spec}\, R$. Thus $\overline{M^{P,\mathcal{L}}_{\mathcal{D}}}$ is proper over S by the valuative criterion of properness. □

5. Examples

In this section, we give several examples of moduli spaces of stable objects determined by a strict ample sequence.

Example 5.1. Let $f : X \to S$ be a flat projective morphism of noetherian schemes and let $\mathcal{O}_X(1)$ be an S-very ample line bundle on X such that $H^i(\mathcal{O}_{X_s}(m)) = 0$ for $i > 0$, $s \in S$ and $m > 0$. Consider the fibered triangulated category $\mathcal{D}_{X/S}$ defined by $(\mathcal{D}_{X/S})_U = D^b(\mathrm{Coh}(X_U/U))$ for $U \in (\mathrm{Sch}/S)$. Then $\mathcal{L} = \{\mathcal{O}_X(-n)\}_{n \geq 0}$ becomes a strict ample sequence in $\mathcal{D}_{X/S}$.

Proof. Definition 3.1 (1),(2),(3) are easy to verify. Let us prove Definition 3.1 (4). Take any $U \in (\mathrm{Sch}/S)$ and any object $E^\bullet \in (\mathcal{D}_{X/S})_U$. We may assume that $E^\bullet$ is given by a complex

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow E^{l_1} \xrightarrow{d^{l_1}} E^{l_1+1} \xrightarrow{d^{l_1+1}} \cdots \xrightarrow{d^{l_2-1}} E^{l_2} \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots,$$

where each E^i is a coherent sheaf on X_U flat over U. By flattening stratification theorem, there is a stratification $U = \coprod_{j=1}^m Y_j$ of U by subschemes Y_j such that each $\mathrm{coker}(d^i)_{Y_j} = \mathrm{coker}(d^i_{Y_j})$ is flat over Y_j for any i and j. Then we can see that $\mathrm{im}(d^i_{Y_j})$ and $\ker(d^i_{Y_j})$ are flat over Y_j for any i and j. For any point $s \in U$, the sequence

$$0 \longrightarrow \mathrm{im}(d^{i-1}_{Y_j}) \otimes k(s) \longrightarrow E^i \otimes k(s) \longrightarrow \mathrm{coker}(d^i_{Y_j}) \otimes k(s) \longrightarrow 0$$

is exact because $\mathrm{coker}(d_{Y_j})$ is flat over Y_j. Then the homomorphism $\mathrm{im}(d^{i-1}_{Y_j}) \otimes k(s) \longrightarrow \ker(d^i_{Y_j}) \otimes k(s)$ is injective for any $s \in Y_j$. Thus the cohomology sheaf $\mathcal{H}^i(E^\bullet_{Y_j}) := \ker(d^i_{Y_j}) / \mathrm{im}(d^{i-1}_{Y_j})$ is flat over Y_j for any i and j. We can take a positive integer n_0 such that for any $n \geq n_0$, $R^p(f_{Y_j})_*(E^i_{Y_j}(n)) = 0$, $R^p(f_{Y_j})_*(\mathrm{im}(d^i_{Y_j})(n)) = 0$ and $R^p(f_{Y_j})_*(\ker d^i_{Y_j}(n)) = 0$ for any $p > 0$ and any i, j. Then we have $R^p(f_{Y_j})_*(\mathcal{H}^i(E^\bullet_{Y_j}(n))) = 0$ for any $p > 0$, any i, j and $n \geq n_0$. From the spectral sequence $R^p(f_{Y_j})_*(\mathcal{H}^q(E^\bullet_{Y_J}(n))) \Rightarrow R^{p+q}(f_{Y_j})_*(E^\bullet_{Y_j}(n))$, we have an isomorphism $R^i(f_{Y_j})_*(E^\bullet_{Y_j}(n)) \cong (f_{Y_j})_*(\mathcal{H}^i(E^\bullet_{Y_j})(n))$ for any i, j and $n \geq n_0$. So we can see that $\mathbf{R}(f_{Y_j})_*(E^\bullet_{Y_j}(n))$ is quasi-isomorphic to the complex

