More on 2-chains with 1-shell boundaries in rosy theories

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Abstract. In [4], B. Kim, and the authors classified 2-chains with 1-shell boundaries into either RN (renamable)-type or NR (non renamable)-type 2-chains up to renamability of support of subsummands of a 2-chain and introduced the notion of chain-walk, which was motivated from graph theory: a directed walk in a directed graph is a sequence of edges with compatible condition on initial and terminal vertices between sequential edges. We consider a directed graph whose vertices are 1-simplices whose supports contain 0 and edges are plus/minus of 2-simplices whose supports contain 0. A chain-walk is a 2-chain induced from a directed walk in this graph. We reduced any 2-chains with 1-shell boundaries into chain-walks having the same boundaries.

In this paper, we reduce any 2-chains of 1-shell boundaries into chain-walks of the same boundary with support of size 3. Using this reduction, we give a combinatorial criterion determining whether a minimal 2-chain is of RN- or NR-type. For a minimal RN-type 2-chains, we show that it is equivalent to a 2-chain of Lascar type (coming from model theory) if and only if it is equivalent to a planar type 2-chain.

1. Introduction

In [2][3], J. Goodrick, B. Kim, and A. Kolesnikov defined homology groups of a strong type \( p \in S(A) \) in any rosy theory \( T \) and they addressed that those groups are related with amalgamation property. More precisely, they proved that the \((n - 1)\)-th homology group of a strong type \( p \) consists of \((n - 1)\)-shells of \( p \) with support \( n + 1 = \{0, \ldots, n\} \) whenever \( T \) has \( n \)-CA over \( A = acl(A) \) \((n \geq 2)\). In particular the first homology group consists of 1-shells of \( p \). Therefore as is known if \( T \) is simple then due to 3-amalgamation the first homology group is trivial. Moreover in [4], B. Kim, and the authors proved that in any rosy theory \( T \), the first homology group of a Lascar type \( p \) is also trivial.

We classified 2-chains with a 1-shell boundary into two types: NR (non-renamable)- and RN (renamable)-types, and we reduce 2-chains with 1-shell boundaries into chain-walks having the same boundaries. Using this classification, we showed that the minimal lengths of 2-chains with 1-shell boundaries are not bounded in rosy theories.

In this paper, we give geometric and combinatorial criteria determining the types of 2-chains. Using the notion of matrix expression, we give a combinatorial criterion for determining whether a given minimal 2-chain having a 1-shell boundary is of RN-type. We deduce that when the length of a given 2-chain is 3 modulo 4, the given chain must be of RN-type. We also show that a Lascar 2-chain (a model theoretic notion crucially used in the proof of Fact 1.9) is equivalent to a planar 2-chain (a geometric notion). We are working in model theoretic setting but our classification results hold in more general categorical setting, that is, an amenable class of functors in [3].

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In the rest of this section, we review some notions and facts from [2],[3] and [4]. We first recall the definitions of simplices and the corresponding homology groups introduced in [2],[3]. Throughout we work with a large saturated model \( \mathcal{M} = \mathcal{M}^{eq} \) whose theory \( T \) is rosy with the thorn-independence relation \( \mathcal{I} \) on the small sets of \( \mathcal{M} \).

From now on, we fix a small algebraically closed set \( A = \text{acl}(A) \) and \( p(x) \in S(A) \) (with possibly infinite \( x \)). Let \( C_A \) denote the category, where

1. objects are small subsets of \( \mathcal{M} \) containing \( A \), and
2. morphisms are elementary maps which fix \( A \) pointwise.

For some finite \( s \subseteq \omega \), the power set of \( s \), \( P(s) \) forms the category as an ordered set:

1. \( \text{Ob}(P(s)) = P(s) \), and
2. for \( u, v \in P(s) \), \( \text{Mor}(u, v) = \{ \iota_{u,v} \} \), where \( \iota_{u,v} \) is the single inclusion map for \( u \subseteq v \), or \( = \emptyset \) otherwise.

For a functor \( f : P(s) \to C_A \) and \( u \subseteq v \in P(s) \), we write \( f^u_v : f(\emptyset) \supseteq A \) and for \( i \in s \), \( f(\{i\}) \) is of the form \( \text{acl}(C_a) \) where \( a(\models p) \) is independent with \( C = f^{\emptyset}_{\{i\}}(\emptyset) \) over \( A \).

**Definition 1.1.** A functor \( f : P(s) \to C_A \) for some finite \( s \subseteq \omega \) is said to be a closed independent (regular) \( n \)-simplex in \( p \) if

1. \( |s| = n + 1 \)
2. \( f(\emptyset) \supseteq A \); and for \( i \in s \), \( f(\{i\}) \) is of the form \( \text{acl}(C_a) \) where \( a(\models p) \) is independent with \( C = f^{\emptyset}_{\{i\}}(\emptyset) \) over \( A \).
3. For all non-empty \( u \in P(s) \), we have

\[
    f(u) = \text{acl}(A \cup \bigcup_{i \in u} f^{\{i\}}_{\{i\}}(\{i\}));
\]

and \( \{ f^{\{i\}}_{\{i\}}(\{i\}) \mid i \in u \} \) is independent over \( f^u_{\emptyset}(\emptyset) \).

We shall call a closed independent \( n \)-simplex simply by an \( n \)-simplex. The set \( s \) is called the support of \( f \), denoted by \( \text{supp}(f) \).

Let \( S_n(p) \) denote the collection of all \( n \)-simplices in \( p \) and \( C_n(p) \) the free abelian group generated by \( S_n(p) \); its elements are called \( n \)-chains in \( p \).

