Dimension formulas of paramodular forms of squarefree level
and comparison with inner twist

By Tomoyoshi Ibukiyama and Hidetaka Kitayama

(Received July 30, 2014)
(Revised June 17, 2015)

Abstract. In this paper, we give an explicit dimension formula for the spaces of Siegel paramodular cusp forms of degree two of squarefree level. As an application, we propose a conjecture on symplectic group version of Eichler-Jacquet-Langlands type correspondence. It is a generalization of the previous conjecture of the first named author for prime levels published in 1985, where inner twists corresponding to binary quaternion hermitian forms over definite quaternion algebras were treated. Our present study contains also the case of indefinite quaternion algebras. Additionally, we give numerical examples of $L$ functions which support the conjecture. These comparisons of dimensions and examples give also evidence for conjecture on a certain precise lifting theory. This is related to the lifting theory from pairs of elliptic cusp forms initiated by Y. Ihara in 1964 in the case of compact twist, but no such construction is known in the case of non-split symplectic groups corresponding to quaternion hermitian groups over indefinite quaternion algebras and this is new in that sense.

1. Introduction

The purpose of this paper is to give two main theorems and two conjectures on Siegel paramodular cusp forms of degree two. The first theorem (Main Theorem 1.1) is an explicit dimension formula for the spaces of Siegel cusp forms with respect to the paramodular group of squarefree level, which was announced in [Kit14]. The second theorem (Main Theorem 1.2) is a comparison of dimensions of paramodular forms and those of automorphic forms belonging to non-split $\mathbb{Q}$-forms of the symplectic group of rank 2. This is derived from the first theorem and other explicit dimension formulas of the authors. We propose Conjecture 1.3 on correspondence between different symplectic $\mathbb{Q}$ forms which is expected from Main Theorem 1.2. This is closely related to Conjecture 1.4 on liftings given next. These two conjectures are not only based on dimension formulas, but also on many explicit examples. These are given as Theorem 4.6 in section 4.

This study is motivated by our attempts to generalize Eichler-Jacquet-Langlands type correspondence to the case of symplectic groups of higher degree. Originally, Eichler [Eic56], [Eic58] proved the correspondence between automorphic forms on $SL(2;\mathbb{R})$.
and on $SU(2)$, and Jacquet-Langlands [JL72] developed it from the viewpoint of representation theory. Langlands [Lan70] gave a general conjecture for any reductive algebraic groups. On the other hand, Ihara [Iha64] raised independently a problem of generalizing Eichler correspondence to symplectic groups of degree two in 1960s. About this problem, the first named author gave several precise conjectures, specifying corresponding discrete subgroups of prime level which are parahoric locally, defining new forms for such cases. These are based on numerical examples [Ibu84] and explicit dimension formulas in [Ibu85], [HI85] and [Ibu07a]. In particular we found there good dimensional relations of automorphic forms on the real symplectic group $Sp(2; \mathbb{R}) \subset M_4(\mathbb{R})$ of rank two and its compact twist which support the conjectures. However, the conjectures were restricted to the case of prime level because explicit dimension formulas for paramodular forms were not known except for prime level. In this paper, we achieve a generalization of the dimension formula of paramodular cusp forms to squarefree level and give further consideration to the conjecture. The generalization to the squarefree level case from the prime level case is not just a patch work of the prime levels. Firstly, the general case contains the case of quaternion hermitian groups over indefinite quaternion algebras, which has not been treated before since each level (discriminant) is a product of an even number of different primes. The nature of automorphic forms in these cases is very different from the compact twist case when the definite quaternion algebras appear, since in the latter, automorphic forms are essentially harmonic polynomials invariant by finite groups, while in the former, they are functions on the Siegel upper half space. Secondly, a global description of new forms in the general case demands more careful study of liftings. Indeed, the new comparison predicts a new conjectural injective liftings in detail.

Our first main theorem is as follows. We denote by $S_{k,j}(K(N))$ the space of Siegel cusp forms of degree two of weight $\det^k \otimes \text{Sym}_j$ for the paramodular group $K(N)$ of level $N$. (See subsection 2.1 for definitions).

**Main Theorem 1.1 (Theorem 3.1).** We assume that $N$ is a squarefree positive integer. If $j \geq 0$ is an even integer and $k \geq 5$, then we have

$$\dim S_{k,j}(K(N)) = \sum_{i=1}^{12} H_i(k, j, N) + \sum_{i=1}^{10} I_i(k, j, N),$$

where $H_i(k, j, N)$ and $I_i(k, j, N)$ are explicit elementary functions of $k$, $j$ and $N$, which are given in Theorem 3.1.

**Remark.** (i) If $k = 3, 4$ and $j = 0$, then the above formula is still applicable as follows:

$$\dim S_{k,0}(K(N)) = \sum_{i=1}^{12} H_i(k, 0, N) + \sum_{i=1}^{10} I_i(k, 0, N) + \begin{cases} 1 \cdots & \text{if } k = 3, \\ 0 \cdots & \text{if } k = 4. \end{cases}$$

See Ibukiyama [Ibu07b]. For $j > 0$ and $k = 3, 4$, we conjecture that the above formula in Theorem 1.1 is valid without any correction terms.

(ii) Main Theorem 1.1 is a generalization of the previous formulas in [Ibu85],[Ibu07a] for prime level, and also [Wak12, Theorem 7.1] for $K(1) = Sp(2; \mathbb{Z})$. 
We prove Main Theorem 1.1 in Section 7 and 8 by using the method which is based on Selberg trace formula. The condition $k \geq 5$ comes from the condition for convergence of the trace formula. Roughly speaking, in order to obtain an explicit dimension formula from Selberg trace formula, we have to calculate the contribution of all $K(N)$-conjugacy classes and sum them up. As for semisimple elements, we can achieve it by careful use of local data in [HI80] and [Ibu85]. On the other hand, for non-semisimple elements, we have to classify $K(N)$-conjugacy classes by direct calculation and evaluate the contributions by using formulas in [Has83] and [Wak12]. For further details, see Section 6.

Our second main theorem is explained briefly as follows. Let $B$ be a quaternion algebra over $\mathbb{Q}$ and let $N$ be the product of primes $p < \infty$ such that $B_p$ is division. We call $N$ the discriminant of $B$. We denote by $\omega(N)$ the number of prime divisors of $N$. Then $B$ is definite if $\omega(N)$ is odd and indefinite if even. We define $G = \{ g \in M(2; B) \mid g^t \overline{g} = n(g) 1_2, n(g) \in \mathbb{Q}^\times \}$, where $\overline{g} = \frac{\overline{a} \overline{d}}{\overline{bc}}$ for $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\overline{\cdot}$ is the canonical involution of $B$. When $B$ is definite, let $G_A$ be its adelization and $U(N)$ be a certain open subgroup of $G_A$ which is defined in subsection 2.2. We denote by $\mathcal{M}_{k+j-3,k-3}(U(N))$ the space of automorphic forms on $U(N)$ of weight $\rho_{k+j-3,k-3}$ corresponding to the Young diagram $(k+j-3, k-3)$ which parametrizes the irreducible representation of the compact symplectic group $Sp(2)$ of rank 2. This space of automorphic forms can be regarded as a direct sum of some harmonic polynomials of eight variables invariant by certain finite groups which are the groups of automorphisms of some binary quaternion hermitian lattices in a fixed genus (See [II87]). On the other hand, when $B$ is indefinite, $G^1 = \{ g \in G \mid n(g) = 1 \}$ is a non-split $\mathbb{Q}$-form of $Sp(2; \mathbb{R})$. Let $U'(N)$ be a certain discrete subgroup of $Sp(2; \mathbb{R})$ which is defined in subsection 2.3. We denote by $S_{k,j}(U'(N))$ the space of Siegel cusp forms with respect to $U'(N)$ of weight $\det^k \otimes \text{Sym}_j$. Then our second main theorem is as follows. We denote by $S_k^{new}(\Gamma_0^{(1)}(N))$ the space of new cusp forms of weight $k$ of level $N$ of one variable with respect to $\Gamma_0^{(1)}(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2; \mathbb{Z}) \mid c \equiv 0 \text{ mod } N \right\}$ in the usual sense.

**Main Theorem 1.2.** (i) We assume that $N$ is a squarefree positive integer and $\omega(N)$ is odd. If $j \geq 2$ is even and $k \geq 5$, or $j = 0$ and $k \geq 3$, then we have

$$\sum_{M \mid N} (-1)^{\omega(M)} 2^{\omega(M)} \dim S_{k,j}(K(N/M)) = \dim \mathcal{M}_{k+j-3,k-3}(U(N)) - \sum_{M \mid N, \omega(M) = \text{odd}} (\dim S_{j+2}^{new}(\Gamma_0^{(1)}(M)) + \delta_{j0}) \times \dim S_{2k+j-2}^{new}(\Gamma_0^{(1)}(N/M)) - \delta_{j0} \delta_{k3},$$

where $\delta_{j0}$ and $\delta_{k3}$ is the Kronecker delta.

(ii) We assume that $N$ is a squarefree positive integer and $\omega(N)$ is even. If $j$ is an even non-negative integer and $k \geq 5$, then we have

$$\sum_{M \mid N} (-1)^{\omega(M)} 2^{\omega(M)} \dim S_{k,j}(K(N/M)) =$$
\[ \dim S_{k,j}(U'(N)) = \sum_{M \mid N, \omega(M) = \text{odd}} (\dim S_{j+2}^{\text{new}}(\Gamma_0(1)(M)) + \delta_{j0}) \times \dim S_{2k+j-2}^{\text{new}}(\Gamma_0(1)(N/M)). \]

We conjecture that the same equality holds also for \( k = 3, 4 \) for any even \( j \geq 0 \) in both cases.

This theorem makes us expect a correspondence between \( S_{k,j}(K(N)) \) and \( \mathfrak{M}_{k+j-3,k-3}(U(N)) \) or \( S_{k,j}(U'(N)) \) according to the parity of \( \omega(N) \). The first named author proved the above equality (i) already in the case of \( \omega(N) = 1 \) and proposed a conjecture of correspondence between spaces of “new forms” of \( S_{k,j}(K(p)) \) and \( \mathfrak{M}_{k+j-3,k-3}(U(p)) \) in [Ibu85] and [Ibu07a]. Main Theorem 1.2 makes us expect that we can propose the following generalized conjecture for subspaces of suitably defined new forms of \( S_{k,j}(K(N)) \), \( \mathfrak{M}_{k+j-3,k-3}(U(N)) \), and \( S_{k,j}(U'(N)) \).

**Conjecture 1.3.** Let \( N \) be a squarefree positive integer. We denote subspaces of “new forms” by \( S_{k,j}^{\text{new}}(K(N)) \), \( \mathfrak{M}_{k+j-3,k-3}^{\text{new}}(U(N)) \), and \( S_{k,j}^{\text{new}}(U'(N)) \). Then for any \( k \geq 3 \) and even \( j \geq 0 \), we have isomorphisms

\[ S_{k,j}^{\text{new}}(K(N)) \cong \begin{cases} \mathfrak{M}_{k+j-3,k-3}^{\text{new}}(U(N)) & \text{if } \omega(N) \text{ is odd,} \\ S_{k,j}^{\text{new}}(U'(N)) & \text{if } \omega(N) \text{ is even} \end{cases} \]

which preserve (the spinor) \( L \) functions.

Definitions of new forms has been given in [Ibu85] when \( N \) is a prime. The general definition for squarefree \( N \) will be explained later in Section 4. Here, to describe such precise isomorphisms, we note that the choice of \( U(N) \) or \( U'(N) \) is quite essential (See section 2 for definition.)

The relation in Main Theorem 1.2 strongly suggests also that there should exist liftings to automorphic forms of both sides from elliptic modular forms. What to expect on such liftings is closely related to Conjecture 1.3 above. To describe the conjecture precisely on such liftings, we must specify the action of the Atkin-Lehner involution. We define

\[ S_{2k-2}^{\text{new}}(\Gamma_0(1)(N)) = \left\{ f \in S_{2k-2}^{\text{new}}(\Gamma_0(1)(N)) \mid u_N f = \mp (-1)^k f \right\}, \]

where \( u_N \) is the Atkin-Lehner involution defined by the action on \( \begin{bmatrix} 0 & -1 \\ N & 0 \end{bmatrix} \). In fact, the sign of the superscript \( \pm \) of \( S_{2k-2}^{\text{new}, \pm}(\Gamma_0(1)(N)) \) is equal to the sign of the functional equation (and not the eigenvalue of the Atkin-Lehner involution), that is, if we put \( \Lambda(s, f) = (2\pi/\sqrt{N})^{-s} \Gamma(s)L(s, f) \) for \( f \in S_{2k-2}^{\text{new}, \pm}(\Gamma_0(1)(N)) \), then \( \Lambda(2k-2-s) = \pm \Lambda(s, f) \).

Careful check of examples and dimensions suggest the existence of the following liftings.

**Conjecture 1.4.** The spaces \( \mathfrak{M}_{k-3,k-3}(U(N)) \) and \( S_{k,0}(U'(N)) \) for \( k \geq 3 \) have liftings from

\[
\begin{cases}
S_2^{\text{new}}(\Gamma_0(1)(M)) \times S_{2k-2}^{\text{new}}(\Gamma_0(1)(N/M)) : M \mid N, \omega(M) = \text{odd}, \\
S_{2k-2}^{\text{new}, +}(\Gamma_0(1)(N/M)) : M \mid N, \omega(M) = \text{odd}, \\
S_{2k-2}^{\text{new}, -}(\Gamma_0(1)(N/M)) : M \mid N, \omega(M) = \text{even}.
\end{cases}
\]
If \( j \geq 2 \), the spaces \( \mathcal{M}_{k+j-3,k-3}(U(N)) \) and \( S_{k,j}(U'(N)) \) for \( k \geq 3 \) have liftings from
\[
S_{j+2}^{\text{new}}(\Gamma_0^{(1)}(M)) \times S_{2k+j-2}^{\text{new}}(\Gamma_0^{(1)}(N/M)) : \ M \mid N, \ \omega(M) = \text{odd}.
\]
All the above lifting maps are injective.

In this Conjecture, the relation of the \( L \) functions should be as follows. For \( g \in S_{j+2}^{\text{new}}(\Gamma_0^{(1)}(M)) \) and \( f \in S_{2k+j-2}^{\text{new}}(\Gamma_0^{(1)}(N/M)) \), the above lift \( \iota(f,g) \in M_{k+j-3,k-3}(U(N)) \) or \( S_{k,j}(U'(N)) \) from the pair \((f,g)\) should satisfy
\[
L(s,\iota(f,g),Sp) = L(s,f)L(s-k+2,g),
\]
up to bad Euler factors where the LHS is the spinor \( L \) function. For \( f \in S_{2k-2}^{\text{new},\pm}(\Gamma_0(M)) \) for an appropriate sign and a lift \( \iota(f) \in \mathcal{M}_{k-3,k-3}(U(N)) \) or \( S_{k,0}(U'(N)) \), we should have
\[
L(s,\iota(f),Sp) = \zeta(s-k+1)\zeta(s-k+2)L(s,f)
\]
up to bad Euler factors.

The last claim on injectivity is important since this is suggested by dimensional relations and a key point for the definition of new forms.

By the way, we recall that for \( \mathcal{M}_{k+j-3,k-3}(U(N)) \), there has been a certain theory of Saito-Kurokawa type or Yoshida type lifting initiated by Ihara [Iha64] and generalized in [II87], but these constructions do not cover all the above cases. Besides, in the case of \( S_{k,j}(U'(N)) \), any construction of Yoshida type lifting is not known as far as the authors know, so it seems very interesting to explore such a concrete theory in this case.

We organize this paper as follows. This paper consists of nine sections. In Section 2, we give preliminaries on Siegel cusp forms, paramodular groups, compact twists and non-split \( \mathbb{Q} \)-forms of \( Sp(2;\mathbb{R}) \). In Section 3, we give Main Theorem 1.1 precisely (Theorem 3.1) and give the proof of Main Theorem 1.2. We postpone the proof of Main Theorem 1.1 until Sections 6–8 because it is very lengthy. Section 6 gives preliminaries to describing the proof, and Sections 7 and 8 give the proof in detail. In Section 4, we explain details on Main Theorem 1.2 and Conjecture 1.3. In particular, we give definitions of new forms and we explain more in detail why our Conjecture 1.3 and 1.4 seem reasonable. Numerical evidence of \( L \) functions which support Conjecture 1.3 and 1.4 is given as Theorem 4.6. In Section 5, we give some details how to calculate the results in Theorem 4.6 and explain more about liftings. Finally, we give numerical tables of dimensions in Section 9.

Acknowledgment: Some calculations of the Euler factors of examples of new forms in Section 4 were done by Cris Poor and David Yuen according to our request. We would like to thank them not only for allowing us to include their results here but also for showing their interest in our works.
2. Preliminaries

2.1. Siegel cusp forms

Let $\text{Sp}(2; \mathbb{R})$ be the real symplectic group of degree two:

$$
\text{Sp}(2; \mathbb{R}) = \left\{ g \in GL(4, \mathbb{R}) \mid g \begin{bmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{bmatrix}^t g = \begin{bmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{bmatrix} \right\}.
$$

Let $\mathfrak{H}_2$ be the Siegel upper half space of degree two:

$$
\mathfrak{H}_2 = \{ Z \in M(2; \mathbb{C}) \mid \text{Im}(Z) \text{ is positive definite} \}.
$$

The group $\text{Sp}(2; \mathbb{R})$ acts on $\mathfrak{H}_2$ by

$$
g(\langle Z \rangle) := \begin{bmatrix} A & B \\ C & D \end{bmatrix} (CZ + D) \begin{bmatrix} A & B \\ C & D \end{bmatrix}^t,
$$

for any $g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \text{Sp}(2; \mathbb{R})$ and $Z \in \mathfrak{H}_2$.

Let $\Gamma$ be a discrete subgroup of $\text{Sp}(2; \mathbb{R})$ such that $\text{vol}(\Gamma \backslash \mathfrak{H}_2) < \infty$. Let $\tau_{k,j} : \text{GL}(2; \mathbb{C}) \to \text{GL}(j+1; \mathbb{C})$ be the irreducible rational representation of the Young diagram parameter $(j+k, k)$ for $k, j \in \mathbb{Z}_{\geq 0}$, i.e. $\tau_{k,j} = \det^k \otimes \text{Sym}_j$, where $\text{Sym}_j$ is the symmetric $j$-th tensor representation of $\text{GL}(2; \mathbb{C})$. We denote by $\text{Sp}_{k,j}(\Gamma)$ the space of Siegel cusp forms of weight $\tau_{k,j}$ with respect to $\Gamma$, i.e. the space which consists of holomorphic function $f : \mathfrak{H}_2 \to \mathbb{C}$ satisfying the following two conditions:

(i) $f(\gamma(Z)) = \tau_{k,j}(CZ + D)f(Z)$, for $\forall \gamma = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Gamma$, $\forall Z \in \mathfrak{H}_2$;

(ii) $|\tau_{k,j}(\text{Im}(Z))^{1/2}f(Z)|_{\mathbb{C}^{j+1}}$ is bounded on $\mathfrak{H}_2$,

where we define $|u|_{\mathbb{C}^{j+1}} = (\sum u_i^2)^{1/2}$ for $u \in \mathbb{C}^{j+1}$.

Let $N$ be a positive integer. We denote the paramodular group of level $N$ by $K(N)$, namely

$$
K(N) = \left[ \begin{bmatrix} Z & N & Z & Z \\ Z & N & Z & Z \\ Z & N & Z & Z \\ NZ & NZ & NZ & Z \end{bmatrix} \right] \cap \text{Sp}(2; \mathbb{Q})
$$

2.2. Compact twist of $\text{Sp}(2, \mathbb{R})$

Let $B$ be a definite quaternion algebra over $\mathbb{Q}$ with discriminant $N$. We put

$$
G = \{ g \in M(2; B) \mid g \cdot i g = n(g)1_2, n(g) \in \mathbb{Q}_+^\times \},
$$

$$
G_\infty = \{ g \in M(2; \mathbb{H}) \mid g \cdot i g = n(g)1_2, n(g) \in \mathbb{R}_+^\times \}
$$

($\mathbb{H}$: the division quaternion algebra over $\mathbb{R}$). Then $G_\infty^1 = \{ g \in G_\infty : n(g) = 1 \}$ is the compact twist of $\text{Sp}(2; \mathbb{R})$. Let $G_A$ be the adelization and $G_p$ the local component at a prime $p$. We fix a maximal order $\mathcal{O}$ of $B$ and let $\mathcal{O}_p$ be the local completion. For $p | N$, we take $\xi \in \text{GL}(2; \mathcal{O}_p) = M_2(\mathcal{O}_p)^\times$ such that $\xi \cdot i \xi = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and let $\pi$ be a prime element
of \( \mathfrak{O}_p \). Put
\[
U_p = \xi^{-1} \left[ \mathfrak{O}_p \pi^{-1} \mathfrak{O}_p \right]^\times \xi \cap G_p.
\]
For any prime \( p \nmid N \), we put \( U_p = GL(2; \mathfrak{O}_p) \cap G_p \). We define an open subgroup \( U(N) \) of \( G_A \) by
\[
U(N) = G_\infty \prod_p U_p.
\]
We take a rational representation \( \rho_{f_1, f_2}, V \) of \( G^1_\infty \) corresponding to the Young diagram parameter \( (f_1, f_2) \) with \( f_1 \geq f_2 \geq 0 \). We assume that \( f_1 \equiv f_2 \mod 2 \). Then \( \rho_{f_1, f_2} \) factors through \( G^1_\infty /\{ \pm 1_2 \} \). We define a representation of \( G_A \) by
\[
G_A \to G_\infty \to G_\infty /\mathbb{R}^{\times} 1_2 \simeq G^1_\infty /\{ \pm 1_2 \} \to GL(V)
\]
and denote this also by \( \rho_{f_1, f_2} \). A \( V \)-valued function \( f(g) \) on \( G_A \) is defined to be an automorphic form of weight \( \rho_{f_1, f_2} \) belonging to \( U(N) \) if \( f(uga) = \rho_{f_1, f_2}(u)f(g) \) for any \( a \in G, u \in U(N) \), and \( g \in G_A \). We denote the space of these automorphic forms by \( \mathfrak{M}_{f_1, f_2}(U(N)) \).

2.3. Non-split \( \mathbb{Q} \)-forms of \( Sp(2, \mathbb{R}) \)

Let \( B \) be an indefinite division quaternion algebra over \( \mathbb{Q} \) with discriminant \( N \). We define an algebraic group \( G^* \) by
\[
G^* = \left\{ g \in M(2; B) \mid g \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mathfrak{B}_g = n(g) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, n(g) \in \mathbb{Q}^{\times} \right\}.
\]
Then \( G^1 = \{ g \in G^* \mid n(g) = 1 \} \) is a non-split \( \mathbb{Q} \) form of \( Sp(2, \mathbb{R}) \) and \( G^1_\infty \simeq Sp(2, \mathbb{R}) \). Let \( \mathfrak{O} \) be a maximal order of \( B \) and \( \mathfrak{P} \) the maximal ideal of \( \mathfrak{O}_p \) when \( p \mid N \). We take the two-sided ideal \( \mathfrak{A} \) of \( \mathfrak{O} \) such that \( \mathfrak{A}_p = \mathfrak{P} \) for \( p \mid N \) and \( \mathfrak{A}_p = \mathfrak{O}_p \) for \( p \nmid N \). We define a discrete subgroup \( U'(N) \) by
\[
U'(N) = \left[ \mathfrak{O} \mathfrak{A}^{-1} \right]^\times \cap G^1.
\]
We can regard this as a discrete subgroup of \( Sp(2, \mathbb{R}) \) with covolume finite through the isomorphism \( G^1_\infty \simeq Sp(2, \mathbb{R}) \).

2.4. Dimension formulas for compact twist and non-split forms

Explicit formulas of \( \text{dim} \mathfrak{M}_{k+j-1, k-3}(U) \) and \( \text{dim} \mathfrak{M}_{k, j}(U') \) were given in \([HI82]\) and \([Kit11]\) for more general \( U \) and \( U' \). Here for \( U(N) \) and \( U'(N) \), we rewrite the formulas below in slightly easier way than the previous papers, which is more suitable for our comparison. We use the following notations. We denote by \( \omega(N) \) the number of prime divisors of \( N \). For natural number \( m \) and \( n \), we denote by \( [a_0, \ldots, a_{m-1}; m]_n \) the function of \( n \) which takes the value \( a_i \) if \( n \equiv i \mod m \). We define the set \( N(m; n) := \{ p \mid N; p \equiv
where integer. Then we have

\[
\left( \frac{-1}{p} \right) = \begin{cases} 
0 & \text{if } p = 2, \\
1 & \text{if } p \equiv 1 \mod 4, \\
-1 & \text{if } p \equiv 3 \mod 4,
\end{cases} \quad \left( \frac{-3}{p} \right) = \begin{cases} 
0 & \text{if } p = 3, \\
1 & \text{if } p \equiv 1 \mod 3, \\
-1 & \text{if } p \equiv 2 \mod 3.
\end{cases}
\]

We define the following functions:

\[
\begin{align*}
C_3(k, j) &= \left[ (k-2)(-1)^{j/2}, -(j+k-1), -(k-2)(-1)^{j/2}, j+k-1; 4 \right]_k, \\
C_4(k, j) &= (j + k - 1) \cdot [1, -1, 0; 3]_k + (k - 2) \cdot [1, 0, -1; 3]_{j+k}, \\
C_5(k, j) &= (j + k - 1) \cdot [-1, -1, 0, 1, 1, 0; 6]_k + (k - 2) \cdot [1, 0, -1, -1, 0; 6]_{j+k}, \\
C_6(k, j) &= \begin{cases} 
[1, 0, 0, -1, -1, -1, 0, 1, 1, 1; 12]_k \cdots \text{if } j \equiv 0 \mod 12, \\
[1, -1, 0, 1, 0, 1, -1, 0, -1, 0; 12]_k \cdots \text{if } j \equiv 2 \mod 12, \\
[1, -1, 0, 0, -1, 1, -1, 1, 0, 1; 12]_k \cdots \text{if } j \equiv 4 \mod 12, \\
[1, -1, 0, 0, 0, 1, -1, 1, 0, 1; 12]_k \cdots \text{if } j \equiv 6 \mod 12, \\
[1, 0, 1, -1, 0, -1, 1, 0, -1, 1; 12]_k \cdots \text{if } j \equiv 8 \mod 12, \\
[1, -1, 0, 0, 1, 1, 1, 0, 0, -1, 1; 12]_k \cdots \text{if } j \equiv 10 \mod 12,
\end{cases}
\]
\[
C_8(k, j) = \begin{cases} 
[0, 0, -1, 0, 0; 6]_k \cdots \text{if } j \equiv 0 \mod 6, \\
[0, -1, 0, 0, 1, 0; 6]_k \cdots \text{if } j \equiv 4 \mod 6, \\
[1, 0, 0, -1, 0; 5]_k \cdots \text{if } j \equiv 2 \mod 6, \\
[1, -1, 0, 0, 0; 5]_k \cdots \text{if } j \equiv 4 \mod 6,
\end{cases}
\]
\[
C_{10}(k, j) = \begin{cases} 
0 \cdots \text{if } j \equiv 0 \mod 10, \\
[0, 0, 0, -1, 1; 5]_k \cdots \text{if } j \equiv 2 \mod 10, \\
[0, -1, 0, 0, 1; 5]_k \cdots \text{if } j \equiv 4 \mod 10, \\
[1, 0, 0, -1, 0; 5]_k \cdots \text{if } j \equiv 6 \mod 10, \\
[1, -1, 0, 0, 0; 5]_k \cdots \text{if } j \equiv 8 \mod 10,
\end{cases}
\]
\[
C_{11}(k, j) = \begin{cases} 
[1, 0, 0, -1; 4]_k \cdots \text{if } j \equiv 0 \mod 8, \\
[1, -1, 0, 0, 0; 4]_k \cdots \text{if } j \equiv 2 \mod 8, \\
[1, -1, 0, 0, 1; 4]_k \cdots \text{if } j \equiv 4 \mod 8, \\
[1, -1, 0, 0; 4]_k \cdots \text{if } j \equiv 6 \mod 8.
\end{cases}
\]

**Theorem 2.1.** Let \( N \) be a squarefree positive integer and \( j \) an even non-negative integer. Then we have

\[
\dim \mathfrak{M}_{k+j-3,k-3}(U(N)) = \sum_{i=1}^{12} H_i'(k, j, N) \quad \text{if } \omega(N) \text{ is odd and } k \geq 3,
\]
\[
\dim S_{k,j}(U'(N)) = \sum_{i=1}^{12} H_i'(k, j, N) + \sum_{i=1}^{10} I_i'(k, j, N) \quad \text{if } \omega(N) \text{ is even and } k \geq 5,
\]

where \( H_i'(k, j, N) \) and \( I_i'(k, j, N) \) are given as follows:

\[
H_1'(k, j, N) = \prod_{p \mid N} (p^2 - 1) \cdot 2^{-7} 3^{-3} 5^{-1} \cdot (j+1)(k-2)(j+k-1)(j+2k-3),
\]
\[
H_2'(k, j, N) = -2^{-7} 3^{-2} (j+k-1)(k-2)(-1)^k \times \begin{cases} 
3 & \text{if } N = 2, \\
0 & \text{if } N \neq 2,
\end{cases}
\]
\[
H'_3(k, j, N) = 2^{-5} 3^{-1} C_3(k, j) \times \begin{cases} 
3 & \text{if } N = 2, \\
0 & \text{if } N \neq 2,
\end{cases}
\]
\[
H'_4(k, j, N) = 2^{-3} 3^{-3} C_4(k, j) \times \begin{cases} 
8 & \text{if } N = 3, \\
0 & \text{if } N \neq 3,
\end{cases}
\]
\[
H'_5(k, j, N) = 0,
\]
\[
H'_6(k, j, N) = H'_{6,1}(N) \times (-1)^{j/2} (2k + j - 3) + H'_{6,2}(N) \times (-1)^{j/2+k}(j + 1),
\]
where
\[
H'_{6,1}(N) = \frac{1}{2^5 \cdot 3} \prod_{p \mid N} \left( p - \left( \frac{-1}{p} \right) \right) + \frac{1}{2^7 \cdot 3} \prod_{p \mid N} \left( p \left( \frac{-1}{p} \right) - 1 \right),
\]
\[
H'_{6,2}(N) = \frac{(-1)^{\omega(N)}}{2^5 \cdot 3} \prod_{p \mid N} \left( p - \left( \frac{-1}{p} \right) \right) - \frac{1}{2^7 \cdot 3} \prod_{p \mid N} \left( p \left( \frac{-1}{p} \right) - 1 \right),
\]
\[
H'_7(k, j, N) = H'_{7,1}(N) \times [1, -1, 0; 3]_j (2k + j - 3) + H'_{7,2}(N) \times [0, 1, -1; 3]_j (2k + j + 1),
\]
where
\[
H'_{7,1}(N) = \frac{1}{2^4 \cdot 3^2} \prod_{p \mid N} \left( p - \left( \frac{-3}{p} \right) \right) + \frac{1}{2^8 \cdot 3^3} \prod_{p \mid N} \left( p \left( \frac{-3}{p} \right) - 1 \right),
\]
\[
H'_{7,2}(N) = \frac{(-1)^{\omega(N)}}{2^4 \cdot 3^2} \prod_{p \mid N} \left( p - \left( \frac{-3}{p} \right) \right) - \frac{1}{2^8 \cdot 3^3} \prod_{p \mid N} \left( p \left( \frac{-3}{p} \right) - 1 \right),
\]
\[
H'_8(k, j, N) = 0,
\]
\[
H'_9(k, j, N) = -2^{-1} 3^{-2} C_9(k, j) \times \begin{cases} 
3 & \text{if } N = 2, \\
0 & \text{if } N \neq 2,
\end{cases}
\]
\[
H'_{10}(k, j, N) = (-1)^{\omega(N)} 2^{-1} 5^{-1} C_{10}(k, j) \prod_{p \mid N} 2 \times \begin{cases} 
0 & \text{if } N(1; 5) \cup N(4; 5) \neq \emptyset, \\
1 & \text{if } N(1; 5) \cup N(4; 5) = \emptyset \text{ and } 5 \mid N, \\
2 & \text{otherwise},
\end{cases}
\]
\[
H'_{11}(k, j, N) = (-1)^{\omega(N)} 2^{-3} C_{11}(k, j) \prod_{p \mid N, p \neq 2} 2 \times \begin{cases} 
0 & \text{if } N(1; 8) \cup N(7; 8) \neq \emptyset, \\
1 & \text{if } N(1; 8) \cup N(7; 8) = \emptyset,
\end{cases}
\]
\[
H'_{12}(k, j, N) = H'_{12,1}(N) \times [1, -1, 0; 3]_j (-1)^{j/2+k} + H'_{12,2}(N) \times [0, -1, 1; 3]_j (2k + (-1)^{j/2}),
\]
where
\[
H'_{12,1}(N) = \frac{(-1)^{\omega(N)}}{2^5 \cdot 3} \prod_{p \mid N} \left( 1 - \left( \frac{3}{p} \right) \right) - \frac{1}{2^8 \cdot 3^3} \prod_{p \mid N} \left( (\frac{-1}{p}) - \left( \frac{-3}{p} \right) \right),
\]
\[
H'_{12,2}(N) = \frac{(-1)^{\omega(N)}}{2^5 \cdot 3} \prod_{p \mid N} \left( 1 - \left( \frac{3}{p} \right) \right) + \frac{(-1)^{\omega(N)}}{2^8 \cdot 3^3} \prod_{p \mid N} \left( (\frac{-1}{p}) - \left( \frac{-3}{p} \right) \right),
\]
\[
I'_1(k, j, N) = 2^{-3} 3^{-1} (j + 1) \prod_{p \mid N} (p - 1),
\]
\[
I'_2(k, j, N) = I'_3(k, j, N) = I'_4(k, j, N) = I'_5(k, j, N) = I'_6(k, j, N) = I'_7(k, j, N) = I'_8(k, j, N) = 0,
\]
\[
I'_9(k, j, N) = -2^{-3} (-1)^{j/2} \prod_{p \mid N} \left( \frac{-1}{p} \right) - 1,
\]
When we use the following formula for \( N \) as in Theorem 2.1 and we denote the cardinality of \( N \) it is very lengthy.

