Inequalities for Eigenvalues of the Biharmonic Operator with Weight on Riemannian Manifolds

Qiaoling Wang, Changyu Xia

Abstract

Given a compact Riemannian manifold M with boundary (possibly empty), we consider the eigenvalues of the biharmonic operator with weight on M, proving a general inequality involving them. Using this inequality, we consider these eigenvalues when M is a compact domain of one of the following three spaces: 1) a complex projective space, 2) a minimal submanifold of a Euclidean space and 3) a minimal submanifold of a unit sphere.

1 Introduction

Let Ω be a connected bounded domain with smooth boundary in an n-dimensional Euclidean space $I\!R^n$. The so called *Dirichlet eigenvalue problem* or the *fixed membrane problem* is stated as:

$$\Delta u = -\lambda u \ \text{ in } \ \Omega, \quad u|_{\partial\Omega} = 0. \tag{1.1}$$

It is well known that this problem has a real and purely discrete spectrum $\{\lambda_i\}_{i=1}^{\infty}$ where

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \to \infty.$$

Here each eigenvalue is repeated according to its multiplicity. In 1955 and 1956, Payne, Pólya and Weinberger [PPW1], [PPW2], proved that

$$\frac{\lambda_2}{\lambda_1} \leq 3 \quad \text{for} \quad \Omega \subset I\!R^2$$

and conjectured that

$$\frac{\lambda_2}{\lambda_1} \leq \frac{\lambda_2}{\lambda_1}\bigg|_{\text{disk}}$$

with equality if and only if Ω is a disk. For general dimension $n \geq 2$, the analogous statements are

$$\frac{\lambda_2}{\lambda_1} \leq 1 + \frac{4}{n} \quad \text{for} \quad \Omega \subset I\!R^n,$$

and the *PPW conjecture*

$$\frac{\lambda_2}{\lambda_1} \leq \frac{\lambda_2}{\lambda_1}\bigg|_{n-\text{ball}},$$

with equality if and only if Ω is an n-ball. This important *PPW conjecture* was proved by Ashbaugh and Benguria (see [AB1], [AB2], [AB3]).

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In [PPW2], Payne, Pólya and Weinberger also proved the bound

$$\lambda_{k+1} - \lambda_k \leq \frac{2}{k} \sum_{k=1}^{k} \lambda_i, \quad k = 1, 2, \cdots, \tag{1.2}$$

for $\Omega \subset I\!R^2$. This result easily extends to $\Omega \subset I\!R^n$ as

$$\lambda_{k+1} - \lambda_k \leq \frac{4}{kn} \sum_{k=1}^{k} \lambda_i, \quad k = 1, 2, \cdots, \tag{1.3}$$

Extending (1.3), Hile and Protter [HP] proved

$$\sum_{i=1}^{k} \frac{\lambda_i}{\lambda_{k+1} - \lambda_i} \geq \frac{kn}{4}, \quad \text{for } k = 1, 2, \cdots. \tag{1.4}$$

In 1991, Yang proved the following much stonger inequality:

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \left(\lambda_{k+1} - \left(1 + \frac{4}{n} \right) \lambda_i \right) \leq 0, \quad \text{for } k = 1, 2, \cdots. \tag{1.5}$$

The inequalities for eigenvalues of the Laplacian on connected bounded domains in $I\!R^n$ obtained by Payne-Pólya-Weinberger, Hile-Protter, Yang have also been extended to some other Riemannian manifolds (cf. [CY1], [CY2], [H1], [HS], [HM1], [HM2], [Leu], [Li], [YY]). Ashbaugh [A2] considered Schrödinger operators with weight on bounded domains in $I\!R^n$ and obtained universal eigenvalue bounds for them. In [CY1], Cheng and Yang studied eigenvalues of the Laplacian on either a bounded connected domain in an n-dimensional unit sphere $S^n(1)$, or a compact homogeneous Riemannain manifold, or an n-dimensional compact minimal submanifold in a unit sphere and proved universal upper bounds for the $(k+1)$-th eigenvalue in terms of the first k eigenvalues. Cheng-Yang [CY2] also obtained a general inequality involving the eigenvalues of the Laplacian on an arbitrary compact manifold with boundary (possibly empty) and used it to derive universal inequalities for these eigenvalues when the manifold is a compact domain or a closed complex hypersurface in a complex projective space. It has been shown by Harrell [H2] and El Soufi et al. [EHI] that eigenvalue inequalities of similar kinds for Schrödinger operators on some special Riemannian manifolds also hold. The work by Ashbaugh, Cheng-Yang in [A2], [CY1], [CY2] has also been extended to Schrödinger operators with weight (see [WX2]).

On the other hand, Payne-Pólya-Weinberger [PPW2] also considered the eigenvalue problem for the *Dirichlet biharmonic operator* or the *clamped plate problem* which describes the characteristic vibrations of a clamped plate. This problem is given by

$$\Delta^2 u = \eta u \quad \text{in } \Omega \subset I\!R^n, \qquad u|_{\partial\Omega} = \left. \frac{\partial u}{\partial \nu} \right|_{\partial\Omega} = 0. \tag{1.6}$$

Payne-Pólya-Weinberger proved in [PPW2] that the eigenvalues $\{\eta_i\}_{i=1}^{\infty}$ of (1.6) satisfy

$$\eta_{k+1} - \eta_k \leq \frac{8(n+2)}{n^2} \frac{1}{k} \sum_{i=1}^{k} \eta_i. \tag{1.7}$$

As a generalization of (1.7), Hile-Yeh obtained [HY]

$$\sum_{i=1}^{k} \frac{\eta_i^{1/2}}{\eta_{k+1} - \eta_i} \geq \frac{n^2 k^{3/2}}{8(n+2)} \left(\sum_{i=1}^{k} \eta_i \right)^{-1/2}. \tag{1.8}$$

Hook [H], Chen-Qian [CQ] proved, independently, the following inequality

$$\frac{n^2k^2}{8(n+2)} \leq \left(\sum_{i=1}^{k} \eta_i^{1/2}\right)\left(\sum_{i=1}^{k} \frac{\eta_i^{1/2}}{\eta_{k+1}-\eta_i}\right). \tag{1.9}$$

Answering a question by Ashbaugh [A1], Cheng-Yang [CY3] proved

$$\eta_{k+1} - \frac{1}{k}\sum_{i=1}^{k}\eta_i \leq \left(\frac{8(n+2)}{n^2}\right)^{1/2} \frac{1}{k}\sum_{i=1}^{k}\left(\eta_i(\eta_{k+1}-\eta_i)\right)^{1/2}. \tag{1.10}$$

As a consequence of (1.10), Cheng and Yang obtained the following upper bound for the $(k+1)$-th eigenvalue in terms of its first k-eigenvalues of the problem (1.6):

$$\begin{aligned} \eta_{k+1} \quad \leq \quad & \left(1+\frac{4(n+2)}{n^2}\right)\frac{1}{k}\sum_{i=1}^{k}\eta_i \\ & + \left\{\left(\frac{4(n+2)}{n^2}\frac{1}{k}\sum_{i=1}^{k}\eta_i\right)^2 - \frac{8(n+2)}{n^2}\frac{1}{k}\sum_{i=1}^{k}\left(\eta_i - \frac{1}{k}\sum_{j=1}^{k}\eta_j\right)^2\right\}^{1/2}. \end{aligned} \tag{1.11}$$

The above inequality has been improved and universal eigenvalue inequalities for the biharmonic operator on compact domains either in a sphere or in a minimal submanifold of a Euclidean space have also been obtained in [WX1].

The purpose of this paper is to extend the work of [CY3] and [WX1] to the biharmonic operator with weight on Riemannian manifolds. Let $(M, \langle,\rangle)$ be a compact Riemannian manifold with boundary ∂M (possibly empty) and denote by ν be the outward unit normal vector field of ∂M. Let V be a nonnegative continuous function on M, and ρ a weight function which is positive and continuous on M. Denote by Δ the Laplacian of M and consider the eigenvalue problem

$$\begin{cases} \Delta^2 u + Vu = \eta\rho u \quad \text{in} \quad M, \\ \qquad u|_{\partial M} = \frac{\partial u}{\partial\nu}\big|_{\partial M} = 0. \end{cases} \tag{1.12}$$

We prove a general inequality involving the eigenvalues of this problem. Using this inequality, we consider these eigenvalues when M is a compact domain of one of the following three spaces: 1) a complex projective space, 2) a minimal submanifold of a Euclidean space and 3) a minimal submanifold of a unit sphere. In these cases we give an explicit upper bound for the $(k+1)$-th eigenvalue in terms of the first k eigenvalues.