A non-zero \( n \)-chain \( c \) is uniquely written (up to permutation of terms) as \( c = \sum_{1 \leq i \leq k} n_i f_i \), where \( n_i \) is a non-zero integer and \( f_1, \ldots, f_k \) are distinct \( n \)-simplices. (This form is called the standard form of the chain \( c \).) We call \( |c| = |n_1| + \cdots + |n_k| \) the length of the chain \( c \), and define the support of \( c \) as the union of \( \text{supp}(f_i) \)'s.

We use \( a, b, c, \ldots, f, g, h, \ldots, \alpha, \beta, \ldots \) to denote simplices and chains. Now we define the boundary operators and using the boundary operators we will define homology groups.
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Definition 1.2. Let \( n \geq 1 \) and \( 0 \leq i \leq n \). The \( i \)-th boundary operator \( \partial_i^n : C_n(p) \to C_{n-1}(p) \) is defined so that if \( f \) is an \( n \)-simplex with domain \( \mathcal{P}(s) \) with \( s = \{s_0, \ldots, s_n\} \), then

\[
\partial_i^n(f) = f \upharpoonright \mathcal{P}(s \setminus \{s_i\})
\]

and extended linearly to all \( n \)-chains in \( C_n(p) \).

The boundary map \( \partial_n : C_n(p) \to C_{n-1}(p) \) is defined by the rule

\[
\partial_n(c) = \sum_{0 \leq i \leq n} (-1)^i \partial_i^n(c).
\]

We write \( \partial_i^{\bullet} \) and \( \partial \) for \( \partial_i^n \) and \( \partial_n \), respectively, if \( n \) is clear from context.

Definition 1.3. The kernel of \( \partial_n \) is denoted \( Z_n(p) \), and its elements are called \( (n-) \)cycles. The image of \( \partial_{n+1} \) in \( C_n(p) \) is denoted by \( B_n(p) \) and its elements are called \( (n-) \)boundaries.

Since \( \partial_n \circ \partial_{n+1} = 0 \), \( B_n(p) \subseteq Z_n(p) \) and we can define simplicial homology groups in \( p \).

Definition 1.4. The \( n \)-th (simplicial) homology group in \( p \) is

\[
H_n(p) := Z_n(p)/B_n(p).
\]

Definition 1.5. For \( n \geq 1 \), an \( n \)-chain \( c \) is called an \( n \)-shell if it is in the form

\[
c = \pm \sum_{0 \leq i \leq n+1} (-1)^i f_i,
\]

where \( f_0, \ldots, f_{n+1} \) are \( n \)-simplices such that whenever \( 0 \leq i < j \leq n+1 \), we have \( \partial_i^{\bullet} f_j = \partial^{j-1} f_i \). Specially, a 1-shell \( c \) is of the form

\[
c = f_0 - f_1 + f_2.
\]

Remark 1.6. The boundary of a 2-simplex is a 1-shell, and the boundary of any 1-shell is 0.

Definition 1.7. Let \( n \geq 0 \).

1. \( p \) has \((n+2)\)-amalgamation if any \( n \)-shell in \( p \) is the boundary of some \((n+1)\)-simplex in \( p \).

2. \( p \) has \((n+2)\)-complete amalgamation (or simply \((n+2)\)-CA) if \( p \) has \( k \)-amalgamation for every \( 2 \leq k \leq n+2 \).

By extension axiom of thorn-independence, whenever \( f : \mathcal{P}(s) \to \mathcal{C}_A \), \( g : \mathcal{P}(t) \to \mathcal{C}_A \in S(p) \) and \( f \upharpoonright \mathcal{P}(s \cap t) = g \upharpoonright \mathcal{P}(s \cap t) \), then \( f \) and \( g \) can be extended to a simplex \( h : \mathcal{P}(s \cup t) \to \mathcal{C}_A \) in \( p \). This property is called strong \((n+2)\)-amalgamation.

The following fact shows why the notion of shells is important.

Fact 1.8. \([2],[3]\) If \( p \) has \((n+1)\)-CA for some \( n \geq 1 \), then
\[ H_n(p) = \{ [c] : c \text{ is an n-shell over } A \text{ with } \text{supp}(c) = \{0, \ldots, n+1\} \}. \]

We have that \( H_1(p) \) is trivial iff any 1-shell in \( p \) is the boundary of some 2-chain in \( p \). Therefore, if \( T \) is simple, due to 3-amalgamation \( H_1(p) \) is trivial. The following shows that the same result holds in any rosy theory.

**FACT 1.9.** [4] Suppose that \( p \) is any Lascar strong type. Then \( H_1(p) = 0 \).

There are two fundamental operations used in the classification of 2-chains in [4]:

- **crossing** and
- **renaming-of-support operations**.

**Remark/Definition 1.10.** Given any bijection \( \sigma : \omega \to \omega \) (not necessarily order-preserving), we may define an automorphism \( \sigma^*_n : C_n(p) \to C_n(p) \) for each \( n \) as follows:

For any \( n \)-chain \( c = \sum_i n_if_i \in C_n(p) \), where each \( f_i \) is an \( n \)-simplex with \( s_i := \text{supp}(f_i) = \{ s_{i,0} < \cdots < s_{i,n} \} \), we let \( \sigma_i := \sigma \upharpoonright s_i \) and \( t_i := \sigma_i(s_i) = \{ t_{i,0} < \cdots < t_{i,n} \} \). We define

\[
\sigma^*(c) := \sum_i n_i|\sigma_i|f_i \circ \sigma_i^{-1}
\]

with \( |\sigma_i| := \text{sign}(\sigma_i') = (\pm 1) \) where \( \sigma_i' \in \text{Sym}(n+1) \) such that for \( j \leq n \), \( \sigma_i(s_{i,j}) = t_{i,\sigma_i'(j)} \). For example

\[
\sigma^*(f_i) = |\sigma_i|f_i \circ \sigma_i^{-1}.
\]

Moreover, \( \sigma^* \) commutes with the boundary map, i.e., \( \partial \circ \sigma^* = \sigma^* \circ \partial \).