$$\cdots \longrightarrow 0 \longrightarrow (f_{Y_j})_*(E^{l_1}_{Y_j}(n)) \longrightarrow (f_{Y_j})_*(E^{l_1+1}_{Y_j}(n)) \longrightarrow \cdots \longrightarrow (f_{Y_j})_*(E^{l_2}_{Y_j}(n)) \longrightarrow 0 \longrightarrow \cdots$$

for any i,j and $n \geq n_0$. Note that there are canonical isomorphisms

$$\mathbf{H}^i(E^\bullet_s(n)) \cong R^i(f_{Y_j})_*(E^\bullet_{Y_j}(n)) \otimes k(s) \cong (f_{Y_j})_*(\mathcal{H}^i(E^\bullet_{Y_j})(n)) \otimes k(s) \cong H^0(X_s, \mathcal{H}^i(E^\bullet_s)(n)).$$

for any i,j, any $s \in Y_j$ and $n \geq n_0$. If we take n_0 sufficiently larger, we may assume that the homomorphism

$$(f_{Y_j})^*(f_{Y_j})_*(\mathcal{H}^i(E^\bullet_{Y_j}(n))) \longrightarrow \mathcal{H}^i(E^\bullet_{Y_j})(n)$$

is surjective for any $n \geq n_0$ and any i,j. Thus there exists a positive integer $N_0 \gg n$ such that

$$(f_{Y_j})_*(\mathcal{O}_{X_{Y_j}}(N-n)) \otimes (f_{Y_j})_*(\mathcal{H}^i(E^\bullet_{Y_j}(n))) \longrightarrow (f_{Y_j})_*(\mathcal{H}^i(E^\bullet_{Y_j})(N))$$

is surjective for any $N \geq N_0$ and any i,j. So we obtain a commutative diagram

$$\begin{array}{ccc}
(f_{Y_j})_*(\mathcal{O}_{X_{Y_j}}(N-n)) \otimes k(s) \otimes (f_{Y_j})_*(\mathcal{H}^i(\mathcal{E}^\bullet_{Y_j}(n))) \otimes k(s) & \longrightarrow & (f_{Y_j})_*(\mathcal{H}^i(E^\bullet_{Y_j}(N)) \otimes k(s) \\
\cong\downarrow & & \cong\downarrow \\
H^0(\mathcal{O}_{X_s}(N-n)) \otimes \mathbf{H}^i(E^\bullet_s(n)) & \longrightarrow & \mathbf{H}^i(E^\bullet_s(N)) \\
\cong\downarrow & & \cong\downarrow \\
\operatorname{Hom}(\mathcal{O}_{X_s}(-N), \mathcal{O}_{X_s}(-n)) \otimes \operatorname{Ext}^i(\mathcal{O}_{X_s}(-n), E^\bullet_s) & \longrightarrow & \operatorname{Ext}^i(\mathcal{O}_{X_s}(-N), E^\bullet_s).
\end{array}$$

for any i,j, any $s \in Y_j$ and $N \geq N_0$. Hence

$$\operatorname{Hom}(\mathcal{O}_{X_s}(-N), \mathcal{O}_{X_s}(-n)) \otimes \operatorname{Ext}^i(\mathcal{O}_{X_s}(-n), E^\bullet_s) \longrightarrow \operatorname{Ext}^i(\mathcal{O}_{X_s}(-N), E^\bullet_s)$$

is surjective for any $s \in U$, any i and $N \geq N_0$ and we have proved Definition 3.1 (4).