2 A General Inequality Involving the Eigenvalues of the Biharmonic Operator with Weight on a Compact Manifold

In this section, we will prove a general result involving the eigenvalues of the biharmonic operator with weight on a compact Riemannian manifold. Namely, we have

Theorem 2.1. *Let η_i be the i-th eigenvalue of (1.12) and u_i be the orthonormal eigenfunction corresponding to η_i, that is,*

$$\Delta^2 u_i + Vu_i = \eta_i\rho u_i \ \text{ in } \ M, \qquad u_i|_{\partial M} = \frac{\partial u_i}{\partial\nu}\bigg|_{\partial M} = 0, \tag{2.1}$$

$$\int_M \rho u_i u_j = \delta_{ij}, \quad \forall \ \ i,j = 1,2,\cdots. \tag{2.2}$$

Then for any function $h \in C^4(M) \cap C^3(\partial M)$ and any integer k, we have

$$\begin{aligned}(2.3)\qquad & \sum_{i=1}^{k}(\eta_{k+1}-\eta_i)^2 \int_M hu_i \left(\Delta(u_i\Delta h) + 2\Delta\langle \nabla h, \nabla u_i\rangle + 2\langle \nabla h, \nabla(\Delta u_i)\rangle + \Delta h \Delta u_i\right) \\ \leq\ & \sum_{i=1}^{k}(\eta_{k+1}-\eta_i) \left|\left| \frac{1}{\sqrt{\rho}} \left(\Delta(u_i\Delta h) + 2\Delta\langle \nabla h, \nabla u_i\rangle + 2\langle \nabla h, \nabla(\Delta u_i)\rangle + \Delta h \Delta u_i\right)\right|\right|^2\end{aligned}$$

and

$$\begin{aligned}(2.4)\qquad & \sum_{i=1}^{k}(\eta_{k+1}-\eta_i)^2 \int_M \left(-hu_i^2\Delta h - 2hu_i\langle \nabla h, \nabla u_i\rangle\right) \\ \leq\ & \delta \sum_{i=1}^{k}(\eta_{k+1}-\eta_i)^2 \int_M hu_i \left(\Delta(u_i\Delta h) + 2\Delta\langle \nabla h, \nabla u_i\rangle + 2\langle \nabla h, \nabla(\Delta u_i)\rangle + \Delta h \Delta u_i\right) \\ & + \sum_{i=1}^{k} \frac{(\eta_{k+1}-\eta_i)}{\delta} \left|\left| \frac{\langle \nabla h, \nabla u_i\rangle + \frac{u_i\Delta h}{2}}{\sqrt{\rho}} \right|\right|^2,\end{aligned}$$

where $||f||^2 = \int_M f^2$.

Remark 2.2. One can check that when $\partial M \neq \emptyset$, the first eigenvalue η_1 of the problem (1.12) is always positive and when $\partial M = \emptyset$, $\eta_1 \geq 0$ with equality holding if and only if $V \equiv 0$. In both cases, we will use the same notations $\eta_1 \leq \eta_2 \leq \cdots \to \infty$ to represent the set of the eigenvalues of this problem.

Proof of Theorem 2.1. Set $S = \Delta^2 + V$ and consider the inner product given by $\langle\langle f, g\rangle\rangle = \int_M \rho f g$. If a nontrivial function ϕ on M satisfying $\phi|_{\partial M} = \frac{\partial u}{\partial \nu}\big|_{\partial M} = 0$ is orthogonal to $u_1, u_2, \cdots, u_k$ with respect to the above inner product, then the Rayleigh-Ritz inequality says that

$$(2.5)\qquad \eta_{k+1} \leq \frac{\int_M \phi(S\phi)}{\int_M \rho\phi^2}.$$

Consider the functions $\phi_i : M \to I\!R$, given by

$$(2.6)\qquad \phi_i = hu_i - \sum_{j=1}^{k} a_{ij}u_j, \qquad i = 1, \cdots, k.$$

where

$$(2.7)\qquad a_{ij} = \int_M \rho h u_i u_j = a_{ji}.$$

Since

$$(2.8)\qquad \phi_i|_{\partial M} = \frac{\partial \phi_i}{\partial \nu}\bigg|_{\partial M} = 0$$

and

$$(2.9)\qquad \int_M \rho u_j \phi_i = 0, \ \ \forall\ i, j = 1, \cdots, k,$$

it follows that

$$(2.10)\qquad \eta_{k+1} \leq \frac{\int_M \phi_i(S\phi_i)}{\int_M \rho\phi_i^2}.$$

We have

$$
\begin{aligned}
(2.11)\qquad & \int_M \phi_i(S\phi_i) \\
= \;& \int_M \phi_i(\Delta(u_i\Delta h) + 2\Delta\langle\nabla h, \nabla u_i\rangle + 2\langle\nabla h, \nabla(\Delta u_i)\rangle + \Delta h\Delta u_i + \eta_i h\rho u_i) \\
= \;& \eta_i \int_M \rho\phi_i^2 + \int_M \phi_i(\Delta(u_i\Delta h) + 2\Delta\langle\nabla h, \nabla u_i\rangle + 2\langle\nabla h, \nabla(\Delta u_i)\rangle + \Delta h\Delta u_i) \\
= \;& \eta_i \int_M \rho\phi_i^2 + \int_M hu_i(\Delta(u_i\Delta h) + 2\Delta\langle\nabla h, \nabla u_i\rangle + 2\langle\nabla h, \nabla(\Delta u_i)\rangle + \Delta h\Delta u_i) - \sum_{j=1}^k a_{ij}b_{ij},
\end{aligned}
$$

where

$$
b_{ij} = \int_M (\Delta(u_i\Delta h) + 2\Delta\langle\nabla h, \nabla u_i\rangle + 2\langle\nabla h, \nabla(\Delta u_i)\rangle + \Delta h\Delta u_i)u_j.
$$

Since

$$
u_i|_{\partial M} = \left.\frac{\partial u_i}{\partial\nu}\right|_{\partial M} = 0, \quad u_j|_{\partial M} = \left.\frac{\partial u_j}{\partial\nu}\right|_{\partial M} = 0,
$$

we have

$$
hu_i|_{\partial M} = \left.\frac{\partial(hu_i)}{\partial\nu}\right|_{\partial M} = 0, \quad hu_j|_{\partial M} = \left.\frac{\partial(hu_j)}{\partial\nu}\right|_{\partial M} = 0
$$

and so we can use the divergence theorem to derive that

$$
\begin{aligned}
(2.12)\qquad b_{ij} &= \int_M (u_i\Delta h + 2\langle\nabla h, \nabla u_i\rangle)\Delta u_j - 2\int_M \Delta u_i \mathrm{div}(u_j\nabla h) + \int_M u_j\Delta h\Delta u_i \\
&= \int_M (u_i\Delta h + 2\langle\nabla h, \nabla u_i\rangle)\Delta u_j - \int_M (u_j\Delta h + 2\langle\nabla h, \nabla u_j\rangle)\Delta u_i \\
&= \int_M \Delta(u_i h)\Delta u_j - \int_M \Delta(u_j h)\Delta u_i \\
&= \int_M u_i h\Delta^2 u_j - \int_M u_j h\Delta^2 u_i \\
&= (\eta_j - \eta_i)a_{ij},
\end{aligned}
$$

where $\mathrm{div}(X)$ denotes the divergence of X. Set

$$
q_i(h) = \Delta(u_i\Delta h) + 2\Delta\langle\nabla h, \nabla u_i\rangle + 2\langle\nabla h, \nabla(\Delta u_i)\rangle + \Delta h\Delta u_i;
$$

then

$$
(2.13)\qquad \int_M \phi_i q_i(h) = \int_M hu_i q_i(h) + \sum_{j=1}^k (\eta_i - \eta_j)a_{ij}^2
$$