**Definition 1.11.** Let \( v \in C_2(p) \) be a 2-chain and let \( w := \epsilon_1\alpha_1 + \epsilon_2\alpha_2 \) be a subsummand of \( v \), where \( \alpha_i \)'s are \( 2 \)-simplices with for \( i = 1, 2 \), \( \epsilon_i = \pm 1 \), \( \text{supp}(\alpha_i) = \{ \ell_1, \ell_2, k_i \} \) (\( k_i, \ell_i \) being all distinct numbers) such that \( \alpha_1 \) and \( \alpha_2 \) agree on the intersection of their domains, namely \( \mathcal{P}(\{ \ell_1, \ell_2 \}) \). Further assume that, if we let \( \gamma := \alpha_i \upharpoonright \mathcal{P}(\{ \ell_1, \ell_2 \}) \), then \( \gamma \) does not appear in \( \partial(v) \), i.e., the two \( \gamma \) terms in \( \partial(v) \) have opposite signs and cancel each other.

Now by strong 2-amalgamation, there exists a 3-simplex \( \mu \) extending both \( \alpha_i \). For \( i = 1, 2 \), let \( \beta_i := \mu \upharpoonright \mathcal{P}(\{ k_1, k_2, \ell_1 \}) \) and

\[
w' := \begin{cases} 
\epsilon_2 \beta_1 + \epsilon_1 \beta_2 & \text{if } \epsilon_1\epsilon_2 = -1, \text{ and exactly one of } k_2, \ell_1 \text{ is in between } k_1, \text{ and } \ell_2 \\
\epsilon_1 \beta_1 + \epsilon_2 \beta_2 & \text{otherwise}.
\end{cases}
\]

Then the operation of replacing the subsummand \( w \) in \( v \) by \( w' \) is called the **crossing operation** (or simply **CR-operation**).

**Definition 1.12.** Let \( c \) be an \( n \)-chain in \( C_n(p) \) and let \( d \) be a subsummand of \( c \). Let \( j \in \text{supp}(d) \) such that \( j \notin \text{supp}(\partial_n(d)) \). (In this situation, we say that \( d \) has a vanishing support, namely \( j \), in its boundary.) Choose any \( k \notin \text{supp}(c) \) and any bijection \( \sigma : \omega \to \omega \) which sends \( j \mapsto k \) but which fixes the rest of the elements in \( \text{supp}(c) \). Then the operation of replacing the subsummand \( d \) in \( c \) by \( \sigma^*_n(d) \) is called the **renaming-of-support operation** (or simply **RS-operation**). (See Remark/Definition 1.10 to recall the definition of \( \sigma^*_n \).)

**Remark/Definition 1.13.** A 2-chain \( c \) is called **proper** if its length \( |c| \) does not change after any finitely many applications of CR/RS-operations to its subsummands.
This allows us to define an equivalence relation \( \sim \) among proper 2-chains by: \( c \sim c' \iff c \) can be obtained from \( c' \) by finitely many applications of the CR/RS-operations to its subsummands. Note that \( c \sim c' \) implies \( \partial(c) = \partial(c') \) and \( |c| = |c'| \). A proper 2-chain \( \alpha \) is said to be minimal if for any proper 2-chain \( \alpha' \) with \( \alpha' \sim \alpha \) there does not exist a subsummand \( \beta \) of \( \alpha' \) such that \( \partial(\beta) = 0 \).

We classify 2-chains into two types.

**Definition 1.14.** Let \( \alpha \) be a 2-chain having a 1-shell boundary. We call \( \alpha \) renamable type (or RN-type) if a subsummand of \( \alpha \) has a vanishing support. If \( \alpha \) is not an RN-type 2-chain (so \( |\supp(\alpha)| = 3 \)) we call \( \alpha \) non-renamable (NR-) type.

**Notation.** Let \( f \) be any simplex. For any subset \( \{j_0, \ldots, j_k\} \subseteq \supp(f) \), we shall abbreviate \( f \upharpoonright \mathcal{P}(\{j_0, \ldots, j_k\}) \) as \( f_{j_0, \ldots, j_k} \). Also, given a chain \( c = \sum_{i \in I} n_i f_i \) (in its standard form), and any subset \( \{j_0, \ldots, j_k\} \subseteq \supp(c) \), we shall write \( c_{j_0, \ldots, j_k} \) to denote the subchain \( \sum_{i \in J} n_i f_i \), where \( J := \{i \in I \mid \supp(f_i) = \{j_0, \ldots, j_k\} \} \).

**Remark 1.15.** If \( \alpha \) is any 2-chain with a 1-shell boundary, then its length is always an odd positive number.

Now we introduce the notion of a chain-walk. At first, we recall the notion of directed walk from graph theory.