Theorem 1.2. We postpone the proof of Main Theorem 1.1 until Sections 6–8 because it is very lengthy.

3. Main Theorem 1.1 and the proof of Main Theorem 1.2

In this section, we give Main Theorem 1.1 precisely and give the proof of Main Theorem 1.2. We assume \( j \) is even and \( k \) is odd, then we have \( \text{dim}_N S_{k,j}(U^{'}(N)) = 2 \) for any \( j \geq 0 \) and \( k \geq 3 \), then we have \( \text{dim}_N S_{k,j}(K(N)) = \{0\} \) for any \( k \) since \( K(N) \) contains \(-14\).

Theorem 3.1 (Main Theorem 1.1). We assume \( N \) is a squarefree positive integer. If \( j \geq 2 \) is even and \( k \geq 5 \), or \( j = 0 \) and \( k \geq 3 \), then we have

\[
\text{dim}_N S_{k,j}(K(N)) = \sum_{i=1}^{12} H_i(k,j,N) + \sum_{i=1}^{10} I_i(k,j,N) + \delta_j \delta_k, \\
\]

where \( H_i(k,j,N) \) and \( I_i(k,j,N) \) are defined in Section 6 and are evaluated explicitly as follows:

\[
H_1(k,j,N) = \prod_{p \mid N} (p^2 + 1) \cdot 2^{-7} \cdot 3^{-3} \cdot 5^{-1} \cdot (j+1)(k-2)(j+k-1)(j+2k-3), \\
H_2(k,j,N) = 2^{-3(N)-8} \cdot 3^{-2} \cdot (j+k-1)(k-2)(-1)^k \cdot \begin{cases} 11 \cdots \text{if } 2 \mid N, \\ 14 \cdots \text{if } 2 \nmid N, \\ \\ 5 \cdots \text{if } 3 \mid N, \\ 2 \cdots \text{if } 3 \nmid N, \\ \\ 5 \cdots \text{if } 13 \mid N, \\ 1 \cdots \text{if } 13 \nmid N, \\ \\ 5 \cdots \text{if } 14 \mid N, \\ 1 \cdots \text{if } 14 \nmid N, \\ \\ \end{cases}, \\
H_3(k,j,N) = 2^{-3(N)-6} \cdot C_3(k,j), \\
H_4(k,j,N) = 2^{-3(N)-3} \cdot C_4(k,j), \\
H_5(k,j,N) = 2^{-3(N)-2} \cdot C_5(k,j), \\
H_6(k,j,N) = H_{6,1}(N) \times (-1)^{j/2}(2k+j-3) + H_{6,2}(N) \times (-1)^{j/2+k}(j+1), \\
\]

where

\[
H_{6,1}(N) = \frac{1}{2^7 \cdot 3} \prod_{p \mid N} \left( p + \left( \frac{-1}{p} \right) \right) + \frac{1}{2^7 \cdot 3} \prod_{p \mid N} \left( p \left( \frac{-1}{p} \right) + 1 \right), \\
\]

and \( H_{6,2}(N) \) is defined as above.
\[ H_{6,2}(N) = \frac{1}{2^4 \cdot 3} \prod_{p \mid N} \left( p + \left( -\frac{1}{p} \right) \right) - \frac{1}{2^3 \cdot 3^2} \prod_{p \mid N} \left( p \left( -\frac{1}{p} \right) + 1 \right), \]

\[ H_7(k, j, N) = H_71(N) \times [1, -1, 0; 3]_j (2k + j - 3) + H_72(N) \times [0, 1, -1; 3]_j + 2k (j + 1), \] where

\[ H_71(N) = \frac{1}{2^4 \cdot 3^2} \prod_{p \mid N} \left( p + \left( -\frac{3}{p} \right) \right) + \frac{1}{2^2 \cdot 3} \prod_{p \mid N} \left( p \left( -\frac{3}{p} \right) + 1 \right), \]

\[ H_72(N) = \frac{1}{2^4 \cdot 3^2} \prod_{p \mid N} \left( p + \left( -\frac{3}{p} \right) \right) - \frac{1}{2^2 \cdot 3^3} \prod_{p \mid N} \left( p \left( -\frac{3}{p} \right) + 1 \right), \]

\[ H_8(k, j, N) = 2^{\omega(N)} - 3^{j - 1} \cdot C_8(k, j), \]

\[ H_9(k, j, N) = 2^{\omega(N)} - 3^{j - 2} \cdot C_9(k, j), \]

\[ H_{10}(k, j, N) = 5^{j - 1} \cdot C_{10}(k, j), \]

\[ H_{11}(k, j, N) = 2^{\omega(N)} - 3^{k - 3} \cdot C_{11}(k, j). \]

\[ H_{12}(k, j, N) = H_{12,1}(N) \times [1, -1, 0; 3]_j (-1)^{j + k} + H_{12,2}(N) \times [0, 1, -1; 3]_j + 2k (-1)^{j + k}, \] where

\[ H_{12,1}(N) = \frac{1}{2^4 \cdot 3} \prod_{p \mid N} \left( 1 + \left( \frac{3}{p} \right) \right) - \frac{1}{2^2 \cdot 3} \prod_{p \mid N} \left( \left( -\frac{1}{p} \right) + \left( -\frac{3}{p} \right) \right), \]

\[ H_{12,2}(N) = \frac{1}{2^4 \cdot 3} \prod_{p \mid N} \left( 1 + \left( \frac{3}{p} \right) \right) + \frac{1}{2^2 \cdot 3} \prod_{p \mid N} \left( \left( -\frac{1}{p} \right) + \left( -\frac{3}{p} \right) \right), \]

\[ I_1(k, j, N) = 2^{-3} 3^{j - 1} \cdot (j + 1) \prod_{p \mid N} (p + 1), \]

\[ I_2(k, j, N) = -2^{\omega(N)} - 3^{j - 1} \cdot (j + 1), \]

\[ I_3(k, j, N) = -2^{\omega(N)} - 3^{j - 2} \cdot N \cdot (j + 1) (2k + j - 3), \]

\[ I_4(k, j, N) = 2^{\omega(N)} - 5^{j - 3} \cdot (-1)^k \cdot \left( 4 - \left( -\frac{1}{N} \right) \right), \]

\[ I_5(k, j, N) = -2^{\omega(N)} - 3^{j - 1} \cdot (-1)^k \cdot (2k + j - 3), \]

\[ I_6(k, j, N) = -2^{\omega(N)} - 3^{j - 2} \cdot \left[ (j - 1)^{j / 2} - 1, -(-1)^{j / 2}, 1, \left( 2k + j - 3 \right) \right], \]

\[ I_7(k, j, N) = -2^{\omega(N)} - 3^{j - 2} \cdot \left[ [3, -3, 0; 3]_k + [3, 0, -3; 3]_k + \cdots \text{if } N \equiv 0 \text{ mod } 3, \right. \]

\[ [5, -1, 4; 3]_k + [1, -4, -5; 3]_k + \cdots \text{if } N \equiv 1 \text{ mod } 3, \]

\[ [1, -5, -4; 3]_k + [5, 4, -1; 3]_k + \cdots \text{if } N \equiv 2 \text{ mod } 3, \]

\[ I_8(k, j, N) = -2^{\omega(N)} - 3^{j - 1} \cdot \left[ [-1, -1, 0, 1, 0; 6]_k + [1, 0, -1, 0, 1; 6]_k + \cdots \right], \]

\[ I_9(k, j, N) = -2^{\omega(N)} - 3^{j - 2} \cdot \left[ (j - 1)^{j / 2} \prod_{p \mid N} \left( 1 + \left( -\frac{1}{p} \right) \right), \right. \]

\[ I_{10}(k, j, N) = -2^{\omega(N)} - 3^{j - 1} \cdot [1, -1, 0; 3]_j \prod_{p \mid N} \left( 1 + \left( -\frac{3}{p} \right) \right). \]

**Proof.** We prove Theorem 3.1 in Sections 6 – 8 by using the method based on the
Selberg trace formula.

3.2. Proof of Main Theorem 1.2

We can prove Main Theorem 1.2 as follows. Let $H_i(k, j, N)$, $H'_i(k, j, N)$, $I_i(k, j, N)$ and $I'_i(k, j, N)$ be the same as in Theorem 2.1 and Theorem 3.1. We put

$$H_i^*(k, j, N) = \sum_{M \mid N} (-1)^{\omega(M)} 2^{\omega(M)} H_i(k, j, N/M) - H'_i(k, j, N),$$

$$I_i^*(k, j, N) = \sum_{M \mid N} (-1)^{\omega(M)} 2^{\omega(M)} I_i(k, j, N/M) - \delta(N) \cdot I'_i(k, j, N),$$

where the sums are taken over positive divisors of $N$, and we put $\delta(N) = 0$ if $\omega(N)$ is odd and $\delta(N) = 1$ if $\omega(N)$ is even.

**Lemma 3.2.** We have

$$H_1^*(k, j, N) = H_2^*(j, k, N) = H_3^*(j, k, N) = H_4^*(j, k, N) = H_5^*(j, k, N),$$

$$= H_7^*(j, k, N) = H_8^*(j, k, N) = H_{10}^*(j, k, N) = H_{11}^*(j, k, N) = 0,$$

$$I_2^*(j, k, N) = I_5^*(j, k, N) = I_6^*(j, k, N) = I_7^*(j, k, N) = 0,$$

regardless of the parity of $\omega(N)$.

**Proof.** The proof can be obtained by straightforward calculations, using the following formula

$$\sum_{M \mid N} (-1)^{\omega(M)} a^{\omega(M)} \prod_{p \mid (N/M)} x_p = \prod_{p \mid N} (x_p - a),$$

which is almost trivial.

As for the others, situations depend on the parity of $\omega(N)$.

**Case 1.** The case $\omega(N) = \text{odd}$: We can obtain the other parts as follows by straightforward calculation.

$H_6^*(k, j, N) = H_{6,1}^*(N) \times (-1)^{j/2(2k + j - 3)} + H_{6,2}^*(N) \times (-1)^{j/2+k(j+1)}$, where

$$H_{6,1}^*(N) = \frac{1}{2^4 \cdot 3} \sum_{M \mid N, \omega(M) = \text{odd}} \prod_{p \mid M} \left(\frac{-1}{p}\right) - 1 \prod_{p \mid (N/M)} (p - 1),$$

$$H_{6,2}^*(N) = \frac{1}{2^4 \cdot 3} \sum_{M \mid N, \omega(M) = \text{even}} \prod_{p \mid M} \left(\frac{-1}{p}\right) - 1 \prod_{p \mid (N/M)} (p - 1),$$

$H_7^*(k, j, N) = H_{7,1}^*(N) \times [1, -1, 0; 3]_j (2k + j - 3) + H_{7,2}^*(N) \times [0, 1, -1; 3]_{j+2k}(j+1)$, where

$$H_{7,1}^*(N) = \frac{1}{2^2 \cdot 3^2} \sum_{M \mid N, \omega(M) = \text{odd}} \prod_{p \mid M} \left(\frac{-3}{p}\right) - 1 \prod_{p \mid (N/M)} (p - 1),$$

$$H_{7,2}^*(N) = \frac{1}{2^2 \cdot 3^2} \sum_{M \mid N, \omega(M) = \text{even}} \prod_{p \mid M} \left(\frac{-3}{p}\right) - 1 \prod_{p \mid (N/M)} (p - 1),$$

$H_{12}^*(k, j, N) = H_{12,1}^*(N) \times [1, -1, 0; 3]_j (-1)^{j/2+k} + H_{12,2}^*(N) \times [0, 1, -1; 3]_{j+2k}(-1)^{j/2}$, where
\[
H_{12,1}^*(N) = -\frac{1}{2^2 \cdot 3} \sum_{M|N, \omega(M) = \text{even}} \prod_{p|N} \left( \left( \frac{-1}{p} \right) - 1 \right) \prod_{p|(N/M)} \left( \frac{-3}{p} \right) - 1,
\]
\[
H_{12,2}^*(N) = -\frac{1}{2^2 \cdot 3} \sum_{M|N, \omega(M) = \text{odd}} \prod_{p|N} \left( \left( \frac{-1}{p} \right) - 1 \right) \prod_{p|(N/M)} \left( \frac{-3}{p} \right) - 1,
\]
\[
I_1^*(k, j, N) = \frac{1}{2^3 \cdot 3} (j + 1) \prod_{p|M} (p - 1),
\]
\[
I_2^*(k, j, N) = -\frac{2^\omega(N)}{2^3 \cdot 3^2} \prod_{p|N} (p - 1) \times (j + 1)(2k + j - 3),
\]
\[
I_3^*(k, j, N) = -\frac{2^\omega(N)}{2^3 \cdot 3^2} \prod_{p|N} \left( \left( \frac{-1}{p} \right) - 1 \right) \times (-1)^k,
\]
\[
I_4^*(k, j, N) = -\frac{2^\omega(N)}{2^3 \cdot 3^2} \times [1, -1, 0; 3]_j \times [0, 1, -1; 3]_{j+2k} \times \prod_{p|N} \left( \frac{-3}{p} \right) - 1,
\]
\[
I_5^*(k, j, N) = -\frac{1}{2^3} \times \prod_{p|N} \left( \left( \frac{-1}{p} \right) - 1 \right) \times (-1)^{3/2},
\]
\[
I_{10}^*(k, j, N) = -\frac{1}{2^3} \times \prod_{p|N} \left( \left( \frac{-3}{p} \right) - 1 \right) \times [1, -1, 0; 3]_j.
\]

By using the following well-known formulas for \( M > 1 \):
\[
\dim S_{j+2}^{\text{new}}(\Gamma_0^{(1)}(M)) = \frac{j + 1}{12} \prod_{p|M} (p - 1) - \frac{(-1)^{j/2}}{4} \prod_{p|M} \left( \left( \frac{-1}{p} \right) - 1 \right)
\]
\[
- \frac{[1, -1, 0; 3]_j}{3} \prod_{p|M} \left( \frac{-3}{p} \right) - 1 + \delta_{j,0} \cdot (-1)^{\omega(M)},
\]
\[
\dim S_{2k+j+2}^{\text{new}}(\Gamma_0^{(1)}(N/M)) = \frac{2k + j - 3}{12} \prod_{p|(N/M)} (p - 1) - \frac{(-1)^{j/2 + k}}{4} \prod_{p|(N/M)} \left( \frac{-1}{p} \right) - 1
\]
\[
- \frac{[0, 1, -1; 3]_{2k+j}}{3} \prod_{p|(N/M)} \left( \frac{-3}{p} \right) - 1 + \delta_{2k+j-2,2} \cdot (-1)^{\omega(N/M)}
\]
\[
- \begin{cases} 
0 \cdots \text{if } N/M > 1 \\
1/2 \cdots \text{if } N/M = 1
\end{cases}
\]

where \( \delta_{*,*} \) means the Kronecker delta, we can verify Main Theorem 1.2 (i) directly from these when \((k, j) \neq (3, 0)\). If \((k, j) = (3, 0)\), the formula for \( \dim S_{3,0}(K(M)) \) is given by adding one to the usual formula ([Ilb07b]). This means that the formula for \( \sum_{M|N}(\omega(M) \dim S_{3,0}(K(N/M)) \) is given by adding \( \sum_{M|N}(\omega(M) \dim S_{3,0}(K(N/M)) \) to the usual formula. Since \( \omega(N) \) is odd now, we should add \(-1\). On the other hand, in the comparison of the theorem, we are subtracting 1 from \( \dim \mathfrak{M}_{0,0} \) in this case. The formula for \( \dim S_{2}^{\text{new}}(\Gamma_0^{(1)}(M)) \) is again given by adding \( (-1)^{\omega(M)} \) to the usual formula for weight bigger than 2, but here we took \( \omega(M) \) to be odd so this is \(-1\), and this cancels with the correction term \( \delta_{0,0} = 1 \), and the formula coming from elliptic modular forms are unchanged. So we obtain the claim also in this case.
In this section, we explain how to interpret Main Theorem 1.2 into Conjecture 1.3 and 1.4 in detail. First, we explain the definition of paramodular old forms and new forms and give an explicit formula of dimensions of paramodular new forms under a certain natural assumption. Then we explain liftings naturally expected by this formula and define new forms belonging to other twists. By these consideration, we explain the precise meaning of Conjecture 1.3 and 1.4 on correspondence between $S_{k,j}(K(N))$ and $M_{k+j-3,k-3}(U(N))$ and $S_{k,j}(U(N))$, and the Ihara-Saito-Kurokawa type lifting and the Yoshida type lifting to these spaces, and also make it clear why these are natural, based on our Main Theorem 1.2.

**Remark.** In case 2 for $(k,j) = (3,0)$, as in case 1, the dimension of LHS is given by adding $(-1)^{\omega(N)} = 1$ to the usual formula. This supports us to conjecture that the formula for $S_{3,0}(U(N))$ is valid also for $k = 3, 4$ for any even $j \geq 0$ if we add one to the usual formula in case $(k,j) = (3,0)$ and the relation in Theorem 1.2 should be valid for any $k \geq 3$ and even $j \geq 0$.

**4. New forms and details of Main Theorem 1.2 and Conjecture 1.3, 1.4**

In this section, we explain how to interpret Main Theorem 1.2 into Conjecture 1.3 and 1.4 in detail. First, we explain the definition of paramodular old forms and new forms and give an explicit formula of dimensions of paramodular new forms under a certain natural assumption. Then we explain liftings naturally expected by this formula and define new forms belonging to other twists. By these consideration, we explain the precise meaning of Conjecture 1.3 and 1.4 on correspondence between $S_{k,j}(K(N))$ and $M_{k+j-3,k-3}(U(N))$ and $S_{k,j}(U(N))$, and the Ihara-Saito-Kurokawa type lifting and the Yoshida type lifting to these spaces, and also make it clear why these are natural, based on our Main Theorem 1.2.
4.1. Paramodular new forms: prime level case

The definition of paramodular new forms was originally given by the first named author in [Ibu85] for prime level case and recently considered by Roberts and Schmidt from the viewpoint of local representation theory (cf. [RS07]). For convenience of readers, we review the prime level case first in detail from [Ibu85] since this case is easy to understand. We explain the general squarefree level case later. For any non-negative integer $m$, we write

$$U_m = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & m & 0 & 0 \\ m & 0 & 0 & 0 \end{pmatrix}.$$ 

By the Iwahori-Bruhat-Tits theory, for a prime $p$, the local split symplectic group $Sp(2, \mathbb{Q}_p)$ of rank 2 has three standard maximal compact subgroups (up to $Sp(2, \mathbb{Q}_p)$ conjugation), one is the localization of $K(p)$ and the other two are $Sp(2, \mathbb{Z}_p)$ and $U_p^{-1}Sp(2, \mathbb{Z}_p)$. The global discrete subgroups corresponding to these are $K(p)$, $K(1) = Sp(2, \mathbb{Z})$ and $U_p^{-1}Sp(2, \mathbb{Z})U_p = U_p^{-1}K(1)U_p$, respectively. There are no inclusion relations between these groups at all. But we can define a mapping between Siegel cusp forms belonging to these discrete groups by taking the traces. We write $\Gamma_0(p) = K(p) \cap Sp(2, \mathbb{Z})$ and $\Gamma'_0(p) = K(p) \cap U_p^{-1}Sp(2, \mathbb{Z})U_p$. Then since $U_p$ normalizes $K(p)$, we have $\Gamma'_0(p) = U_p^{-1}\Gamma_0(p)U_p$. We can define two mappings $T(1, p)$ and $T'(1, p)$ to $S_{k,j}(K(p))$, one is from $S_{k,j}(Sp(2, \mathbb{Z}))$ and the other from $S_{k,j}(U_p^{-1}Sp(2, \mathbb{Z})U_p)$, by

$$F|T(1, p) = \sum_{g \in \Gamma_0(p) \setminus K(p)} F|_{k,j}[g], \quad G|T'(1, p) = \sum_{g \in \Gamma'_0(p) \setminus K(p)} G|_{k,j}[g]$$

for any $F \in S_{k,j}(Sp(2, \mathbb{Z}))$ and $G \in S_{k,j}(U_p^{-1}Sp(2, \mathbb{Z})U_p)$. We obviously have $S_{k,j}(U_p^{-1}Sp(2, \mathbb{Z})U_p) = S_{k,j}(Sp(2, \mathbb{Z}))|_{k,j}U_p$ as spaces. If $K(p) = \bigcup \Gamma_0(p)\gamma_i$, then we have $K(p) = \bigcup \Gamma'_0(p)\gamma_i^{-1}U_p$. So we have $T'(1, p) = U_p^{-1}T(1, p)U_p$ and

$$S_{k,j}(U_p^{-1}Sp(2, \mathbb{Z})U_p)|_{k,j}T'(1, p) = S_{k,j}(Sp(2, \mathbb{Z}))|_{k,j}T(1, p)U_p.$$ 

So it is natural to define the space of old forms $S_{k,j}^{\text{old}}(K(p))$ of $S_{k,j}(K(p))$ by the subspace

$$S_{k,j}^{\text{old}}(K(p)) = S_{k,j}(K(1))/T(1, p) + S_{k,j}(K(1))/T(1, p)U_p \subset S_{k,j}(K(p)),$$

where we note that $K(1) = Sp(2, \mathbb{Z})$. We define the space $S_{k,j}^{\text{new}}(K(p))$ of new forms as the orthogonal complement of $S_{k,j}^{\text{old}}(K(p))$ in $S_{k,j}(K(p))$ with respect to the usual Petersson inner product. The dimensional relation of Theorem 1.2 for prime level case reads

$$(1) \quad \dim S_{k,j}(K(p)) - 2 \dim S_{k,j}(K(1)) = 
\dim \mathfrak{M}_{k+j-3, k-3}(U(p)) - \left( \dim S_{k,j+2}^{\text{new}}(\Gamma_0^{(1)}(p)) + \delta_{j0} \right) \times \dim S_{2k+j-2}(SL_2(\mathbb{Z})) - \delta_{j0} \delta_{k3}.$$
Now we will see how this fits our Conjecture 1.3. For $j = 0$ and even $k \geq 10$, we know that there exists the Saito-Kurokawa lifting from $S_{2k-2}(\text{SL}_2(\mathbb{Z}))$ to $S_k(\text{Sp}(2, \mathbb{Z}))$. This lifting is known to be injective, that is, if we denote the image of this lifting by $SK_k$, then we have $\dim SK_k = \dim S_{2k-2}(\text{SL}_2(\mathbb{Z}))$. Since the two trace maps coincide on $SK_k$ (which was a conjecture in [Iha85] and now it is a fact by [RS06]), we can expect that the correct dimension of old paramodular forms should be

$$\dim S_{k,j}(\text{Sp}(2, \mathbb{Z})) + \dim S_{k,j}(U_p^{-1}\text{Sp}(2, \mathbb{Z})U_p) - \delta_{j0} \dim SK_k = 2 \dim S_{k,j}(K(1)) - \delta_{j0} \delta_{k,\text{even}} \dim S_{2k-2}(\text{SL}_2(\mathbb{Z}))$$

where we put $\delta_{k,\text{even}} = 1$ or $0$ according to the case when $k$ is even or odd. When $j = 0$ and $k$ is even, it is more natural to move $-\dim S_{2k-2}(\text{SL}_2(\mathbb{Z}))$ in RHS of (1) to $+ \dim S_{2k-2}(\text{SL}_2(\mathbb{Z}))$ in LHS of (1), expecting that the two trace maps are independent on non Saito-Kurokawa liftings. For $\epsilon = \pm$, we denote by $S_{2k-2}^{\text{new},\epsilon}(\text{SL}_2(\mathbb{Z}))$ the space of forms whose sign of the functional equation is $\epsilon$. It is well known that $S_{2k-2}(\text{SL}_2(\mathbb{Z})) = \bigoplus_{\epsilon} S_{2k-2}^{\text{new},\epsilon}(\text{SL}_2(\mathbb{Z}))$, and the first space $S_{2k-2}^{\text{new},+}(\text{SL}_2(\mathbb{Z}))$ is the space of those for which $\epsilon = +$.

Next we see the RHS of (1). First of all, there exists a theory of lifting from $S_{j+2}^p(\Gamma_0^1(p)) \times S_{2k+j-2}(\text{SL}_2(\mathbb{Z}))$ to $\mathcal{M}_{k+j-3,k-3}(U(p))$ for general $j \geq 0$ and from $S_{2k-2}(\text{SL}_2(\mathbb{Z}))$ in case $j = 0$, as is shown in [Iha64] and [II87]. We will call these Ihara liftings. The former lift is a kind of Yoshida-type, and the latter one is a kind of Saito-Kurokawa type. The above relation suggests that the Yoshida type is always injective, and the Saito-Kurokawa type is injective when $k$ is odd but vanishes when $k$ is even. Though these conjectural injectivities have never been proved in general, all the examples that we have support this property. So when $(k, j) \neq (3, 0)$, we define the space of old forms of $\mathcal{M}_{k+j-3,k-3}(U(p))$ by the space obtained by the above Ihara liftings. If $(k, j) = (3, 0)$, then $\mathcal{M}_{k+j-3,k-3}(U(p)) = \mathcal{M}_{0,0}(U(p))$ contains the constant functions, and we have $S_3(\text{SL}_2(\mathbb{Z})) = 0$, so we define old forms to be the constant functions. The space $\mathcal{M}_{k+j-3,k-3}(U(p))$ of new forms is defined to be the orthogonal complement of old forms in $\mathcal{M}_{k+j-3,k-3}(U(p))$.

Summing up, we should have

$$\dim S_{k,j}(K(p)) = \dim S_{k,j}(K(1)) + 2 \dim S_{k,j}(U_p^{-1}\text{Sp}(2, \mathbb{Z})U_p) - \delta_{j0} \dim S_{2k-2}^{\text{new},-}(\text{SL}_2(\mathbb{Z})).$$

We must note that our definition of new forms does not mean that new forms are not lifts. In fact, if $j = 0$, there exists an injective Saito-Kurokawa type lifting from $S_{2k-2}^{\text{new},-}(\Gamma_0^1(p))$ to $S_k(K(p))$ defined through Jacobi forms of index $p$ by Gritsenko and these are new forms in the above sense. We are not specifying the local representations at $p$ corresponding to paramodular new forms and in fact several different local representations appear. Our conjecture also indicates that there should be the corresponding injective lifting from $S_{2k-2}^{\text{new},-}(\Gamma_0^1(p))$ to $\mathcal{M}_{k+j-3,k-3}(U(p))$. There are many experimental examples of such liftings, but we cannot give a general construction of this lift by the known method of Ihara liftings. So this gives an interesting new problems. But anyway, we are saying that new forms of both groups contain certain liftings also and that liftings...
should correspond to liftings in our conjecture.

Roberts and Schmidt also considered some operators from \( S_{k,j}(K(N)) \) to \( S_{k,j}(K(pN)) \) in [RS06]. In next section, we show that these are really the same operators as ours at least for the squarefree level case.

4.2. Paramodular old forms: squarefree level case

Next we consider the general squarefree level case. Throughout this section, we assume that \( N \) is squarefree for simplicity, though some part can be easily generalized for non squarefree case. First, for any \( 0 < N_1 < N \), we define the trace operator \( T(N_1, N) : S_{k,j}(K(N_1)) \to S_{k,j}(K(N)) \) by

\[
F|T(N_1, N) = \sum_{g \in K(N_1) \cap K(N) \setminus K(N)} F|_{k,j}[g]
\]

for any \( F \in S_{k,j}(K(N_1)) \). It is easy to see that a complete set of representatives of \( K(N_1) \cap K(N) \setminus K(N) \) is given by

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & a & 0 & N_1^{-1}b \\
0 & 0 & 1 & 0 \\
0 & cN & 0 & d
\end{pmatrix}
\]

where \((a : b) \mod (N/N_1)\) runs over elements of

\[
\mathbb{P}^1(\mathbb{Z}/(N/N_1)\mathbb{Z}) = \prod_{p|N/N_1, p \text{ prime}} \mathbb{P}^1(\mathbb{F}_p)
\]

where \(\mathbb{P}^1\) means the projective space of lines and \(\mathbb{F}_p\) is the finite field of \(p\) elements. We also define the Atkin-Lehner operator \(u_m\) on \( S_{k,j}(K(N)) \) as in [RS06] as follows. For any \( 0 < m | N \), we put

\[
(2) \quad \eta_m = \begin{pmatrix}
m & 0 & 0 & 0 \\
0 & m & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

and let \( \gamma_m \) be any element in \( Sp(2, \mathbb{Z}) \) such that

\[
\gamma_m \equiv \begin{pmatrix}
1 & 4 & \mod N/m \\
0 & 0 & -1 \\
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix} \mod m.
\]

Then the action \( F|_{k,j}[\gamma_m \eta_m] \) for \( F \in S_{k,j}(K(N)) \) does not depend on the choice of \( \gamma_m \) since the principal congruence subgroup \( \Gamma(N) \) of level \( N \) is contained in \( K(N) \). We write this operator on \( S_{k,j}(K(N)) \) by \( u_m \). In particular, on \( S_{k,j}(K(N)) \), the action of \( u_N \) and \( U_N \) are the same. The operator \( u_m \) apparently depends on the choice of \( N \), but we omit \( N \) in the notation since this does not matter so much.
The proof needs slightly lengthy arguments but essentially elementary and we omit the proofs here. See also [RS06] for a similar argument. There they defined two operators \( \theta_p \) and \( \theta_p' \) from \( S_k(K(N)) \) to \( S_k(K(pN)) \). We see easily from (2) that \( \theta_p = T(N,pN) = U_N \theta_p U_{pN} \).