Now we prove Definition 3.1 (5). Assume that an object $E \in (\mathcal{D}_{X/S})_U$ and integers i, n_0 are given such that $\operatorname{Ext}^i(\mathcal{O}_{X_s}(-n), E^\bullet_s) = 0$ for any $s \in U$ and $n \geq n_0$. Replacing n_0 by a sufficiently large integer, we have

$$\operatorname{Ext}^i(\mathcal{O}_{X_s}(-n), E^\bullet_s) \cong \mathbf{H}^i(E^\bullet_s(n)) \cong H^0(X_s, \mathcal{H}^i(E^\bullet_s)(n)) = 0$$

for any $s \in U$ and any $n \geq n_0$. Then we have $\mathcal{H}^i(E^\bullet_s) = 0$. If $E^\bullet$ is given by

$$E^{l_1} \xrightarrow{d^{l_1}} E^{l_1+1} \xrightarrow{d^{l_1+1}} \cdots \xrightarrow{d^{l_2-1}} E^{l_2},$$

such that each E^j is flat over U, then the induced homomorphism $\operatorname{coker}(d^{i-1}) \otimes k(s) \to E^{i+1} \otimes k(s)$ is injective for any $s \in U$. Then $\operatorname{coker}(d^i)$ is flat over U and $\operatorname{coker}(d^{i-1}) \to E^{i+1}$ is injective. Let $F^\bullet$ be the complex given by

$$\cdots \longrightarrow 0 \longrightarrow \operatorname{coker}(d^i) \longrightarrow E^{i+2} \xrightarrow{d^{i+2}} \cdots \xrightarrow{d^{l_2-1}} E^{l_2} \longrightarrow 0 \longrightarrow \cdots.$$

Then there is a canonical morphism $u : E^\bullet \to F^\bullet$. Note that

$$R^j \operatorname{Hom}_f(\mathcal{O}_{X_U}(-n), E^\bullet) = R^j(f_U)_*(E^\bullet(n)) \cong (f_U)_*(\mathcal{H}^j(E^\bullet)(n))$$

for $n \gg 0$. So u induces isomorphisms

$$R^j \operatorname{Hom}_f(\mathcal{O}_{X_U}(-n), E^\bullet) \xrightarrow{\sim} (f_U)_*(\mathcal{H}^j(E^\bullet)(n)) \xrightarrow{\sim} (f_U)_*(\mathcal{H}^j(F^\bullet)(n)) \xrightarrow{\sim} R^j \operatorname{Hom}_f(\mathcal{O}_{X_U}(-n), F^\bullet)$$

for $j > i$ and $n \gg 0$. By definition we have $R^j \operatorname{Hom}_f(\mathcal{O}_{X_U}(-n), F^\bullet) = (f_U)_*(\mathcal{H}^j(F^\bullet(n))) = 0$ for $j \leq i$ and $n \gg 0$. Thus we have proved Definition 3.1 (5).

Finally, let us prove Definition 3.1 (6). Let $E^\bullet$ and $F^\bullet$ be objects of $(\mathcal{D}_{X/S})_U$. Assume that $R^j(f_U)_*(E^\bullet(n)) = 0$ for $j \geq 0$ and $n \gg 0$ and that $R^j(f_U)_*(F^\bullet(n)) = 0$ for $j < 0$ and $n \gg 0$. Since $R^j(f_U)_*(E^\bullet(n)) \cong (f_U)_*(\mathcal{H}^j(E^\bullet)(n))$ for $n \gg 0$, we have $\mathcal{H}^j(E^\bullet) = 0$ for $j \geq 0$. Then $E^\bullet$ is quasi-isomorphic to the complex given by

$$\cdots \longrightarrow 0 \longrightarrow E^{l_1} \xrightarrow{d_E^{l_1}} E^{l_1+1} \longrightarrow \cdots \longrightarrow E^{-2} \longrightarrow \ker(d_E^{-1}) \longrightarrow 0 \longrightarrow \cdots.$$

On the other hand, we have $\mathcal{H}^j(F^\bullet) = 0$ for $j < 0$, because $R^j(f_U)_*(F^\bullet(n)) \cong (f_U)_*(\mathcal{H}^j(F^\bullet)(n))$ for $n \gg 0$. Then $F^\bullet$ is quasi-isomorphic to the complex given by

$$\cdots \longrightarrow 0 \longrightarrow \operatorname{coker} d_F^{-1} \longrightarrow F^1 \xrightarrow{d_F^1} \cdots \longrightarrow F^{m_2} \longrightarrow 0 \longrightarrow \cdots.$$