**Definition 1.16.** A directed (multi) graph, abbreviated to digraph, \( G \) is a pair \((V, E)\) of disjoint sets (of vertices and edges) together with two maps \( \init : E \to V \) and \( \ter : E \to V \) assigning to every edge \( e \in E \) an initial vertex \( \init(e) \) and a terminal vertex \( \ter(e) \). The edge \( e \) is said to be (directed) from \( \init(e) \) to \( \ter(e) \) (or is written as \( e : \init(e) \to \ter(e) \)). Indeed, we loosely write \( e \) as a triple \((\init(e), \ter(e), \init(e))\). For any \( e \in E \), let \( -e \) denote the triple \((\ter(e), \init(e), \init(e))\). We may write \( G = (V, \{\pm e \mid e \in E\}) \).

A directed walk in \( G \) from a vertex \( v_0 \) to a vertex \( v_{n+1} \) is a sequence of the form \((e_0v_0, \ldots, e_nv_n)\) where \( e_i = \pm 1 \) and \( e_i \in E \) for each \( i \), such that \( v_0 = \init(e_0v_0) \), \( v_{n+1} = \ter(e_nv_n) \) and \( \ter(e_{i+1}) = \init(e_{i+1}v_{i+1}) \) for all \( i = 0, \ldots, n-1 \).

**Remark/Definition 1.17.** Let \( (x_i)_{i=0}^n \) be a finite sequence of natural numbers with \( n \geq 0 \). For \( k \leq n \), the \( k \)-th sign of this sequence is the number \((-1)^k \), where \( \lfloor k \rfloor \) is the number of places \( x_j < x_{j+1} \) in the sequence of \( (x_i)_{i=0}^n \) with \( x_{n+1} = x_0 \), denoted by \( \sign_k(x_i)_{i=0}^n \).

Fix \( n_0 \geq 0 \). Consider a digraph \( G_{n_0} = (V_{n_0}, E_{n_0}) \), where \( V_{n_0} \) is the set of 1-simplices with support containing \( n_0 \). Let \( s_j \in V_{n_0}(j = 1, 2) \) be with \( \supp(s_j) = \{n_0, n_j\} \). An edge from \( s_0 \) to \( s_1 \) is the 2-chain \( ea \), where \( a \) is a 2-simplex with \( \supp(a) = \{n_0, n_1, n_2\} \) such that \( a_{n_0, n_1} = s_j \), and \( \epsilon = \sign(n_i)_{i=0}^3 \). In this case, if \( n_1 = n_2 \), there are no edges between \( s_1 \) and \( s_2 \). For \( s, t \in V_{n_0} \), a 2-chain \( \alpha \) is called a chain-walk from \( s \) to \( t \) if it is of the form \( \sum \epsilon a_i \), where \( (\epsilon_i a_i) \) is a directed walk in \( G_{n_0} \) from \( s \) to \( t \). So, if \( \supp(s) = \{n_0, n_1\} \) and \( \supp(t) = \{n_0, n_2\} \), then \( (\partial \alpha)_{n_0, n_1} = -\sign_0(n_0, n_1)s \) and \( (\partial \alpha)_{n_0, n_2} = \sign_0(n_0, n_2)t \) (*).

To emphasize (*), we call \( a \) a chain-walk from \( -\sign_0(n_0, n_1)s \) to \( \sign_0(n_0, n_2)t \) instead of a chain-walk from \( s \) to \( t \).

Let \( \beta \) be a 2-chain which is a chain-walk. Then there is a sum \( \sum_{i=0}^m \epsilon_i b_i \) with respect
to the order of indices, which ensures the propery of a chain-walk; for $0 \leq i < m$, $(\partial \epsilon_i b_i)^{n,n_{i+1}} + (\partial \epsilon_{i+1} b_{i+1})^{n,n_{i+1}} = 0$, where each supp$(b_i) = \{n, n_i, n_{i+1}\}$. We call such sum a chain-walk representation of $\beta$, simply a representation. By a section of the chain-walk $\beta$, we shall mean a subchain of $\beta$ of the form

$$\beta' := \sum_{i=j}^{m'} \epsilon_i b_i \quad \text{for some } 0 \leq j < m' \leq m.$$ 

Let $\beta$ be a 2-chain which is a chain-walk. Then there is a sum $\sum_{i=0}^m \epsilon_i b_i$ making $\beta$ a chain-walk, and this sum is called a chain-walk representation of $\beta$, simply a representation. By a section of the chain-walk $\beta$, we shall mean a subchain of $\beta$ of the form

$$\beta' := \sum_{i=j}^{m'} \epsilon_i b_i \quad \text{for some } 0 \leq j < m' \leq m.$$ 

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**Figure 1:** An example of a chain-walk 2-chain

For the rest of this paper, we fix a 1-shell boundary $f_{12} - f_{02} + f_{01}$ with supp$(f_{jk}) = \{j < k\}$. We reduce 2-chains having 1-shell boundaries into chain-walks with the same boundaries.

**Fact 1.18.** [4] Let $\alpha$ be a minimal 2-chain with the boundary $f_{12} - f_{02} + f_{01}$.

1. Assume $\alpha$ is of NR-type. Then $|\alpha| = 1$ or $|\alpha| \geq 5$. If $|\alpha| \geq 5$ then any chain-walk in $\alpha$ from $f_{01}$ to $-f_{02}$ is of the form $\sum_{i=0}^{2n} (-1)^i a_i$ which is as a chain equal to $\alpha$ such that $f_{12} = a_j^{1/2}$ for some $1 \leq j \leq n - 1$.