The following corollary is obvious.

**Corollary 4.2.** For any \( N_1 | N \), write \( N/N_1 = p_1 \cdots p_r \), where \( p_i \) are distinct primes. We assume that \( e_i = 0 \) or 1 for each \( i \) with \( 1 \leq i \leq r \). Then we have

\[
T(N_1, p_1 N_1) u_{p_1} \cdots T(p_1 N_1, p_1 p_2 N_1) u_{p_2} \cdots T(N_1 p_1 \cdots p_{r-1} N, N) u_{p_r} = T(N_1, N) u_m
\]

where \( m = p_1^{e_1} \cdots p_r^{e_r}. \) In particular, this depends only on \( e_i \) for each \( p_i \) and does not depend on the order of the action of \( u_{p_i} \).

Then by induction, we can easily see that the space of old forms can be written by

\[
S_{k,j}^{\text{old}}(K(N)) = \sum_{M \mid N} \sum_{M \neq N} S_{k,j}^{\text{new}}(K(M)) T(M,N) u_m.
\]

Here we must note that the sum is not necessary the direct sum. To figure out what is likely to happen for this sum and to make a necessary assumption to get the conjectural dimension formulas of new forms from the formula for the total dimensions, we need explanation on liftings, which we do in the next subsection.

### 4.3. Dimensions of new paramodular forms and liftings

A kind of Saito-Kurokawa lifting is constructed to \( S_q(K(N)) \) (i.e. the case \( j = 0 \)) as a lifting from the space of Jacobi forms \( J_{k,N}(\Gamma'_1) \) of weight \( k \) and of index \( N \) for the Jacobi modular group \( \Gamma'_1 \) by Gritsenko [Gri95] (See also Schmidt [S07]). By Skoruppa and Zagier [SZ88], the space of Jacobi cusp forms \( J_{k,N}^{\text{cusp}}(\Gamma'_1) \) is isomorphic to \( \bigoplus_{M \mid N} S_{2k-2}^{\text{new}}(\Gamma_0^1(M)) \),
for squarefree $N$, where the latter notation with $\pm$ is as in the introduction. So it seems natural to expect that the multiplicity of the lifting from $S_{2k-2}^{\text{new,-}}(\Gamma_0^{(1)}(M))$ to $S_k^{\text{new}}(K(M))$ is one for each $M|N$. Actually, if $F \in S_k(K(M))$ is a Gr"{i}tsenko lift, then $F[T(M,pM) = F[T(M,pM)u_p$ by [RS06], so this means that the RHS of (3) is not a direct sum. We denote by $S_k^{\text{new,L}}(K(N))$ the space of Gr"{i}tsenko lifts from Jacobi forms corresponding to $S_{2k-2}^{\text{new,-}}(\Gamma_0^{(1)}(N))$.

We consider the following assumption.

**Assumption 4.3.** (i) For each $N$, we have the direct sum decomposition

$$\dim S_{k,j}^{\text{odd}}(K(N)) = \bigoplus_{M|N} \left( \sum_{m | (N/M)} S_{k,j}^\text{new}(K(M))|T(M,N)u_m \right).$$

(ii) For each $M|N$, $T(M,N)$ is injective on $S_{k,j}^\text{new}(K(M))$.

(iii) For each $M|N$ and for any $m, m' | (N/M)$, we have

$$S_{k,j}^\text{new}(K(M))|T(M,N)u_m \cap S_{k,j}^\text{new}(K(M))|T(M,N)u_{m'} = \delta_{j0} S_k^{\text{new,L}}(K(M))|T(M,N).$$

We believe that it is reasonable to expect that this assumption is always satisfied.

**Proposition 4.4.** Under the above assumption 4.3, we have

(4) \[ \dim S_{k,j}(K(N)) = \sum_{M|N} \{2^{\omega(M)} \dim S_{k,j}^\text{new}(K(N/M)) - (2^{\omega(M)} - 1)\delta_{j0} \dim S_{2k-2}^{\text{new,-}}(\Gamma_0^{(1)}(N/M)) \}. \]

(5) \[ \dim S_{k,j}^\text{new}(K(N)) = \sum_{M|N} (-2)^{\omega(M)} \dim S_{k,j}(K(N/M)) \]

\[ - \delta_{j0} \sum_{M|N, M \neq 1} (-1)^{\omega(M)} \dim S_{2k-2}^{\text{new,-}}(\Gamma_0^{(1)}(N/M)). \]

**Proof.** The formula (4) is a direct consequence from Assumption (4.3). The formula (5) is proved easily by M"{o}bius inversion formula applied to (4). Indeed, if we put

$$f(M) = 2^{-\omega(M)} \left( \dim S_{k,j}(K(M)) - \delta_{j0} \sum_{d|M} \dim S_{2k-2}^{\text{new,-}}(\Gamma_0^{(1)}(d)) \right),$$

$$g(M) = 2^{-\omega(M)} \left( \dim S_{k,j}^\text{new}(K(M)) - \delta_{j0} \dim S_{2k-2}^{\text{new,-}}(\Gamma_0^{(1)}(M)) \right),$$

then the formula (4) reads

$$f(N) = \sum_{M|N} g(M).$$

So we have $g(N) = \sum_{M|N} \mu(N/M) f(M)$ where $\mu(*)$ is the M"{o}bius function, and since
\[ \sum_{d|N} (-2)^{(N/d)} \sum_{d|M} \dim S^{new}_{2k-2} (\Gamma_0^{(1)}(d)) = \sum_{d|N} (-1)^{(N/d)} \dim S^{new}_{2k-2} (\Gamma_0^{(1)}(d)), \]

we have (5).

Now we see what is the consequence of this result, compared with our dimensional relation Main Theorem 1.2. If we modify the LHS of Main Theorem 1.2 (i) and (ii) to \( \dim S^{new}_{k,j} (K(N)) \) by using the above formula, then the RHS should be the conjectural dimension of new forms of the non-split group. This gives an idea how to define new forms for non-split group and how is the lifting there. So we should have

\[ \dim \mathcal{M}^{new}_{k+j-3,k-3}(U(N)) = \dim \mathcal{M}^{new}_{k+j-3,k-3}(U(N)) - d^{odd}(N) \text{ if } \omega(N) \text{ is odd} \]

\[ \dim S^{new}_{k,j}(U'(N)) = \dim S^{new}_{k,j}(U'(N)) - d^{odd}(N) \text{ if } \omega(N) \text{ is even}, \]

where

\[ d^{odd}(N) = \sum_{\substack{M|N \omega(M) = odd}} \dim S^{new}_{j+2}(\Gamma_0^{(1)}(M)) \times \dim S^{new}_{2k+j-2}(\Gamma_0^{(1)}(N/M)) \]

\[ + \delta_0 \sum_{\substack{M|N \omega(M) = odd}} S^{new,new}_{2k-2}(\Gamma_0^{(1)}(N/M)) + \delta_0 \sum_{\substack{M|N \omega(M) = even}} S^{new,new}_{2k-2}(\Gamma_0^{(1)}(N/M)) + \delta_0 \delta_3 \omega(N); odd, \]

where we put \( \delta_\omega(N); odd = 1 \text{ if } \omega(N) \text{ is odd and } = 0 \text{ otherwise.} \)

Here the definition and liftings are explained in the following conjecture.

**Conjecture 4.5.** For each \( M|N, \) where \( N \) is squarefree, we have the following lifts.

1. When \( \omega(M) \) is odd, there should exist an injective lifting:
   \[ S^{new}_{j+2}(\Gamma_0^{(1)}(M)) \times \dim S^{new}_{2k+j-2}(\Gamma_0^{(1)}(N/M)) \rightarrow \begin{cases} \mathcal{M}^{new}_{k+j-3,k-3}(U(N)) & \text{if } \omega(N) \text{ is odd}, \\ S^{new}_{k,j}(U'(N)) & \text{if } \omega(N) \text{ is even}. \end{cases} \]

2. When \( \omega(M) \) is odd, then there should exist an injective lifting:
   \[ S^{new,new}_{2k-2}(\Gamma_0^{(1)}(N/M)) \rightarrow \begin{cases} \mathcal{M}^{new}_{k-3,k-3}(U(N)) & \text{if } \omega(N) \text{ is odd}, \\ S^{new}_{k,j}(U'(N)) & \text{if } \omega(N) \text{ is even}. \end{cases} \]

3. When \( \omega(M) \) is even, there should exist an injective lifting:
   \[ S^{new,new}_{2k-2}(\Gamma_0^{(1)}(N/M)) \rightarrow \begin{cases} \mathcal{M}^{new}_{k-3,k-3}(U(N)) & \text{if } \omega(N) \text{ is odd}, \\ S^{new}_{k,j}(U'(N)) & \text{if } \omega(N) \text{ is even}. \end{cases} \]

4. If we define new forms by the orthogonal complement of the space spanned by the images of all the above liftings in \( \mathcal{M}^{new}_{k+j-3,k-3}(U(N)) \) or \( S^{new}_{k,j}(U'(N)) \) and also by the constant functions in \( \mathcal{M}_{0,0}(N) \) in case \( (k,j) = (3,0) \) and \( \omega(N) \) is odd, then \( \dim \mathcal{M}^{new}_{k+j-3,k-3}(U(N)) \) or \( \dim S^{new}_{k,j}(U'(N)) \) is given by (6).

This fits dimensional relations and examples we give later. So we propose Conjecture 1.3 and 1.4.

By the way, if \( f \in S^{new,new}_{2k-2}(\Gamma_0^{(1)}(M)) \) is lifted through Jacobi forms to \( \iota(f) \in \)
$S_{k,j}(K(N))$, then the spinor $L$ function is written by
\[
L(s,\iota(f), Sp) = \zeta(s-k+1)\zeta(s-k+2) L(s,f)
\]
up to bad Euler factors.

### 4.4. Numerical examples of non-lift

Since the description is fairly complicated, examples seem indispensable. In this subsection, we shortly give examples of non-lift in Conjecture 1.3 for $N = 6$. The examples of liftings and Conjecture 1.4 will be explained more in detail in next section.

The dimensions of $\dim S_{2k-2}^{new,\pm}(\Gamma_0^{(1)}(M))$ can be calculated by the trace formula of the Atkin-Lehner involution in [Y73]. We remark also that $\dim S_{2k-2}(SL_2(\mathbb{Z})) = \dim S_{2k-2}^{new,(-1)^{k-1}}(SL_2(\mathbb{Z}))$.

Then together with our new dimension formulas and [Kit11], we obtain the following table of dimensions.

**Total dimension of** $\dim S_k(K(N))$.

<table>
<thead>
<tr>
<th>$k$</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K(6)$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>$K(3)$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$K(2)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$K(1)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

Dimension of $S_{2k-2}^{new,\pm}(\Gamma_0(N))$. The first one is for $+$ and the second for $-$.  

<table>
<thead>
<tr>
<th>$2k-2$</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
<th>14</th>
<th>16</th>
<th>18</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma_0^{(1)}(6)$</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$\Gamma_0^{(1)}(3)$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$\Gamma_0^{(1)}(2)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$\Gamma_0^{(1)}(1)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

By our conjecture and the fact that $S_2(\Gamma_0^{(1)}(2)) = S_2(\Gamma_0^{(1)}(3)) = 0$, the dimensions of new forms of $K(6)$ or $U'(6)$ are given for $j = 0$ and $N = 6$ by
\[
\dim S_k^{new}(K(6)) = \dim S_k(K(6)) - 2 \dim S_k(K(2)) - 2 \dim S_k(K(3)) + 4 \dim S_k(K(1)) + \dim S_{2k-2}^{new,\pm}(\Gamma_0^{(1)}(2)) + \dim S_{2k-2}^{new,\pm}(\Gamma_0^{(1)}(3)) - \dim S_{2k-2}^{new,\pm}(SL_2(\mathbb{Z})) = \dim S_{k,0}(U'(6)) - \dim S_{2k-2}^{new,\pm}(\Gamma_0^{(1)}(2)) - \dim S_{2k-2}^{new,\pm}(\Gamma_0^{(1)}(3)) - \dim S_{2k-2}^{new,\pm}(SL_2(\mathbb{Z}))
\]
so we obtain the following table.

**The dimension of new forms** $\dim S_k^{new}(K(N))$.

<table>
<thead>
<tr>
<th>$k$</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K(6)$</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
</tr>
<tr>
<td>$K(3)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$K(2)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$K(1)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>
The dimension of new forms of $K(N)$ which are not Gritsenko lift.

<table>
<thead>
<tr>
<th>level $k$</th>
<th>3 4 5 6 7 8 9 10 11</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K(6)$</td>
<td>0 0 0 0 0 0 1 1 2</td>
</tr>
<tr>
<td>$K(3)$</td>
<td>0 0 0 0 0 0 0 0 0</td>
</tr>
<tr>
<td>$K(2)$</td>
<td>0 0 0 0 0 0 0 0 0</td>
</tr>
<tr>
<td>$K(1)$</td>
<td>0 0 0 0 0 0 0 0 0</td>
</tr>
</tbody>
</table>

The dimension of $S_k(U'(6))$ and $S_k^{\text{new}}(U(6))$

<table>
<thead>
<tr>
<th>$k$</th>
<th>3 4 5 6 7 8 9 10 11</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_k(U'(6))$</td>
<td>0 1 2 2 3 4 6 6</td>
</tr>
<tr>
<td>$S_k^{\text{new}}(U(6))$</td>
<td>0 0 0 1 0 2 2 3</td>
</tr>
</tbody>
</table>

We see the case $k = 10$ more in detail. The space $S_{10}^{\text{new}}(K(3))$ consists of the lift from $S_{10}^{\text{new},-}(\Gamma_0^{(1)}(3))$, and $S_{10}^{\text{old}}(K(3))$ consists of the lift form $S_{10}(SL_2(\mathbb{Z}))$, both one dimensional. $S_{10}^{\text{new}}(K(2)) = 0$ and $S_{10}^{\text{old}}(K(2))$ and $S_{10}(K(1))$ consists of the lift from $S_{18}(SL_2(\mathbb{Z}))$. The basis of $S_{10}(K(6))$ consists of a non-lift and the lift from $S_{18}^{\text{new},-}(\Gamma_0^{(1)}(6)), S_{18}^{\text{new},-}(\Gamma_0^{(3)}), S_{18}^{\text{new},-}(SL_2(\mathbb{Z}))$. Here we note that the lift from $S_{18}^{\text{new},-}(\Gamma_0^{(1)}(6))$ is regarded as a new form. On the other hand, the basis of $S_{10}(U'(6))$ consists of a non-lift, one lift from $S_{18}^{\text{new},-}(\Gamma_0^{(1)}(6))$ (which is a new form), two lifts from $S_{18}^{\text{new},+}(\Gamma_0^{(1)}(2))$, one lift from $S_{18}^{\text{new},+}(\Gamma_0^{(1)}(2))$, and one lift from $S_{18}^{\text{new},-}(SL_2(\mathbb{Z}))$.

The following numerical examples up to $k = 11$, which is independent of any assumptions, support all these theoretical observations.

**Theorem 4.6.** (i) If $k \leq 8$, then the space $S_k(K(6)) = S_{k,0}(K(6))$ is generated by Gritsenko lifts from Jacobi cusp forms of weight $k$ and index 6. The lifts from $S_{2k-2}^{\text{new},-}(\Gamma_0^{(1)}(N))$ ($N = 1, 2, 3, 6$) to $S_k(K(6))$ are 2, 2, and 3 dimensional for $k = 9, 10$, and 11 respectively and the codimensions in $S_k(K(6))$ are 1, 1, and 2.

(ii) Assume $k \leq 11$. Then for each cusp eigenform $g$ in $S_{2k-2}^{\text{new},-}(SL(2;\mathbb{Z}))$, $S_{2k-2}^{\text{new},+}(\Gamma_0^{(1)}(2))$, $S_{2k-2}^{\text{new},-}(\Gamma_0^{(1)}(3))$ or $S_{2k-2}^{\text{new},-}(\Gamma_0^{(1)}(6))$, there is a cusp form $F$ in $S_{k,0}(U'(6))$ such that $H_5(t, F) = (t - 5^{k-2})(t - 5^{k-1})h_5(t, g)$, where $h_5(t, g)$ is the Hecke polynomial of $g$ at $5$. If $k \leq 8$, then there are no other cusp eigenforms of $U_k(U'(6))$. If $k = 9, 10, 11$, then such cusp forms generate 3, 5, and 4-dimensional subspace of $S_k(U'(6))$ respectively. The codimension is 1 for $k = 9$ and 10 and 2 for $k = 11$.

(iii) If we take the Hecke eigenforms $\phi_k \in S_k(K(6))$ for $k = 9, 10$ and $\phi_{11, a}, \phi_{11, b} \in S_{11}(K(6))$, and $\psi_k \in S_k(U'(6))$ for $k = 9, 10$, $\psi_{11, a}, \psi_{11, b} \in S_{11}(U'(6))$ in a suitable order which are not obtained by lifting in the above sense, then eigenvalues of the Hecke operators $T_k(5)$ and $T_k(5^2)$ for $S_k(K(6))$ and $S_k(U'(6))$ coincide with each other. The Hecke polynomials of these forms in $S_k(U'(6))$ at 5 are given by

$$H_5(t, \psi_0) = t^4 + 33540t^3 + 15293281250t^2 + 33540 \cdot 5^{15}t + 5^{30},$$
$$H_5(t, \psi_{10}) = t^4 + 88980t^3 + 1170167968750t^2 + 88980 \cdot 5^{17}t + 5^{34},$$
$$H_5(t, \psi_{11, a}) = t^4 - 5029620t^3 + 11396230468750t^2 - 5029620 \cdot 5^{19}t + 5^{38},$$
$$H_5(t, \psi_{11, b}) = t^4 + 222420t^3 + 21376386718750t^2 + 222420 \cdot 5^{19}t + 5^{38}. $$
All absolute values of the zeros of these polynomials are $5^{(2k-3)/2}$. We have $H_5(t, \phi_9) = H_5(t, \psi_9)$, $H_5(t, \phi_{10}) = H_5(t, \psi_{10})$, $H_5(t, \phi_{11,a}) = H_5(t, \psi_{11,a})$, $H_5(t, \phi_{11,b}) = H(t, \psi_{11,b})$.

It is not easy at all to construct explicitly the non-lifted Hecke eigenforms and calculate the Hecke polynomials. We constructed $\psi \in S_{k,0}(U'(6))$ for $k \leq 11$ and gave $H_5(t, g)$. But on the other hand, the results on non-lift paramodular forms in $S_{k,0}(K(6))$ for $9 \leq k \leq 11$ are due to Cris Poor and David Yuen. They kindly informed the authors of the results on $\phi$ and $H_5(t, \phi)$ obtained by their original technique in their book [PSY] in preparation.

5. Numerical examples of non-split group and liftings

In the case when $N$ is a prime, where the non-split group is the compact twist of $Sp(2, \mathbb{R})$, we have several numerical examples in [Ibu84] for $N = 2$ and similar calculation is possible for further $N$. In fact there are several related experimental results in this case, but we omit them here. But the case when $N$ is a product of two distinct primes is completely different from the case $N$ is a prime, since the group in question is a non-split $\mathbb{Q}$ form of $Sp(2, \mathbb{R})$. So in this section, we sketch the calculation in this case more in detail.

5.1. Hecke operators

We review briefly Hecke operators on $S_{k,0}(U'(N))$ (cf. [Sug84]). We put $\Gamma = U'(N)$. For $m \in \mathbb{N}$ and $f \in S_{k,0}(\Gamma)$, we define

$$(T_k(m)f)(Z) := m^{2k-3} \sum_{\sigma \in \Gamma \backslash S_m} \det(CZ + D)^{-k}f(\sigma(Z)),$$

where we put

$$S_m := \left\{ g = \left[ \begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right] \bigg| \alpha, \delta \in \mathbb{O}, \beta, \gamma \in \mathbb{A}, \alpha \beta + \beta \overline{\alpha} = \gamma \delta + \delta \overline{\gamma} = 0, \alpha \overline{\delta} + \beta \overline{\alpha} = m \right\}$$

and $\sigma \in \Gamma \backslash S_m$ runs over a complete system of representatives of $\Gamma \backslash S_m$. We fix a prime number $p \nmid N$. For Hecke eigenform $f \in S_{k,0}(\Gamma)$, we put

$$T_k(p)f = \lambda(p)f, \quad T_k(p^2)f = \lambda(p^2)f$$

Then (for $j = 0$) we define

$$H_p(t, f) := t^4 - \lambda(p)t^3 + (\lambda(p)^2 - \lambda(p^2) - p^{2k-4})t^2 - \lambda(p)p^{2k-3}t + p^{4k-6}.$$  

Then $H_p(p^{-s}, f)$ is the Euler $p$ factor of the spinor $L$ function of $f$. For any $X \subset B$, we write $X^0 = \{ x \in X; Tr(x) = 0 \}$. By easy consideration, we can give complete systems of representatives of $U'(N) \backslash S_p$ and $U'(N) \backslash S_p^2$ for a prime number $p \nmid N$ as follows. We put $\mathbb{D}^{x,1} = \{ x \in \mathbb{D}^x \mid x\bar{x} = 1 \}$. 

$U'(N) \backslash S_p$
(1) \[
\begin{bmatrix}
 1 & x \\
 0 & p
\end{bmatrix},
\]
where \( x \) ranges over a complete system of representatives of \((\mathfrak{A}^{-1})^0/p(\mathfrak{A}^{-1})^0\).

(2) \[
\begin{bmatrix}
  a_i & \alpha_{i,j} \\
  0 & a_i
\end{bmatrix},
\]
where \( a_i \) ranges over a complete system of representatives of \(\mathfrak{D}^{x,1}\)-equivalent classes of elements of \(\mathfrak{D}\) with norm \( p \). For each \( a_i, \alpha_{i,j} \) ranges over a complete system of representatives of \( \{ x \in \mathfrak{A}^{-1} \mid \text{tr}(a_i^{-1}x) = 0 \} / ((a_i\mathfrak{A}^{-1})^0) \).

(3) \[
\begin{bmatrix}
  b_i & \beta_{i,j} \\
  0 & p^{-1}b_i
\end{bmatrix},
\]
where \( b_i \) ranges over a complete system of representatives of \(\mathfrak{D}^{x,1}\)-equivalent classes of elements of \(\mathfrak{D}\) with norm \( p^2 \). For each \( b_i, \beta_{i,j} \) ranges over a complete system of representatives of \( \{ x \in \mathfrak{A}^{-1} \mid \text{tr}(b_i^{-1}x) = 0 \} / (p^{-1}b_i(\mathfrak{A}^{-1})^0) \).

\[ U'(N)\backslash S_{p^2} \]

(1) \[
\begin{bmatrix}
  1 & x \\
  0 & p^2
\end{bmatrix},
\]
where \( x \) ranges over a complete system of representatives of \((\mathfrak{A}^{-1})^0/p^2(\mathfrak{A}^{-1})^0\).

(2) \[
\begin{bmatrix}
  a_i & \alpha_{i,j} \\
  0 & pa_i
\end{bmatrix},
\]
where \( a_i \) ranges over a complete system of representatives of \(\mathfrak{D}^{x,1}\)-equivalent classes of elements of \(\mathfrak{D}\) with norm \( p \). For each \( a_i, \alpha_{i,j} \) ranges over a complete system of representatives of \( \{ x \in \mathfrak{A}^{-1} \mid \text{tr}(a_i^{-1}x) = 0 \} / (pa_i(\mathfrak{A}^{-1})^0) \).

(3) \[
\begin{bmatrix}
  b_i & \beta_{i,j} \\
  0 & b_i
\end{bmatrix},
\]
where \( b_i \) ranges over a complete system of representatives of \(\mathfrak{D}^{x,1}\)-equivalent classes of elements of \(\mathfrak{D}\) with norm \( p^2 \). For each \( b_i, \beta_{i,j} \) ranges over a complete system of representatives of \( \{ x \in \mathfrak{A}^{-1} \mid \text{tr}(b_i^{-1}x) = 0 \} / (b_i(\mathfrak{A}^{-1})^0) \).

(4) \[
\begin{bmatrix}
  c_i & \gamma_{i,j} \\
  0 & p^{-1}c_i
\end{bmatrix},
\]
where \( c_i \) ranges over a complete system of representatives of \(\mathfrak{D}^{x,1}\)-equivalent classes of elements of \(\mathfrak{D}\) with norm \( p^3 \). For each \( c_i, \gamma_{i,j} \) ranges over a complete system of representatives of \( \{ x \in \mathfrak{A}^{-1} \mid \text{tr}(c_i^{-1}x) = 0 \} / (p^{-1}c_i(\mathfrak{A}^{-1})^0) \).

(5) \[
\begin{bmatrix}
  d_i & \delta_{i,j} \\
  0 & p^{-2}d_i
\end{bmatrix},
\]
where \( d_i \) ranges over a complete system of representatives of \(\mathfrak{D}^{x,1}\)-equivalent classes of elements of \(\mathfrak{D}\) with norm \( p^4 \). For each \( d_i, \delta_{i,j} \) ranges over a complete system of representatives of \( \{ x \in \mathfrak{A}^{-1} \mid \text{tr}(d_i^{-1}x) = 0 \} / (p^{-2}d_i(\mathfrak{A}^{-1})^0) \).
5.2. Discriminant 6

Our lifting conjecture to $S_{k,0}(U'(6))$ from $S_{2k-2}^{\text{new},k}(\Gamma_0^{(1)}(N))$ with $N = 1, 2, 3, 6$ seem to be a part of the lifting theory of Sugano or Oda, though there is no definite result on the image. Anyway we would like to see $L$ functions more closely by experiment in order to say something more definite.

The second named author constructed the graded ring of Siegel modular forms $M_{k,0}(U'(6))$ explicitly. We can use the following theorem for our experiment.

**Theorem 5.1** ([Kit12]). The graded ring of Siegel modular forms with respect to $U'(6)$ is given explicitly by

$$
\bigoplus_{k=0}^{\infty} M_{k,0}(U'(6)) = \mathbb{C}[E_2, E_4, \chi_{5a}, E_6] \oplus \chi_{5b}\mathbb{C}[E_2, E_4, \chi_{5a}, E_6]
$$

where we denote by $E_k$ ($k = 2, 4, 6$) the Eisenstein series of weight $k$ which were defined in [Hir99], and denote by $\chi_{5a}, \chi_{5b}$ and $\chi_{15}$ the Siegel cusp forms of weight 5, 5 and 15 respectively, which are defined in [Kit12]. The four modular forms $E_2, E_4, \chi_{5a}$ and $E_6$ are algebraically independent over $\mathbb{C}$. The fundamental relations among the generators are also given in [Kit12].

For any $f(Z) \in M_{k,0}(U'(6))$, we have the following Fourier expansion

$$
f(Z) = C(0) + \sum_{\eta \in L^*, \eta J > 0} C(\eta) \mathbf{e}[\text{Tr}(\eta Z J)] \quad (\mathbf{e}[z] = e^{2\pi iz}),
$$

where

$$
L^* = \left\{ \left[ \frac{(5x + 10z)\sqrt{6}}{60}, \frac{(x + 5y)\sqrt{5} + x\sqrt{30}}{60}, \frac{-(5x + 10z)\sqrt{6}}{60} \right] \mid x, y, z \in \mathbb{Z}, \eta J > 0 \right\}, \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},
$$

and $\eta J > 0$ means $\eta J$ is positive definite. See [Kit12, §2.4 and §3.1] for details. Note that $f(Z)$ is a cusp form if and only if $C(0) = 0$.

Now we consider Fourier coefficients of $(T_k(n)f)(Z)$ for $f(Z) \in S_{k,0}(U'(6))$. For any cusp form $f(Z) = \sum_{\eta \in L^*, \eta J > 0} C(\eta) \mathbf{e}[\text{Tr}(\eta Z J)] \in S_{k,0}(U'(6))$, we put

$$(T_k(n)f)(Z) = \sum_{\eta \in L^*, \eta J > 0} C(n; \eta) \mathbf{e}[\text{Tr}(\eta Z J)].$$

Calculating carefully by taking explicit representatives of $U'(6) \backslash S_5$ and $U'(6) \backslash S_5^2$, we obtain the following relations:

**Proposition 5.2.**

1. $C(5;[2,1,-1]) = C([10,5,-5]),$
2. $C(5;[2,0,-1]) = C([10,0,-5]) + 2 \cdot 5^{k-2} C([2,0,-1]),$
3. $C(5;[4,2,-2]) = C([20,10,-10]),$
(4) \( C(5; [4, 0, -2]) = C([20, 0, -10]) + 2 \cdot 5^{k-2} C([4, 0, -2]), \)
(5) \( C(5; [4, 1, -2]) = C([20, 5, -10]) + 2 \cdot 5^{k-2} C([4, 1, -2]), \)
(6) \( C(5; [5, 1, -2]) = C([25, 5, -10]) + 2 \cdot 5^{k-2} C([7, -1, -3]), \)
(7) \( C(5^2; [2, 1, -1]) = C([50, 25, -25]), \)
(8) \( C(5^2; [2, 0, -1]) = C([50, 0, -25]) + 2 \cdot 5^{k-2} C([10, 0, -5]) + 2 \cdot 5^{2k-4} C([2, 0, -1]), \)
(9) \( C(5^2; [4, 1, -2]) = C([100, 25, -50]) + 2 \cdot 5^{k-2} C([20, 5, -10]) + 2 \cdot 5^{2k-4} C([4, 1, -2]). \)

In the next subsection, we calculate Hecke eigenvalues for \( S_{k; 0}(U') \) with \( 4 \leq k \leq 11 \) by using Proposition 5.2. A basis of each space is obtained by the ring generators of Theorem 5.1 and their Fourier coefficients can be calculated in the manner of [Kit12].

5.3. Results of numerical experiment

As before we define
\[
S_{2k-2}^{new, \pm}(\Gamma_0^{(1)}(N)) = \left\{ f \in S_{2k-2}^{new}(\Gamma_0^{(1)}(N)) \mid u_N f = \pm(-1)^k f \right\},
\]
where \( u_N \) is the Atkin-Lehner involution defined by the action of \( \begin{bmatrix} 0 & -1 \\ N & 0 \end{bmatrix} \).