and we have from (2.10) and (2.11) that

$$
(2.14)\qquad (\eta_{k+1} - \eta_i)\int_M \rho\phi_i^2 \le \int_M \phi_i q_i(h)
$$

which implies that

$$
(2.15)\quad (\eta_{k+1} - \eta_i)\left(\int_M \phi_i q_i(h)\right)^2 = (\eta_{k+1} - \eta_i)\left(\int_M \sqrt{\rho}\phi_i\left(\frac{q_i(h)}{\sqrt{\rho}} - \sum_{j=1}^k b_{ij}\sqrt{\rho}u_j\right)\right)^2
$$

$$
\begin{aligned}
&\le (\eta_{k+1}-\eta_i)\left(\int_M \rho\phi_i^2\right)\left(\int_M\left(\frac{q_i(h)}{\sqrt{\rho}}-\sum_{j=1}^k b_{ij}\sqrt{\rho}u_j\right)^2\right)\\
&= (\eta_{k+1}-\eta_i)\left(\int_M \rho\phi_i^2\right)\left(\left\|\frac{q_i(h)}{\sqrt{\rho}}\right\|^2-\sum_{j=1}^k b_{ij}^2\right)\\
&\le \left(\int_M \phi_i q_i(h)\right)\left(\left\|\frac{q_i(h)}{\sqrt{\rho}}\right\|^2-\sum_{j=1}^k b_{ij}^2\right).
\end{aligned}
$$

Hence, we have

$$
(\eta_{k+1}-\eta_i)\int_M \phi_i q_i(h) \le \left\|\frac{q_i(h)}{\sqrt{\rho}}\right\|^2-\sum_{j=1}^k(\eta_i-\eta_j)^2a_{ij}^2. \tag{2.16}
$$

Multiplying (2.13) by $(\eta_{k+1}-\eta_i)^2$ and summing on i, we get

$$
\begin{aligned}
&\sum_{i=1}^k(\eta_{k+1}-\eta_i)^2\int_M \phi_i q_i(h) \\
=& \sum_{i=1}^k(\eta_{k+1}-\eta_i)^2\int_M hu_iq_i(h)+\sum_{i,j=1}^k(\eta_{k+1}-\eta_i)^2(\eta_i-\eta_j)a_{ij}^2\\
=& \sum_{i=1}^k(\eta_{k+1}-\eta_i)^2\int_M hu_iq_i(h)-\sum_{i,j=1}^k(\eta_{k+1}-\eta_i)(\eta_i-\eta_j)^2a_{ij}^2.
\end{aligned} \tag{2.17}
$$

On the other hand, one obtains by multiplying (2.16) by $(\eta_{k+1}-\eta_i)$ and summing on i that

$$
\begin{aligned}
&\sum_{i=1}^k(\eta_{k+1}-\eta_i)^2\int_M \phi_i q_i(h) \\
\le& \sum_{i=1}^k(\eta_{k+1}-\eta_i)\left\|\frac{q_i(h)}{\sqrt{\rho}}\right\|^2-\sum_{i,j=1}^k(\eta_{k+1}-\eta_i)(\eta_i-\eta_j)^2a_{ij}^2.
\end{aligned} \tag{2.18}
$$

Combining (2.17) and (2.18), we obtain (2.3).

Now set

$$
c_{ij}=\int_M u_j\left(\langle\nabla h,\nabla u_i\rangle+\frac{u_i\Delta h}{2}\right); \tag{2.19}
$$

then $c_{ij}+c_{ji}=0$ and

$$
\int_M(-2)\phi_i\left(\langle\nabla h,\nabla u_i\rangle+\frac{u_i\Delta h}{2}\right)=w_i+2\sum_{j=1}^k a_{ij}c_{ij}, \tag{2.20}
$$

where

$$
w_i=\int_M(-hu_i^2\Delta h-2hu_i\langle\nabla h,\nabla u_i\rangle). \tag{2.21}
$$

Multiplying (2.20) by $(\eta_{k+1}-\eta_i)^2$ and using Schwarz inequality and (2.14), we get

$$
\begin{aligned}
(2.22)\qquad & (\eta_{k+1}-\eta_i)^2\left(w_i+2\sum_{j=1}^{k}a_{ij}c_{ij}\right)\\
= \ & (\eta_{k+1}-\eta_i)^2\int_M(-2)\phi_i\sqrt{\rho}\left(\frac{\langle\nabla h,\nabla u_i\rangle+\frac{u_i\Delta h}{2}}{\sqrt{\rho}}-\sum_{j=1}^{k}c_{ij}\sqrt{\rho}u_j\right)\\
\leq \ & \delta(\eta_{k+1}-\eta_i)^3\|\sqrt{\rho}\phi_i\|^2+\frac{(\eta_{k+1}-\eta_i)}{\delta}\int_M\left|\frac{\langle\nabla h,\nabla u_i\rangle+\frac{u_i\Delta h}{2}}{\sqrt{\rho}}-\sum_{j=1}^{k}c_{ij}\sqrt{\rho}u_j\right|^2\\
= \ & \delta(\eta_{k+1}-\eta_i)^3\|\sqrt{\rho}\phi_i\|^2+\frac{(\eta_{k+1}-\eta_i)}{\delta}\left(\left\|\frac{\langle\nabla h,\nabla u_i\rangle+\frac{u_i\Delta h}{2}}{\sqrt{\rho}}\right\|^2-\sum_{j=1}^{k}c_{ij}^2\right)\\
\leq \ & \delta(\eta_{k+1}-\eta_i)^2\left(\int_M hu_iq_i(h)+\sum_{j=1}^{k}(\eta_i-\eta_j)a_{ij}^2\right)\\
& +\frac{(\eta_{k+1}-\eta_i)}{\delta}\left(\left\|\frac{\langle\nabla h,\nabla u_i\rangle+\frac{u_i\Delta h}{2}}{\sqrt{\rho}}\right\|^2-\sum_{j=1}^{k}c_{ij}^2\right).
\end{aligned}
$$

Summing over i and noticing $a_{ij}=a_{ji},\ c_{ij}=-c_{ji}$, we obtain

$$
\begin{aligned}
& \sum_{i=1}^{k}(\eta_{k+1}-\eta_i)^2w_i-2\sum_{i,j=1}^{k}(\eta_{k+1}-\eta_i)(\eta_i-\eta_j)a_{ij}c_{ij}\\
\leq \ & \delta\sum_{i=1}^{k}(\eta_{k+1}-\eta_i)^2\int_M hu_iq_i(h)+\sum_{i=1}^{k}\frac{(\eta_{k+1}-\eta_i)}{\delta}\left\|\frac{\langle\nabla h,\nabla u_i\rangle+\frac{u_i\Delta h}{2}}{\sqrt{\rho}}\right\|^2\\
& -\sum_{i,j=1}^{k}(\eta_{k+1}-\eta_i)\delta(\eta_i-\eta_j)^2a_{ij}^2-\sum_{i,j=1}^{k}\frac{(\eta_{k+1}-\eta_i)}{\delta}c_{ij}^2.
\end{aligned}
$$

Hence (2.4) is true. □

We end this section by proving an algebraic lemma which will be needed in the next section.