2. $\alpha$ is of RN-type iff $\alpha$ is equivalent to a 2-chain

$$\alpha' = a_0 + \sum_{i=1}^{2n-1} \epsilon_i a_i + a_{2n}$$

($n \geq 1$) which is a chain-walk from $f_{01}$ to $-f_{02}$ such that supp$(a_{2n}) = \{0, 1, 2\}$ and $\partial^0 a_{2n} = f_{12}, \partial^1(a_{2n}) = -f_{02}$. 
2. Reduction to 2-chains with support \(\{0, 1, 2\}\)

Next we show that any 2-chain having a 1-shell boundary can be reduced to a 2-chain with support \(\{0, 1, 2\}\). From this, any 2-chain having the 1-shell boundary \(f_{12} - f_{02} + f_{01}\) is reduced to a 2-chain of the form \(\sum_{i=0}^{2n} (-1)^i a_i\), which is a chain-walk from \(f_{01}\) to \(-f_{02}\), and its support is \(\{0, 1, 2\}\). This reduction is crucially used to develop a combinatorial criterion for RN-type 2-chains in the next section.

First we digress to recall some graph theoretical notions related with directed walk, and we consider a directed walk in a digraph, related with a given chain-walk, which is different from one in Remark/Definition 1.17.

**Definition 2.1.** Let \(G = (V, E)\) be a digraph and consider a directed walk in \(G\) from a vertex \(v_0\) to a vertex \(v_{n+1}\) of the form \((\epsilon_0 e_0, \ldots, \epsilon_n e_n)\) where \(\epsilon_i = \pm 1\) and \(\epsilon_i \in E\) for each \(i\). A directed walk from \(v_0\) to \(v_{n+1}\) is closed if \(v_0 = v_{n+1}\); is reduced to \(\epsilon_0 e_0\) if the commutative sum \(\sum_{i \leq n} \epsilon_i e_i = \epsilon_0 e_0 (0 \leq i_0 \leq n)\); and is a balanced walk (from \(v_0\) to \(v_0\)) if \(\sum_{i \leq n} \epsilon_i e_i = 0\). A balanced walk is closed.

**Remark 2.2.** Let \(\beta = \sum_{i \in J} \epsilon_i b_i\) be the chain-walk in Remark/Definition 1.17 from \(f_{01}\) to \(-f_{02}\), and let \(\beta' = \sum_{i \in J} \epsilon_i b_i\) be a section of \(\beta\). Then \(\beta'\) canonically induces a directed walk in \((V, E)\) where \(V = \text{supp}(\beta') \setminus \{0\}\) and \(E = \{\pm b_i^{k_i} k_{i+1} \mid i \in J\}\) with

\[
(\partial \epsilon_i b_i)^{k_i} : k_i \rightarrow k_{i+1}.
\]

(We may also loosely write \(\epsilon_i b_i : k_i \rightarrow k_{i+1}\).) Such a directed walk is called the induced walk from \(\beta'\).

Hence the directed walk induced from the chain-walk \(\sum_{i \leq 2n} (-1)^i a_i\) in Fact 1.18(1), is reduced to the edge \(f_{12}\). The induced walk from the section \(a_0 + \sum_{i=1}^{2n-1} \epsilon_i a_i\) of \(\alpha'\) in Fact 1.18(2), is balanced.

We aim now to further reduce a given minimal 2-chain with a 1-shell boundary to an equivalent one having a support of size 3.

**Theorem 2.3.** Let \(\alpha\) be a minimal 2-chain having the 1-shell boundary \(f_{12} - f_{02} + f_{01}\). Then \(\alpha\) is of RN-type iff \(\alpha\) is equivalent to a 2-chain

\[
\alpha' = \sum_{i=0}^{2n} (-1)^i a_i
\]

\((n \geq 1)\) which is a chain-walk from \(f_{01}\) to \(-f_{02}\) such that \(\text{supp}(\alpha') = \{0, 1, 2\}\) and \(\partial \alpha' a_{2n} = f_{12}, \partial^I (a_{2n}) = -f_{02}\).

**Proof.** Suppose that \(\alpha = \sum_{i \leq 2n} \epsilon_i b_i + a_{2n}\) (\(\epsilon_0 = 1\)) is a chain-walk from \(f_{01}\) to \(-f_{02}\) such that \(\text{supp}(a_{2n}) = \{0, 1, 2\}\) and \(\partial \partial a_{2n} = f_{12}, \partial^I (a_{2n}) = -f_{02}\). Due to Fact 1.18, it suffices to show that there is \(\alpha' = \sum_{i \leq 2n} \epsilon'_i a_i + a_{2n}\) \(\sim \alpha\) as in the theorem (then it automatically follows \(\epsilon'_i = (-1)^i\)). We show this using induction on \(|s(= s_\alpha)|\) where \(s := \text{supp}(\beta) \setminus \{0\}\) with \(\beta := \alpha - a_{2n}\). Note that \(1 \in s\) and \(0 \notin s\). As was pointed out in Remark 2.2, the induced walk from \(\beta\) is a balanced walk in \(V = s\) from 1 to 1 (*). If
$|s| = 2$ then $s = \{1 < k\}$ and $k$ vanishes in $\partial(\beta)$. Hence by the RS-operation we rename $k$ to 2 and we are done. Now let $|s| = m + 1$ with the induction hypothesis for $m \geq 2$ (**).