**weight 4:** \( \dim S_{4, 0}(U'(6)) = 1 \). If we put
\[
\phi_1 = E_2^2 - E_4,
\]
then \( \phi_1 \in S_{4, 0}(U'(6)) \), and we have
\[
T_4(5)\phi_1 = 156\phi_1, \quad T_4(5^2)\phi_1 = 16561\phi_1
\]
by Proposition 5.2 (1), (7) and Table 1. We obtain
\[
H_5(t, \phi_1) = (t - 5^2)(t - 5^3)(t^2 - 6t + 5^5).
\]

<table>
<thead>
<tr>
<th>Table 1.</th>
<th>( \eta )</th>
<th>( E_2^2 )</th>
<th>( E_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>([0, 0, 0])</td>
<td>([2, 1, -1])</td>
<td>([10, 5, -5])</td>
<td>([50, 25, -25])</td>
</tr>
<tr>
<td>([0, 0, 0])</td>
<td>([2, 1, -1])</td>
<td>([10, 5, -5])</td>
<td>([50, 25, -25])</td>
</tr>
</tbody>
</table>

On the other hand, the spaces of cusp forms of one variable are given as follows:
\[
\dim S_{6, +}^{new}(\Gamma_0^{(1)}(2)) = \dim S_{6, -}^{new}(\Gamma_0^{(1)}(2)) = \dim S_{6, +}^{new}(\Gamma_0^{(1)}(3)) = \dim S_{6, -}^{new}(\Gamma_0^{(1)}(3)) = 0, \quad \text{and} \quad S_{6, +}^{new}(\Gamma_0^{(1)}(6)) = \mathbb{C}f_1, \quad S_{6, -}^{new}(\Gamma_0^{(1)}(6)) = \mathbb{C}f_2,
\]
where
\[
f_1 = q - 6q^2 + 9q^3 + 4q^4 + 6q^5 + O(q^6),
\]
\[
f_2 = q + 5q^2 + 9q^3 + 6q^4 + 2q^5 + O(q^6).
\]
and

\[ f_2 = q + 4q^2 - 9q^3 + 16q^4 - 66q^5 + O(q^6), \]

respectively.

Hence we find \( H_5(t, \phi_1) = (t - 5^3)(t - 5^3)h_5(t, f_1) \), where \( h_5(t, f_1) \) is the Hecke polynomial of \( f_1 \) at 5.

**weight 5**: \( \dim S_{5,0}(U'(6)) = 2 \). If we put

\[ \phi_1 = \chi_{5a}, \quad \phi_2 = \chi_{5b}, \]

then we see that \( \phi_1 \) and \( \phi_2 \) are Hecke eigenbasis of \( S_{5,0}(U'(6)) \) and

\[
\begin{align*}
T_5(5)\phi_1 &= 1140\phi_1, \\
T_5(5^2)\phi_1 &= 835225\phi_1, \\
T_5(5)\phi_2 &= 540\phi_2, \\
T_5(5^2)\phi_2 &= 277225\phi_2
\end{align*}
\]

by Proposition 5.2 (1),(2),(7),(8) and Table 2. We obtain

\[
\begin{align*}
H_5(t, \phi_1) &= (t - 5^3)(t - 5^4)(t^2 - 390t + 5^7) \\
H_5(t, \phi_2) &= (t - 5^3)(t - 5^4)(t^2 + 210t + 5^7).
\end{align*}
\]

<table>
<thead>
<tr>
<th>( \eta )</th>
<th>( \chi_{5a} )</th>
<th>( \chi_{5b} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>[0, 0, 0]</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>[2, 1, -1]</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>[2, 0, -1]</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>[10, 5, -5]</td>
<td>0</td>
<td>540</td>
</tr>
<tr>
<td>[10, 0, -5]</td>
<td>890</td>
<td>0</td>
</tr>
<tr>
<td>[50, 25, -25]</td>
<td>0</td>
<td>277225</td>
</tr>
<tr>
<td>[50, 0, -25]</td>
<td>581475</td>
<td>0</td>
</tr>
</tbody>
</table>

On the other hand, the spaces of cusp forms of one variable are given as follows:

\( \dim S_{8,0}^{new,+}(1_0(1)(2)) = \mathbb{C}f_1 \), where

\[ f_1 = q - 8q^2 + 12q^3 + 64q^4 - 210q^5 + O(q^6). \]

\( \dim S_{8,0}^{new,-}(1_0(1)(2)) = 0. \)

\( \dim S_{8,0}^{new,+}(1_0(1)(3)) = \mathbb{C}f_2 \), where

\[ f_2 = q + 6q^2 - 27q^3 - 92q^4 + 390q^5 + O(q^6). \]

\( \dim S_{8,0}^{new,-}(1_0(1)(3)) = 0. \)

\( \dim S_{8,0}^{new,+}(1_0(1)(6)) = \mathbb{C}f_3 \), where

\[ f_3 = q + 8q^2 + 27q^3 + 64q^4 - 114q^5 + O(q^6). \]

\( \dim S_{8,0}^{new,-}(1_0(1)(6)) = 0. \)
Hence we find $H_5(t, \phi_1) = (t - 5^3)(t - 5^4)h_5(t, f_2)$ and $H_5(t, \phi_2) = (t - 5^3)(t - 5^4)h_5(t, f_1)$.

**weight 6**: $\dim S_{6,0}(U'(6)) = 2$. If we put

$$\phi_1 = 225E_2^3 - 247E_2E_4 + 22E_6, \quad \phi_2 = 70E_2^3 - 39E_2E_4 - 31E_6,$$

then we see that $\phi_1$ and $\phi_2$ are Hecke eigenbasis of $S_{6,0}(U'(6))$ and

$$T_6(5)\phi_1 = 4620\phi_1, \quad T_6(5^2)\phi_1 = 13785025\phi_1,$$
$$T_6(5)\phi_2 = 2220\phi_2, \quad T_6(5^2)\phi_2 = 6369025\phi_2$$

by Proposition 5.2 (1),(2),(7) and Table 3. We obtain

$$H_5(t, \phi_1) = (t - 5^4)(t - 5^5)(t^2 - 870t + 5^9),$$
$$H_5(t, \phi_2) = (t - 5^4)(t - 5^5)(t^2 + 1530t + 5^9).$$

<table>
<thead>
<tr>
<th>$\eta$</th>
<th>$E_4$</th>
<th>$E_2E_4$</th>
<th>$E_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[0, 0, 0]$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$[2, 1, -1]$</td>
<td>144</td>
<td>1584/13</td>
<td>2016/341</td>
</tr>
<tr>
<td>$[2, 0, -1]$</td>
<td>216</td>
<td>3096/13</td>
<td>7560/341</td>
</tr>
<tr>
<td>$[10, 5, -5]$</td>
<td>11848896</td>
<td>153119808/13</td>
<td>3945062016/341</td>
</tr>
<tr>
<td>$[10, 0, -5]$</td>
<td>43651440</td>
<td>43758000</td>
<td>14784532560/341</td>
</tr>
<tr>
<td>$[50, 25, -25]$</td>
<td>22596758341776</td>
<td>293755169840688/13</td>
<td>770522882562016/341</td>
</tr>
</tbody>
</table>

Table 3.

On the other hand, the spaces of cusp forms of one variable are given as follows:

$\dim S_{10}^{new, +}(I_0^{(1)}(2)) = \mathbb{C}f_1$, where

$$f_1 = q + 16q^2 - 156q^3 + 256q^4 + 870q^5 + O(q^6).$$

$\dim S_{10}^{new, -}(I_0^{(1)}(2)) = 0$.

$\dim S_{10}^{new, +}(I_0^{(1)}(3)) = \mathbb{C}f_2$, where

$$f_2 = q + 18q^2 + 81q^3 - 188q^4 - 1530q^5 + O(q^6).$$

$\dim S_{10}^{new, -}(I_0^{(1)}(3)) = \mathbb{C}f_3$, where

$$f_3 = q - 36q^2 - 81q^3 + 784q^4 - 1314q^5 + O(q^6).$$

$\dim S_{10}^{new, +}(I_0^{(1)}(6)) = \mathbb{C}f_4$, where

$$f_4 = q - 16q^2 + 81q^3 + 256q^4 + 2694q^5 + O(q^6).$$

$\dim S_{10}^{new, -}(I_0^{(1)}(6)) = 0$.

Hence we find $H_5(t, \phi_1) = (t - 5^4)(t - 5^5)h_5(t, f_1)$ and $H_5(t, \phi_2) = (t - 5^4)(t - 5^5)h_5(t, f_2)$. 
weight 7: \( \dim S_{7,0}(U'(6)) = 2 \). If we put
\[
\phi_1 = E_2 \chi_5 a, \quad \phi_2 = E_2 \chi_5 b,
\]
then we see that \( \phi_1 \) and \( \phi_2 \) are Hecke eigenbasis of \( S_{7,0}(U'(6)) \) and
\[
T_7(5)\phi_1 = 13380\phi_1, \quad T_7(5^2)\phi_1 = 172290025\phi_1, \\
T_7(5)\phi_2 = 7020\phi_2, \quad T_7(5^2)\phi_2 = 161796025\phi_2
\]
by Proposition 5.2 (1),(2),(7),(8) and Table 4. We obtain
\[
H_5(t;\phi_1) = (t^5 - 5)(t^5 - 5^2)(t^2 + 5370t + 5^{11}) , \\
H_5(t;\phi_2) = (t^5 - 5)(t^5 - 5^2)(t^2 + 11730t + 5^{11}).
\]

<table>
<thead>
<tr>
<th>( \eta )</th>
<th>( E_2 \chi_5^a )</th>
<th>( E_2 \chi_5^b )</th>
</tr>
</thead>
<tbody>
<tr>
<td>[0, 0, 0]</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>[2, 1, -1]</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>[2, 0, -1]</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>[10, 5, -5]</td>
<td>0</td>
<td>7020</td>
</tr>
<tr>
<td>[10, 0, -5]</td>
<td>7130</td>
<td>0</td>
</tr>
<tr>
<td>[50, 25, -25]</td>
<td>0</td>
<td>161796025</td>
</tr>
<tr>
<td>[50, 0, -25]</td>
<td>108196275</td>
<td>0</td>
</tr>
</tbody>
</table>

On the other hand, the spaces of cusp forms of one variable are given as follows:
\[
\dim S_{12,0}^{\text{new},+}(\Gamma_0^{(1)}(2)) = 0, \\
\dim S_{12,0}^{\text{new},-}(\Gamma_0^{(1)}(2)) = 0, \\
\dim S_{12,0}^{\text{new},+}(\Gamma_0^{(1)}(3)) = 1 \text{ and } S_{12,0}^{\text{new},+}(\Gamma_0^{(1)}(3)) = \mathbb{C}f_1, \text{ where} \\
f_1 = q + 78q^2 - 243q^3 + 4036q^4 - 5370q^5 + O(q^6),
\]
\[
\dim S_{12,0}^{\text{new},-}(\Gamma_0^{(1)}(3)) = 0, \\
\dim S_{12,0}^{\text{new},+}(\Gamma_0^{(1)}(6)) = 2 \text{ and } S_{12,0}^{\text{new},+}(\Gamma_0^{(1)}(6)) = \mathbb{C}f_2 \oplus \mathbb{C}f_3, \text{ where} \\
f_2 = q + 32q^2 + 243q^3 + 1024q^4 + 3630q^5 + O(q^6), \\
f_3 = q - 32q^2 - 243q^3 + 1024q^4 + 5766q^5 + O(q^6),
\]
\[
\dim S_{12,0}^{\text{new},-}(\Gamma_0^{(1)}(6)) = 1 \text{ and } S_{12,0}^{\text{new},-}(\Gamma_0^{(1)}(6)) = \mathbb{C}f_4, \text{ where} \\
f_4 = q - 32q^2 + 243q^3 + 1024q^4 - 11730q^5 + O(q^6).
\]

Hence we find \( H_5(t, \phi_1) = (t - 5^3)(t - 5^6)h_5(t, f_1) \) and \( H_5(t, \phi_2) = (t - 5^3)(t - 5^6)h_5(t, f_4) \).
weight 8: \( \dim S_{8,0}(U'(6)) = 3 \). If we put
\[
\phi_1 = 110043E_2^4 - 218010E_2^2E_4 - 2728E_2E_6 + 110695E_4^2, \\
\phi_2 = -4(69719 - 6427\sqrt{1969})E_2^4 + 13(59939 - 3733\sqrt{1969})E_2^2E_4 \\
+ 682(13 + \sqrt{1969})E_2E_6 - 22139(23 - \sqrt{1969})E_4^2, \\
\phi_3 = -4(69719 + 6427\sqrt{1969})E_2^4 + 13(59939 + 3733\sqrt{1969})E_2^2E_4 \\
+ 682(13 - \sqrt{1969})E_2E_6 - 22139(23 + \sqrt{1969})E_4^2,
\]
then we see that \( \phi_1, \phi_2, \phi_3 \) are Hecke eigenbasis of \( S_{8,0}(U'(6)) \) and
\[
T_6(5)\phi_1 = 36300\phi_1, \quad T_6(5^2)\phi_1 = 4018080625\phi_1, \\
T_6(5)\phi_2 = (114108 + 384\sqrt{1969})\phi_2, \quad T_6(5^2)\phi_2 = (8716867153 + 51634944\sqrt{1969})\phi_2, \\
T_6(5)\phi_3 = (114108 - 384\sqrt{1969})\phi_3, \quad T_6(5^2)\phi_3 = (8716867153 - 51634944\sqrt{1969})\phi_3
\]
by Proposition 5.2 (1),(2),(3),(7) and Table 5 and 6. We obtain
\[
H_5(t, \phi_1) = (t - 5^6)(t - 5^7)(t^2 + 57450t + 5^{13}), \\
H_5(t, \phi_2) = (t - 5^6)(t - 5^7)(t^2 - (20358 + 384\sqrt{1969})t + 5^{13}), \\
H_5(t, \phi_3) = (t - 5^6)(t - 5^7)(t^2 - (20358 - 384\sqrt{1969})t + 5^{13}).
\]

<table>
<thead>
<tr>
<th>( \eta )</th>
<th>( E_2^5 )</th>
<th>( E_2^2E_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>([0, 0, 0])</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>([2, 1, -1])</td>
<td>192</td>
<td>2208/13</td>
</tr>
<tr>
<td>([2, 0, -1])</td>
<td>288</td>
<td>4032/13</td>
</tr>
<tr>
<td>([4, 2, -2])</td>
<td>14592</td>
<td>162024/13</td>
</tr>
<tr>
<td>([10, 5, -5])</td>
<td>1308602888</td>
<td>1673186688/13</td>
</tr>
<tr>
<td>([10, 0, -5])</td>
<td>780547392</td>
<td>10190996352/13</td>
</tr>
<tr>
<td>([20, 10, -10])</td>
<td>975896195784</td>
<td>12683189263104/13</td>
</tr>
<tr>
<td>([50, 25, -25])</td>
<td>144288528492774208</td>
<td>1857548634717301408/13</td>
</tr>
<tr>
<td>([50, 0, -25])</td>
<td>942999141117375072</td>
<td>1225899176588909632/13</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \eta )</th>
<th>( E_2E_6 )</th>
<th>( E_2^4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>([0, 0, 0])</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>([2, 1, -1])</td>
<td>18384/341</td>
<td>1920/13</td>
</tr>
<tr>
<td>([2, 0, -1])</td>
<td>32112/341</td>
<td>4320/13</td>
</tr>
<tr>
<td>([4, 2, -2])</td>
<td>1228704/341</td>
<td>1845120/169</td>
</tr>
<tr>
<td>([10, 5, -5])</td>
<td>41501838528/341</td>
<td>21377180160/169</td>
</tr>
<tr>
<td>([10, 0, -5])</td>
<td>264658285152/341</td>
<td>1330222800960/169</td>
</tr>
<tr>
<td>([20, 10, -10])</td>
<td>332756404101504/341</td>
<td>164856944171520/169</td>
</tr>
<tr>
<td>([50, 25, -25])</td>
<td>4920214392528644104/341</td>
<td>1875746380332904320/13</td>
</tr>
<tr>
<td>([50, 0, -25])</td>
<td>321562470788343886032/341</td>
<td>159366930121797852960/169</td>
</tr>
</tbody>
</table>

On the other hand, the spaces of cusp forms of one variable are given as follows:
dimension formula for Siegel paramodular cusp forms

$$\dim S_{14}^{new,+}(\Gamma_0^{(1)}(2)) = 1$$ and $$S_{14}^{new,+}(\Gamma_0^{(1)}(2)) = \mathbb{C}f_1,$$ where
$$f_1 = q + 64q^2 + 1236q^3 + 4096q^4 - 57450q^5 + O(q^6).$$

$$\dim S_{14}^{new,-}(\Gamma_0^{(1)}(2)) = 1$$ and $$S_{14}^{new,-}(\Gamma_0^{(1)}(2)) = \mathbb{C}f_2,$$ where
$$f_2 = q - 64q^2 - 1836q^3 + 4096q^4 + 3990q^5 + O(q^6).$$

$$\dim S_{14}^{new,+}(\Gamma_0^{(1)}(3)) = 2$$ and $$S_{14}^{new,+}(\Gamma_0^{(1)}(3)) = \mathbb{C}f_3 \oplus \mathbb{C}f_4,$$ where
$$f_3 = q + (-27 + 3\sqrt{1969})q^2 + 729q^3 + (10258 - 162\sqrt{1969})q^4 + (20358 + 384\sqrt{1969})q^5 + O(q^6),$$
$$f_4 = q + (-27 - 3\sqrt{1969})q^2 + 729q^3 + (10258 + 162\sqrt{1969})q^4 + (20358 - 384\sqrt{1969})q^5 + O(q^6).$$

$$\dim S_{14}^{new,-}(\Gamma_0^{(1)}(3)) = 1$$ and $$S_{14}^{new,-}(\Gamma_0^{(1)}(3)) = \mathbb{C}f_5,$$ where
$$f_5 = q - 12q^2 - 729q^3 - 8048q^4 - 30210q^5 + O(q^6).$$

$$\dim S_{14}^{new,+}(\Gamma_0^{(1)}(6)) = 1$$ and $$S_{14}^{new,+}(\Gamma_0^{(1)}(6)) = \mathbb{C}f_6,$$ where
$$f_6 = q + 64q^2 - 729q^3 + 4096q^4 + 54654q^5 + O(q^6).$$

$$\dim S_{14}^{new,-}(\Gamma_0^{(1)}(6)) = 0.$$

Hence we find $$H_5(t, \phi_1) = (t - 5^3)(t - 5^7)h_5(t, f_1)$$ and $$H_5(t, \phi_2) = (t - 5^6)(t - 5^7)h_5(t, f_3)$$ and $$H_5(t, \phi_3) = (t - 5^6)(t - 5^7)h_5(t, f_4).$$

**weight 9:** $$\dim S_{9,0}(U'(6)) = 4.$$ If we put

$$\phi_1 = -13(E_4 - E_2^2)\chi_{5b} + 2E_2\chi_{5a}, \quad \phi_2 = -52(E_4 - E_2^2)\chi_{5a} + 3E_2^2\chi_{5a},$$
$$\phi_3 = 169(E_4 - E_2^2)\chi_{5a} + 16E_2^2\chi_{5a}, \quad \phi_4 = 65(E_4 - E_2^2)\chi_{5b} + 3E_2^2\chi_{5b},$$

then we see that $$\phi_1, \phi_2, \phi_3, \phi_4$$ are Hecke eigenbasis of $$S_{9,0}(U'(6))$$ and

$$T_9(5)\phi_1 = 559260\phi_1 \quad T_9(5)\phi_2 = 749460\phi_2,$$
$$T_9(5)\phi_3 = 353940\phi_3 \quad T_9(5)\phi_4 = -33540\phi_4,$$
$$T_9(5^2)\phi_1 = 203206513225\phi_1, \quad T_9(5^2)\phi_2 = 362968807225\phi_2,$$
$$T_9(5^2)\phi_3 = 111952039225\phi_3, \quad T_9(5^2)\phi_4 = 10314697225\phi_4,$$

by Proposition 5.2 (1),(2),(3),(4),(7),(8) and Table 7. We obtain

$$H_5(t, \phi_1) = (t - 5^7)(t - 5^3)(t^2 - 90510t + 5^{15})$$
$$H_5(t, \phi_2) = (t - 5^7)(t - 5^3)(t^2 - 280710t + 5^{15})$$
$$H_5(t, \phi_3) = (t - 5^7)(t - 5^3)(t^2 + 114810t + 5^{15})$$
$$H_5(t, \phi_4) = t^4 + 33540t^3 - 15293281250t^2 + 33540 \cdot 5^{15}t + 5^{30}.$$  

<table>
<thead>
<tr>
<th>$\eta$</th>
<th>$E_2^\chi_{30}$</th>
<th>$E_2^\chi_{26}$</th>
<th>$E_4^\chi_{30}$</th>
<th>$E_4^\chi_{26}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>[0, 0, 0]</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>[2, 1, -1]</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>[2, 0, -1]</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>[4, 2, -2]</td>
<td>0</td>
<td>112</td>
<td>0</td>
<td>1168/13</td>
</tr>
<tr>
<td>[4, 0, -2]</td>
<td>150</td>
<td>0</td>
<td>2238/13</td>
<td>0</td>
</tr>
<tr>
<td>[10, 5, -5]</td>
<td>0</td>
<td>422460</td>
<td>0</td>
<td>5218380/13</td>
</tr>
<tr>
<td>[10, 0, -5]</td>
<td>347450</td>
<td>0</td>
<td>4332530/13</td>
<td>0</td>
</tr>
<tr>
<td>[20, 10, -10]</td>
<td>0</td>
<td>112979520</td>
<td>0</td>
<td>1408351680/13</td>
</tr>
<tr>
<td>[20, 0, -10]</td>
<td>-5390340</td>
<td>0</td>
<td>29991180</td>
<td>0</td>
</tr>
<tr>
<td>[50, 25, -25]</td>
<td>0</td>
<td>158693017225</td>
<td>0</td>
<td>1973982231925/13</td>
</tr>
<tr>
<td>[50, 0, -25]</td>
<td>140501129475</td>
<td>0</td>
<td>133718153475</td>
<td>0</td>
</tr>
</tbody>
</table>

On the other hand, the spaces of cusp forms of one variable are given as follows:

$\dim S_{16}^{\text{new},+}(\Gamma_0^{(1)}(2)) = 1$ and $S_{16}^{\text{new},+}(\Gamma_0^{(1)}(2)) = \mathbb{C}f_1$, where

$$f_1 = q - 128q^2 + 6252q^3 + 16384q^4 + 90510q^5 + O(q^6).$$

$\dim S_{16}^{\text{new},-}(\Gamma_0^{(1)}(2)) = 0.$

$\dim S_{16}^{\text{new},+}(\Gamma_0^{(1)}(3)) = 1$ and $S_{16}^{\text{new},+}(\Gamma_0^{(1)}(3)) = \mathbb{C}f_2$, where

$$f_2 = q - 234q^2 - 2187q^3 + 21988q^4 + 280710q^5 + O(q^6).$$

$\dim S_{16}^{\text{new},-}(\Gamma_0^{(1)}(3)) = 1$ and $S_{16}^{\text{new},-}(\Gamma_0^{(1)}(3)) = \mathbb{C}f_3$, where

$$f_3 = q - 72q^2 + 2187q^3 - 27584q^4 - 221490q^5 + O(q^6).$$

$\dim S_{16}^{\text{new},+}(\Gamma_0^{(1)}(6)) = 2$ and $S_{16}^{\text{new},+}(\Gamma_0^{(1)}(6)) = \mathbb{C}f_4 \oplus \mathbb{C}f_5$, where

$$f_4 = q + 128q^2 + 2187q^3 + 16384q^4 + 77646q^5 + O(q^6),$$

$$f_5 = q - 128q^2 - 2187q^3 + 16384q^4 - 314490q^5 + O(q^6).$$

$\dim S_{16}^{\text{new},-}(\Gamma_0^{(1)}(6)) = 1$ and $S_{16}^{\text{new},-}(\Gamma_0^{(1)}(6)) = \mathbb{C}f_6$, where

$$f_6 = q + 128q^2 - 2187q^3 + 16384q^4 - 114810q^5 + O(q^6).$$

Hence we find $H_5(t, \phi_1) = (t - 5^7)(t - 5^5)h_5(t, f_1), H_5(t, \phi_2) = (t - 5^7)(t - 5^5)h_5(t, f_2)$ and $H_5(t, \phi_3) = (t - 5^7)(t - 5^5)h_5(t, f_6)$, and $H_5(t, \phi_4)$ does not have such factorization. All absolute values of zeros of $H_5(t, \phi_4)$ are $5^{15}/2$.

**weight 10:** $\dim S_{10, \overline{1}}(U'(6)) = 6.$ If we put

$$\phi_1 = -1141885E_2^5 + 2578420E_2^3E_4 - 395560E_2^2E_6 - 1413347E_2E_4^2 + 372372E_4E_6 + 6519398400\chi_{5a}^2$$
\[ \phi_2 = \chi_{5a} \chi_{5b} \]
\[ \phi_3 = -11881 E_2^{5} - 16523 E_2^{3} E_4 + 67859 E_2^{2} E_6 + 27040 E_2 E_4^{2} - 66495 E_4 E_6 + 358566912 \chi_{5a}^{2} \]
\[ \phi_4 = -1005445 E_2^{5} + 2058550 E_2^{3} E_4 - 56265 E_2^{2} E_6 - 1045603 E_2 E_4^{2} + 48763 E_4 E_6 - 814924800 \chi_{5a}^{2} \]
\[ \phi_5 = 5(8553171360955 - 88807370233 \sqrt{14569}) E_2^{5} + 65(-1258629706843 + 16378550233 \sqrt{14569}) E_2^{3} E_4 + 1705(-1426727601 - 93536405 \sqrt{14569}) E_2^{2} E_6 + 1352(29037896401 - 460871627 \sqrt{14569}) E_2 E_4^{2} + 31031(71490083 + 5220879 \sqrt{14569}) E_4 E_6 + 814924800(-57532981 + 1170775 \sqrt{14569}) \chi_{5a}^{2} \]
\[ \phi_6 = 5(8553171360955 + 88807370233 \sqrt{14569}) E_2^{5} + 65(-1258629706843 - 16378550233 \sqrt{14569}) E_2^{3} E_4 + 1705(-1426727601 + 93536405 \sqrt{14569}) E_2^{2} E_6 + 1352(29037896401 + 460871627 \sqrt{14569}) E_2 E_4^{2} + 31031(71490083 - 5220879 \sqrt{14569}) E_4 E_6 + 814924800(-57532981 - 1170775 \sqrt{14569}) \chi_{5a}^{2} \]

then we see that \( \phi_i \) with \( 1 \leq i \leq 6 \) are Hecke eigenbasis of \( S_{10,0}(U'(6)) \) and

\[ T_{10}(5)\phi_1 = 3598860\phi_1, \quad T_{10}(5^2)\phi_1 = 8331662440225\phi_1, \]
\[ T_{10}(5)\phi_2 = 2988900\phi_2, \quad T_{10}(5^2)\phi_2 = 5742986100625\phi_2, \]
\[ T_{10}(5)\phi_3 = 1317900\phi_3, \quad T_{10}(5^2)\phi_3 = 246272950625\phi_3, \]
\[ T_{10}(5)\phi_4 = -88980\phi_4, \quad T_{10}(5^2)\phi_4 = -1314838418975\phi_4, \]
\[ T_{10}(5)\phi_5 = (2535180 + 10560 \sqrt{14569})\phi_5, \]
\[ T_{10}(5^2)\phi_5 = (5924648411425 + 28793001600 \sqrt{14569})\phi_5, \]
\[ T_{10}(5)\phi_6 = (2535180 - 10560 \sqrt{14569})\phi_6, \]
\[ T_{10}(5^2)\phi_6 = (5924648411425 - 28793001600 \sqrt{14569})\phi_6 \]
by Proposition 5.2 (1),(2),(3),(4),(5),(6),(7),(9) and Table 8, 9 and 10. We obtain

\[ H_5(t, \phi_1) = (t - 5^{8})(t - 5^{9})(t^2 - 1255110t + 5^{17}), \]
\[ H_5(t, \phi_2) = (t - 5^{8})(t - 5^{9})(t^2 - 645150t + 5^{17}), \]
\[ H_5(t, \phi_3) = (t - 5^{8})(t - 5^{9})(t^2 + 1025850t + 5^{17}), \]
\[ H_5(t, \phi_4) = t^4 + 88980t^3 + 1170167968750t^2 + 88980 \cdot 5^{17}t + 5^{34}, \]
\[ H_5(t, \phi_5) = (t - 5^8)(t - 5^9)(t^2 - (191430 + 10560\sqrt{14569})t + 5^{17}), \]
\[ H_5(t, \phi_6) = (t - 5^8)(t - 5^9)(t^2 - (191430 - 10560\sqrt{14569})t + 5^{17}). \]
On the other hand, the spaces of cusp forms of one variable are given as follows:

\[ \dim S_{18}^{\text{new},+}(\Gamma_0(2)) = 1 \] and \( S_{18}^{\text{new},+}(\Gamma_0(3)) = \mathbb{C}f_1, \]

where

\[
f_1 = q + 256q^2 + 6084q^3 + 65536q^4 + 1255110q^5 + O(q^6).\]

\[ \dim S_{18}^{\text{new},-}(\Gamma_0(2)) = 0. \]

\[ \dim S_{18}^{\text{new},+}(\Gamma_0(3)) = 2 \] and \( S_{18}^{\text{new},+}(\Gamma_0(3)) = \mathbb{C}f_2 \oplus \mathbb{C}f_3, \)

where

\[
f_2 = q + (297 - 3\sqrt{14569})q^2 + 6561q^3 + (88258 - 1782\sqrt{14569})q^4 + (191430 + 10560\sqrt{14569})q^5 + O(q^6), \]

\[ \dim S_{18}^{\text{new},-}(\Gamma_0(3)) = 1 \] and \( S_{18}^{\text{new},-}(\Gamma_0(3)) = \mathbb{C}f_4, \)

where

\[
f_3 = q + (297 + 3\sqrt{14569})q^2 + 6561q^3 + (88258 + 1782\sqrt{14569})q^4 + (191430 - 10560\sqrt{14569})q^5 + O(q^6). \]

\[ \dim S_{18}^{\text{new},+}(\Gamma_0(6)) = 2 \] and \( S_{18}^{\text{new},+}(\Gamma_0(6)) = \mathbb{C}f_5 \oplus \mathbb{C}f_6, \)

where

\[
f_5 = q - 256q^2 + 6561q^3 + 65536q^4 - 72186q^5 + O(q^6), \]

\[ \dim S_{18}^{\text{new},-}(\Gamma_0(6)) = 1 \] and \( S_{18}^{\text{new},-}(\Gamma_0(6)) = \mathbb{C}f_7, \)

where

\[
f_6 = q + 256q^2 - 6561q^3 + 65536q^4 - 199650q^5 + O(q^6). \]

\[ \dim S_{18}(SL_2(\mathbb{Z})) = 1 \] and \( S_{18}(SL_2(\mathbb{Z})) = \mathbb{C}f_8, \)

where

\[
f_8 = q - 528q^2 - 4284q^3 + 147712q^4 - 1025850q^5 + O(q^6). \]

Hence we find \( H_5(t, \phi_1) = (t - 5^8)(t - 5^9)h_5(t, f_1), \) \( H_5(t, \phi_2) = (t - 5^6)(t - 5^7)h_5(t, f_7), \)

\( H_5(t, \phi_3) = (t - 5^8)(t - 5^9)h_5(t, f_8), \) \( H_5(t, \phi_5) = (t - 5^8)(t - 5^9)h_5(t, f_2) \) and \( H_5(t, \phi_6) = \)

<table>
<thead>
<tr>
<th>( \eta )</th>
<th>( E_{i_{18}} )</th>
<th>( \chi_{5^2} )</th>
<th>( \chi_{5^3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>([0, 0, 0])</td>
<td>( \chi_{1} )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>([2, 1, -1])</td>
<td>( 353568/4433 )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>([2, 0, -1])</td>
<td>( 834840/4433 )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>([4, 2, -2])</td>
<td>( 2791172/4433 )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>([4, 0, -2])</td>
<td>( 91053720/4433 )</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>([4, 1, -2])</td>
<td>( 149770080/4433 )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>([5, 1, -2])</td>
<td>( 417610080/4433 )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>([7, -1, -3])</td>
<td>( 417610080/4433 )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>([10, 5, -5])</td>
<td>( 2347962277248/4433 )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>([10, 0, -5])</td>
<td>( 21815524301040/4433 )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>([20, 10, -10])</td>
<td>( 228989195274897792/4433 )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>([20, 0, -10])</td>
<td>( 2639604828357339120/4433 )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>([20, 5, -10])</td>
<td>( 11347079657888387520/4433 )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>([25, 5, -10])</td>
<td>( 82828080448639358400/4433 )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>([50, 25, -25])</td>
<td>( 123906229611846391572768/4433 )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>([100, 25, -50])</td>
<td>( 8657611757586543477529978410960/4433 )</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
(t - 5^8)(t - 5^9)h_6(t, f_3), and \( H_5(t, \phi_4) \) does not have such factorization. All absolute values of zeros of \( H_5(t, \phi_4) \) are 5^{17/2}.