Lemma 2.2. Let $\{a_i\}_{i=1}^m$, $\{b_i\}_{i=1}^m$ and $\{c_i\}_{i=1}^m$ be three sequences of non-negative real numbers with $\{a_i\}_{i=1}^m$ decreasing and $\{b_i\}_{i=1}^m$ and $\{c_i\}_{i=1}^m$ increasing. Then the following inequality holds:

$$
(2.23)\qquad \left(\sum_{i=1}^{m}a_i^2b_i\right)\left(\sum_{i=1}^{m}a_ic_i\right)\leq\left(\sum_{i=1}^{m}a_i^2\right)\left(\sum_{i=1}^{m}a_ib_ic_i\right).
$$

Proof. When $m=1$, (2.23) holds trivially. Suppose that (2.23) holds when $m=k$, that is

$$
(2.24)\qquad \left(\sum_{i=1}^{k}a_i^2b_i\right)\left(\sum_{i=1}^{k}a_ic_i\right)\leq\left(\sum_{i=1}^{k}a_i^2\right)\left(\sum_{i=1}^{k}a_ib_ic_i\right).
$$

Then when $m=k+1$, we have from (2.24) that

$$
(2.25)\qquad \left(\sum_{i=1}^{k+1}a_i^2\right)\left(\sum_{i=1}^{k+1}a_ib_ic_i\right)-\left(\sum_{i=1}^{k+1}a_i^2b_i\right)\left(\sum_{i=1}^{k+1}a_ic_i\right)
$$

$$
\begin{aligned}
&= \left(\sum_{i=1}^{k} a_i^2\right)\left(\sum_{i=1}^{k} a_i b_i c_i\right) - \left(\sum_{i=1}^{k} a_i^2 b_i\right)\left(\sum_{i=1}^{k} a_i c_i\right) + a_{k+1}^2 \sum_{i=1}^{k} a_i b_i c_i \\
&\quad - a_{k+1}^2 b_{k+1} \sum_{i=1}^{k} a_i c_i + a_{k+1} b_{k+1} c_{k+1} \sum_{i=1}^{k} a_i^2 - a_{k+1} c_{k+1} \sum_{i=1}^{k} a_i^2 b_i \\
&\geq a_{k+1}^2 \sum_{i=1}^{k} a_i b_i c_i - a_{k+1}^2 b_{k+1} \sum_{i=1}^{k} a_i c_i + a_{k+1} b_{k+1} c_{k+1} \sum_{i=1}^{k} a_i^2 - a_{k+1} c_{k+1} \sum_{i=1}^{k} a_i^2 b_i \\
&= -a_{k+1}^2 \sum_{i=1}^{k} (b_{k+1} - b_i) a_i c_i + a_{k+1} c_{k+1} \sum_{i=1}^{k} a_i^2 (b_{k+1} - b_i) \\
&= \sum_{i=1}^{k} a_{k+1} a_i (b_{k+1} - b_i)(c_{k+1} a_i - a_{k+1} c_i) \\
&\geq 0.
\end{aligned}
$$

Where in the last inequality we have used the fact that

$$
a_{k+1} a_i (b_{k+1} - b_i)(c_{k+1} a_i - a_{k+1} c_i) \geq 0, \quad i = 1, \cdots, k. \tag{2.26}
$$

Thus (2.23) holds for $m = k+1$. This completes the proof of Lemma 2.2. □

3 Inequalities for Eigenvalues of the Biharmonic Operator with Weight on Compact Domains in $CP^n(4)$, in a Minimal Submanifold of $I\!R^m$ or of $S^m(1)$

In this section, we will prove universal bounds on eigenvalues of the biharmonic operator in Weighted Sobolev space on compact domains of any of the following three manifolds: a complex projective space, a minimal submanifold of a Euclidean space or of a unit sphere.

Theorem 3.1. *Let M be an n-dimensional complete Riemannian manifold and let Ω be a connected bounded domain with smooth boundary $\partial\Omega$ in M. Denote by ν the outward unit normal of $\partial\Omega$. Let V be a nonnegative continuous function on $\overline{\Omega}$ and ρ a positive continuous function on $\overline{\Omega}$. Set $V_0 = \min_{x\in\overline{\Omega}} V(x)$, $P_1 = \max_{x\in\overline{\Omega}} \rho(x)$ and $P_2 = \min_{x\in\overline{\Omega}} \rho(x)$. Let Δ be the Laplacian of M and denote by η_i the i-th eigenvelue of the problem:*

$$
\begin{cases}
(\Delta^2 + V)u = \eta\rho u & \text{in } \Omega, \\
u|_{\partial\Omega} = \frac{\partial u}{\partial\nu}\big|_{\partial\Omega} = 0.
\end{cases}
$$

i) If M is a minimal submanifold of $I\!R^m$, then

$$
\begin{aligned}
&\sum_{i=1}^{k} (\eta_{k+1} - \eta_i)^2 \\
&\leq \frac{2P_1(2(n+2))^{1/2}}{nP_2} \left\{ \sum_{i=1}^{k} (\eta_{k+1} - \eta_i)^2 \left(\eta_i - \frac{V_0}{P_1}\right)^{1/2} \right)^{1/2} \left(\sum_{i=1}^{k} (\eta_{k+1} - \eta_i) \left(\eta_i - \frac{V_0}{P_1}\right)^{1/2} \right\}^{1/2}.
\end{aligned} \tag{3.1}
$$

ii) If M is a minimal submanifold in a unit m-sphere $S^m(1)$, then

$$
\sum_{i=1}^{k} (\eta_{k+1} - \eta_i)^2 \leq \frac{P_1}{nP_2} \left\{ \sum_{i=1}^{k} (\eta_{k+1} - \eta_i)^2 \left(\frac{n^2}{P_2^{1/2}} + (2n+4)\left(\eta_i - \frac{V_0}{P_1}\right)^{1/2} \right) \right\}^{1/2} \tag{3.2}
$$

$$\times \left\{ \sum_{i=1}^{k} (\eta_{k+1} - \eta_i) \left(\frac{n^2}{P_2^{1/2}} + 4 \left(\eta_i - \frac{V_0}{P_1} \right)^{1/2} \right) \right\}^{1/2}.$$

iii) If M is a complex projetive space $CP^n(4)$ of complex dimension n and of holomorphic sectional curvature 4, then

$$\sum_{i=1}^{k} (\eta_{k+1} - \eta_i)^2 \leq \frac{2(n+1)^{1/2} P_1}{n P_2} \left\{ \sum_{i=1}^{k} (\eta_{k+1} - \eta_i)^2 \left(\frac{2n}{P_2^{1/2}} + \left(\eta_i - \frac{V_0}{P_1} \right)^{1/2} \right) \right\}^{1/2} \times \left\{ \sum_{i=1}^{k} (\eta_{k+1} - \eta_i) \left(\frac{8n(n+1)}{P_2^{1/2}} + \left(\eta_i - \frac{V_0}{P_1} \right)^{1/2} \right) \right\}^{1/2}. \tag{3.3}$$

Corollary 3.2. *Let the assupmtions be as in Theorem 3.1.*
i) If Ω is a domian in an n-dimensional complete minimal submanifold of $I\!R^m$, then

$$\eta_{k+1} \leq -\frac{SV_0}{P_1} + (1+S)\frac{1}{k}\sum_{i=1}^{k} \eta_i + \left\{ S^2 \left(\frac{1}{k}\sum_{i=1}^{k} \eta_i - \frac{V_0}{P_1} \right)^2 - (1+2S)\frac{1}{k}\sum_{j=1}^{k} \left(\eta_j - \frac{1}{k}\sum_{i=1}^{k}\eta_i \right)^2 \right\}^{1/2}, \tag{3.4}$$

where

$$S = \frac{4P_1^2(n+2)}{n^2 P_2^2}.$$

ii) If Ω is a domain in a minimal submanifold of $S^m(1)$, then

$$\eta_{k+1} \leq A_{k+1} + \sqrt{A_{k+1}^2 - B_{k+1}}, \tag{3.5}$$

where

$$A_{k+1} = \frac{1}{k}\sum_{i=1}^{k} \eta_i + \frac{P_1^2}{2kn^2P_2^2} \sum_{i=1}^{k} \left(\frac{n^2}{P_2^{1/2}} + 4\left(\eta_i - \frac{V_0}{P_1} \right)^{1/2} \right) \left(\frac{n^2}{P_2^{1/2}} + (2n+4)\left(\eta_i - \frac{V_0}{P_1} \right)^{1/2} \right),$$

$$B_{k+1} = \frac{1}{k}\sum_{i=1}^{k} \eta_i^2 + \frac{P_1^2}{kn^2P_2^2} \sum_{i=1}^{k} \left(\frac{n^2}{P_2^{1/2}} + 4\left(\eta_i - \frac{V_0}{P_1} \right)^{1/2} \right) \left(\frac{n^2}{P_2^{1/2}} + (2n+4)\left(\eta_i - \frac{V_0}{P_1} \right)^{1/2} \right) \eta_i.$$

iii) If Ω is a domain in $CP^n(4)$, then

$$\eta_{k+1} \leq C_{k+1} + \sqrt{C_{k+1}^2 - D_{k+1}}, \tag{3.6}$$

where

$$C_{k+1} = \frac{1}{k}\sum_{i=1}^{k} \eta_i + \frac{2(n+1)P_1^2}{kn^2P_2^2} \sum_{i=1}^{k} \left(\frac{2n}{P_2^{1/2}} + \left(\eta_i - \frac{V_0}{P_1} \right)^{1/2} \right) \left(\frac{8n(n+1)}{P_2^{1/2}} + \left(\eta_i - \frac{V_0}{P_1} \right)^{1/2} \right),$$

$$D_{k+1} = \frac{1}{k}\sum_{i=1}^{k} \eta_i^2 + \frac{4(n+1)P_1^2}{kn^2P_2^2} \sum_{i=1}^{k} \left(\frac{2n}{P_2^{1/2}} + \left(\eta_i - \frac{V_0}{P_1} \right)^{1/2} \right) \left(\frac{8n(n+1)}{P_2^{1/2}} + \left(\eta_i - \frac{V_0}{P_1} \right)^{1/2} \right) \eta_i.$$