Fix $j(\neq 1) \in s$. We are now going to inductively remove all the edges induced from $\beta$, connecting 1 and $j$. For $s_0 \neq s_1 \in s$, let

$$I^\alpha_{s_0, s_1} := \{i < 2n \mid \epsilon_i b_i: s_0 \to s_1 \text{ or } \epsilon_i b_i: s_1 \to s_0\}$$

and

$$I^\alpha_{s_0} := \{i < 2n \mid \text{for some } k \in s, \epsilon_i b_i: s_0 \to k \text{ or } \epsilon_i b_i: k \to s_0\}$$

Let $I := I_{1,j}$. We use induction on $|I|$. Due to (*), $|I|$ is even. Assume $|I| = 0$. Then 1 $\notin I$ and in particular 1 $\notin \text{supp}(\beta_j)$ where $\beta_j := \sum_{i \in E} \epsilon_i b_i$. Now due to (*), it follows $j$ vanishes in $\partial(\beta_j)$. Then by applying the RS-operation to $\beta_j$, we rename $j$ to 1, and obtain $\alpha'' \sim \alpha$ with $|s_{\alpha''}| = m$. Then due to (**), there is a desired $\alpha' \sim \alpha''$.

Now let $|I| = m' + 2$ with the induction hypothesis that if $|I| = m'$ then we can find a desired $\alpha' \sim \alpha$ (**). Since $|s| \geq 3$, there are $j(\neq 1, j) \in s$ and $\ell \in I$ such that either of $\varepsilon_{\ell+1} b_{\ell+1}: j \to j', \varepsilon_{\ell+1} b_{\ell+1}: 1 \to j'$, $\varepsilon_{\ell-1} b_{\ell-1}: j' \to j$, or $\varepsilon_{\ell-1} b_{\ell-1}: j' \to 1$. We will only show for the case $\varepsilon_{\ell+1} b_{\ell+1}: j \to j'$ (the other cases can be checked by similar arguments). Now due to (*), we have $\epsilon_0 b_0: 1 \to j$; and for some $t \in I$, $\epsilon_t b_t: j \to 1$, and $(\partial \epsilon_t b_t)^{1,j} + (\partial \epsilon_t b_t)^{1,j} = 0$.

We now can apply the CR-operation to $\epsilon_t b_t + \epsilon_{t+1} b_{t+1}$ and obtain $\epsilon_t' b_t' + \epsilon_{t+1}' b_{t+1}'$ preserving the boundary such that $\text{supp}(b_t') = \{0, 1, j'\}$ and $\text{supp}(b_{t+1}') = \{1, j, j'\}$. Hence we replace $\epsilon_t b_t + \epsilon_{t+1} b_{t+1}$ in $\beta$ by $\epsilon_t b_t'$, and obtain $\beta'$. Then $\beta' + a_{2n}$ is still a chain-walk, while the term $\epsilon_t' b_t' b_{t+1}'$ is left. We use it as follows. Since $(\partial \epsilon_t b_t)^{1,j} = (\partial \epsilon_{t+1} b_{t+1})^{1,j}$, we can apply the CR-operation to $\epsilon_t b_t + \epsilon_{t+1} b_{t+1}$, and obtain $\epsilon_t b_t' + \epsilon_{t+1}' b_{t+1}'$ with $\text{supp}(b_t') = \{0, 1, j'\}$ and $\text{supp}(b_{t+1}') = \{0, 1, j'\}$. Then we replace $\epsilon_t b_t$ in $\beta$ by $\epsilon_t b_t'$ and obtain $\beta''$. Thus $\alpha' := \beta' + a_{2n}$ is a chain-walk equivalent to $\alpha$. Note now $|I^{\alpha''}_{1,j}| = m'$. Hence due to the induction hypothesis (**), there is a desired $\alpha' \sim \alpha''$. \hfill \qed

**Corollary 2.4.** Suppose that $\alpha$ is a minimal 2-chain with the 1-shell boundary $f_{12} - f_{02} + f_{01}$. Then there is an equivalent minimal 2-chain $\alpha' = \sum_{i = 0}^{2n} (-1)^i a_i$ which is a chain-walk from $f_{01}$ to $-f_{02}$, and supp($\alpha'$) = $\{0, 1, 2\}$.

**Remark 2.5.** In this remark we bring to our attention an issue of counting number of possible directed walks induced from chain-walks of the form $\alpha'$ in Corollary 2.4. We use the same notation. Let $\alpha'$ be an RN-type 2-chain (so $n \geq 1$) such that $\partial^1 a_{2n} = f_{12}$, $\partial^1 a_{2n} = -f_{02}$. Let $\beta := \alpha' - a_{2n}$. Then $\beta$ induces a balanced walk $w$ (from 1 to 1) of length $2n$ on $V = \{1, 2\}$ with $E_+^\beta = E^+ := \{a_i^{1,2} \mid i < 2n\}$, the set of positive edges. So $|E^+| \leq n$, and $a_2^{1,2} : 1 \to 2$ for $a_2^{1,2} \in E^+$. One may ask how many such balanced walks of length $2n$ exist with a fixed set $E^+$. Let $|E^+| = m \leq n$, and rewrite $E^+ = \{b_1, \ldots, b_m\}$. Recall that the walk $w$ induced form $\beta$ with $E^+ = E_+^\beta$ is a sequence of the form $w^+ := \langle -1 \rangle e_i (i < 2n)$. So $w^+ := \langle e_0, e_2, \ldots, e_{2n-2} \rangle$ is an enumeration of $E^+$ using all the members. Also $w^- := \langle -e_1, -e_3, \ldots, -e_{2n-1} \rangle$ is an enumeration of
$E^- := \{-b_1, \ldots, -b_m\}$. Now the number of occurrence of a given $b_i$ in $w^+$ and $b_i$ in $w^-$ are the same, say $(0 <) r_i \leq n$. Hence given an assignment $r : E^+ \to \omega$ with $r_i = r(b_i)$ and $r_1 + \cdots + r_m = n$, there are at most
\[
\left( \begin{array}{c}
\binom{n}{r_1} \\
\binom{n-r_1}{r_2} \\
\vdots \\
\binom{n-(r_1+\cdots+r_{m-1})}{r_m}
\end{array} \right)^2 = \left( \frac{n!}{r_1! \cdots r_m!} \right)^2
\]
many possible such walks. Thus if we only fix $E^+$ (so $n, m$ fixed while $r$ and $r_1 > 0$ vary) then for $R := \{ r : E^+ \to \{1, \ldots, n\} \mid r_1 + \cdots + r_m = n \}$ there are at most
\[
\sum_{r \in R} \left( \frac{n!}{r_1! \cdots r_m!} \right)^2
\]
many possible induced walks.