**weight 11:** \( \dim S_{11,0}(U'(6)) = 6 \). If we put

\[
\begin{align*}
\phi_1 &= (-168651E_2^3 + 209937E_2E_4 - 40238E_6)\chi_{56}, \\
\phi_2 &= (284240E_2^3 - 307983E_2E_4 + 27280E_6)\chi_{55}, \\
\phi_3 &= (38070E_2^3 - 38727E_2E_4 + 1705E_6)\chi_{54}, \\
\phi_4 &= (-5160E_2^3 + 3848E_2E_4 + 1705E_6)\chi_{54}, \\
\phi_5 &= (15(648516161263 + 489095155\sqrt{87481})E_2^3 \\
&\qquad + 13(925073695727 + 448963853\sqrt{87481})E_2E_4 \\
&\qquad + 3410(-470385337 - 3317995\sqrt{87481})E_6)\chi_{54}, \\
\phi_6 &= (15(648516161263 - 489095155\sqrt{87481})E_2^3 \\
&\qquad + 13(925073695727 - 448963853\sqrt{87481})E_2E_4 \\
&\qquad + 3410(-470385337 + 3317995\sqrt{87481})E_6)\chi_{54},
\end{align*}
\]

then we see that \( \phi_i \) with \( 1 \leq i \leq 6 \) are Hecke eigenbasis of \( S_{11,0}(U'(6)) \) and

\[
\begin{align*}
T_{11}(5)\phi_1 &= 18265500\phi_1, & T_{11}(5)\phi_1 &= 214947093765625\phi_1, \\
T_{11}(5)\phi_2 &= 5869260\phi_2, & T_{11}(5)\phi_2 &= 61035253963225\phi_2, \\
T_{11}(5)\phi_3 &= 5029620\phi_3, & T_{11}(5)\phi_3 &= 10086149610025\phi_3, \\
T_{11}(5)\phi_4 &= -222420\phi_4, & T_{11}(5)\phi_4 &= -25141613327975\phi_4, \\
T_{11}(5)\phi_5 &= (14726820 + 12480\sqrt{87481})\phi_5, \\
T_{11}(5)\phi_6 &= (1494777240554800 + 221331427200\sqrt{87481})\phi_6, \\
T_{11}(5)\phi_6 &= (14726820 - 12480\sqrt{87481})\phi_6, \\
T_{11}(5)\phi_6 &= (1494777240554800 - 221331427200\sqrt{87481})\phi_6.
\end{align*}
\]

by Proposition 5.2 (1),(2),(3),(4),(5),(6),(8),(9) and Table 11 and 12. We obtain

\[
\begin{align*}
H_5(t, \phi_1) &= (t - 5^3)(t - 5^9)(t^2 - 6546750t + 5^{19}), \\
H_5(t, \phi_2) &= (t - 5^3)(t - 5^9)(t^2 + 5849490t + 5^{19}), \\
H_5(t, \phi_3) &= t^4 - 5029620t^3 + 11396230468750t^2 - 5029620 \cdot 5^{19}t + 5^{38}, \\
H_5(t, \phi_4) &= t^4 + 222420t^3 + 21376386718750t^2 + 222420 \cdot 5^{19}t + 5^{38}, \\
H_5(t, \phi_6) &= (t - 5^3)(t - 5^9)(t^2 - 3008070 + 12480\sqrt{87481})t + 5^{19}). \\
H_5(t, \phi_6) &= (t - 5^3)(t - 5^9)(t^2 - 3008070 - 12480\sqrt{87481})t + 5^{19}).
\end{align*}
\]
On the other hand, the spaces of cusp forms of one variable are given as follows:

\[
\dim S_{20}^{\text{new},+}(\Gamma_0^{(1)}(2)) = 1 \text{ and } S_{20}^{\text{new},+}(\Gamma_0^{(1)}(2)) = \mathbb{C} f_1, \text{ where} \\
f_1 = q - 512q^2 - 13092q^3 + 262144q^4 + 6546750q^5 + O(q^6).
\]

\[
\dim S_{20}^{\text{new},-}(\Gamma_0^{(1)}(2)) = 1 \text{ and } S_{20}^{\text{new},-}(\Gamma_0^{(1)}(2)) = \mathbb{C} f_2, \text{ where} \\
f_2 = q + 512q^2 - 53028q^3 + 262144q^4 - 5556930q^5 + O(q^6).
\]

\[
\dim S_{20}^{\text{new},+}(\Gamma_0^{(1)}(3)) = 2 \text{ and } S_{20}^{\text{new},+}(\Gamma_0^{(1)}(3)) = \mathbb{C} f_3 \oplus \mathbb{C} f_4, \text{ where} \\
f_3 = q + (351 + 3\sqrt{87481})q^2 - 19683q^3 + (386242 + 2106\sqrt{87481})q^4 \\
+ (3008070 + 12480\sqrt{87481})q^5 + O(q^6),
\]

\[
\dim S_{20}^{\text{new},-}(\Gamma_0^{(1)}(3)) = 1 \text{ and } S_{20}^{\text{new},-}(\Gamma_0^{(1)}(3)) = \mathbb{C} f_4, \text{ where} \\
f_4 = q - (351 + 3\sqrt{87481})q^2 - 19683q^3 + (386242 - 2106\sqrt{87481})q^4 \\
- (3008070 - 12480\sqrt{87481})q^5 + O(q^6).
\]
\[ f_4 = q + (351 - 3\sqrt{78481})q^2 - 19683q^3 + (386242 - 2106\sqrt{78481})q^4 + (3008070 - 12480\sqrt{78481})q^5 + O(q^6). \]

\[ \dim S^{new,-}_{20}(\Gamma_0^{(1)}(3)) = 1 \text{ and } S^{new,-}_{20}(\Gamma_0^{(1)}(3)) = \mathbb{C}f_5, \text{ where} \]

\[ f_5 = q - 1104q^2 + 19683q^3 + 694528q^4 + 3516270q^5 + O(q^6). \]

\[ \dim S^{new,+}_{20}(\Gamma_0^{(1)}(6)) = 2 \text{ and } S^{new,+}_{20}(\Gamma_0^{(1)}(6)) = \mathbb{C}f_6 \oplus \mathbb{C}f_7, \text{ where} \]

\[ f_6 = q - 512q^2 - 19683q^3 + 262144q^4 - 3732474q^5 + O(q^6), \]

\[ f_7 = q + 512q^2 + 19683q^3 + 262144q^4 + 1953390q^5 + O(q^6). \]

\[ \dim S^{new,-}_{20}(\Gamma_0^{(1)}(6)) = 1 \text{ and } S^{new,-}_{20}(\Gamma_0^{(1)}(6)) = \mathbb{C}f_8, \text{ where} \]

\[ f_8 = q - 512q^2 + 19683q^3 + 262144q^4 - 5849490q^5 + O(q^6). \]

Hence we find \( H_5(t, \phi_1) = (t-5^9)(t-5^{10})h_5(t, f_1), H_5(t, \phi_2) = (t-5^9)(t-5^{10})h_5(t, f_8), \)
\( H_5(t, \phi_3) = (t-5^9)(t-5^{10})h_5(t, f_3) \) and \( H_5(t, \phi_6) = (t-5^9)(t-5^{10})h_5(t, f_4) \). All absolute values of zeros of \( H_5(t, \phi_3) \) and \( H_5(t, \phi_4) \) are \( 5^{10}/2 \).

6. Preliminaries to proving Main Theorem 1.1

We describe a brief outline of the method for proving Main Theorem 1.1 (Theorem 3.1). The details of the proof are described in Section 7 and 8. We use the Selberg trace formula. Roughly speaking, we have to calculate the sum of contributions of all \( K(N) \)-conjugacy classes of \( K(N) \). The method for evaluating each contribution depends on the types of conjugacy classes. We divide \( \Gamma = K(N) \) into disjoint union of three subsets \( \Gamma^{(c)}, \Gamma^{(p)} \) and \( \Gamma^{(h)} \) as follows:

(i) \( \Gamma^{(c)} \) consists of torsion elements of \( \Gamma \).

(ii) \( \Gamma^{(p)} \) consists of non-semi-simple elements of \( \Gamma \) whose semi-simple factors belong to \( \Gamma^{(c)} \).

(iii) \( \Gamma^{(h)} \) consists of the other elements of \( \Gamma \) than the above two types.

We denote the contribution of each set by \( I(\Gamma^{(c)})_{k,j}, I(\Gamma^{(p)})_{k,j} \) and \( I(\Gamma^{(h)})_{k,j} \). Then it is known that \( I(\Gamma^{(h)})_{k,j} = 0 \) and

\[ \dim S_{k,j}(\Gamma) = I(\Gamma^{(c)})_{k,j} + I(\Gamma^{(p)})_{k,j}. \]

The methods for evaluating \( I(\Gamma^{(c)})_{k,j} \) and \( I(\Gamma^{(p)})_{k,j} \) are quite different. We evaluate \( I(\Gamma^{(c)})_{k,j} \) in Section 7 and \( I(\Gamma^{(p)})_{k,j} \) in Section 8.

6.1. Semi-simple elements

We explain briefly how to evaluate \( I(\Gamma^{(e)})_{k,j} \) explicitly. This method was developed in [Has80], [HI80], [HI82], [HI83], [Has83], [Has84], [Ibu85].
We define an element $\alpha(\mu, \nu)$ of $Sp(2; \mathbb{R})$ by

$$\alpha(\mu, \nu) = \begin{bmatrix} \cos \mu & 0 & \sin \mu & 0 \\ 0 & \cos \nu & 0 & \sin \nu \\ -\sin \mu & 0 & \cos \mu & 0 \\ 0 & -\sin \nu & 0 & \cos \nu \end{bmatrix}.$$ 

We define a function $J_0(\alpha(\mu, \nu))$ and a subgroup $C_0(\alpha(\mu, \nu); Sp(2; \mathbb{R}))$ of $Sp(2; \mathbb{R})$ as in [Wak12, (a), (b-1)-(b-5) on pages 206, 207].

Let $\gamma$ be an element of $\Gamma^{(c)}$. Then there exists $g \in Sp(2; \mathbb{R})$ such that $g^{-1} \gamma g = \alpha(\mu, \nu)$ for some $\mu$ and $\nu$. We define

$$J_0(\gamma) = J_0(\alpha(\mu, \nu)) \text{ and } C_0(\gamma; Sp(2; \mathbb{R})) = gC_0(\alpha(\mu, \nu); Sp(2; \mathbb{R}))g^{-1},$$

and

$$J'_0(\gamma) = \begin{cases} J_0(\gamma) & \text{if } -14 \not\mid C_0(\gamma; Sp(2; \mathbb{R})), \\ \frac{1}{2}J_0(\gamma) & \text{if } -14 \mid C_0(\gamma; Sp(2; \mathbb{R})). \end{cases}$$

We put $G := GSp(2; \mathbb{Q})^+ = \left\{ g \in GL(4, \mathbb{Q}) \mid g \begin{bmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{bmatrix}, g = \lambda(g) \begin{bmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{bmatrix}, \lambda(g) > 0 \right\}$

and

$$R_p := M(4; \mathbb{Z}_p) \text{ if } p \nmid N \text{ and } R_p := \begin{bmatrix} \mathbb{Z}_p & p\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p & p^{-1}\mathbb{Z}_p \\ \mathbb{Z}_p & p\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & p\mathbb{Z}_p & p\mathbb{Z}_p & \mathbb{Z}_p \end{bmatrix} \text{ if } p \mid N.$$

We define $c_{k,j} = 2^{-6} \cdot \pi^{-3} \cdot (k-2)(j+k-1)(j+2k-3)$.

**Theorem 6.1.** The contribution $I(\Gamma^{(c)})_{k,j}$ can be obtained by

$$I(\Gamma^{(c)})_{k,j} = c_{k,j} \sum_{\{g\}_G} J'_0(g) \sum_{\Lambda} M_G(\Lambda) \prod_p c_p(g, R_p, \Lambda_p).$$

See [Has83, Theorem 2-4] and [Wak12, Theorem B.1] for details. In this section, we explain only rough procedure.

The first sum of Theorem 6.1 is taken over $G$-conjugacy classes $\{g\}_G$ of torsion elements of $G$ which satisfies $\{g\}_G \cap \Gamma \neq \emptyset$. Each $H_i(k, j; N)$ in Theorem 3.1 means the partial sum taken over all $G$-conjugacy classes whose characteristic polynomials are of the form $f_i(\pm x)$.
\[ f_2(x) = (x^2 + x + 1)^2, \quad f_7(-x) = (x^2 - x + 1)^2, \]
\[ f_3(x) = (x^2 + 1)(x^2 + x + 1), \quad f_8(-x) = (x^2 + 1)(x^2 - x + 1), \]
\[ f_5(x) = (x^2 + x + 1)(x^2 - x + 1), \quad f_9(-x) = x^4 - x^3 + x^2 - x + 1, \]
\[ f_{10}(x) = x^4 + x^3 + x^2 + x + 1, \quad f_{10}(-x) = x^4 - x^3 + x^2 - x + 1, \]
\[ f_{11}(x) = x^4 + 1, \quad f_{12}(x) = x^4 - x^2 + 1. \]

The second sum in Theorem 6.1 for each \( \{g\}_G \) is taken over \( G \)-genus of \( \mathbb{Z} \)-orders in \( \mathbb{Z}(g) \), the commutator algebra of \( g \) in \( \mathbb{M}(4; \mathbb{Q}) \). The third product runs over all prime numbers. We give \( M_G(\Lambda) \) and \( c_p(g, R_p, \Lambda_p) \) explicitly separately for each characteristic polynomial in Section 7.

The advantage of Theorem 6.1 is that all necessary data are obtained by local calculations. All local data \( c_p(g, R_p, \Lambda_p) \) have already been obtained in [HI80] if \( p \nmid N \) and in [Ibu85] if \( p \mid N \). So what we have to do is to put the local data together carefully by considering a gap between global and local conjugacy, which is summarized as follows. For each \( f(x) = f_i(x) \), let \( G[f] \) be the set of torsion elements of \( G \) whose characteristic polynomials are \( f(\pm x) \), and consider the decomposition of \( G[f] \) by \( G \)-conjugation. Let \( G_A \) be the adelization of \( G \), and similarly consider the decomposition of \( G_A[f] \) by \( G_A \)-conjugation. It was proved by Asai [Asa76] that the following natural map \( \phi \) is injective.

\[ \phi : G[f]/ \sim_G \rightarrow G_A[f]/ \sim_{G_A}. \]

However, it is not necessarily surjective.

**Theorem 6.2** (cf. [HI80], [Has83]). (i) If \( f(x) = f_1(\pm x), f_2(x), f_3(\pm x), f_4(\pm x), f_5(\pm x), f_8(\pm x) \) or \( f_{10}(\pm x) \), then the map \( \phi \) is surjective. Thus we have only to put the local data together in all the combinations of local conjugacy.

(ii) If \( f(x) = f_6(x) \) or \( f_7(\pm x) \), then the map \( \phi \) is not surjective. We put \( F = \mathbb{Q}(\sqrt{a^2 - 4b}) \) for \( f(x) = (x^2 + ax + b)^2 \). Then there is a correspondence between \( \{g\}_G \in G[f]/ \sim_G \) and the set of isomorphic classes of quaternion algebras \( Z_0(g) \) over \( \mathbb{Q} \) which is contained in \( Z(g) \). This correspondence is one-to-one or two-to-one according as \( Z_0(g) \) is indefinite or definite. If \( Z_0(g) \) is indefinite, then \( g \) is \( \text{Sp}(2; \mathbb{R}) \)-conjugate to \( (\alpha, \mu, -\mu) \). If \( Z_0(g) \) is definite, then \( g \) is \( \text{Sp}(2; \mathbb{R}) \)-conjugate to \( (\alpha, \mu, \mu) \) and \( \{g\}_G \) and \( \{g^{-1}\}_G \) correspond to \( Z_0(g) \). In addition, a \( G \)-conjugacy class \( \{g\}_G \) appears in the first sum of Theorem 6.1 if and only if \( D(Z_0(g)) \) \( | d_F \), where \( D(Z_0(g)) \) and \( d_F \) are the discriminants of \( Z_0(g) \) and \( F \).

(iii) If \( f(x) = f_9(x), f_{11}(x) \) or \( f_{12}(x), \) then the map \( \phi \) is not surjective.

**Remark.** In the case (iii) above, for each \( \{g\}_G \in G[g]/ \sim_G \), the characteristic polynomial of \( g^2 \) is \( f_6(x) \) or \( f_7(\pm x) \). Hence we can sometimes use \( Z_0(g^2) \) in (ii) for the case (iii) to judge if some \( G_A \)-conjugacy class is in the image of \( \phi \).

### 6.2. Non-semi-simple elements

We explain briefly how to evaluate \( I(T^{(p)})_{k,j} \). Main tools were contained in [Has83] and [Wak12], and for prime levels the explicit results are known in [Ibu85] and [Ibu07a].
Let $\gamma$ be an element of $\Gamma^{(p)}$. Then there exists $g \in \text{Sp}(2; \mathbb{R})$ such that $g^{-1}\gamma g$ is one of the following types.

**Unipotent**: $f(x) = f_1(\pm x)$

\[
(1) \pm \begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}, \quad (2) \pm \begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}, \quad (3) \pm \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix},
\]

**\(\delta\)-parabolic**: $f(x) = f_2(x)$

\[
(4) \begin{bmatrix}
1 & 0 & s_1 & 0 \\
0 & -1 & 0 & s_2 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix}; \quad (s_1, s_2) = (\pm 1, \pm 1), \quad (5) \begin{bmatrix}
1 & 0 & s & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix}; \quad s = \pm 1.
\]

**Elliptic/parabolic**: $f(x) = f_3(\pm x), f_4(\pm x), f_5(\pm x)$

\[
(6) \pm \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}; \quad \lambda = \pm 1, \quad (7) \pm \begin{bmatrix}
-1/2 & 0 & \sqrt{3}/2 & 0 \\
0 & 1 & 0 & \lambda \\
-\sqrt{3}/2 & 0 & -1/2 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}; \quad \lambda = \pm 1,
\]

\[
(8) \pm \begin{bmatrix}
1/2 & 0 & \sqrt{3}/2 & 0 \\
0 & 1 & 0 & \lambda \\
-\sqrt{3}/2 & 0 & 1/2 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}; \quad \lambda = \pm 1,
\]

**Paraelliptic**: $f(x) = f_6(x), f_7(\pm x)$

\[
(9) \begin{bmatrix}
0 & 1 & 0 & s \\
-1 & 0 & s & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1
\end{bmatrix}; \quad s = \pm 1, \quad (10) \pm \begin{bmatrix}
1/2 & \sqrt{3}/2 & s/2 & \sqrt{3}s/2 \\
-\sqrt{3}/2 & 1/2 & -\sqrt{3}s/2 & s/2 \\
0 & 0 & 1/2 & \sqrt{3}/2 \\
0 & 0 & -\sqrt{3}/2 & 1/2
\end{bmatrix}; \quad s = \pm 1.
\]

We denote by $I_i(k, j, N) (i = 1, 2, \ldots, 10)$ the contribution of all elements of type (1)–(10) respectively.

Let $C(\gamma; \text{Sp}(2; \mathbb{R}))$ be the centralizer of $\gamma$ in $\text{Sp}(2; \mathbb{R})$. We define a function $J_0(\gamma; s)$ and a subgroup $C_0(\gamma; \text{Sp}(2; \mathbb{R}))$ of $C(\gamma; \text{Sp}(2; \mathbb{R}))$ as in [Wak12, (e) and (f) on page 207–211].

**Definition 6.3.** For each $\gamma \in \Gamma$, we set

$[\gamma]_\Gamma = \{ \gamma' \in \Gamma \mid \gamma_s = \gamma'_s, \ C_0(\gamma'; \text{Sp}(2; \mathbb{R})) = C_0(\gamma; \text{Sp}(2; \mathbb{R})), \ C(\gamma'; \text{Sp}(2; \mathbb{R})) \simeq C(\gamma; \text{Sp}(2; \mathbb{R})) \}$,

where $\gamma_s$ (resp. $\gamma'_s$) is the semisimple factor of the Jordan decomposition of $\gamma$ (resp. $\gamma'$).

We call the set $[\gamma]_\Gamma$ the family represented by $\gamma$. We use the notation $C_0(F; \text{Sp}(2; \mathbb{R}))$ for a family $F$. We define $C_0(F; \Gamma) = C_0(F; \text{Sp}(2; \mathbb{R})) \cap \Gamma$.

For a subgroup $C$ of $\text{Sp}(2; \mathbb{R})$, we define $\mathcal{C} = \{ \pm 1_4 \cdot C \}/\{ \pm 1_4 \}$. We define $c_{k,j} =$
$$2^{-6} \cdot \pi^{-3} \cdot (k - 2)(j + k - 1)(j + 2k - 3).$$

**Theorem 6.4.** The contribution $I(\Gamma(p))_{k,j}$ can be obtained by

$$I(\Gamma(p))_{k,j} = \frac{c_{k,j}}{2} \sum_F \text{vol}(C_0(F;\Gamma) \setminus C_0(F; Sp(2;\mathbb{R}))) \cdot \lim_{s \to +0} \sum_{\gamma \in F/\sim} \frac{J_0(\gamma; s)}{[C(\Gamma; F) : C_0(F;\Gamma)]}$$

$$+ \frac{c_{k,j}}{2} \sum_F \frac{\text{vol}(C_0(F;\Gamma) \setminus C_0(F; Sp(2;\mathbb{R})))}{[C(\Gamma; F) : C_0(F;\Gamma)]} \cdot \lim_{s \to +0} \sum_{\gamma \in F} J_0(\gamma; s).$$

See [Has83] and [Wak12] for details. The first sum in Theorem 6.4 is taken over a complete system of representatives of $\Gamma$-conjugacy classes of families of $Sp(2;\mathbb{R})$-conjugacy types (1), (2) and (3) above, and the notation $F/\sim$ means a complete system of representatives of $\Gamma$-conjugacy classes of elements of each family $F$. The second sum is taken over a complete system of representatives of $\Gamma$-conjugacy classes of families of $Sp(2;\mathbb{R})$-conjugacy types (4)–(10) above. Here $\gamma \in F$ is taken over all elements of each family $F$. Explicit formulas for the limits in each sum are given in [Wak12, page 207–212].

We calculate the first and the second sums of Theorem 6.4 separately for each $Sp(2;\mathbb{R})$-conjugacy types (1)–(10) in Section 8. In this section, we summarize necessary lemmas which will be used repeatedly.

In order to use Theorem 6.4, we should classify $\Gamma$-conjugacy classes of families which are extended over in the first and the second sum. It requires elaborate works, but the following proposition on the classification of equivalence classes of cusps gives a lot of help to us. The one-dimensional cusps of the Satake compactification of $\mathbb{H}^2$ correspond bijectively with $\mathbb{H}(Q_{2,1})$, and zero-dimensional cusps with $\mathbb{H}(Q_{2,0})$.

**Proposition 6.5 (Poor and Yuen [PY13]).** Let $N$ be a squarefree positive integer. Then we have

$$Sp(2;\mathbb{Q}) = K(N)P_{2,0}(\mathbb{Q})$$

$$= \bigcup_{m|N, m > 0} K(N) \begin{bmatrix} 1 & m & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -m \end{bmatrix} P_{2,1}(\mathbb{Q}),$$

**Corollary 6.6.** Let $g$ be an element of $\Gamma(p)$. Then $g$ is $K(N)$-conjugate to an
element in $P_{2,0}(\mathbb{Q})$ or $x_mP_{2,1}(\mathbb{Q})x_m^{-1}$ for some $x_m = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -m \end{bmatrix}$.

Let $G$ be a group and $H$ a subgroup of $G$. There are many cases where we should consider the splitting of a $G$-conjugacy class into $H$-conjugacy classes, that is, classify $H$-conjugacy classes of $\{\gamma\}_G \cap H$ for each $G$-conjugacy class $\{\gamma\}_G$. Let $C(\gamma; G)$ be the centralizer of $\gamma$ in $G$ and $M(\gamma; G/H) = \{x \in G; x^{-1}\gamma x \in H\}$.

**Proposition 6.7.** We have a bijective map

$$
(\{\gamma\}_G \cap H)/\sim_H \overset{\sim}{\longrightarrow} C(\gamma; G) \backslash M(\gamma; G/H)/H,
$$

which sends the $H$-conjugacy class $\{x^{-1}\gamma x\}_H$ to the double coset $C(\gamma; G)xH$.

**7. Proof of Main Theorem 1.1: semi-simple elements**

In this section, we give a proof to obtain the contribution $I(\Gamma^{(c)})_{k,j}$ in detail. Notations are the same as in Section 6.

**7.1. The contribution $H_1(k, j, N)$**

We consider the contribution $H_1(k, j, N)$ of $\pm 1_4$, namely the partial sum of Theorem 6.1 for $\{1_4\}_G$ and $\{-1_4\}_G$. As is well-known, it corresponds to the volume of the fundamental domain. If we put

$$
B(N) = \begin{bmatrix} Z & NZ & Z & Z \\ Z & Z & Z & Z \\ NZ & NZ & Z & Z \\ NZ & NZ & NZ & Z \end{bmatrix} \cap \text{Sp}(2; \mathbb{Q}),
$$

then we have

$$
[\text{Sp}(2; \mathbb{Z}) : B(N)] = \prod_{p \mid N} (p^2 + 1)(p + 1)^2 \quad \text{and} \quad [K(N) : B(N)] = \prod_{p \mid N} (p + 1)^2.
$$

This can be proved, for example, by Bruhat-Tits theory. Now we have $H_1(k, j, N) = H_1^{\text{Sp}(2; \mathbb{Z})}(k, j) \times \prod_{p \mid N} (p^2 + 1)$, where $H_1^{\text{Sp}(2; \mathbb{Z})}(k, j) = 2^{-7}3^{-3}5^{-1}(j + 1)(k - 2)(j + k - 1)(j + 2k - 3)$. (cf. [Wak12, Theorem 7.1])

**7.2. The contribution $H_2(k, j, N)$**

We consider the contribution of the elliptic elements whose characteristic polynomial is $f_2(x) = (x - 1)^2(x + 1)^2$. The set of such elements of $G = \text{GSp}(2; \mathbb{Q})^+$ consists of a single $G$-conjugacy class $\{g\}_G$, and the contribution $H_2(k, j, N)$ is given by

$$
H_2(k, j, N) = c_{k,j}J'_0(g) \sum_{\Lambda} M_G(\Lambda) \prod_p c_p(g, R_p, \Lambda_p).
$$
We have
\[
C_0(g; Sp(2; \mathbb{R})) = \left\{ \begin{bmatrix} a & b \\ 0 & a' \end{bmatrix} \begin{bmatrix} a' b' \\ c d \end{bmatrix} : \begin{bmatrix} a b \\ c d \end{bmatrix}, \begin{bmatrix} a' b' \\ c' d' \end{bmatrix} \in SL(2; \mathbb{R}) \right\} \ni -14
\]
and
\[
J_0(g) = \frac{1}{2} J_0(g) = c_k j^{-1} 2^{-7} \pi^{-4} (-1)^k (j + k - 1)(k - 2)
\]
by [Wak12, (b-5)]. We can use the local data \(c_p(g, R_p, \Lambda_p)\) from [HI80, Proposition 13] for prime numbers \(p \mid N\). On the other hand, the local data for \(p \nmid N\) need to be calculated newly since in [Ibu85] the global conjugacy classes are given directly and local data have not been used.

**Proposition 7.1.** For a prime number \(p\) with \(p \mid N\), we have
\[
c_p(g, R_p, \Lambda_p) = \begin{cases} 2 & \text{if } \Lambda_p \sim Z(g)_p \cap R_p, \\ 1 & \text{if } \Lambda_p \sim Z(g)_p \cap xR_p x^{-1} \text{ and } p = 2, \\ 0 & \text{otherwise}, \end{cases}
\]
where \(x = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\), and where \(\sim\) means conjugation by \(Z(g)_p^\times \cap G_p\).

**Proof.** We define \(M_p(g, R_p) = \{x \in G_p \mid x^{-1}gx \in R_p\}\). By direct calculations, a complete system of representatives of \((Z(g)_p^\times \cap G_p)\) is given by \\{\(x_1, x_2\)\} if \(p \neq 2\), and \\{\(x_1, x_2, x_3\)\} if \(p = 2\), where
\[
x_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.
\]
We can check \(Z(g)_p \cap x_1R_px_1^{-1} \sim Z(g)_p \cap x_2R_px_2^{-1}\) and \(Z(g)_p \cap x_1R_px_1^{-1} \not\sim Z(g)_p \cap x_3R_px_3^{-1}\) if \(p = 2\).

Putting all local data together, we have \(\prod_p c_p(g, R_p, \Lambda_p)\) and \(M_G(\Lambda)\) as follows:
\[
\prod_p c_p(g, R_p, \Lambda_p) = \begin{cases} 2^{-\langle N \rangle} & \text{if } \Lambda \sim L_1, \\ 2^{-\langle N \rangle} & \text{if } \Lambda \sim L_2 \text{ and } 2 \nmid N, \\ 2^{-\langle N \rangle - 1} & \text{if } \Lambda \sim L_3 \text{ and } 2 \mid N, \\ 0 & \text{otherwise}, \end{cases}
\]
where \(L_1, L_2\) and \(L_3\) are certain \(\mathbb{Z}\)-orders in \(Z(g)\) such that \(M_G(L_1) = \pi^4/9\), \(M_G(L_2) = 2\pi^4/3\) and \(M_G(L_3) = \pi^4\). Here \(\Lambda \sim L\) means that two \(\mathbb{Z}\)-orders \(\Lambda\) and \(L\) belong to the
same $G$-genus, that is, they are locally conjugate by $Z(g)^g \cap G_p$ for all $p$. Hence the sum of (8) runs over these three $\mathbb{Z}$-orders, and we obtain $H_2(k, j, N)$.