We can take a complex

$$\cdots \longrightarrow 0 \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots$$

such that each I^j is an injective sheaf on X_U and that $I^\bullet$ is quasi-isomorphic to $F^\bullet$. Then we have $\mathrm{Hom}_{(\mathcal{D}_{X/S})_U}(E^\bullet, F^\bullet) \cong H^0(\mathrm{Hom}^\bullet(E^\bullet, I^\bullet)) = 0$. So we have proved Definition 3.1 (6). □

For an object $E \in (\mathcal{D}_{X/S})_U$, $\mathrm{Ext}^i(\mathcal{O}_{X_s}(-n), E_s) = 0$ for $n \gg 0$, $i \neq 0$ and $s \in U$ if and only if $E^\bullet$ is quasi-isomorphic to a coherent sheaf on X_U flat over U. Hence, for a numerical polynomial P, the moduli space $M^{P,\mathcal{L}}_{\mathcal{D}_{X/S}}$ (resp. $\overline{M^{P,\mathcal{L}}_{\mathcal{D}_{X/S}}}$) is just the usual moduli space of $\mathcal{O}_X(1)$-stable sheaves (resp. moduli space of S-equivalence classes of $\mathcal{O}_X(1)$-semistable sheaves) on X over S.

Example 5.2. Let X, S, $\mathcal{O}_X(1)$ and $\mathcal{D}_{X/S}$ be as in Example 5.1. Take a vector bundle G on X. Replacing $\mathcal{O}_X(1)$ by some multiple, $\mathcal{L}_G = \{G \otimes \mathcal{O}_X(-n)\}_{n\geq 0}$ also becomes a strict ample sequence in $\mathcal{D}_{X/S}$ and the moduli space $M^{P,\mathcal{L}_G}_{\mathcal{D}_{X/S}}$ (resp. $\overline{M^{P,\mathcal{L}_G}_{\mathcal{D}_{X/S}}}$) is the moduli space of G-twisted $\mathcal{O}_X(1)$-stable sheaves (resp. moduli space of S-equivalence classes of G-twisted $\mathcal{O}_X(1)$-semistable sheaves) on X over S.

Example 5.3. Let X, Y be projective schemes over an algebraically closed field k and let $\mathcal{O}_X(1)$ be a very ample line bundle on X such that $H^i(X, \mathcal{O}_X(m)) = 0$ for $i > 0$ and $m > 0$. Assume that a Fourier-Mukai transform

$$\Phi : D^b_c(X) \xrightarrow{\sim} D^b_c(Y)$$
$$E \mapsto \mathbf{R}(p_Y)_*(p_X^*(E) \otimes \mathcal{P})$$

with the kernel $\mathcal{P} \in D^b_c(X \times Y)$ is given. Then Φ extends to an equivalence of fibered triangulated categories

$$\Phi : \mathcal{D}_{X/k} \xrightarrow{\sim} \mathcal{D}_{Y/k}.$$

Since $\mathcal{L} = \{\mathcal{O}_X(-n)\}_{n\geq 0}$ is a strict ample sequence in $\mathcal{D}_{X/k}$, $\mathcal{L}^\Phi = \{\Phi(\mathcal{O}_X(-n))\}_{n\geq 0}$ is a strict ample sequence in $\mathcal{D}_{Y/k}$. Moreover Φ determines an isomorphism

$$\Phi : M^{P,\mathcal{L}}_{\mathcal{D}_{X/k}} \xrightarrow{\sim} M^{P,\mathcal{L}^\Phi}_{\mathcal{D}_{Y/k}}$$

of the moduli space of stable sheaves on X to the moduli space of stable objects in $D^b_c(Y)$.