A similar computation can be made when $\alpha'$ is an NR-type 2-chain.

3. Combinatorial criterion for RN-type 2-chains

In this section, by introducing the notion of *matrix expression*, we shall give a criterion for determining whether a given minimal 2-chain having a 1-shell boundary is of RN-type. We also give sufficient conditions for 2-chains to be of RN-type. According to Corollary 2.4, we are mainly interested in minimal 2-chains having 1-shell boundaries with support $\{0, 1, 2\}$, which are chain-walks. Let $\alpha$ be a chain-walk of length $2n + 1$ having the 1-shell boundary $f_{12} - f_{02} + f_{01}$ with $\text{supp}(\alpha) = \{0, 1, 2\}$. For $\{0, 1, 2\} = \{i, j, k\}$, $f'_i$ denotes $f_{jk}$ (j < k). Fix $I = \{0, 1, 2\}$ and $J = \{0, \ldots, n\}$. Also, we write $\epsilon a \in \alpha (\epsilon = \pm 1)$ to denote that a 2-simplex term $\epsilon a$ is in $\alpha$.

Now we assign a $3 \times (n + 1)$ matrix to the 2-chain as follows:

**Definition 3.1.** Let $\sum_{j=0}^{2n} (-1)^j a_j$ be a representation of the given $\alpha$ which itself is a chain-walk from $f'_2$ to $-f'_1$. By a *matrix expression* of (the representation of) $\alpha$, we mean a function $M : I \times J \to J$ such that

1. for each $i \in I$, $M \mid \{i\} \times J : (\{i\} \times) J \to J$ is a permutation of $J$;
2. for each $i \in I$, $M(i, 0)$ is an index such that $f'_i = \partial^i a_{2M(i, 0)}$;
3. for each $i \in I$, $j \in J \setminus \{0\}$, $M(i, j)$ is an index such that $\partial^i a_{2j-1} = \partial^i a_{2M(i, j)}$.

Interpret $M(i, j)$ as an entry $m_{ij}$ of a matrix in the $(i+1)$-th row and the $(j+1)$-th column, then $M = (m_{ij})_{i,j}$ is a $3 \times (n + 1)$ matrix. Obviously, given a chain-walk representation of $\alpha$, there is at least one (possibly more than one) matrix expression.

We may and will use $M(i, j)$ to represent both the image of $(i, j)$ under the function $M$, and the $(i+1, j+1)$-entry of the matrix $M$.

**Example 3.2.** Let $\alpha = \sum_{j=0}^{8} (-1)^j a_j$ be a chain-walk from $f_{01}$ to $-f_{02}$ such that $\partial^0 a_2 = f_{12}$, $\partial^1 a_8 = f_{02}$, $\partial^2 a_0 = f_{01}$, and $\partial^0 a_0 = \partial^0 a_4 = \partial^0 a_5 = \partial^0 a_7$, $\partial^0 a_1 =
\[ \partial^0 a_8, \partial^0 a_3 = \partial^0 a_6. \]  
Then both 
\[
M_1 = \begin{pmatrix} 1 & 4 & 3 & 0 & 2 \\ 4 & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 4 & 3 & 2 & 0 \\ 4 & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix}
\]
are matrix expressions of \( \alpha \) and \((1, 4), (1, 5)\) entries are swapped between \( M_1 \) and \( M_2 \).

Notice that matrix expressions are determined according to the *choices* of pairs of terms which cancel out each other. Therefore the second (similarly the third) row of a matrix expression need not always be of the form \((n \ 0 \ 1 \ \cdots \ n-1)\); even the \((2, 1)\) entry can be \( n'(n < n) \), if \( f_{02} = \partial^1 a_{2n'}(= \partial^1 a_{2n}) \).

Now we are ready to state a criterion for determining the type of \( \alpha \).

**Theorem 3.3.** Let \( \alpha \) be a minimal 2-chain of length \( 2n + 1 \) having the 1-shell boundary \( f_{12} - f_{02} + f_{01} \) with \( \text{supp}(\alpha) = \{0, 1, 2\} \), which is a chain-walk. Then \( \alpha \) is of RN-type if and only if there is a matrix expression \( M \) for a representation \( \sum_{j=0}^{2n} (-1)^j a_j \) such that for some \( 0 \leq i_0 < i_1 \leq 2 \), and non-empty \( J_0 \subseteq \{1, \ldots, n\} \), \( M(\{i_0\} \times J_0) = M(\{i_1\} \times J_0) \) as image set under the function \( M \).