### 7.3. The contribution $H_3(k, j, N)$, $H_4(k, j, N)$ and $H_5(k, j, N)$

We consider the contributions of the elliptic elements with characteristic polynomials

\[
\begin{align*}
  f_3(x) &= (x - 1)^2(x^2 + 1), \\
  f_4(x) &= (x - 1)^2(2x^2 + x + 1), \\
  f_5(x) &= (x - 1)^2(2x^2 - x + 1).
\end{align*}
\]

For each $i$, the set of such elements of $G = GSp(2; \mathbb{Q})^+$ with $f_i(\pm x)$ consists of four $G$-conjugacy classes represented by $g$, $g^{-1}$, $-g$ and $-g^{-1}$ for some $g$, and the contribution $H_i(k, j, N)$ is given by

\[ H_i(k, j, N) = 2 \cdot c_{k, j} \cdot (J_0(g) + J_0(g^{-1})) \sum_p M_G(\Lambda) \prod_p c_p(g, R_p, \Lambda_p). \tag{9} \]

For each $i$, $g$ is $Sp(2; \mathbb{R})$-conjugate to $\alpha(\theta, 0)$ where $\theta = \pi/2$, $2\pi/3$, $\pi/3$ respectively. We have

\[
C_0(g; Sp(2; \mathbb{R})) = \begin{cases} 
  \left[ \begin{array}{cc}
  1 & 0 \\
  0 & 0 \\
  a & b \\
  0 & c \\
  0 & d 
  \end{array} \right] & \in SL(2; \mathbb{R}) \\
  \text{if} \ g = \begin{pmatrix} 0 & 1 \\
  1 & 0 \\
  a & 0 \\
  0 & 1 \\
  0 & 0 
  \end{pmatrix} \\
  \not\in \mathfrak{sl}(2, \mathbb{R}) 
\end{cases}
\]

and

\[
J_0(g) + J_0(g^{-1}) = J_0(g) + J_0(g)^{-1} = \frac{(j + k - 1)\sin(k - 2)\theta - (k - 2)\sin(j + k - 1)\theta}{2^4\pi^2\sin\theta(1 - \cos\theta)}
\]

by [Wak12, (b-4)]. We quote necessary local data from [HI80, Proposition 14] and [Ibu85, Proposition 2.3] below.

\[
\begin{align*}
  H_3(k, j, N) : \prod_p c_p(g, R_p, \Lambda_p) &= \begin{cases} 
    2^{c(N)} & \text{if } \Lambda \sim L_1, \\
    2^{c(N)-1} & \text{if } \Lambda \sim L_2 \text{ and } 2 \mid N, \\
    0 & \text{otherwise},
  \end{cases} \\
  H_4(k, j, N) : \prod_p c_p(g, R_p, \Lambda_p) &= \begin{cases} 
    2^{c(N)} & \text{if } \Lambda \sim L_3, \\
    2^{c(N)-1} & \text{if } \Lambda \sim L_4 \text{ and } 3 \mid N, \\
    0 & \text{otherwise},
  \end{cases} \\
  H_5(k, j, N) : \prod_p c_p(g, R_p, \Lambda_p) &= \begin{cases} 
    2^{c(N)} & \text{if } \Lambda \sim L_5, \\
    0 & \text{otherwise},
  \end{cases}
\end{align*}
\]

where $L_1$, $L_2$, $L_3$, $L_4$ and $L_5$ are certain $\mathbb{Z}$-orders in $Z(g)$ such that $M_G(L_1) = \pi^2/12$, $M_G(L_2) = \pi^2/4$, $M_G(L_3) = \pi^2/18$, $M_G(L_4) = \pi^2/3$ and $M_G(L_5) = \pi^2/18$. Hence we obtain $H_3(k, j, N)$, $H_4(k, j, N)$ and $H_5(k, j, N)$ by (9).
7.4. The contribution $H_6(k, j, N)$ and $H_7(k, j, N)$

We consider the contributions of elliptic elements whose characteristic polynomials are $f_6(x) = (x^2 + 1)^2$ and $f_7(x) = (x^2 + x + 1)^2$. The set of such elements of $G = \text{GSp}(2; \mathbb{Q})^+$ consists of infinitely many $G$-conjugacy classes, but only finite numbers of $G$-conjugacy classes appear in the first sum of Theorem 6.1.

First we consider $H_6(k, j, N)$. In this case, three $G$-conjugacy classes $\{g_1\}_G$, $\{g_2\}_G$ and $\{g_2^{-1}\}_G$ appear, where

$$g_1 = \alpha \left( \frac{\pi}{2}, -\frac{\pi}{2} \right) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad g_2 = \alpha \left( \frac{\pi}{2}, \frac{\pi}{2} \right) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and the contribution $H_6(k, j, N)$ is given by

$$H_6(k, j, N) = c_{k,j} \cdot J_0(g_1) \sum_{\Lambda} M_G(\Lambda) \prod_p c_p(g_1, R_p, \Lambda_p) + c_{k,j} \cdot \left( J_0'(g_2) + J_0'(g_2^{-1}) \right) \sum_{\Lambda} M_G(\Lambda) \prod_p c_p(g_2, R_p, \Lambda_p).$$

We have

$$C_0(g_1; \text{Sp}(2; \mathbb{R})) = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}, a \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \in \text{SL}(2; \mathbb{R}) \right\} \ni -1_4$$

and

$$C_0(g_2; \text{Sp}(2; \mathbb{R})) = \{1_4\} \ni -1_4$$

and

$$J_0'(g_1) = 2^{-1} J_0(g_1) = c_{k,j}^{-1} 2^{-5} \pi^{-2} (j + 2k - 3) (-1)^{j/2}$$

$$J_0'(g_2) + J_0'(g_2^{-1}) = J_0(g_2) + J_0(g_2^{-1}) = c_{k,j}^{-1} 2^{-2} (j + 1)(-1)^{\frac{j+1}{2}}$$

by [Wak12, (b-3),(b-2)]. The remaining parts of (10) depend on the conditions of $N$. We collect necessary local data as follows from [HI80, Proposition 15,16] and [Ibu85, Proposition 2.4, 2.5].

(i) If $2 \mid N$ and $\# \{ p \mid N; \left( \frac{-1}{p} \right) = -1 \} = \text{even}$, we have

$$\prod_p c_p(g_1, R_p, \Lambda_p) = \begin{cases} 1 & \text{if } \Lambda \sim L_1, \\ 1 & \text{if } \Lambda \sim L_2, \\ 0 & \text{otherwise}, \end{cases} \quad \prod_p c_p(g_2, R_p, \Lambda_p) = \begin{cases} 1 & \text{if } \Lambda \sim L_3, \\ 1 & \text{if } \Lambda \sim L_4, \\ 0 & \text{otherwise}, \end{cases}$$

and $M_G(L_i) = a_i \prod_{p \mid N} \left( 1 + \left( \frac{-1}{p} \right) \right)$, where $a_i = \pi^2/8$ if $i = 1$, $\pi^2/4$ if $i = 2$, $1/32$ if $i = 3$, and $1/192$ if $i = 4$. 

(ii) If \( 2 \mid N \) and \( 2 \nmid \{ p \mid N ; \left( \frac{-1}{p} \right) = -1 \} \) = odd, we have

\[
\prod_p c_p(g_1, R_p, \Lambda_p) = \begin{cases} 
1 & \text{if } \Lambda \sim L_1, \\
1 & \text{if } \Lambda \sim L_2, \\
0 & \text{otherwise,}
\end{cases}
\prod_p c_p(g_2, R_p, \Lambda_p) = \begin{cases} 
1 & \text{if } \Lambda \sim L_3, \\
1 & \text{if } \Lambda \sim L_4, \\
0 & \text{otherwise,}
\end{cases}
\]

and \( M_G(L_i) = a_i \prod_{p \mid N} (p + \left( \frac{-1}{p} \right)) \) where \( a_i = \pi^2/4 \) if \( i = 1 \), \( \pi^2/24 \) if \( i = 2 \), \( 1/64 \) if \( i = 3 \), and \( 1/32 \) if \( i = 4 \).

(iii) If \( 2 \nmid N \) and \( 2 \nmid \{ p \mid N ; \left( \frac{-1}{p} \right) = -1 \} \) = even, we have

\[
\prod_p c_p(g_1, R_p, \Lambda_p) = \begin{cases} 
1 & \text{if } \Lambda \sim L_1, \\
1 & \text{if } \Lambda \sim L_2, \\
0 & \text{otherwise,}
\end{cases}
\prod_p c_p(g_2, R_p, \Lambda_p) = \begin{cases} 
1 & \text{if } \Lambda \sim L_3, \\
0 & \text{otherwise,}
\end{cases}
\]

and \( M_G(L_i) = a_i \prod_{p \mid N} (p + \left( \frac{-1}{p} \right)) \) where \( a_i = \pi^2/4 \) if \( i = 1 \), \( \pi^2/6 \) if \( i = 2 \), and \( 1/32 \) if \( i = 3 \).

(iv) If \( 2 \nmid N \) and \( 2 \mid \{ p \mid N ; \left( \frac{-1}{p} \right) = -1 \} \) = odd, we have

\[
\prod_p c_p(g_1, R_p, \Lambda_p) = \begin{cases} 
1 & \text{if } \Lambda \sim L_1, \\
0 & \text{otherwise,}
\end{cases}
\prod_p c_p(g_2, R_p, \Lambda_p) = \begin{cases} 
1 & \text{if } \Lambda \sim L_2, \\
1 & \text{if } \Lambda \sim L_3, \\
0 & \text{otherwise,}
\end{cases}
\]

and \( M_G(L_i) = a_i \prod_{p \mid N} (p + \left( \frac{-1}{p} \right)) \) where \( a_i = \pi^2/4 \) if \( i = 1 \), \( 1/32 \) if \( i = 2 \), and \( 1/48 \) if \( i = 3 \).

Summing up all the cases, we obtain \( H_6(k, j, N) \) by (10) as follows:

\[
H_6(k, j, N) = (\alpha \cdot (j + 2k - 3)(-1)^{j/2} + \beta \cdot (j + 1)(-1)^{k+j/2}) \cdot \prod_{p \mid N} (p + \left( \frac{-1}{p} \right)),
\]

where

\[
\alpha = 2^{-8} \cdot 3, \quad \beta = 2^{-8} \cdot 3^{-1} \cdot 7, \quad \text{if } 2 \mid N \text{ and } 2 \nmid \{ p \mid N ; \left( \frac{-1}{p} \right) = -1 \} \text{ is even,}
\]

\[
\alpha = 2^{-8} \cdot 3^{-1} \cdot 7, \quad \beta = 2^{-8} \cdot 3, \quad \text{if } 2 \mid N \text{ and } 2 \nmid \{ p \mid N ; \left( \frac{-1}{p} \right) = -1 \} \text{ is odd,}
\]

\[
\alpha = 2^{-7} \cdot 3^{-1} \cdot 5, \quad \beta = 2^{-7}, \quad \text{if } 2 \nmid N \text{ and } 2 \nmid \{ p \mid N ; \left( \frac{-1}{p} \right) = -1 \} \text{ is even,}
\]

\[
\alpha = 2^{-7}, \quad \beta = 2^{-7} \cdot 3^{-1} \cdot 5, \quad \text{if } 2 \nmid N \text{ and } 2 \nmid \{ p \mid N ; \left( \frac{-1}{p} \right) = -1 \} \text{ is odd.}
\]

By careful calculation, we can rewrite \( H_6(k, j, N) \) as the simple formula in Theorem 3.1.

Next we consider \( H_7(k, j, N) \). This is calculated in the same way as \( H_6(k, j, N) \). In this case, six \( G \)-conjugacy classes \( \{ g_1 \}_G, \{ g_2 \}_G, \{ g_2^{-1} \}_G, \{ -g_1 \}_G, \{ -g_2 \}_G, \{ -g_2^{-1} \}_G \) appear in the first sum of Theorem 6.1, where \( g_1 \) and \( g_2 \) are some elements of \( G \) such that \( g_1 \sim \alpha(2\pi/3, -2\pi/3), g_2 \sim \alpha(2\pi/3, 2\pi/3) \) by \( Sp(2; \mathbb{R}) \)-conjugation. The contribution is
given by

\[
H_7(k, j, N) = 2 \cdot c_{k,j} \cdot J'_0(g_1) \sum_{\Lambda} M_G(\Lambda) \prod_p c_p(g_1, R_p, \Lambda_p) \\
+ 2 \cdot c_{k,j} \cdot \left( J'_0(g_2) + J'_0(g_2^{-1}) \right) \sum_{\Lambda} M_G(\Lambda) \prod_p c_p(g_2, R_p, \Lambda_p).
\]

We have the same definitions \(C_0(g_1; Sp(2; \mathbb{R}))\) and \(C_0(g_2; Sp(2; \mathbb{R}))\) as in \(H_6(k, j, N)\), and

\[
J'_0(g_1) = 2^{-1} J_0(g_1) = c_{k,j}^{-1} 2^{-3} 3^{-1} \pi^{-2} (j + 2k - 3)[1, -1, 0; 3]_j,
\]

\[
J'_0(g_2) + J'_0(g_2^{-1}) = J_0(g_2) + J_0(g_2^{-1}) = c_{k,j}^{-1} 3^{-1} (j + 1)[0, 1, -1; 3]_{j+2k}.
\]

The local data are given as follows:

(i) If \(3 \nmid N\) and \(\sharp \{ p \mid N; \left( \frac{a}{p} \right) = -1 \} = \text{even} \), we have

\[
\prod_p c_p(g_1, R_p, \Lambda_p) = \begin{cases} 
1 & \text{if } \Lambda \sim L_1, \\
1 & \text{if } \Lambda \sim L_2, \\
0 & \text{otherwise},
\end{cases} 
\quad \prod_p c_p(g_2, R_p, \Lambda_p) = \begin{cases} 
1 & \text{if } \Lambda \sim L_3, \\
0 & \text{otherwise},
\end{cases}
\]

and \(M_G(L_i) = a_i \prod_{p \nmid N} (p + \left( \frac{-4}{p} \right))\), where \(a_i = \pi^2/27\) if \(i = 1\), \(4\pi^2/27\) if \(i = 2\), and \(1/54\) if \(i = 3\).

(ii) If \(3 \nmid N\) and \(\sharp \{ p \mid N; \left( \frac{a}{p} \right) = -1 \} = \text{odd} \), we have

\[
\prod_p c_p(g_1, R_p, \Lambda_p) = \begin{cases} 
1 & \text{if } \Lambda \sim L_1, \\
0 & \text{otherwise},
\end{cases} 
\quad \prod_p c_p(g_2, R_p, \Lambda_p) = \begin{cases} 
1 & \text{if } \Lambda \sim L_3, \\
0 & \text{otherwise},
\end{cases}
\]

and \(M_G(L_i) = a_i \prod_{p \nmid N} (p + \left( \frac{-4}{p} \right))\), where \(a_i = 4\pi^2/27\) if \(i = 1\), \(1/216\) if \(i = 2\), and \(1/54\) if \(i = 3\).

(iii) If \(3 \nmid N\) and \(\sharp \{ p \mid N; \left( \frac{a}{p} \right) = -1 \} = \text{even} \), we have

\[
\prod_p c_p(g_1, R_p, \Lambda_p) = \begin{cases} 
1 & \text{if } \Lambda \sim L_1, \\
0 & \text{otherwise},
\end{cases} 
\quad \prod_p c_p(g_2, R_p, \Lambda_p) = \begin{cases} 
1 & \text{if } \Lambda \sim L_2, \\
0 & \text{otherwise},
\end{cases}
\]

and \(M_G(L_i) = a_i \prod_{p \nmid N} (p + \left( \frac{-4}{p} \right))\), where \(a_i = 2\pi^2/9\) if \(i = 1\), and \(1/72\) if \(i = 2\).

(iv) If \(3 \nmid N\) and \(\sharp \{ p \mid N; \left( \frac{a}{p} \right) = -1 \} = \text{odd} \), we have

\[
\prod_p c_p(g_1, R_p, \Lambda_p) = \begin{cases} 
1 & \text{if } \Lambda \sim L_1, \\
0 & \text{otherwise},
\end{cases} 
\quad \prod_p c_p(g_2, R_p, \Lambda_p) = \begin{cases} 
1 & \text{if } \Lambda \sim L_2, \\
0 & \text{otherwise},
\end{cases}
\]

and \(M_G(L_i) = a_i \prod_{p \nmid N} (p + \left( \frac{-4}{p} \right))\), where \(a_i = \pi^2/9\) if \(i = 1\), and \(1/36\) if \(i = 2\).

We can obtain \(H_7(k, j, N)\) by (11) by the same way as \(H_6(k, j, N)\).
7.5. The contribution $H_S(k, j, N)$

We consider the contribution of the elliptic elements whose characteristic polynomials are $f_S(x) = (x^2 + 1)(x^2 + x + 1)$ or $f_S(-x) = (x^2 + 1)(x^2 - x + 1)$. The set of such elements of $G = GSp(2; \mathbb{Q})^+$ consists of eight $G$-conjugacy classes. Let $g_1$ be an element of $G$ such that $g_1$ is $Sp(2; \mathbb{R})$-conjugate to $\alpha(\pi/2, 2\pi/3)$, and $h$ an element of $Sp(2; \mathbb{R})$ such that $g_1 = h^{-1}\alpha(\pi/2, 2\pi/3)h$. Then the $G$-conjugacy classes are represented by $g_1 = h^{-1}\alpha(\pi/2, 2\pi/3)h, g_2 = h^{-1}\alpha(\pi/2, -2\pi/3)h, g_3 = h^{-1}\alpha(-\pi/2, 2\pi/3)h, g_4 = h^{-1}\alpha(-\pi/2, -2\pi/3)h$, and $-g_1, -g_2, -g_3, -g_4$. The contribution $H_S(k, j, N)$ is given by

\begin{equation}
H_S(k, j, N) = 2 \cdot c_{k, j} \cdot (J'_0(g_1) + J'_0(g_2) + J'_0(g_3) + J'_0(g_4)) \cdot \sum_{\Lambda \in \mathcal{C}(\Lambda)} M_G(\Lambda) \prod_p c_p(g, R_p, \Lambda_p)
\end{equation}

by Theorem 6.1. We have

$C_0(g_1; Sp(2; \mathbb{R})) = C_0(g_2; Sp(2; \mathbb{R})) = C_0(g_3; Sp(2; \mathbb{R})) = C_0(g_4; Sp(2; \mathbb{R})) = \{14\} \neq 14$

and

$J'_0(g_1) + J'_0(g_2) + J'_0(g_3) + J'_0(g_4) = J_0(g_1) + J_0(g_2) + J_0(g_3) + J_0(g_4) = c_{k, j}^{-1} \cdot C_8(k, j)$

by [Wak12, (b-1)], where $C_8(k, j)$ is given in subsection 2.4. The local data are given by

\[ \prod_p c_p(g, R_p, \Lambda_p) = \begin{cases} 2^{2N} & \text{if } \Lambda \sim L, \\ 0 & \text{otherwise}, \end{cases} \]

where $L$ is a certain $\mathbb{Z}$-order in $Z(g)$ such that $M_G(L) = 1/24$. Hence we obtain $H_S(k, j, N)$ by (12).

7.6. The contribution $H_S(k, j, N)$

We consider the contribution of the elliptic elements whose characteristic polynomials are $f_S(x) = (x^2 + x + 1)(x^2 - x + 1)$. The set of such elements of $G = GSp(2; \mathbb{Q})^+$ consists of infinitely many $G$-conjugacy classes, which are $Sp(2; \mathbb{R})$-conjugate to $g_1 = \alpha(\pi/3, 2\pi/3), g_1^{-1}, g_2 = \alpha(\pi/3, -2\pi/3)$ and $g_2^{-1}$, but exactly four $G$-conjugacy classes out of them appear in the first sum of Theorem 6.1 since they are the only $G$-conjugacy classes which have nonempty intersections with $K(N)$. Since they are $Sp(2; \mathbb{R})$-conjugate to $g_1, g_1^{-1}, g_2$ and $g_2^{-1}$, the contribution $H_S(k, j, N)$ is given by

\begin{equation}
H_S(k, j, N) = c_{k, j} \cdot \left\{ J'_0(g_1) + J'_0(g_1^{-1}) + J'_0(g_2) + J'_0(g_2^{-1}) \right\} \prod \Lambda M_G(\Lambda) \prod_p c_p(g, R_p, \Lambda_p).
\end{equation}

We have $C_0(g_1; Sp(2; \mathbb{R})) = C_0(g_2; Sp(2; \mathbb{R})) = \{14\}$ and

$J'_0(g_1) + J'_0(g_1^{-1}) + J'_0(g_2) + J'_0(g_2^{-1}) = J_0(g_1) + J_0(g_1^{-1}) + J_0(g_2) + J_0(g_2^{-1}) = c_{k, j}^{-1} \cdot C_9(k, j)$

dimension formula for Siegel paramodular cusp forms 49
by [Wak12, (b-1)] where $C_0(k, j)$ is given in subsection 2.4. We can see from [HI80, Proposition 18] and [Ibu85, Proposition 2.8] that

$$\prod_p c_p(g, R_p, \Lambda_p) = \begin{cases} 2^{s(N)} & \text{if } \Lambda \sim L_1, \\ 2^{s(N)} & \text{if } \Lambda \sim L_2 \text{ and } 2 \nmid N, \\ 0 & \text{otherwise}, \end{cases}$$

where $L_1$ and $L_2$ are certain $\mathbb{Z}$-orders in $Z(g)$ such that $M_G(L_1) = 1/36$ and $M_G(L_2) = 1/12$. Thus we can obtain $H_9(k, j, N)$ by (13).

7.7. The contribution $H_{10}(k, j, N)$

We consider the contribution of the elliptic elements whose characteristic polynomials are $f_{10}(x) = x^4 + x^3 + x^2 + x + 1$ or $f_{10}(-x) = x^4 - x^3 + x^2 - x + 1$. The set of such elements of $G = GSp(2; \mathbb{Q})^+$ consists of infinitely many $G$-conjugacy classes, which are $Sp(2; \mathbb{R})$-conjugate to $g = \alpha(2\pi/5, -4\pi/5)$, $g_2^2$, $g_3^3$ and $g_4^4$, but only four $G$-conjugacy classes out of them can appear in the first sum of Theorem 6.1 since they can have nonempty intersections with $K(N)$. Since they are $Sp(2; \mathbb{R})$-conjugate to $g$, $g_2^2$, $g_3^3$ and $g_4^4$, the contribution $H_{10}(k, j, N)$ is given by

$$H_{10}(k, j, N) = c_{k, j} \left\{ J_0'(g) + J_0'(g_2^2) + J_0'(g_3^3) + J_0'(g_4^4) \right\} \prod_{\Lambda} M_G(\Lambda) \prod_p c_p(g, R_p, \Lambda_p).$$

We have $C_0(g; Sp(2; \mathbb{R})) = \{14\}$ and

$$J_0'(g) + J_0'(g_2^2) + J_0'(g_3^3) + J_0'(g_4^4) = J_0(g) + J_0(g_2^2) + J_0(g_3^3) + J_0(g_4^4) = c_{k, j}^{-1} \cdot C_{10}(k, j)$$

by [Wak12, (b-1)] where $C_{10}(k, j)$ is given in subsection 2.4. We can see from [HI80, Proposition 18] that

$$\prod_p c_p(g, R_p, \Lambda_p) = \begin{cases} 2^{s(N) + 1/5} & \text{if } \Lambda \sim L \text{ and } \{p \mid N; p \equiv 2, 3 \text{ mod } 5\} = \emptyset, \\ 0 & \text{otherwise}, \end{cases}$$

where $L$ is a certain $\mathbb{Z}$-orders in $Z(g)$ such that $M_G(L) = 1/10$. Thus we can obtain $H_{10}(k, j, N)$ by (14).

7.8. The contribution $H_{11}(k, j, N)$

We consider the contribution of the elliptic elements whose characteristic polynomials are $f_{11}(x) = x^4 + 1$. The set of such elements of $G = GSp(2; \mathbb{Q})^+$ consists of infinitely many $G$-conjugacy classes, which are $Sp(2; \mathbb{R})$-conjugate to $\gamma_1 = \alpha(\pi/4, 3\pi/4)$, $\gamma_1^{-1}$, $\gamma_2 = \alpha(\pi/4, -3\pi/4)$ and $\gamma_2^{-1}$, but only four $G$-conjugacy classes out of them can appear in the first sum of Theorem 6.1 since they can have nonempty intersections with $K(N)$. They are $Sp(2; \mathbb{R})$-conjugate to $\gamma_1$, $\gamma_1^{-1}$, $\gamma_2$ and $\gamma_2^{-1}$. We need to take notice of Theorem 6.2 (iii) and the Remark after that. There is a single $G_p$-conjugacy class $\{g_p\}_{G_p}$ with $Z_0(g_p^2) =$split if $p \equiv 1, 3, 5 \text{ mod } 8$, and there are two $G_p$-conjugacy classes $\{g_p\}_{G_p}$ and $\{h_p\}_{G_p}$ with $Z_0(g_p^2) =$split and $Z_0(h_p^2) =$division if $p \equiv 2$ or $p \equiv 7 \text{ mod } 8$. For a given combination of local conjugacy classes $\{g_v\}_{G_v}$ for each place $v$, the necessary
and sufficient condition for existence of the corresponding $G$-conjugacy is

(i) the number of prime numbers such that $Z_0(g_p^2)$ is division finite, and

(ii) if the number of prime numbers such that $Z_0(g_p^2)$ is division even (resp. odd),

then $g_\infty$ is $Sp(2;\mathbb{R})$-conjugate to $g_1$ or $g_1^{-1}$ (resp. $g_2$ or $g_2^{-1}$).

As a result, for a fixed conjugacy class $g_1^{\pm 1}$ or $g_2^{\pm 1}$ at $\infty$, we can take any combination of local conjugacy classes except for $p=2$ and choose one of two $G_2$-conjugacy classes to make the above condition (ii) hold. For each $G_p$-conjugacy classes, $c_p(g_p, R_p, \mathcal{O}_p)$ are given as follows.

(i) if $p \equiv 1 \mod 8$,

$$c_p(g_p, R_p, \mathcal{O}_p) = \begin{cases} 2 & \text{if } p \mid N \\ 1 & \text{if } p \nmid N \end{cases}$$

(ii) if $p \equiv 3, 5 \mod 8$,

$$c_p(g_p, R_p, \mathcal{O}_p) = \begin{cases} 0 & \text{if } p \mid N \\ 1 & \text{if } p \nmid N \end{cases}$$

(iii) if $p \equiv 7 \mod 8$,

$$c_p(g_p, R_p, \mathcal{O}_p) + c_p(h_p, R_p, \mathcal{O}_p) = \begin{cases} 2 & \text{if } p \mid N \\ 1 & \text{if } p \nmid N \end{cases}$$

(iv) if $p = 2$,

$$c_p(g_p, R_p, \mathcal{O}_p) = c_p(h_p, R_p, \mathcal{O}_p) = 1$$

It follows that $H_{11}(k, j, N) = 0$ if $\{p \mid N; p \equiv 3, 5 \mod 8\} \neq \mathcal{O}$. Otherwise,

$$H_{11}(k, j, N) = 2^{-3} \cdot c_{k, j}^{-1} \cdot (J_0'(\gamma_1) + J'_0(\gamma_1^{-1}) + J_0'(\gamma_2) + J'_0(\gamma_2^{-1}))$$

$$\cdot \prod_{p \nmid N, p \equiv 2} c_p(g_p, R_p, \mathcal{O}_p) \cdot \prod_{p \equiv 1(8)} c_p(g_p, R_p, \mathcal{O}_p)$$

$$\cdot \prod_{p \equiv 1(8)} \left( c_p(g_p, R_p, \mathcal{O}_p) + c_p(h_p, R_p, \mathcal{O}_p) \right)$$

$$\cdot \left\{ c_2(g_2, R_2, \mathcal{O}_2) \text{ or } c_2(h_2, R_2, \mathcal{O}_2) \right\}$$

$$= 2^{-3} \cdot c_{k, j}^{-1} \cdot (J_0'(\gamma_1) + J'_0(\gamma_1^{-1}) + J_0'(\gamma_2) + J'_0(\gamma_2^{-1})) \cdot 2^f(p|N; p \equiv 1(8)).$$

We have $C_0(\gamma_1; Sp(2;\mathbb{R})) = C_0(\gamma_2; Sp(2;\mathbb{R})) = \{14\}$ and

$$J_0'(\gamma_1) + J'_0(\gamma_1^{-1}) + J_0'(\gamma_2) + J'_0(\gamma_2^{-1}) = J_0(\gamma_1) + J_0(\gamma_1^{-1}) + J_0(\gamma_2) + J_0(\gamma_2^{-1})$$

$$= c_{k, j} \cdot C_{11}(k, j).$$

Thus we obtain $H_{11}(k, j, N)$. 
7.9. The contribution \( H_{12}(k, j, N) \)

We consider the contribution of the elliptic elements whose characteristic polynomials are \( f_{12}(x) = x^4 - x^2 + 1 \). The set of such elements of \( G = GSp(2; \mathbb{Q})^+ \) consists of infinitely many \( G \)-conjugacy classes, which are \( Sp(2; \mathbb{R}) \)-conjugate to \( \gamma_1 = \alpha(\pi/6, 5\pi/6), \) \( \gamma_2 = \alpha(\pi/6, -5\pi/6), \) \( \gamma_1^{-1} \) and \( \gamma_2^{-1} \), but only four \( G \)-conjugacy classes out of them can appear in the first sum of Theorem 6.1 since they can have nonempty intersections with \( K(N) \). They are \( Sp(2; \mathbb{R}) \)-conjugate to \( \gamma_1, \gamma_2, \gamma_1^{-1} \) and \( \gamma_2^{-1} \). We need to take notice of Theorem 6.2 (iii) and use Remark after that. There is a single \( G_p \)-conjugacy class \( \{g_p\}_{G_p} \) if \( p \equiv 1, 5, 7 \mod 12 \), and there are two \( G_p \)-conjugacy classes \( \{g_p\}_{G_p} \) and \( \{h_p\}_{G_p} \) otherwise. For each \( g_p \in G_p \), necessary local data are given as follows.

(i) if \( p \equiv 1 \mod 12 \),
\[
c_p(g_p, R_p, \mathcal{O}_p) = \begin{cases} 2 & \text{if } p \mid N, \\ 0 & \text{if } p \nmid N, \end{cases} \quad Z_0(g_p^2) = \text{split}
\]

(ii) if \( p \equiv 5, 7 \mod 12 \),
\[
c_p(g_p, R_p, \mathcal{O}_p) = \begin{cases} 0 & \text{if } p \mid N, \\ 1 & \text{if } p \nmid N, \end{cases} \quad Z_0(g_p^2) = \text{split}
\]

(iii) if \( p \equiv 11 \mod 12 \),
\[
c_p(g_p, R_p, \mathcal{O}_p) = \begin{cases} 0 & \text{if } p \mid N, \\ 1 & \text{if } p \nmid N, \end{cases} \quad Z_0(g_p^2) = \text{split}
\]
\[
c_p(h_p, R_p, \mathcal{O}_p) = \begin{cases} 2 & \text{if } p \mid N, \\ 0 & \text{if } p \nmid N, \end{cases} \quad Z_0(h_p^2) = \text{division}
\]

(iv) if \( p = 2 \),
\[
c_p(g_p, R_p, \mathcal{O}_p) = \begin{cases} 0 & \text{if } p \mid N, \\ 1 & \text{if } p \nmid N, \end{cases} \quad Z_0(g_p^2) = \text{split}
\]
\[
c_p(h_p, R_p, \mathcal{O}_p) = \begin{cases} 1 & \text{if } p \mid N, \\ 0 & \text{if } p \nmid N, \end{cases} \quad Z_0(h_p^2) = \text{division}
\]

(v) if \( p = 3 \),
\[
c_p(g_p, R_p, \mathcal{O}_p) = \begin{cases} 0 & \text{if } p \mid N, \\ 1 & \text{if } p \nmid N, \end{cases} \quad Z_0(g_p^2) = \text{division}
\]

\[
c_p(h_p, R_p, \mathcal{O}_p) = \begin{cases} 0 & \text{if } p \mid N, \\ 1 & \text{if } p \nmid N, \end{cases} \quad Z_0(h_p^2) = \text{division}
\]
\[ c_p(h_p, R_p, O_p) = \begin{cases} 1 & \text{if } p \mid N, \\ 0 & \text{if } p \nmid N, \end{cases} \quad Z_0(h_p^2) = \text{split} \]

It follows that \( H_{12}(k, j, N) = 0 \) if \( \{p \mid N; p \equiv 5, 7 \mod 12\} \neq \emptyset \). Otherwise, \( H_{12}(k, j, N) \) is given as follows:

(I) if “2 \mid N” and “3 \mid N” or “2 \nmid N” and “3 \nmid N”,

(i) if \( p \mid N; p \equiv 11 \mod 12 \) = even,

\[ H_{12}(k, j, N) = 2^{-23} \cdot (J_0'((\gamma_2) + J_0'((\gamma_2)^{-1})) \cdot 2^{f(p/N; p \equiv \pm 1(12))}. \]

(ii) if \( p \mid N; p \equiv 11 \mod 12 \) = odd,

\[ H_{12}(k, j, N) = 2^{-23} \cdot (J_0'((\gamma_1) + J_0'((\gamma_1)^{-1})) \cdot 2^{f(p/N; p \equiv \pm 1(12))}. \]

(II) if “2 \nmid N” and “3 \mid N” or “2 \mid N” and “3 \nmid N”,

(i) if \( p \mid N; p \equiv 11 \mod 12 \) = even,

\[ H_{12}(k, j, N) = 2^{-23} \cdot (J_0'((\gamma_1) + J_0'((\gamma_1)^{-1})) \cdot 2^{f(p/N; p \equiv \pm 1(12))}. \]

(ii) if \( p \mid N; p \equiv 11 \mod 12 \) = odd,

\[ H_{12}(k, j, N) = 2^{-23} \cdot (J_0'((\gamma_2) + J_0'((\gamma_2)^{-1})) \cdot 2^{f(p/N; p \equiv \pm 1(12))}. \]

We have \( C_0(\gamma_1; Sp(2; \mathbb{R})) = C_0(\gamma_2; Sp(2; \mathbb{R})) = \{14\} \) and

\[ J_0'(\gamma_1) + J_0'((\gamma_1)^{-1}) = J_0(\gamma_1) + J_0((\gamma_1)^{-1}) = [1, -1, 0; 3], \]
\[ J_0'(\gamma_2) + J_0'((\gamma_2)^{-1}) = J_0(\gamma_2) + J_0((\gamma_2)^{-1}) = [0, -1, 1; 3]. \]

Thus we obtain \( H_{12}(k, j, N) \). By careful calculation, we can rewrite \( H_{12}(k, j, N) \) as in Theorem 3.1.