Example 5.4. Let G be a finite group and X be a projective variety over $\mathbf{C}$ on which G acts. Take a G-linearized very ample line bundle $\mathcal{O}_X(1)$ on X such that $H^i(X, \mathcal{O}_X(m)) = 0$ for $i > 0$ and $m > 0$. Let $\rho_0, \rho_1, \dots, \rho_s$ be the irreducible representations of G. Consider the fibered triangulated category $\mathcal{D}^G_{X/\mathbf{C}}$ defined by $(\mathcal{D}^G_{X/\mathbf{C}})_U = D^G(\mathrm{Coh}(X_U/U))$, for $U \in (\mathrm{Sch}/\mathbf{C})$, where $D^G(\mathrm{Coh}(X_U/U))$ is the full subcategory of the derived category of bounded complexes of G-equivariant coherent sheaves on X_U consisting of the objects of finite Tor-dimension over U. For positive integers $r_0, r_1, \dots, r_s$, $\mathcal{L}^G_{(r_0,\dots,r_s)} = \left\{\mathcal{O}_X(-n) \otimes (\rho_0^{\oplus r_0} \oplus \cdots \oplus \rho_s^{\oplus r_s})\right\}_{n\geq 0}$ becomes a strict ample sequence in $\mathcal{D}^G_{X/\mathbf{C}}$. The moduli space $M^{P,\mathcal{L}^G_{(r_0,\dots,r_s)}}_{\mathcal{D}^G_{X/\mathbf{C}}}$ is just the moduli space of G-equivariant sheaves $\mathcal{E}$ on X satisfying the stability condition: $\mathcal{E}$ is of pure dimension $d = \deg P$ and for any G-equivariant subsheaf $0 \neq \mathcal{F} \subsetneq \mathcal{E}$, the inequality

$$\frac{\mathrm{Hom}_G\left(\rho_0^{\oplus r_0} \oplus \cdots \oplus \rho_s^{\oplus r_s}, H^0(X, \mathcal{F} \otimes \mathcal{O}_X(n))\right)}{a_0(\mathcal{F})} < \frac{\mathrm{Hom}_G\left(\rho_0^{\oplus r_0} \oplus \cdots \oplus \rho_s^{\oplus r_s}, H^0(X, \mathcal{E} \otimes \mathcal{O}_X(n))\right)}{a_0(\mathcal{E})}$$

holds for $n \gg 0$, where we define

$$\chi(\mathcal{E}(m)) = \sum_{i=0}^{d} a_i(\mathcal{E}) \binom{m+d-i}{d-i} \quad \text{and} \quad \chi(\mathcal{F}(m)) = \sum_{i=0}^{d} a_i(\mathcal{F}) \binom{m+d-i}{d-i}$$

and so on.

Example 5.5. Let X be a projective variety over $\mathbf{C}$ and let $\mathcal{O}_X(1)$ be a very ample line bundle on X such that $H^i(X, \mathcal{O}_X(m)) = 0$ for $i > 0$ and $m > 0$. For a torsion class $\alpha \in H^2(X, \mathcal{O}_X^\times)$, consider the fibered triangulated category $\mathcal{D}^\alpha_{X/\mathbf{C}}$ over $(\mathrm{Sch}/\mathbf{C})$ defined by $(\mathcal{D}^\alpha_{X/\mathbf{C}})_U := D^b(\mathrm{Coh}(X_U/U), \alpha_U)$, where $D^b(\mathrm{Coh}(X_U/U), \alpha_U)$ is the derived category of bounded complexes of coherent α_U-twisted sheaves on $X \times U$ of finite Tor-dimension over U and α_U is the image of α in $H^2(X_U, \mathcal{O}_{X_U}^\times)$. For a locally free α-twisted sheaf G of finite rank on X, $\mathcal{L}^\alpha_G = \{G \otimes \mathcal{O}_X(-n)\}_{n\geq 0}$ becomes a strict ample sequence in $\mathcal{D}^\alpha_{X/\mathbf{C}}$, after replacing $\mathcal{O}_X(1)$ by some multiple. The moduli space $M^{P,\mathcal{L}^\alpha_G}_{\mathcal{D}^\alpha_{X/\mathbf{C}}}$ is just the moduli space of G-twisted stable α-twisted sheaves on X in the sense of [10].

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