Proof of Theorem 3.1. Let ∇ be the gradient operator on Ω and let u_i be the i-th orthonormal eigenfunction corresponding to the eigenvalue η_i, $i = 1, 2, \cdots$, that is,

$$(\Delta^2 + V)u_i = \eta_i \rho u_i \ \text{ in } \ \Omega, \qquad u_i|_{\partial\Omega} = \left.\frac{\partial u_i}{\partial \nu}\right|_{\partial\Omega} = 0, \tag{3.7}$$

$$\int_\Omega \rho u_i u_j = \delta_{ij}, \quad \forall\, i, j; \tag{3.8}$$

then

$$\frac{1}{P_1} \le \int_\Omega u_i^2 \le \frac{1}{P_2}. \tag{3.9}$$

Multiplying (3.7) by u_i and integrating on Ω, one has

$$\begin{aligned} \int_\Omega (\Delta u_i)^2 &= \eta_i \int_\Omega \rho u_i^2 - \int_\Omega V u_i^2 \\ &\le \eta_i - \frac{V_0}{P_1}. \end{aligned} \tag{3.10}$$

It follows from the Schwarz inequality that

$$\int_\Omega |\nabla u_i|^2 = \int_\Omega (-u_i \Delta u_i) \le \left(\int_\Omega u_i^2\right)^{1/2} \left(\int_\Omega (\Delta u_i)^2\right)^{1/2} \le \frac{1}{{P_2}^{1/2}} \left(\eta_i - \frac{V_0}{P_1}\right)^{1/2}. \tag{3.11}$$

i) Suppose that Ω is a domain in an n-dimensional minimal submanifold in $\mathbb{R}^m$. Let $x_1, x_2, \cdots, x_m$ be the standard coordinate functons of $\mathbb{R}^m$; then

$$\Delta x_\alpha = 0, \quad \alpha = 1, \cdots, m. \tag{3.12}$$

Taking $h = x_\alpha$ in (2.4), we get

$$\begin{aligned} &\sum_{i=1}^k (\eta_{k+1} - \eta_i)^2 \int_\Omega (-2x_\alpha u_i \langle \nabla x_\alpha, \nabla u_i \rangle) \\ \le\ & \delta \sum_{i=1}^k (\eta_{k+1} - \eta_i)^2 \int_\Omega x_\alpha u_i (2\Delta \langle \nabla x_\alpha, \nabla u_i \rangle + 2 \langle \nabla x_\alpha, \nabla(\Delta u_i) \rangle) \\ & + \sum_{i=1}^k \frac{(\eta_{k+1} - \eta_i)}{\delta} \left\| \frac{1}{\sqrt{\rho}} \langle \nabla x_\alpha, \nabla u_i \rangle \right\|^2. \end{aligned} \tag{3.13}$$

Summing over α, one has

$$\begin{aligned} &\sum_{i=1}^k (\eta_{k+1} - \eta_i)^2 \sum_{\alpha=1}^m \int_\Omega (-2x_\alpha u_i \langle \nabla x_\alpha, \nabla u_i \rangle) \\ \le\ & \delta \sum_{i=1}^k (\eta_{k+1} - \eta_i)^2 \sum_{\alpha=1}^m \int_\Omega x_\alpha u_i (2\Delta \langle \nabla x_\alpha, \nabla u_i \rangle + 2 \langle \nabla x_\alpha, \nabla(\Delta u_i) \rangle) \\ & + \sum_{i=1}^k \frac{(\eta_{k+1} - \eta_i)}{\delta} \sum_{\alpha=1}^m \left\| \frac{1}{\sqrt{\rho}} \langle \nabla x_\alpha, \nabla u_i \rangle \right\|^2. \end{aligned} \tag{3.14}$$

Observe that

$$\sum_{\alpha=1}^m \langle \nabla x_\alpha, \nabla u_i \rangle^2 = \sum_{\alpha=1}^m (\nabla u_i(x_\alpha))^2 = |\nabla u_i|^2, \quad \sum_{\alpha=1}^m |\nabla x_\alpha|^2 = n. \tag{3.15}$$

We have from (3.9), (3.11), (3.12) and (3.15) that

$$\sum_{\alpha=1}^{m}\int_\Omega(-2x_\alpha u_i\langle\nabla x_\alpha,\nabla u_i\rangle)=\frac{1}{2}\sum_{\alpha=1}^{m}\int_\Omega u_i^2\Delta x_\alpha^2=n\int_\Omega u_i^2\geq\frac{n}{P_1}, \tag{3.16}$$

$$\begin{aligned}
&\sum_{\alpha=1}^{m}\int_\Omega x_\alpha u_i\left(2\Delta\langle\nabla x_\alpha,\nabla u_i\rangle+2\langle\nabla x_\alpha,\nabla(\Delta u_i)\rangle\right) \qquad (3.17)\\
=\;& 2\sum_{\alpha=1}^{m}\int_\Omega\Delta\left(x_\alpha u_i\right)\langle\nabla x_\alpha,\nabla u_i\rangle-2\sum_{\alpha=1}^{m}\int_\Omega\Delta u_i\ \mathrm{div}\left(x_\alpha u_i\nabla x_\alpha\right)\\
=\;& \sum_{\alpha=1}^{m}\int_\Omega\left(4\langle\nabla x_\alpha,\nabla u_i\rangle^2+2x_\alpha(\Delta u_i)\langle\nabla x_\alpha,\nabla u_i\rangle\right)-2\sum_{\alpha=1}^{m}\int_\Omega\Delta u_i\left(|\nabla x_\alpha|^2u_i+\langle x_\alpha\nabla x_\alpha,\nabla u_i\rangle\right)\\
=\;& 4\int_\Omega|\nabla u_i|^2-2n\int_\Omega u_i\Delta u_i\\
=\;& (4+2n)\int_\Omega|\nabla u_i|^2\leq(4+2n)\frac{1}{{P_2}^{1/2}}\left(\eta_i-\frac{V_0}{P_1}\right)^{1/2}
\end{aligned}$$

and

$$\sum_{\alpha=1}^{m}\left|\left|\frac{1}{\sqrt{\rho}}\langle\nabla x_\alpha,\nabla u_i\rangle\right|\right|^2=\int_\Omega\frac{|\nabla u_i|^2}{\rho}\leq\frac{1}{P_2}\int_\Omega|\nabla u_i|^2\leq\frac{1}{{P_2}^{3/2}}\left(\eta_i-\frac{V_0}{P_1}\right)^{1/2}. \tag{3.18}$$

Substituting (3.16)-(3.18) into (3.13), we have

$$\begin{aligned}
\frac{n}{P_1}\sum_{i=1}^{k}(\eta_{k+1}-\eta_i)^2\quad\leq\quad&\delta\frac{4+2n}{{P_2}^{1/2}}\sum_{i=1}^{k}(\eta_{k+1}-\eta_i)^2\left(\eta_i-\frac{V_0}{P_1}\right)^{1/2} \qquad (3.19)\\
&+\frac{1}{\delta}\frac{1}{{P_2}^{3/2}}\sum_{i=1}^{k}(\eta_{k+1}-\eta_i)\left(\eta_i-\frac{V_0}{P_1}\right)^{1/2}.
\end{aligned}$$