8. Proof of Main Theorem 1.1: non-semi-simple elements

8.1. Unipotent elements

In this subsection, we evaluate the contributions \( I_1(k, j, N) \), \( I_2(k, j, N) \), and \( I_3(k, j, N) \). Let \( \gamma \) be a unipotent element of \( K(N) \). We write \( SM(2; \mathbb{R}) = \{X \in M(2; \mathbb{R}); \quad tX = X\} \). Then by virtue of [Mor74], \( \gamma \) has a contribution to the dimension formula only when \( \gamma \) is \( Sp(2; \mathbb{Q}) \)-conjugate to an element of the form:

\[ \begin{bmatrix} 1 & S \\ 0 & 1 \end{bmatrix}, \quad S \in SM(2; \mathbb{R}). \]

Since \( K(N) \) has a single equivalence class of 0-dimensional cusps (Proposition 6.5), \( \gamma \) is \( K(N) \)-conjugate to an element of the above form. We divide it into the following four cases:

(i) \( S \): definite
(ii) $S$: indefinite, $\det S \neq 0$, and $-\det S \in (\mathbb{Q}^\times)^2$

(iii) $S$: indefinite, $\det S \neq 0$, and $-\det S \notin (\mathbb{Q}^\times)^2$

(iv) $\det S = 0$.

As for (iii), it is known that such $\gamma$ has no contribution to the dimension formula ([Wak12, Section 4.13]). We denote by $I_1(k, j, N)$, $I_2(k, j, N)$, $I_3(k, j, N)$ the contribution of all elements where $\pm \gamma$ correspond to (i), (ii) and (iv) respectively.

**Case 1.** We assume that $S$ is definite. We can evaluate the contribution $I_1(k, j, N)$ as follows. (cf. [Wak12, (e-2) on page 207]) There is a single $K(N)$-conjugacy class of family, which is given by

$$F = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in K(N) \middle| T : \text{definite} \right\}.$$  

We define

$$C_0(F; Sp(2; \mathbb{R})) = \left\{ \begin{bmatrix} 1 & X \\ 0 & 1 \end{bmatrix} \middle| X \in SM(2; \mathbb{R}) \right\},$$  

and $C_0(F; K(N)) = C_0(F; Sp(2; \mathbb{R})) \cap K(N)$.

We see

$$\left\{ \begin{bmatrix} A & 0_2 \\ 0_2 & A^{-1} \end{bmatrix} \middle| A \in GL(2; \mathbb{Q}) \right\} \cap K(N)$$  

$$= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -c & d \end{bmatrix} \middle| a, c, d \in \mathbb{Z}, b \in N\mathbb{Z}, ad - bc = \pm 1 \right\}$$  

$$\simeq \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2; \mathbb{Z}) \middle| c \in N\mathbb{Z} \right\} =: \Gamma_0(N).$$

We define a lattice $L$ in $SM(2; \mathbb{R})$ by

$$L = \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} \middle| a, b, c \in \mathbb{Z} \right\}.$$  

Then, by Theorem 6.4 and the formula in [Wak12, (e-2) on page 207], we have

(15) $$I_1(k, j, N) = \text{vol}(C_0(F; K(N)) \backslash C_0(F; Sp(2; \mathbb{R}))) \cdot \frac{j + 1}{2^{2\pi}} \cdot \frac{1}{[GT_0(N) : \Gamma_0(N)]} \cdot \frac{1}{\text{vol}(\Gamma_0(N) \backslash \mathfrak{D}_1)} \cdot \frac{\text{vol}(\Gamma_0(N) \backslash \mathfrak{D}_1)}{\text{vol}(L)},$$

where $\text{vol}(L) = \int_{L \setminus SM(2; \mathbb{R})} dX$. It is easy to see that

$$\text{vol}(C_0(F; K(N)) \backslash C_0(F; Sp(2; \mathbb{R}))) = \text{vol}(L) = \frac{1}{N}.$$  

Since we have $[GT_0(N) : \Gamma_0(N)] = 2$ and $\text{vol}(\Gamma_0(N) \backslash \mathfrak{D}_1) = [SL(2; \mathbb{Z}) : \Gamma_0(N)]$. 

vol(SL(2; Z)\Δ1) = \prod_{p|N} (p + 1) \cdot \frac{\pi}{3}, we obtain by (15)

\[ I_1(k, j, N) = 2^{-3}j^{-1} (j + 1) \prod_{p|N} (p + 1). \]

**Case 2.** We assume that \( S \) is indefinite, \( \det S \neq 0 \), and \( \det S \in (\mathbb{Q}^\times)^2 \). We can evaluate the contribution \( I_2(k, j, N) \) as follows (cf. [Wak12, (e-3) on page 207]). There is a single \( K(N) \)-conjugacy class of family, which is given by

\[ F = \left\{ \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \in K(N) \mid T : \text{indefinite and } - \det S \in (\mathbb{Q}^\times)^2 \right\}. \]

We define

\[ C_0(F; Sp(2; \mathbb{R})) = \left\{ \begin{bmatrix} 1 & X \\ 0 & 1 \end{bmatrix} \mid X \in SM(2; \mathbb{R}) \right\}, \]

and \( C_0(F; K(N)) = C_0(F; Sp(2; \mathbb{R})) \cap K(N) \). Let \( \beta_1, \ldots, \beta_t \) be the set of non-equivalent cusps of \( \Gamma_0(N) \) in \( \mathbb{H}_1 \), and \( V_j \) be the element of \( SL(2; \mathbb{Q}) \) which satisfies \( V_j(\beta_j) = \infty \).

Then \( \left\{ \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\} \cap V_j L \backslash V_j \) has a unique basis of the form:

\[ \begin{bmatrix} t_1(j) \\ 0 \\ 0 \\ d_j \end{bmatrix}, \begin{bmatrix} t_2(j) \\ d_j \\ 0 \\ 0 \end{bmatrix}; \quad d_j > 0, \quad t_1(j) > |t_2(j)| \geq 0. \]

Let \( B_j \) be the parabolic subgroup of \( \Gamma_0(N) \) which stabilizes \( \beta_j \), and we define \( c_j \) by

\[ B_j = V_j^{-1} \left\{ \pm \begin{bmatrix} 1 & nt_1(j)c_j/2d_j \\ 0 & 1 \end{bmatrix} \mid n \in \mathbb{Z} \right\} V_j. \]

Then, by Theorem 6.4 and the formula in [Wak12, (e-3) on page 207], we have (16)

\[ I_2(k, j, N) = -\vol(C_0(F; K(N)) \backslash C_0(F; Sp(2; \mathbb{R}))) \cdot \frac{(j + 1)}{2^4 \cdot 3} \cdot \frac{1}{\Gamma_0(N) : \Gamma_0(N)} \cdot \sum_{j=1}^t \frac{c_j}{d_j^3}. \]

In our case, a complete system of representatives of non-equivalent cusp of \( \Gamma_0(N) \) is given by \( \left\{ \begin{bmatrix} 1/n \\ 0 \end{bmatrix} \mid n \text{ is a positive divisor of } N \right\} \). We put \( \beta_n = \begin{bmatrix} 1/n \\ 0 \end{bmatrix} \), then we can take \( V_n = \begin{bmatrix} 1 \\ -n \end{bmatrix} \). We can easily obtain

\[ \left\{ \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\} \cap V_j L \backslash V_j = \mathbb{Z} \begin{bmatrix} 1/n \\ 0 \end{bmatrix} \oplus \mathbb{Z} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]

and

\[ V_n B_n V_n^{-1} = \left\{ \begin{bmatrix} b \\ 0 \end{bmatrix} \mid b \in \frac{N}{n} \mathbb{Z} \right\}. \]
hence we get \( c_n = 2N \) and \( d_n = 1 \). Hence we obtain by (16)
\[
I_2(k,j,N) = -2^{\omega(N)-4} \cdot 3^{-1} \cdot (j + 1),
\]
where \( \omega(N) \) is the number of prime divisors of \( N \).

**Case 3.** We assume that \( \det S = 0 \). We can evaluate the contribution \( I_3(k,j,N) \) as follows (cf. [Wak12, (e-4) on page 208]). In this case, the elements in question are \( Sp(2; \mathbb{Q}) \) conjugate to \( \begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix} \) with \( S = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \). Using Proposition 6.5 for \( P_{2,1}(\mathbb{Q}) \), we see that each unipotent element of \( K(N) \) of this type is \( K(N) \)-conjugate to an element of the form:
\[
\begin{pmatrix}
1 & 0 & nm & n \\
0 & 1 & n/m \\
0 & 0 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]
for a positive divisor \( m \) of \( N \) and non-zero integer \( n \). We can take a complete system of representatives of \( K(N) \)-conjugacy classes of families as
\[
F_m = \left\{ \begin{pmatrix} 1 & 0 & nm & n \\ 0 & 1 & n/m \\ 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid n \in \mathbb{Z} - \{0\} \right\}
= g_m \cdot \left\{ \begin{pmatrix} 1 & 0 & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \mid n \in \mathbb{Z} - \{0\} \right\} \cdot g_m^{-1},
\]
where
\[
g_m = \begin{pmatrix}
\sqrt{m} & 0 & 0 & 0 \\
1/\sqrt{m} & 1/\sqrt{m} & 0 & 0 \\
0 & 0 & 1/\sqrt{m} & -1/\sqrt{m} \\
0 & 0 & 0 & \sqrt{m}
\end{pmatrix}.
\]
So there are \( 2^{\omega(N)} \) families since \( 2^{\omega(N)} \) is the number of divisors of \( N \). For each family \( F_m \), we define \( C_0(F_m; Sp(2; \mathbb{R})) \) as in [Wak12, (e-4) on page 208]. Then we have
\[
\text{vol}(C_0(F_m; K(N))) \cap C_0(F_m; Sp(2; \mathbb{R}))) = \frac{\pi^2}{3} N.
\]
Hence, by Theorem 6.4 and the formula in [Wak12, (e-4) on page 209], we obtain
\[
I_3(k,j,N) = -2^{\omega(N)} \cdot \text{vol}(C_0(F_m; K(N))) \cap C_0(F_m; Sp(2; \mathbb{R}))) \cdot \frac{(j + 1)(j + 2k - 3)}{2^5 \cdot 3 \cdot \pi^2} = -2^{\omega(N)-5} 3^{-2} N (j + 2k - 3).
8.2. $\delta$-parabolic elements

In this subsection, we evaluate $I_4(k,j,N)$ and $I_5(k,j,N)$. Let $\gamma$ be a $\delta$-parabolic element of $K(N)$, that is, $\gamma$ is $Sp(2;\mathbb{R})$-conjugate to an element of the form:

$$\delta(s_1, s_2) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & s_1 & 0 \\ 0 & 1 & 0 & s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (s_1, s_2) = (\pm 1, \pm 1) \text{ or } (\pm 1, 0).$$

We denote by $I_4(k,j,N)$ the contribution for $\gamma$ with $(s_1, s_2) = (\pm 1, \pm 1)$ and by $I_5(k,j,N)$ for those with $(s_1, s_2) = (\pm 1, 0)$.

We can see from Corollary 6.6 that such elements are $K(N)$-conjugate to elements of the form:

$$\begin{bmatrix} A & B \\ 0_2 & A^{-1} \end{bmatrix} \quad (A \cdot {}^t B = B \cdot {}^t A).$$

Since $A \in M(2;\mathbb{Z})$ is of order two and $Tr A = 0$, it is well-known that $A$ is $GL(2;\mathbb{Z})$-conjugate to $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Thus $\gamma$ is $\iota(GL(2;\mathbb{Z}))$-conjugate to

$$X_1(S) = \iota \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \iota \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

where $\iota$ is a map defined by $\iota(A) = \begin{bmatrix} A & 0_2 \\ 0_2 & A^{-1} \end{bmatrix}$ and $S$ is a symmetric matrix.

We define $\Gamma' = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2;\mathbb{Z}) \mid b \in NZ \right\}$, then $\iota(GL(2;\mathbb{Z})) \cap K(N) = \iota(\Gamma')$. We try to determine how $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}_{GL(2;\mathbb{Z})} \cap \Gamma'$ and $\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\}_{GL(2;\mathbb{Z})} \cap \Gamma'$ split into $\Gamma'$-conjugacy classes. By Proposition 6.7, we can achieve this purpose by determining complete systems of representatives of the double coset decompositions:

$$C \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} ; GL(2;\mathbb{Z}) \right) \backslash M \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} ; GL(2;\mathbb{Z})/\Gamma' \right) / \Gamma', \quad (18)$$

$$C \left( \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} ; GL(2;\mathbb{Z}) \right) \backslash M \left( \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} ; GL(2;\mathbb{Z})/\Gamma' \right) / \Gamma'. \quad (19)$$

We consider the cases $(s_1, s_2) = (\pm 1, \pm 1)$ and $(\pm 1, 0)$ in (17) separately.

**Case 1.** We consider the case of $(s_1, s_2) = (\pm 1, \pm 1)$ and $2 \nmid N$. For each positive divisor $d$ of $N$, we fix elements $x_d$ and $y_d$ of $GL(2;\mathbb{Z})$ which satisfies

$$x_d \equiv \begin{bmatrix} 10 \\ 01 \\ 01 \\ 10 \end{bmatrix} \mod p \text{ if } p \mid d, \quad y_d \equiv \begin{bmatrix} 10 \\ 01 \end{bmatrix} \mod p \text{ if } p \nmid d, \quad \begin{bmatrix} 10 \\ p-21 \end{bmatrix} \mod p \text{ if } p \nmid d.$$
Then, by straightforward calculation, we see that complete systems of representatives of (18) and (19) are given by $x_d \,(d | N, d > 0)$ and $y_d \,(d | N, d > 0)$ respectively. Hence a complete system of representatives of $\iota(\Gamma')$-conjugacy classes is given by $\iota(x_d)^{-1}X_1(S)\iota(x_d)$ and $\iota(y_d)^{-1}X_2(S)\iota(y_d)$ \,(d \mid N, d > 0). Since $\iota(\Gamma') \subset K(N)$, we need to examine which $\iota(\Gamma')$-conjugacy classes are the same as $K(N)$-conjugacy classes. By careful calculation, we see that a complete system of representatives of $K(N)$-conjugacy classes of elements in question is given by

$$\iota(x_d)^{-1}X_1 \left( \begin{bmatrix} m/d & 0 \\ 0 & n \end{bmatrix} \right) \iota(x_d), \quad \iota(x_d)^{-1}X_1 \left( \begin{bmatrix} m/d & 1 \\ 1 & n \end{bmatrix} \right) \iota(x_d),$$

$$\iota(y_d)^{-1}X_2 \left( \begin{bmatrix} m + n/d - m & -m \\ -m & 2m \end{bmatrix} \right) \iota(y_d), \quad \iota(y_d)^{-1}X_2 \left( \begin{bmatrix} m + n/d & -m \\ -m & 2m - 1 \end{bmatrix} \right) \iota(y_d), \quad (d \mid N, d > 0).$$

Finally, we determine $K(N)$-conjugacy classes of families and evaluate each contribution. For each positive divisor $d \, \text{of } N$, we define

$$g_{1,d} = \begin{bmatrix} \sqrt{d}0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \quad g_{2,d} = \begin{bmatrix} \sqrt{d}0 & \sqrt{d}/2 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$g_{3,d} = \begin{bmatrix} \sqrt{2d} & \sqrt{d}/2 & 0 & 0 \\ 0 & -1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & -\sqrt{2} \end{bmatrix}, \quad \quad g_{4,d} = \begin{bmatrix} \sqrt{2d} & \sqrt{d}/2 & \sqrt{2d}/8 - \sqrt{2d}/4 \\ 0 & -1/\sqrt{2} & \sqrt{2}/8 & 0 \\ 0 & 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & -\sqrt{2} \end{bmatrix}.$$

Then we obtain the following result by careful calculation.

**Proposition 8.1.** A complete system of representatives of $K(N)$-conjugacy classes of families is given as

$$F_{1,d} = \iota(x_d)^{-1}g_{1,d}^{-1} \cdot \delta(m, n) \begin{cases} m, n \in \mathbb{Z} \\ m, n \neq 0 \end{cases} \cdot g_{1,d} \cdot \iota(x_d),$$

$$F_{2,d} = \iota(x_d)^{-1}g_{2,d}^{-1} \cdot \delta(m, n) \begin{cases} m, n \in \mathbb{Z} \\ m, n \neq 0 \end{cases} \cdot g_{2,d} \cdot \iota(x_d),$$

$$F_{3,d} = \iota(y_d)^{-1}g_{3,d}^{-1} \cdot \delta(Nm + 2n, m) \begin{cases} m, n \in \mathbb{Z} \\ m \neq 0, \ Nm + 2n \neq 0 \end{cases} \cdot g_{3,d} \cdot \iota(y_d),$$

$$F_{4,d} = \iota(y_d)^{-1}g_{4,d}^{-1} \cdot \delta(Nm + 2n - N/2, m - 1/2) \begin{cases} m, n \in \mathbb{Z} \\ Nm + 2n - N/2 \neq 0 \end{cases} \cdot g_{4,d} \cdot \iota(y_d)$$

for positive divisors $d \, \text{of } N$, where $\iota$ is the map defined by $\iota(A) = \begin{bmatrix} A & 0_2 \\ 0_2 & A^{-1} \end{bmatrix}$.

For each family $F$, we define $C_0(F; \text{Sp}(2; \mathbb{R}))$ as in [Wak12, (E-2) on page 210].

**Proposition 8.2.** Necessary data for each family are as follows.
Proof. Direct calculations and [Wak12, (f-2) on page 210]. In (iii), \( \left( \frac{-1}{N} \right) \) appears since
\[
\lim_{s \to +0} \sum_{\gamma' \in \Gamma} J_0(\gamma'; s) = c_{k,j}^{-1} \cdot 2^{-5} \cdot (-1)^k \cdot \sum_{t=0}^{1} \left( 1 + i \cdot \cot \left( \frac{t+1/2}{2} \pi \right) \right) \left( 1 - i \cdot \cot \frac{-N(t+1/2)\pi}{2} \right)
\]
\[
= c_{k,j}^{-1} \cdot 2^{-5} \cdot (-1)^k \cdot \begin{cases} 0 & \text{if } N \equiv 1 \text{ mod } 4 \\ 4 & \text{if } N \equiv 3 \text{ mod } 4. \end{cases}
\]
It follows that the total contribution of all families is
\[
I_q(k, j, N) = \left( 4 - \left( \frac{-1}{N} \right) \right) \cdot 2^{5(N-5)} \cdot (-1)^k.
\]

**Case 2.** We consider the case of \((s_1, s_2) = (\pm 1, \pm 1)\) and \(2 \mid N\). For each positive divisor \(d\) of \(N/2\), we fix elements \(x_d^{(1)}, x_d^{(2)}, x_d^{(3)}\) and \(y_d\) of \(GL(2; \mathbb{Z})\) which satisfies
\[
x_d^{(1)} \equiv \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \text{mod } p \text{ if } p \mid d \text{ or } p = 2, \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \text{mod } p \text{ if } p \nmid d, \end{cases}
\]
\[
x_d^{(2)} \equiv \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \text{mod } p \text{ if } p \mid d, \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \text{mod } p \text{ if } p \nmid d \text{ or } p = 2, \end{cases}
\]
for positive divisors $d$ of families is given as

$$x_d^{(3)} \equiv \begin{cases}
10 \mod p \text{ if } p \mid d, \\
01 \mod p \text{ if } p \nmid d, \\
01 \mod p \text{ if } p = 2, \\
10 \mod p \text{ if } p = 2,
\end{cases}$$

$$y_d \equiv \begin{cases}
10 \mod p \text{ if } p \mid d \text{ or } p = 2, \\
01 \mod p \text{ if } p \mid d.
\end{cases}$$

By similar argument as in Case 1, we obtain the following result.

**Proposition 8.3.** A complete system of representatives of $K(N)$-conjugacy classes of families is given as

$$F_{1,d} = \iota(x_d^{(i)})^{-1} g_{1,d}^{-1} \cdot \left\{ \delta(m,n) \middle| m,n \in \mathbb{Z} \right\} \cdot g_{1,d} \cdot \iota(x_d^{(i)}),$$

$$F_{2,d} = \iota(x_d^{(i)})^{-1} g_{2,d}^{-1} \cdot \left\{ \delta(m,n) \middle| m,n \in \mathbb{Z} \right\} \cdot g_{2,d} \cdot \iota(x_d^{(i)}),$$

$$F_{3,d} = \iota(x_d^{(i)})^{-1} g_{3,d}^{-1} \cdot \left\{ \delta(Nm + 2n, m) \middle| m,n \in \mathbb{Z}, Nm + 2n \neq 0 \right\} \cdot g_{3,d} \cdot \iota(x_d^{(i)}),$$

$$F_{4,d} = \iota(y_d)^{-1} g_{4,d}^{-1} \cdot \left\{ \delta(Nm + 2n - N/2, m - 1/2) \middle| m,n \in \mathbb{Z}, Nm + 2n - N/2 \neq 0 \right\} \cdot g_{4,d} \cdot \iota(y_d)$$

for positive divisors $d$ of $N$ and $i = 1, 2, 3$, where $g_{1,d}, g_{2,d}, g_{3,d}, g_{4,d}$ are the same as in Case 1.

For each family $F$, we define $C_0(F; Sp(2; \mathbb{R}))$ as in [Wak12, (f-2) on page 210].

**Proposition 8.4.** Necessary data for each family are as follows:

(i) If $F = F_{1,d}^{(i)}, F_{2,d}^{(i)}$, or $F_{3,d}^{(i)}$, then

$$\text{vol} \left( \overline{C_0(F; K(N)) \setminus C_0(F; Sp(2; \mathbb{R}))} \right) = 1, \quad [C(F; K(N)) : C_0(F; K(N))] = 2$$

and $\lim_{s \to +0} \sum_{\gamma \in F} J_0(\gamma'; s) = c_{k,j}^{-1} \cdot 2^{-3} \cdot (-1)^k$, hence the contribution of $F$ is $2^{-5} \cdot (-1)^k$.

(ii) If $F = F_{4,d}$, then

$$\text{vol} \left( \overline{C_0(F; K(N)) \setminus C_0(F; Sp(2; \mathbb{R}))} \right) = 2, \quad [C(F; K(N)) : C_0(F; K(N))] = 2$$

and $\lim_{s \to +0} \sum_{\gamma \in F} J_0(\gamma'; s) = c_{k,j}^{-1} \cdot 2^{-4} \cdot (-1)^k$, hence the contribution of $F$ is $2^{-5} \cdot (-1)^k$.

**Proof.** Direct calculations and [Wak12, (f-2) on page 210].

It follows that the total contribution of all families is

$$I_4(k,j, N) = 4 \cdot 2^{-4} \cdot (-1)^k.$$
Case 3. We consider the case \((s_1, s_2) = (\pm 1, 0)\). For each positive integer \(d\) of \(N\), we define

\[
g_{1,d} = \begin{bmatrix} \sqrt{d} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad g_{2,d} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},
\]

\[
g_{3,d} = \begin{bmatrix} \sqrt{d} & 0 & 0 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \end{bmatrix}, \quad g_{4,d} = \begin{bmatrix} 0 & 1 & -1/2 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.
\]

Then we obtain the following result.

**Proposition 8.5.** A complete system of representatives of \(K(N)\)-conjugacy classes of families is given as

\[
F_{1,d} = g_{1,d}^{-1} \cdot \left\{ \delta(m, 0) \mid m \in \mathbb{Z}, m \neq 0 \right\} \cdot g_{1,d}, \quad F_{2,d} = g_{2,d}^{-1} \cdot \left\{ \delta(m, 0) \mid m \in \mathbb{Z}, m \neq 0 \right\} \cdot g_{2,d},
\]

\[
F_{3,d} = g_{3,d}^{-1} \cdot \left\{ \delta(m, 0) \mid m \in \mathbb{Z}, m \neq 0 \right\} \cdot g_{3,d}, \quad F_{4,d} = g_{4,d}^{-1} \cdot \left\{ \delta(m, 0) \mid m \in \mathbb{Z}, m \neq 0 \right\} \cdot g_{4,d},
\]

\[
F_{5,d} = -F_{1,d}, \quad F_{6,d} = -F_{2,d}, \quad F_{7,d} = -F_{3,d}, \quad F_{8,d} = -F_{4,d}
\]

for positive divisors \(d\) of \(N\).

For each family \(F\), we define \(C_0(F; \text{Sp}(2; \mathbb{R}))\) as in [Wak12, (f-2) on page 210].

**Proposition 8.6.** Necessary data for each family are as follows:

(i) If \(F = F_{1,d}, F_{2,d}, F_{5,d}\) or \(F_{6,d}\), then

\[
\text{vol}\left(C_0(F; K(N)) \setminus C_0(F; \text{Sp}(2; \mathbb{R}))\right) = \frac{\pi^2}{3}, \quad [C(F; K(N)) : C_0(F; K(N))] = 1
\]

\[
\lim_{s \to +0} \sum_{\gamma' \in F} J_0(\gamma'; s) = -c_{k,j}^{-1} \cdot 2^{-6} \cdot \pi^{-2} \cdot (-1)^k \cdot (2k + j - 3),
\]

hence the contribution of \(F\) is \(-2^{-7} \cdot 3^{-1} \cdot (-1)^k \cdot (2k + j - 3)\).

(ii) If \(F = F_{3,d}, F_{4,d}, F_{7,d}\) or \(F_{8,d}\), then

\[
\text{vol}\left(C_0(F; K(N)) \setminus C_0(F; \text{Sp}(2; \mathbb{R}))\right) = \pi^2, \quad [C(F; K(N)) : C_0(F; K(N))] = 1
\]

\[
\lim_{s \to +0} \sum_{\gamma' \in F} J_0(\gamma'; s) = -c_{k,j}^{-1} \cdot 2^{-6} \cdot \pi^{-2} \cdot (-1)^k \cdot (2k + j - 3),
\]

hence the contribution of \(F\) is \(-2^{-7} \cdot (-1)^k \cdot (2k + j - 3)\).
It follows that the total contribution of all families is
\[ I_5(k, j, N) = -2^a(N)^{-4} \cdot 3^{-1} \cdot (-1)^k \cdot (2k + j - 3). \]

### 8.3. Elliptic/parabolic elements

In this subsection, we evaluate \( I_6(k, j, N), I_7(k, j, N) \) and \( I_8(k, j, N) \). Let \( \gamma \) be an elliptic/parabolic element of \( K(N) \), that is, \( \gamma \) is \( Sp(2; \mathbb{R}) \)-conjugate to an element of the form:
\[
\tilde{\gamma}(\theta, \lambda) = \begin{bmatrix}
\cos \theta & 0 & \sin \theta & 0 \\
0 & 1 & 0 & 0 \\
-\sin \theta & 0 & \cos \theta & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}, \quad \theta = \frac{\pi}{2}, \frac{\pi}{3}, \lambda \neq 0.
\]

Let \( m \) be a positive divisor of \( N \) and \( m' = N/m \). Since \( N \) is squarefree, \( m \) and \( m' \) are mutually prime. We fix \( \alpha, \beta \in \mathbb{Z} \) such that \( \alpha m + m' \beta = -1 \). We define matrices of \( Sp(2; \mathbb{R}) \) as follows:
\[
g_m = \begin{bmatrix}
-\beta \sqrt{m'} & \sqrt{m} & 0 & 0 \\
\alpha \sqrt{m'} & 1 & 0 & 0 \\
0 & 0 & \sqrt{m'} & -\alpha \sqrt{m} \\
0 & 0 & -m \sqrt{m'} & -\beta \sqrt{m}
\end{bmatrix}, \quad T_1 = \begin{bmatrix}1 & 0 & 0 & 1/2 \\
-1/2 & -1 & -1/2 & 0 \\
0 & 0 & 1 & -1/2 \\
0 & 0 & 0 & -1 \end{bmatrix},
\]
\[
T_3 = \begin{bmatrix}2c & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
c & 0 & \sqrt{3}c & 0 \\
0 & 0 & 0 & 1 \end{bmatrix}, \quad T_4 = \begin{bmatrix}2c & 0 & 0 & 1/3 \\
c & 1 & 0 & 0 \\
c & 0 & \sqrt{3}c & -1/3 \\
0 & 0 & 0 & 1 \end{bmatrix}, \quad (c^2 = \sqrt{3}/6)
\]

**Case 1:** We consider elements of \( K(N) \) whose characteristic polynomial is \( (x-1)^2(x^2+1) \) and double the contribution. We denote by \( I_6(k, j, N) \) the total of such contribution.

**Proposition 8.7.** A complete system of representatives of \( K(N) \)-conjugacy classes of families are given as follows:
\[
F_{1,m} = g_m T_1 \{ \tilde{\gamma}(\pi/2, n) \mid n \in \mathbb{Z}, n \neq 0 \} T_1^{-1} g_m^{-1}
\]
\[
F_{2,m} = g_m T_1 \{ \tilde{\gamma}(-\pi/2, n) \mid n \in \mathbb{Z}, n \neq 0 \} T_1^{-1} g_m^{-1}
\]
\[
F_{3,m} = g_m T_2 \{ \tilde{\gamma}(\pi/2, n - 1/2) \mid n \in \mathbb{Z} \} T_2^{-1} g_m^{-1}
\]
\[
F_{4,m} = g_m T_2 \{ \tilde{\gamma}(-\pi/2, n + 1/2) \mid n \in \mathbb{Z} \} T_2^{-1} g_m^{-1}
\]

for positive divisors \( m \) of \( N \).

For each family \( F \), we define \( C_0(F; Sp(2; \mathbb{R})) \) as in [Wak12, (f-4) on page 212].

**Proposition 8.8.** Necessary data for each family are as follows.
Case 2: We consider elements of $x$ of families are given as follows:

$$\text{vol}(C_0(F; K(N)) \setminus C_0(F; Sp(2; R))) = 1, \quad [C(F; K(N)) : C_0(F; K(N))] = 4$$

and

$$\lim_{s \to 0} \sum_{\gamma' \in F} J_0(\gamma'; s) = -e_k^{-1} \cdot 2^{-3} \cdot [i + (1)^{1/2}, -1 + i(-1)^{1/2}, -i - (-1)^{1/2}, 1 - i(-1)^{1/2}; 4]_k,$$

hence the contribution of $F$ is $2^{-6} \cdot [i + (1)^{1/2}, -1 + i(-1)^{1/2}, -i - (-1)^{1/2}, 1 - i(-1)^{1/2}; 4]_k$.

(ii) If $F = F_{2,m}$ or $F_{4,m}$,

$$\text{vol}(C_0(F; K(N)) \setminus C_0(F; Sp(2; R))) = 1, \quad [C(F; K(N)) : C_0(F; K(N))] = 4$$

and

$$\lim_{s \to 0} \sum_{\gamma' \in F} J_0(\gamma'; s) = -e_k^{-1} \cdot 2^{-3} \cdot [-i + (1)^{1/2}, -1 - i(-1)^{1/2}, i - (-1)^{1/2}, 1 + i(-1)^{1/2}; 4]_k,$$

hence the contribution of $F$ is $2^{-6} \cdot [-i + (1)^{1/2}, -1 - i(-1)^{1/2}, i - (-1)^{1/2}, 1 + i(-1)^{1/2}; 4]_k$.

Proof. Direct calculations and [Wak12, (f-4) on page 212].

It follows that the total contribution of all families is

$$I_0(k, j, N) = -2^{\omega(N)-3} \cdot [(-1)^{1/2}, -1, -(-1)^{1/2}, 1; 4]_k.$$

Case 2: We consider elements of $K(N)$ whose characteristic polynomial is $(x - 1)^2(x^2 + x + 1)$ and double the contribution. We denote by $I_2(k, j, N)$ the total of such contribution.

Proposition 8.9. A complete system of representatives of $K(N)$-conjugacy classes of families are given as follows:

$$F_{1,m} = g_m T_3 \left\{ \hat{\beta}(2\pi/3, n) \mid n \in \mathbb{Z}, n \neq 0 \right\} T_3^{-1} g_m^{-1}$$

$$F_{2,m} = g_m T_3 \left\{ \hat{\beta}(-2\pi/3, n) \mid n \in \mathbb{Z}, n \neq 0 \right\} T_3^{-1} g_m^{-1}$$

$$F_{3,m} = g_m T_4 \left\{ \hat{\beta}(2\pi/3, n + 2N/3) \mid n \in \mathbb{Z} \right\} T_4^{-1} g_m^{-1}$$

$$F_{4,m} = g_m T_4 \left\{ \hat{\beta}(-2\pi/3, n + N/3) \mid n \in \mathbb{Z} \right\} T_4^{-1} g_m^{-1}$$

for positive divisors $m$ of $N$.

For each family $F$, we define $C_0(F; Sp(2; R))$ as in [Wak12, (f-4) on page 212].

Proposition 8.10. Necessary data for each family are as follows.

(i) If $F = F_{1,m}$ or $F_{2,m}$,

$$\text{vol}(C_0(F; K(N)) \setminus C_0(F; Sp(2; R))) = 1, \quad [C(F; K(N)) : C_0(F; K(N))] = 6,$$
and $\sum_{i=1}^{2} \lim_{s \to +0} \sum_{\gamma \in F_{1,m}} J_0(\gamma'; s) = -c_{k,j}^{-1} \cdot 2^{-1} \cdot 3^{-1} \cdot \left( [1, -1, 0; 3]_k + [1, 0, -1; 3]_{j+k} \right)$, hence the sum of the contributions of $F_{1,m}$ and $F_{2,m}$ is $-2^{-3} \cdot 3^{-2} \cdot \left( [1, -1, 0; 3]_k + [1, 0, -1; 3]_{j+k} \right)$.

(ii) If $F = F_{3,m}$ or $F_{4,m}$,

$$\text{vol} \left( C_0(F; K(N)) \setminus C_0(F; Sp(2; \mathbb{R})) \right) = 1, \quad [C(F; K(N)) : C_0(F; K(N))] = 3$$

and $\sum_{i=3}^{4} \lim_{s \to +0} \sum_{\gamma \in F_{1,m}} J_0(\gamma'; s)
\begin{align*}
&= -c_{k,j}^{-1} \begin{cases} 
2^{-1} \cdot 3^{-1} \cdot \left( [1, -1, 0; 3]_k + [1, 0, -1; 3]_{j+k} \right) & \text{if } N \equiv 0 \text{ mod } 3, \\
3^{-2} \cdot \left( [1, -2, 1; 3]_k + [2, -1, -1; 3]_{j+k} \right) & \text{if } N \equiv 1 \text{ mod } 3, \\
3^{-2} \cdot \left( [2, -1, -1; 3]_k + [1, 1, -2; 3]_{j+k} \right) & \text{if } N \equiv 2 \text{ mod } 3,
\end{cases}
\end{align*}$

hence the sum of the contributions of $F_{3,m}$ and $F_{4,m}$ is

$$\begin{cases} 
-2^{-2} \cdot 3^{-2} \cdot \left( [1, -1, 0; 3]_k + [1, 0, -1; 3]_{j+k} \right) & \text{if } N \equiv 0 \text{ mod } 3, \\
-2^{-1} \cdot 3^{-3} \cdot \left( [1, -2, 1; 3]_k + [2, -1, -1; 3]_{j+k} \right) & \text{if } N \equiv 1 \text{ mod } 3, \\
-2^{-1} \cdot 3^{-3} \cdot \left( [2, -1, -1; 3]_k + [1, 1, -2; 3]_{j+k} \right) & \text{if } N \equiv 2 \text{ mod } 3,
\end{cases}$$

PROOF. Direct calculations and [Wak12, (f-4) on page 212]. \qed

It follows that the total contribution of all families is

$$I_T(k, j, N) = -2^{\omega(N)-2} \cdot 3^{-2} \cdot \begin{cases} 
[3, -3, 0; 3]_k + [3, 0, -3; 3]_{j+k} & \cdots \text{if } N \equiv 0 \text{ mod } 3, \\
[5, -1, 4; 3]_k + [1, -4, 5; 3]_{j+k} & \cdots \text{if } N \equiv 1 \text{ mod } 3, \\
[1, -5, -4; 3]_k + [5, 4, -1; 3]_{j+k} & \cdots \text{if } N \equiv 2 \text{ mod } 3.
\end{cases}$$

Case 3: We consider elements of $K(N)$ whose characteristic polynomial is $(x - 1)^2(x^2 - x + 1)$ and double the contribution. We denote by $I_6(k, j, N)$ the total of such contribution.

**Proposition 8.11.** A complete system of representatives of $K(N)$-conjugacy classes of families are given as follows:

$$F_{1,m} = g_m T_3 \left\{ \beta(\pi/3, n) \mid n \in \mathbb{Z}, n \neq 0 \right\} T_3^{-1} g_m^{-1}$$

$$F_{2,m} = g_m T_3 \left\{ \beta(-\pi/3, n) \mid n \in \mathbb{Z}, n \neq 0 \right\} T_3^{-1} g_m^{-1}$$

for positive divisors $m$ of $N$.

For each family $F$, we define $C_0(F; Sp(2; \mathbb{R}))$ as in [Wak12, page 212].

**Proposition 8.12.** Necessary data for each family are as follows:

$$\text{vol} \left( C_0(F_{1,m}; K(N)) \setminus C_0(F_{1,m}; Sp(2; \mathbb{R})) \right) = \text{vol} \left( C_0(F_{2,m}; K(N)) \setminus C_0(F_{2,m}; Sp(2; \mathbb{R})) \right) = 1,$$
and \[ \lim_{s \to 0} \sum_{\gamma \in F_{1,m}} J_0(\gamma'; s) = -2^{-1} \cdot ([-1, -1, 0, 1, 1, 0; 6]_k + [1, 0, -1, -1, 0, 1; 6]_{j+k}), \]

hence the sum of the contributions of \( F_{1,m} \) and \( F_{2,m} \) is \(-2^{-3} \cdot 3^{-1} \cdot ([-1, -1, 0, 1, 1, 0; 6]_k + [1, 0, -1, -1, 0, 1; 6]_{j+k})\).

**Proof.** Direct calculations and [Wak12, (f-4) on page 212]. \( \square \)

It follows that the total contribution of all families is

\[
I_8(k, j, N) = -2^{\omega(N)-2} \cdot 3^{-1} \cdot ([-1, -1, 0, 1, 1, 0; 6]_k + [1, 0, -1, -1, 0, 1; 6]_{j+k}).
\]

### 8.4. Paraeliotic elements

In this subsection, we prove \( I_9(k, j, N) \) and \( I_{10}(k, j, N) \) in Theorem 3.1. Let \( \gamma \) be a paraelptic element of \( K(N) \), that is, \( \gamma \) is \( \text{Sp}(2; \mathbb{R}) \)-conjugate to an element of the form:

\[
(21) \quad \gamma(\theta, s) = \begin{bmatrix}
\cos \theta & \sin \theta & 0 & 0 \\
-\sin \theta \cos \theta & \cos \theta & 0 & 0 \\
0 & 0 & \cos \theta & \sin \theta \\
0 & 0 & -\sin \theta \cos \theta & \cos \theta
\end{bmatrix} \begin{bmatrix} 100 & 0 \\ 010 & s \\ 001 & 0 \\ 000 & 1 \end{bmatrix}, \quad \theta = \pi \frac{2}{3} \text{ or } \pm \frac{2\pi}{3}, s = \pm 1.
\]

We consider the cases \( \theta = \pi/2 \) and \( \theta = \pm 2\pi/3 \) in (21) separately.

**Case 1.** We consider the case \( \theta = \pi/2 \). By similar calculation as in subsection 8.2 by using Corollary 6.6 and Proposition 6.7, we obtain the following result. We define

\[
x_d = \begin{bmatrix} 10 \\ d \\ 1 \end{bmatrix}, \quad (d = 0, 1, \ldots, N - 1, d^2 + 1 \in \mathbb{N}), \quad g_1 = 1_4,
\]

\[
g_2 = \begin{bmatrix} 10 & 0 & -1/2 \\ 01 & -1/2 & 0 \\ 00 & 1 & 0 \\ 00 & 0 & 1 \end{bmatrix}, \quad g_3 = \begin{bmatrix} 100 & 0 \\ 010 & -1/2 \\ 001 & 0 \\ 000 & 1 \end{bmatrix}, \quad g_4 = \begin{bmatrix} 10 & 0 & -1/2 \\ 01 & -1/2 & -1/2 \\ 00 & 1 & 0 \\ 00 & 0 & 1 \end{bmatrix}.
\]

**Proposition 8.13.** A complete system of representatives of \( K(N) \)-conjugacy classes of families is given by

\[
F_1 = \iota(x_d)^{-1} g_1^{-1} \cdot \{ \gamma(\pi/2, n) \mid n \in \mathbb{Z}, n \neq 0 \} \cdot g_1 \iota(x_d)^{-1},
\]

\[
F_2 = \iota(x_d)^{-1} g_2^{-1} \cdot \{ \gamma(\pi/2, n) \mid n \in \mathbb{Z}, n \neq 0 \} \cdot g_2 \iota(x_d)^{-1},
\]

\[
F_3 = \iota(x_d)^{-1} g_3^{-1} \cdot \{ \gamma(\pi/2, n + 1/2) \mid n \in \mathbb{Z} \} \cdot g_3 \iota(x_d)^{-1},
\]

\[
F_4 = \iota(x_d)^{-1} g_4^{-1} \cdot \{ \gamma(\pi/2, n + 1/2) \mid n \in \mathbb{Z} \} \cdot g_4 \iota(x_d)^{-1},
\]

where \( d \in \{ 0, 1, \ldots, N - 1 \} \) such that \( d^2 + 1 \in \mathbb{N} \).

For each family \( F \), we define \( C_0(F; \text{Sp}(2; \mathbb{R})) \) as in [Wak12, page 211].
where $d$.

We consider the case $\text{Case 2.}$

$$\lim_{s \to +0} \sum_{\gamma' \in F} J_0(\gamma'; s) = -c_{k,j}^{-1} \cdot 2^{-3} \cdot (-1)^{j/2},$$

hence the contribution of $F$ is $-2^{-5} \cdot (-1)^{j/2}$.

PROOF. Direct calculations and [Wak12, (f-3) on page 211].

It follows that the total contribution of all families is

$$I_9(k,j,N) = -2^{-3} \cdot (-1)^{j/2} \cdot \prod_{p | N} \left( 1 + \left( \frac{-1}{p} \right) \right).$$

Case 2. We consider the case $\theta = \pm 2\pi/3$. We need only to double the contribution of the case $\theta = 2\pi/3$ to obtain $I_{10}(k,j,N)$. We define

$$x_d = \begin{bmatrix} 1 & 0 \\ d & 1 \end{bmatrix} \quad \left( d = 0, 1, \ldots, N - 1 \right) \quad \left( d^2 + d + 1 \in \mathbb{N} \right),$$

$$g_1 = \begin{bmatrix} 1/\sqrt{2} & 0 & 0 & 0 \\ 1/\sqrt{6} - 2\sqrt{3}/3 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & 0 & -\sqrt{3/2} \end{bmatrix},$$

$$g_2 = \begin{bmatrix} 1/\sqrt{2} & 0 & 0 & -\sqrt{2/12} \\ 1/\sqrt{6} - 2\sqrt{3}/3 \sqrt{6/18} & 7\sqrt{6/36} & 0 & 0 \\ 0 & 0 & \sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & 0 & -\sqrt{3/2} \end{bmatrix},$$

$$g_3 = \begin{bmatrix} 1/\sqrt{2} & 0 & 0 & -\sqrt{2/6} \\ 1/\sqrt{6} - 2\sqrt{3}/3 \sqrt{6/18} & 7\sqrt{6/18} & 0 & 0 \\ 0 & 0 & \sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & 0 & -\sqrt{3/2} \end{bmatrix}.$$
PROOF. Direct calculations and [Wak12, (f-3) on page 211]. □

It follows that the total contribution of all families is

\[ I_{10}(k, j, N) = -2^{-1} \cdot 3^{-1} \cdot [1, -1, 0; 3]_j \prod_{p|N} \left( 1 + \left( \frac{-3}{p} \right) \right). \]

9. Appendix: tables of dimensions

9.1. Scalar valued case

We give a table of dimensions of \( S_{k, 0}(K(N)) \) by Main Theorem 1.1 (Theorem 3.1) and its remark. Concerning Theorem 4.6 (i), we also give a table of dimensions of \( J_{cusp}^{k, m} \), the space of Jacobi cusp forms of weight \( k \) and index \( m \), by a well-known formula. As is well-known, Gritsenko [Gri95] proved a lifting theory from \( J_{cusp}^{k, N} \) to \( S_{k, 0}(K(N)) \).

\[
\dim S_{k, 0}(K(N))
\]

\[
\begin{array}{cccccccccccccccc}
N \setminus k & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 2 & 0 & 2 & 0 & 3 \\
2 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 2 & 0 & 2 & 1 & 4 & 1 & 4 & 2 & 7 \\
3 & 0 & 0 & 1 & 0 & 1 & 1 & 2 & 1 & 4 & 1 & 4 & 3 & 6 & 3 & 10 & 4 & 11 \\
4 & 0 & 1 & 1 & 1 & 2 & 2 & 4 & 4 & 6 & 5 & 9 & 8 & 13 & 12 & 18 & 16 & 25 \\
5 & 0 & 0 & 0 & 1 & 1 & 2 & 3 & 4 & 5 & 10 & 6 & 11 & 15 & 20 & 17 & 30 & 27 & 40 \\
6 & 0 & 1 & 1 & 2 & 2 & 4 & 4 & 7 & 7 & 11 & 11 & 16 & 16 & 24 & 24 & 33 & 33 & 45 \\
7 & 0 & 1 & 1 & 2 & 4 & 6 & 7 & 12 & 15 & 21 & 23 & 32 & 38 & 52 & 55 & 72 & 81 & 103 \\
8 & 0 & 1 & 2 & 3 & 3 & 6 & 7 & 12 & 14 & 20 & 22 & 32 & 36 & 48 & 54 & 69 & 76 & 97 \\
9 & 0 & 1 & 2 & 3 & 5 & 7 & 10 & 13 & 19 & 23 & 31 & 37 & 48 & 56 & 72 & 82 & 102 & 115 & 140 \\
10 & 0 & 1 & 2 & 3 & 5 & 10 & 12 & 19 & 26 & 36 & 42 & 58 & 70 & 92 & 105 & 132 & 132 & 189 \\
11 & 0 & 1 & 2 & 5 & 5 & 10 & 14 & 21 & 26 & 40 & 44 & 62 & 74 & 96 & 109 & 144 & 156 & 197 \\
13 & 0 & 1 & 2 & 3 & 4 & 6 & 7 & 12 & 14 & 20 & 22 & 32 & 36 & 48 & 54 & 69 & 76 & 97 & 127 \\
14 & 0 & 1 & 2 & 3 & 5 & 7 & 10 & 13 & 19 & 23 & 31 & 37 & 48 & 56 & 72 & 82 & 102 & 115 & 140 \\
15 & 0 & 1 & 2 & 5 & 5 & 10 & 14 & 21 & 26 & 40 & 44 & 62 & 74 & 96 & 109 & 144 & 156 & 197 & 232 \\
\end{array}
\]

\[
\dim J_{cusp}^{k, m}
\]

\[
\begin{array}{cccccccccccccccccccc}
m \setminus k & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 2 & 0 & 2 & 0 & 2 \\
2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 2 & 0 & 2 & 1 & 3 & 1 & 3 & 1 & 4 \\
3 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 2 & 1 & 3 & 1 & 3 & 2 & 4 & 2 & 5 & 2 & 5 \\
4 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 3 & 5 & 4 & 6 & 5 & 7 & 5 & 8 \\
5 & 0 & 0 & 0 & 0 & 1 & 2 & 6 & 7 & 8 & 9 & 10 & 12 & 12 & 15 & 16 & 18 & 18 & 19 & 20 \\
6 & 0 & 0 & 0 & 0 & 1 & 2 & 2 & 3 & 3 & 5 & 3 & 5 & 5 & 7 & 5 & 6 & 9 & 6 & 9 \\
7 & 0 & 1 & 1 & 1 & 2 & 6 & 7 & 8 & 9 & 10 & 11 & 11 & 13 & 12 & 15 & 15 & 17 & 18 & 20 \\
8 & 0 & 1 & 1 & 2 & 3 & 4 & 6 & 6 & 8 & 7 & 9 & 9 & 12 & 10 & 13 & 12 & 15 & 16 & 17 & 18 & 20 \\
9 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 6 & 7 & 8 & 9 & 10 & 11 & 13 & 12 & 15 & 16 & 17 & 18 & 19 & 21 \\
10 & 0 & 1 & 1 & 2 & 3 & 4 & 6 & 6 & 8 & 9 & 11 & 10 & 13 & 13 & 16 & 15 & 18 & 17 & 20 & 21 & 22 \\
12 & 0 & 1 & 2 & 3 & 4 & 6 & 6 & 8 & 11 & 10 & 13 & 13 & 16 & 15 & 18 & 17 & 20 & 21 & 22 \\
14 & 0 & 1 & 2 & 3 & 4 & 6 & 6 & 8 & 9 & 11 & 10 & 13 & 13 & 16 & 15 & 18 & 17 & 20 & 21 & 22 \\
15 & 0 & 1 & 2 & 4 & 4 & 6 & 7 & 9 & 9 & 12 & 11 & 14 & 14 & 17 & 16 & 20 & 18 & 20 & 21 & 22 \\
\end{array}
\]

9.2. Vector valued case

We give a table of dimensions of \( S_{k, j}(K(N)) \) with \( j \geq 2 \) by Main Theorem 1.1 (Theorem 3.1). The theorem is valid for \( k \geq 5 \), but we believe that the formula is still
applicable for $k = 3$ and 4. The values below for $k = 3$ and 4 are conjectural in most cases.

$$\dim S_{k,2}(K(N))$$

<table>
<thead>
<tr>
<th>$N \times k$</th>
<th>3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0 0 0 0 0 0 0 0 0 0 0 0 1 0 2 0 2 0 3</td>
</tr>
<tr>
<td>2</td>
<td>0 0 0 0 0 0 1 1 1 1 1 4 4 6 4 9 7 12</td>
</tr>
<tr>
<td>3</td>
<td>0 0 0 0 0 1 1 2 2 4 4 8 8 13 11 18 16 26</td>
</tr>
<tr>
<td>5</td>
<td>0 0 0 0 1 2 3 5 7 10 13 19 22 31 33 44 48 64</td>
</tr>
<tr>
<td>6</td>
<td>0 0 0 1 2 4 6 11 15 20 25 38 46 59 65 87 96 121</td>
</tr>
<tr>
<td>7</td>
<td>0 0 1 1 3 5 8 12 16 22 29 39 47 61 70 88 100 124</td>
</tr>
<tr>
<td>10</td>
<td>0 0 1 3 7 12 19 30 42 56 73 99 123 154 181 226 262 316</td>
</tr>
<tr>
<td>11</td>
<td>0 1 2 4 9 14 21 31 43 57 75 97 119 151 178 217 254 304</td>
</tr>
<tr>
<td>13</td>
<td>0 1 3 5 12 19 29 43 59 79 104 134 166 208 248 301 354 422</td>
</tr>
<tr>
<td>14</td>
<td>0 1 4 8 17 27 42 63 87 115 151 197 245 304 363 443 520 618</td>
</tr>
<tr>
<td>15</td>
<td>0 1 4 8 17 29 44 65 89 121 157 205 253 318 377 461 538 646</td>
</tr>
</tbody>
</table>

$$\dim S_{k,4}(K(N))$$

<table>
<thead>
<tr>
<th>$N \times k$</th>
<th>3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0 0 0 0 0 0 0 0 1 0 1 0 2 1 1 1 1 1 1 2 6</td>
</tr>
<tr>
<td>2</td>
<td>0 0 0 0 0 1 1 1 3 3 5 4 8 9 13 11 17 18 26</td>
</tr>
<tr>
<td>3</td>
<td>0 0 0 0 1 0 2 3 5 5 10 8 16 17 21 24 35 43 49</td>
</tr>
<tr>
<td>5</td>
<td>0 1 1 3 4 7 10 16 18 27 31 43 50 65 73 92 102 127</td>
</tr>
<tr>
<td>6</td>
<td>0 0 1 4 6 12 17 25 34 49 54 78 95 117 135 170 191 239</td>
</tr>
<tr>
<td>7</td>
<td>0 1 1 5 7 12 18 28 34 49 59 79 95 120 138 172 196 237</td>
</tr>
<tr>
<td>10</td>
<td>0 3 5 13 22 35 50 74 96 131 161 209 256 317 370 450 524 623</td>
</tr>
<tr>
<td>11</td>
<td>1 4 6 15 22 35 51 73 93 127 157 201 245 302 355 430 498 589</td>
</tr>
<tr>
<td>13</td>
<td>1 5 9 20 31 49 71 101 131 176 220 280 342 420 497 598 696 820</td>
</tr>
<tr>
<td>14</td>
<td>1 6 12 27 45 70 101 145 191 255 319 407 500 613 725 872 1020 1201</td>
</tr>
<tr>
<td>15</td>
<td>0 5 11 28 42 71 104 148 194 264 326 422 515 632 750 907 1049 1246</td>
</tr>
</tbody>
</table>
### 9.3. Paramodular new forms

We give tables of dimensions of $S_{k,j}^{\text{new}}(K(N))$ for $N \geq 2$ given in Section 4 by Proposition 4.4. (For $N = 1$, we have $S_{k,j}^{\text{new}}(K(1)) = S_{k,j}(K(1))$ by definition.) Since the space of new forms contain the lifts, we denote by $S_{k}^{\text{new}}(K(N))$ the orthogonal complement in $S_{k,j}^{\text{new}}(K(N))$ of the space spanned by the lifts from $S_{2k-2}^{\text{new},-}(\Gamma_0^1(N))$ and we also give a table of $\dim S_{k}^{\text{new}}(K(N))$.

#### $\dim S_{k,0}^{\text{new}}(K(N))$

<table>
<thead>
<tr>
<th>$N \backslash k$</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
<th>19</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>9</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>11</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>12</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>13</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>14</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>15</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

#### $\dim S_{k}^{\text{new}}(K(N))$

<table>
<thead>
<tr>
<th>$N \backslash k$</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
<th>19</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>9</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>11</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>12</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>13</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>14</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>15</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
\[ \dim S^\text{new}_{k,2}(K(N)) \]

<table>
<thead>
<tr>
<th>N &amp; k</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
<th>19</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>4</td>
<td>2</td>
<td>4</td>
<td>4</td>
<td>7</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>4</td>
<td>4</td>
<td>6</td>
<td>8</td>
<td>9</td>
<td>11</td>
<td>14</td>
<td>16</td>
<td>20</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>7</td>
<td>10</td>
<td>13</td>
<td>17</td>
<td>22</td>
<td>27</td>
<td>33</td>
<td>40</td>
<td>48</td>
<td>57</td>
<td>57</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>4</td>
<td>5</td>
<td>9</td>
<td>15</td>
<td>18</td>
<td>22</td>
<td>29</td>
<td>35</td>
<td>41</td>
<td>50</td>
<td>57</td>
<td>57</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>8</td>
<td>12</td>
<td>16</td>
<td>22</td>
<td>29</td>
<td>37</td>
<td>47</td>
<td>57</td>
<td>70</td>
<td>84</td>
<td>100</td>
<td>118</td>
</tr>
<tr>
<td>10</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>8</td>
<td>13</td>
<td>18</td>
<td>26</td>
<td>34</td>
<td>45</td>
<td>57</td>
<td>71</td>
<td>88</td>
<td>107</td>
<td>128</td>
<td>152</td>
<td>178</td>
</tr>
<tr>
<td>11</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>9</td>
<td>14</td>
<td>21</td>
<td>31</td>
<td>43</td>
<td>57</td>
<td>75</td>
<td>95</td>
<td>119</td>
<td>147</td>
<td>178</td>
<td>213</td>
<td>254</td>
<td>298</td>
</tr>
<tr>
<td>13</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>12</td>
<td>19</td>
<td>29</td>
<td>43</td>
<td>57</td>
<td>79</td>
<td>104</td>
<td>132</td>
<td>166</td>
<td>204</td>
<td>248</td>
<td>297</td>
<td>354</td>
<td>416</td>
</tr>
<tr>
<td>14</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>6</td>
<td>11</td>
<td>17</td>
<td>26</td>
<td>37</td>
<td>53</td>
<td>69</td>
<td>91</td>
<td>115</td>
<td>143</td>
<td>178</td>
<td>215</td>
<td>257</td>
<td>306</td>
<td>358</td>
</tr>
<tr>
<td>15</td>
<td>0</td>
<td>1</td>
<td>4</td>
<td>8</td>
<td>15</td>
<td>23</td>
<td>36</td>
<td>51</td>
<td>71</td>
<td>93</td>
<td>123</td>
<td>155</td>
<td>193</td>
<td>238</td>
<td>289</td>
<td>345</td>
<td>410</td>
<td>480</td>
</tr>
</tbody>
</table>

\[ \dim S^\text{new}_{k,4}(K(N)) \]

| N \& k | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|--------|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|----|
| 2      | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 3 | 3 | 4 | 4 | 7 | 7 | 9 | 9 | 14 | 14 |
| 3      | 0 | 0 | 0 | 1 | 0 | 2 | 3 | 3 | 5 | 8 | 8 | 12 | 15 | 17 | 22 | 27 | 37 |
| 5      | 0 | 1 | 1 | 3 | 4 | 7 | 10 | 14 | 18 | 25 | 31 | 39 | 48 | 57 | 79 | 94 | 98 | 115 |
| 6      | 0 | 1 | 2 | 4 | 6 | 9 | 13 | 18 | 23 | 30 | 38 | 47 | 57 | 69 | 82 | 97 | 113 |
| 7      | 1 | 1 | 5 | 7 | 12 | 18 | 26 | 34 | 47 | 59 | 75 | 93 | 114 | 136 | 164 | 192 | 225 |
| 10     | 0 | 1 | 3 | 7 | 12 | 19 | 28 | 40 | 54 | 71 | 91 | 115 | 142 | 173 | 208 | 248 | 292 | 341 |
| 11     | 1 | 4 | 6 | 15 | 23 | 35 | 51 | 71 | 93 | 125 | 157 | 197 | 234 | 296 | 353 | 422 | 494 | 577 |
| 13     | 1 | 5 | 9 | 20 | 31 | 49 | 71 | 99 | 131 | 174 | 220 | 276 | 340 | 414 | 495 | 590 | 692 | 808 |
| 14     | 1 | 4 | 10 | 17 | 29 | 44 | 63 | 87 | 117 | 151 | 193 | 241 | 296 | 359 | 431 | 510 | 600 | 699 |
| 15     | 0 | 3 | 9 | 20 | 34 | 53 | 78 | 110 | 148 | 194 | 248 | 312 | 385 | 468 | 562 | 669 | 787 | 918 |

\[ \dim S^\text{new}_{k,6}(K(N)) \]

| N \& k | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|--------|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|----|
| 2      | 0 | 0 | 0 | 0 | 0 | 2 | 2 | 4 | 3 | 5 | 7 | 10 | 9 | 13 | 14 | 20 | 19 |
| 3      | 0 | 0 | 1 | 2 | 3 | 7 | 7 | 10 | 14 | 16 | 21 | 27 | 30 | 37 | 45 | 51 | 59 |
| 5      | 0 | 0 | 3 | 4 | 7 | 11 | 17 | 22 | 31 | 39 | 50 | 63 | 77 | 92 | 112 | 131 | 154 | 179 |
| 6      | 0 | 0 | 4 | 7 | 12 | 15 | 23 | 30 | 40 | 51 | 62 | 74 | 94 | 111 | 133 | 154 | 180 |
| 10     | 0 | 4 | 7 | 11 | 18 | 26 | 38 | 50 | 67 | 85 | 105 | 133 | 162 | 194 | 232 | 272 | 319 | 369 |
| 13     | 0 | 4 | 10 | 15 | 26 | 37 | 53 | 75 | 94 | 126 | 160 | 195 | 240 | 281 | 343 | 407 | 476 | 550 |
| 14     | 0 | 4 | 15 | 29 | 48 | 75 | 105 | 145 | 193 | 249 | 315 | 390 | 477 | 575 | 686 | 810 | 945 | 1098 |
| 15     | 2 | 10 | 23 | 43 | 70 | 105 | 147 | 201 | 265 | 341 | 429 | 530 | 647 | 779 | 926 | 1092 | 1275 | 1478 |

**References**


[PSY] C. Poor, J. Shurman and D. Yuen, Computations with Siegel modular forms and an introduction to the geometry of numbers, in preparation.


Tomoyoshi Ibukiyama
Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka, Osaka, 560-0043 Japan
E-mail: ibukiyam@math.sci.osaka-u.ac.jp

Hidetaka Kitayama
Department of Mathematics, Faculty of Education, Wakayama University, Sakaedani 930, Wakayama-City, 640-8510 Japan
E-mail: hkitayam@center.wakayama-u.ac.jp