Taking

$$\delta=\left\{\frac{\sum_{i=1}^{k}(\eta_{k+1}-\eta_i)\left(\eta_i-\frac{V_0}{P_1}\right)^{1/2}}{(4+2n)P_2\sum_{i=1}^{k}(\eta_{k+1}-\eta_i)^2\left(\eta_i-\frac{V_0}{P_1}\right)^{1/2}}\right\}^{1/2},$$

one obtains (3.1).

ii) Assume now that Ω is a domain in an n-dimensional minimal submanifold of $S^m(1)$. Let $x_1,x_2,\cdots,x_{m+1}$ be the standard coordinate functons of the Euclidean space $I\!R^{m+1}$; then

$$S^m(1)=\left\{(x_1,\ldots,x_{m+1})\in I\!R^{m+1};\sum_{\alpha=1}^{m+1}x_\alpha^2=1\right\}.$$

It is well known that

$$\Delta x_\alpha=-nx_\alpha,\quad\alpha=1,\cdots,m+1. \tag{3.20}$$

For any $\delta>0$, by taking $h=x_\alpha$ in (2.4) and using (3.20), we get

$$\sum_{i=1}^{k}(\eta_{k+1}-\eta_i)^2\int_\Omega\left(nx_\alpha^2u_i^2-2x_\alpha u_i\langle\nabla x_\alpha,\nabla u_i\rangle\right)$$

$$\leq \delta\sum_{i=1}^{k}(\eta_{k+1}-\eta_i)^2\int_\Omega x_\alpha u_i(-n\Delta(u_ix_\alpha)+2\Delta\langle\nabla x_\alpha,\nabla u_i\rangle+2\langle\nabla x_\alpha,\nabla(\Delta u_i)\rangle-nx_\alpha\Delta u_i)$$
$$+\sum_{i=1}^{k}\frac{(\eta_{k+1}-\eta_i)}{\delta}\left\|\frac{\langle\nabla x_\alpha,\nabla u_i\rangle-\frac{nx_\alpha u_i}{2}}{\sqrt{\rho}}\right\|^2.$$

Summing over α, we get

$$\sum_{i=1}^{k}(\eta_{k+1}-\eta_i)^2\sum_{\alpha=1}^{m+1}\int_\Omega\left(nx_\alpha^2u_i^2-2x_\alpha u_i\langle\nabla x_\alpha,\nabla u_i\rangle\right) \tag{3.21}$$
$$\leq \delta\sum_{i=1}^{k}(\eta_{k+1}-\eta_i)^2\sum_{\alpha=1}^{m+1}\int_\Omega x_\alpha u_i\left(-n\Delta(u_ix_\alpha)+2\Delta\langle\nabla x_\alpha,\nabla u_i\rangle+2\langle\nabla x_\alpha,\nabla(\Delta u_i)\rangle-nx_\alpha\Delta u_i\right)$$
$$+\sum_{i=1}^{k}\frac{(\eta_{k+1}-\eta_i)}{\delta}\sum_{\alpha=1}^{m+1}\left\|\frac{\langle\nabla x_\alpha,\nabla u_i\rangle-\frac{nx_\alpha u_i}{2}}{\sqrt{\rho}}\right\|^2.$$

Using $\sum_{\alpha=1}^{m+1}x_\alpha^2=1$ and (3.9), we have

$$\sum_{\alpha=1}^{m+1}\int_\Omega\left(nx_\alpha^2u_i^2-2x_\alpha u_i\langle\nabla x_\alpha,\nabla u_i\rangle\right) \tag{3.22}$$
$$=\int_\Omega\left(nu_i^2-u_i\left\langle\nabla\left(\sum_{\alpha=1}^{m+1}x_\alpha^2\right),\nabla u_i\right\rangle\right)=\int_\Omega nu_i^2\geq\frac{n}{P_1}.$$

Let us calculate

$$\begin{aligned}\sum_{\alpha=1}^{m+1}\int_\Omega x_\alpha u_i(-n\Delta(u_ix_\alpha)) &= -n\sum_{\alpha=1}^{m+1}\int_\Omega x_\alpha u_i(-nu_ix_\alpha+2\langle\nabla x_\alpha,\nabla u_i\rangle+x_\alpha\Delta u_i)\\ &= -n\left(-n\int_\Omega u_i^2-\int_\Omega|\nabla u_i|^2\right)\\ &= n^2\int_\Omega u_i^2+n\int_\Omega|\nabla u_i|^2,\end{aligned} \tag{3.23}$$

$$\begin{aligned}\sum_{\alpha=1}^{m+1}\int_\Omega x_\alpha u_i(2\Delta\langle\nabla x_\alpha,\nabla u_i\rangle) &= 2\sum_{\alpha=1}^{m+1}\int_\Omega\Delta(x_\alpha u_i)\langle\nabla x_\alpha,\nabla u_i\rangle\\ &= 2\sum_{\alpha=1}^{m+1}\int_\Omega(-nx_\alpha u_i+2\langle\nabla x_\alpha,\nabla u_i\rangle+x_\alpha\Delta u_i)\langle\nabla x_\alpha,\nabla u_i\rangle\\ &= 4\int_\Omega\sum_{\alpha=1}^{m+1}\langle\nabla x_\alpha,\nabla u_i\rangle^2\\ &= 4\int_\Omega\sum_{\alpha=1}^{m+1}(\nabla u_i(x_\alpha))^2=4\int_\Omega|\nabla u_i|^2,\end{aligned} \tag{3.24}$$

$$\sum_{\alpha=1}^{m+1}\int_\Omega x_\alpha u_i(2\langle\nabla x_\alpha,\nabla(\Delta u_i)\rangle)=\int_\Omega u_i\left\langle\nabla\left(\sum_{\alpha=1}^{m+1}x_\alpha^2\right),\nabla(\Delta u_i)\right\rangle=0 \tag{3.25}$$

and

$$\sum_{\alpha=1}^{m+1}\int_\Omega x_\alpha u_i(-nx_\alpha\Delta u_i) = -n\int_\Omega u_i\Delta u_i = n\int_\Omega |\nabla u_i|^2. \tag{3.26}$$

Therefore, we obtain from (3.9), (3.11), (3.22)-(3.26) that

$$\begin{aligned}(3.27)\quad &\sum_{\alpha=1}^{m+1}\int_\Omega x_\alpha u_i(-n\Delta(u_ix_\alpha)+2\Delta\langle\nabla x_\alpha,\nabla u_i\rangle+2\langle\nabla x_\alpha,\nabla(\Delta u_i)\rangle-nx_\alpha\Delta u_i)\\ = \ & n^2\int_\Omega u_i^2+n\int_\Omega|\nabla u_i|^2+4\int_\Omega|\nabla u_i|^2+n\int_\Omega|\nabla u_i|^2\\ = \ & n^2\int_\Omega u_i^2+(2n+4)\int_\Omega|\nabla u_i|^2\\ \le \ & \frac{n^2}{P_2}+\frac{2n+4}{P_2^{1/2}}\left(\eta_i-\frac{V_0}{P_1}\right)^{1/2}\end{aligned}$$

and

$$\begin{aligned}(3.28)\quad &\sum_{\alpha=1}^{m+1}\left\|\frac{\langle\nabla x_\alpha,\nabla u_i\rangle+\frac{u_i\Delta x_\alpha}{2}}{\sqrt{\rho}}\right\|^2\\ = \ & \int_\Omega\frac{1}{\rho}\sum_{\alpha=1}^{n+1}\left(\langle\nabla x_\alpha,\nabla u_i\rangle^2-n\langle\nabla x_\alpha,\nabla u_i\rangle u_ix_\alpha+\frac{n^2u_i^2x_\alpha^2}{4}\right)\\ = \ & \frac{n^2}{4}\int_\Omega\frac{u_i^2}{\rho}+\int_\Omega\frac{|\nabla u_i|^2}{\rho}\\ \le \ & \frac{n^2}{4P_2^2}+\frac{\left(\eta_i-\frac{V_0}{P_1}\right)^{1/2}}{P_2^{3/2}}.\end{aligned}$$

Substituting (3.22), (3.27) and (3.28) into (3.21), we have

$$\begin{aligned}(3.29)\quad &\frac{n}{P_1}\sum_{i=1}^k(\eta_{k+1}-\eta_i)^2\\ \le \ & \delta\sum_{i=1}^k(\eta_{k+1}-\eta_i)^2\left(\frac{n^2}{P_2}+\frac{2n+4}{P_2^{1/2}}\left(\eta_i-\frac{V_0}{P_1}\right)^{1/2}\right)+\sum_{i=1}^k\frac{(\eta_{k+1}-\eta_i)}{\delta}\left(\frac{n^2}{4P_2^2}+\frac{\left(\eta_i-\frac{V_0}{P_1}\right)^{1/2}}{P_2^{3/2}}\right).\end{aligned}$$

Taking

$$\delta=\left\{\frac{\sum_{i=1}^k(\eta_{k+1}-\eta_i)\left(\frac{n^2}{4P_2^2}+\frac{\left(\eta_i-\frac{V_0}{P_1}\right)^{1/2}}{P_2^{\frac{3}{2}}}\right)}{\sum_{i=1}^k(\eta_{k+1}-\eta_i)^2\left(\frac{n^2}{P_2}+\frac{2n+4}{P_2^{1/2}}\left(\eta_i-\frac{V_0}{P_1}\right)^{1/2}\right)}\right\}^{1/2}$$

and simplifying, we get (3.2).

iii) Finally, assume that Ω is a domain in $CP^n(4)$. Let $z=(z_0,z_1,\cdots,z_n)$ be a homogeneous coordinate system of $CP^n(4)$, $(z_i\in C)$ and consider the functions

$$h_{p\bar{q}}=\frac{z_p\overline{z_q}}{\sum_{r=0}^n z_r\overline{z_r}},\quad p,q=0,1,\cdots,n. \tag{3.30}$$

Setting $g_{p\bar q}=\mathrm{Re}(h_{p\bar q})$ and $f_{p\bar q}=\mathrm{Im}(h_{p\bar q})$, $p,q=0,1,\cdots,n$, we have (cf. [CY2])

$$\sum_{p,q=0}^{n}(g_{p\bar q}^2+f_{p\bar q}^2)=1, \tag{3.31}$$

$$\sum_{p,q=0}^{n}(g_{p\bar q}\nabla g_{p\bar q}+f_{p\bar q}\nabla f_{p\bar q})=0, \tag{3.32}$$

$$\sum_{p,q=0}^{n}(\langle\nabla g_{p\bar q},\nabla g_{p\bar q}\rangle+\langle\nabla f_{p\bar q},\nabla f_{p\bar q}\rangle)=-\sum_{p,q=0}^{n}(g_{p\bar q}\Delta g_{p\bar q}+f_{p\bar q}\Delta f_{p\bar q})=4n, \tag{3.33}$$

$$\sum_{p,q=0}^{n}(\Delta g_{p\bar q}\nabla g_{p\bar q}+\Delta f_{p\bar q}\nabla f_{p\bar q})=0, \tag{3.34}$$

$$\sum_{p,q=0}^{n}(\Delta g_{p\bar q}\Delta g_{p\bar q}+\Delta f_{p\bar q}\Delta f_{p\bar q})=16n(n+1), \tag{3.35}$$

$$\sum_{p,q=0}^{n}(\langle\nabla g_{p\bar q},\nabla u_i\rangle^2+\langle\nabla f_{p\bar q},\nabla u_i\rangle^2)=2|\nabla u_i|^2. \tag{3.36}$$

Applying (2.4) to the functions $g_{p\bar q}$ and $f_{p\bar q}$ and summing over p and q, we obtain

$$\begin{aligned}
&\sum_{i=1}^{k}(\eta_{k+1}-\eta_i)^2\int_\Omega\sum_{p,q=0}^{n}(-u_i^2g_{p\bar q}\Delta g_{p\bar q}-2u_i\langle g_{p\bar q}\nabla g_{p\bar q},\nabla u_i\rangle-u_i^2f_{p\bar q}\Delta f_{p\bar q}-2u_i\langle f_{p\bar q}\nabla f_{p\bar q},\nabla u_i\rangle)\\
\le\ &\delta\sum_{i=1}^{k}(\eta_{k+1}-\eta_i)^2\Bigg\{\int_\Omega\sum_{p,q=0}^{n}g_{p\bar q}u_i(\Delta(u_i\Delta g_{p\bar q})+2\Delta\langle\nabla g_{p\bar q},\nabla u_i\rangle+2\langle\nabla g_{p\bar q},\nabla(\Delta u_i)\rangle+\Delta g_{p\bar q}\Delta u_i)\\
&+\int_\Omega\sum_{p,q=0}^{n}f_{p\bar q}u_i(\Delta(u_i\Delta f_{p\bar q})+2\Delta\langle\nabla f_{p\bar q},\nabla u_i\rangle+2\langle\nabla f_{p\bar q},\nabla(\Delta u_i)\rangle+\Delta f_{p\bar q}\Delta u_i)\Bigg\}\\
&+\frac{1}{\delta}\sum_{i=1}^{k}(\eta_{k+1}-\eta_i)\sum_{p,q=0}^{n}\left(\left\|\frac{1}{\sqrt\rho}\left(\langle\nabla g_{p\bar q},\nabla u_i\rangle+\frac{u_i\Delta g_{p\bar q}}{2}\right)\right\|^2\right.\\
&\left.+\left\|\frac{1}{\sqrt\rho}\left(\langle\nabla f_{p\bar q},\nabla u_i\rangle+\frac{u_i\Delta f_{p\bar q}}{2}\right)\right\|^2\right),
\end{aligned} \tag{3.37}$$

where δ is any positive constant.

From (3.9), (3.32) and (3.33), we have

$$\begin{aligned}
&\int_\Omega\sum_{p,q=0}^{n}(-u_i^2g_{p\bar q}\Delta g_{p\bar q}-2u_i\langle g_{p\bar q}\nabla g_{p\bar q},\nabla u_i\rangle-u_i^2f_{p\bar q}\Delta f_{p\bar q}-2u_i\langle f_{p\bar q}\nabla f_{p\bar q},\nabla u_i\rangle)\\
&=4n\int_\Omega u_i^2\ge\frac{4n}{P_1}.
\end{aligned} \tag{3.38}$$

Observe from (3.9), (3.11), (3.32)-(3.36) that

$$\begin{aligned}
&\int_\Omega\sum_{p,q=0}^{n}(g_{p\bar q}u_i\Delta(u_i\Delta g_{p\bar q})+f_{p\bar q}u_i\Delta(u_i\Delta f_{p\bar q}))\\
=\ &\int_\Omega\sum_{p,q=0}^{n}(\Delta(g_{p\bar q}u_i)u_i\Delta g_{p\bar q}+\Delta(f_{p\bar q}u_i)u_i\Delta f_{p\bar q})
\end{aligned} \tag{3.39}$$

$$
\begin{aligned}
&= \int_\Omega \sum_{p,q=0}^{n} \left\{ \left((\Delta g_{p\bar q})^2 + (\Delta f_{p\bar q})^2\right) u_i^2 + 2\left\langle \Delta g_{p\bar q}\nabla g_{p\bar q} + \Delta f_{p\bar q}\nabla f_{p\bar q}, \nabla u_i\right\rangle + (g_{p\bar q}\Delta g_{p\bar q} + f_{p\bar q}\Delta f_{p\bar q}) u_i \Delta u_i \right\} \\
&= 16n(n+1)\int_\Omega u_i^2 - 4n\int_\Omega u_i\Delta u_i \\
&\le \frac{16n(n+1)}{P_2} + \frac{4n}{{P_2}^{1/2}}\left(\eta_i - \frac{V_0}{P_1}\right)^{1/2},
\end{aligned}
$$

$$
\begin{aligned}
(3.40)\quad & \int_\Omega \sum_{p,q=0}^{n} \left(g_{p\bar q}u_i(\Delta\langle\nabla g_{p\bar q},\nabla u_i\rangle + \langle \nabla g_{p\bar q}, \nabla(\Delta u_i)\rangle) + f_{p\bar q}u_i(\Delta\langle\nabla f_{p\bar q},\nabla u_i\rangle + \langle\nabla f_{p\bar q},\nabla(\Delta u_i)\rangle)\right) \\
&= \int_\Omega \sum_{p,q=0}^{n} \left(\Delta(g_{p\bar q}u_i)\langle\nabla g_{p\bar q},\nabla u_i\rangle + \Delta(f_{p\bar q}u_i)\langle\nabla f_{p\bar q},\nabla u_i\rangle\right) \\
&= 2\int_\Omega \sum_{p,q=0}^{n} \left(\langle\nabla g_{p\bar q},\nabla u_i\rangle^2 + \langle\nabla f_{p\bar q},\nabla u_i\rangle^2\right) \\
&= 4\int_\Omega |\nabla u_i|^2 \le \frac{4}{{P_2}^{1/2}}\left(\eta_i - \frac{V_0}{P_1}\right)^{1/2},
\end{aligned}
$$

$$
(3.41)\quad \int_\Omega \sum_{p,q=0}^{n} (g_{p\bar q}u_i\Delta g_{p\bar q}\Delta u_i + f_{p\bar q}u_i\Delta f_{p\bar q}\Delta u_i) = -4n\int_\Omega u_i\Delta u_i \le \frac{4n}{{P_2}^{1/2}}\left(\eta_i - \frac{V_0}{P_1}\right)^{1/2}
$$

and

$$
\begin{aligned}
(3.42)\quad & \sum_{p,q=0}^{n}\left(\left\|\frac{1}{\sqrt{\rho}}\left(\langle\nabla g_{p\bar q},\nabla u_i\rangle + \frac{u_i\Delta g_{p\bar q}}{2}\right)\right\|^2 + \left\|\frac{1}{\sqrt{\rho}}\left(\langle\nabla f_{p\bar q},\nabla u_i\rangle + \frac{u_i\Delta f_{p\bar q}}{2}\right)\right\|^2\right) \\
&= \int_\Omega \frac{1}{\rho}\sum_{p,q=0}^{n}\left(\langle\nabla g_{p\bar q},\nabla u_i\rangle^2 + \langle\nabla f_{p\bar q},\nabla u_i\rangle^2\right) \\
&\quad + \int_\Omega \frac{1}{\rho}\sum_{p,q=0}^{n}\left(\langle\Delta g_{p\bar q}\nabla g_{p\bar q} + \Delta f_{p\bar q}\nabla f_{p\bar q}, u_i\nabla u_i\rangle + \frac{u_i^2}{4}((\Delta g_{p\bar q})^2 + (\Delta f_{p\bar q})^2)\right) \\
&= 2\int_\Omega \frac{|\nabla u_i|^2}{\rho} + 16n(n+1)\int_\Omega \frac{u_i^2}{\rho} \le \frac{2}{P_2^{3/2}}\left(\eta_i - \frac{V_0}{P_1}\right)^{1/2} + \frac{16n(n+1)}{P_2^2}.
\end{aligned}
$$

Substituting (3.38)-(3.42) into (3.37), we get

$$
\begin{aligned}
(3.43)\quad \frac{4n}{P_1}\sum_{i=1}^{k}(\eta_{k+1}-\eta_i)^2 \quad &\le \quad \delta\sum_{i=1}^{k}(\eta_{k+1}-\eta_i)^2\left(\frac{16n(n+1)}{P_2} + \frac{8n+8}{{P_2}^{1/2}}\left(\eta_i - \frac{V_0}{P_1}\right)^{1/2}\right) \\
&\quad + \frac{1}{\delta}\sum_{i=1}^{k}(\eta_{k+1}-\eta_i)\left(\frac{16n(n+1)}{P_2^2} + \frac{2\left(\eta_i - \frac{V_0}{P_1}\right)^{1/2}}{P_2^{3/2}}\right).
\end{aligned}
$$

Taking

$$
\delta = \left\{ \frac{\sum_{i=1}^{k}(\eta_{k+1}-\eta_i)\left(\frac{16n(n+1)}{P_2^2} + \frac{2\left(\eta_i - \frac{V_0}{P_1}\right)^{1/2}}{P_2^{3/2}}\right)}{\sum_{i=1}^{k}(\eta_{k+1}-\eta_i)^2\left(\frac{16n(n+1)}{P_2} + \frac{8n+8}{{P_2}^{1/2}}\left(\eta_i - \frac{V_0}{P_1}\right)^{1/2}\right)} \right\}^{1/2}
$$

and simplifying, one gets (3.3). This completes the proof of Theorem 3.1. □

Proof of Corollary 3.2. Consider first the case that Ω is a domian in an n-dimensional complete minimal submanifold of $I\!R^m$. In this case, we have from lemma 2.2 and (3.1) that

$$\sum_{i=1}^{k}(\eta_{k+1}-\eta_i)^2 \leq \frac{2P_1(2(n+2))^{1/2}}{nP_2}\left(\sum_{i=1}^{k}(\eta_{k+1}-\eta_i)^2\right)^{1/2}\left(\sum_{i=1}^{k}(\eta_{k+1}-\eta_i)\left(\eta_i-\frac{V_0}{P_1}\right)\right)^{1/2}, \tag{3.44}$$

which gives

$$\sum_{i=1}^{k}(\eta_{k+1}-\eta_i)^2 \leq \frac{8P_1^2(n+2)}{n^2P_2^2}\sum_{i=1}^{k}(\eta_{k+1}-\eta_i)\left(\eta_i-\frac{V_0}{P_1}\right). \tag{3.45}$$

Set

$$\nu_i=\eta_i-\frac{V_0}{P_1};$$

then

$$\sum_{i=1}^{k}(\nu_{k+1}-\nu_i)^2 \leq \frac{8P_1^2(n+2)}{n^2P_2^2}\sum_{i=1}^{k}(\nu_{k+1}-\nu_i)\,\nu_i. \tag{3.46}$$

Setting

$$S=\frac{4P_1^2(n+2)}{n^2P_2^2}$$

and solving the quadratic inequality (3.46), we get

$$\nu_{k+1}\leq(1+S)\frac{1}{k}\sum_{i=1}^{k}\nu_i+\left\{\left(\frac{S}{k}\sum_{i=1}^{k}\nu_i\right)^2-(1+2S)\frac{1}{k}\sum_{j=1}^{k}\left(\nu_j-\frac{1}{k}\sum_{i=1}^{k}\nu_i\right)\right\}^{\frac{1}{2}}. \tag{3.47}$$

This is the inequality (3.4).

Assume now that Ω is a domain in a minimal submanifold of $S^m(1)$. It follows from (3.2) and lemma 2.2 that

$$\sum_{i=1}^{k}(\eta_{k+1}-\eta_i)^2 \leq \frac{P_1}{nP_2}\left\{\sum_{i=1}^{k}(\eta_{k+1}-\eta_i)^2\right\}^{\frac{1}{2}}\left\{\sum_{i=1}^{k}(\eta_{k+1}-\eta_i)\left(\frac{n^2}{P_2^{1/2}}+4\left(\eta_i-\frac{V_0}{P_1}\right)^{1/2}\right)\left(\frac{n^2}{P_2^{1/2}}+(2n+4)\left(\eta_i-\frac{V_0}{P_1}\right)^{1/2}\right)\right\}^{\frac{1}{2}},$$

which gives

$$\sum_{i=1}^{k}(\eta_{k+1}-\eta_i)^2 \leq \frac{P_1^2}{n^2P_2^2}\sum_{i=1}^{k}(\eta_{k+1}-\eta_i)\left(\frac{n^2}{P_2^{1/2}}+4\left(\eta_i-\frac{V_0}{P_1}\right)^{1/2}\right)\left(\frac{n^2}{P_2^{1/2}}+(2n+4)\left(\eta_i-\frac{V_0}{P_1}\right)^{1/2}\right). \tag{3.48}$$

Solving this quadratic inequality about η_{k+1}, one obtains (3.5). The proof of (3.6) is similar and will be omitted. This completes the proof of Corollary 3.2. □

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Qiaoling Wang
Departamento de Matemática, Universidade de Brasília
70910-900 Brasília-DF, Brazil. e-mail: wang@mat.unb.br

Changyu Xia
Departamento de Matemática, Universidade de Brasília, 70910-900 Brasília-DF, Brazil and
MPI for Mathematics in the Sciences, Inselstr. 22 D-04103 Leipzig, Germany. e-mail: xia@mat.unb.br