**Proof.** \((\Rightarrow)\) Assume that \( \alpha = \sum_{j=0}^{2n} (-1)^j a_j \) is of RN-type. Then there is a subchain \( \alpha_1 \) of \( \sum_{j=0}^{2n} (-1)^j a_j \) having a vanishing support \( i_* \in \{0, 1, 2\} \). Hence \( \partial^0 \alpha_1 = 0 \) for \( i \in I \setminus \{i_*\} \)
and \( |\alpha_1| = 2m \). Therefore when we choose a matrix expression \( M \) satisfying Definition 3.1(3), we can let \( a_{2M(i,j)} \in \alpha_1 \) for each \( i \in I \setminus \{i_*\} \) and \( j \in J_0 := \{j \in J : -a_{2j-1} \in \alpha_1\} \).

Then by the choice, \( M(\{i_0\} \times J_0) = M(\{i_1\} \times J_0) \), where \( \{i_0, i_1, i_*\} = I \), as desired.

\((\Leftarrow)\) Let \( M \) be a given matrix expression of \( \alpha \) so that for \( J_0 \subseteq J \setminus \{0\} \) and \( 0 \leq i_0 < i_1 \leq 2 \), we have \( M(\{i_0\} \times J_0) = M(\{i_1\} \times J_0) \), say \( J_1 \). Let \( \alpha_1 := \sum_{j \in J_1} a_{2j} + \sum_{j \in J_0} -a_{2j-1} \), a subchain of \( \alpha \). Then for \( i_0, i_1 \), we have \( \partial^m \alpha_1 = \partial^0 \alpha_1 = 0 \), so \( \alpha_1 \) has a vanishing support \( i_* \), where \( \{i_0, i_1, i_*\} = I \). Therefore \( \alpha \) is of RN-type.

**Remark 3.4.** The matrix expression \( M_1 \) from Example 3.2 does not satisfy the right-hand side in the above theorem. However \( M_2 \) is also a matrix expression of \( \alpha = \sum_{j=0}^{2n} (-1)^j a_j \) as well and the fact that \( M_2(0, 3) = M_2(1, 3) \) ensures that the 2-chain \( \alpha \) is of RN-type.

Let \( M : I \times J \rightarrow J \) be a matrix expression. Then \( M \) induces a triple \((\sigma_{01}, \sigma_{12}, \sigma_{02})\) of permutations of \( J \) such that \( \sigma_{ik} \) is a map sending the \((i+1)\)-th row to the \((k+1)\)-th row, i.e., \( \sigma_{ik}(m_{ij}) = m_{kj} \) for \( j \in J \), and \( 0 \leq i < k \leq 2 \). Notice that \( \sigma_{02} = \sigma_{12} \circ \sigma_{01} \).

As is well-known that every element of the symmetric group \( S_{J_1} \) has the unique cycle decomposition (up to obvious permutations), where each \( j \in J \) appears exactly once in the decomposition (so it may contain a 1-cycle). Therefore we have the following fact: \( M \) is a matrix expression of an RN-type 2-chain \( \alpha \) described in Theorem 3.3 if and only if there is a permutation \( \sigma_{ik} \) from the triple of \( M \) whose cycle decomposition has (more
than) two disjoint cycles. On the other hand, any $\sigma_{ik}$ from any matrix expressions of NR-type 2-chains cannot be disjointly decomposed.

Now we recall some basic facts about permutations which will be used in the proof of upcoming theorems.

**FACT 3.5.** Let $A = \{a_0, \ldots, a_m\}$ and $B = \{b_0, \ldots, b_k\}$ be disjoint sets. Then for $0 < i \leq m$,

1. $(a_0 a_i) \circ (a_0 a_1 \cdots a_i \cdots a_m) = (a_0 a_1 \cdots a_{i-1}) \circ (a_i a_{i+1} \cdots a_m)$;
2. $(a_0 a_1 \cdots a_i a_m) \circ (a_0 a_i) = (a_0 a_{i+1} a_{i+2} \cdots a_m) \circ (a_1 a_2 \cdots a_i)$;
3. $(a_0 b_0) \circ (a_0 a_1 \cdots a_m) \circ (b_0 b_1 \cdots b_k) = (a_0 a_1 \cdots a_m b_0 b_1 \cdots b_k)$;

**Notation.** For a permutation $\tau$, let $\#_c(\tau)$ denote the number of disjoint cycles in the cycle decomposition of $\tau$.

**Theorem 3.6.** Let $\alpha$ be a minimal 2-chain having the 1-shell boundary $f_{12} - f_{02} + f_{01}$. If the length of $\alpha$ is 3 modulo 4, then $\alpha$ is always of RN-type.

**Proof.** Suppose the length of $\alpha$ is $2n + 1$ for odd $n$. Moreover, by Corollary 2.4, we may assume that $\alpha$ is a chain-walk from $f_{01}$ to $-f_{02}$. Let $(\sigma_{01}, \sigma_{12}, \sigma_{02})$ be a triple induced from some matrix expression of $\alpha$. If the cycle decomposition of $\sigma_{01}$ or $\sigma_{12}$ has disjoint cycles, then we are done. Otherwise, we can assume that $\sigma_{01} = (j_0 j_1 \cdots j_n)$ and $\sigma_{12} = (0 k_1 \cdots k_n)$, where $\{j_0, \ldots, j_n\} = \{0, k_1, \ldots, k_n\} = J$. For $i \geq 1$, let $\tau_i = (0 k_i) \circ (0 k_{i-1}) \circ \cdots \circ (0 k_1) \circ \sigma_{01}$. Then $\tau_n = \sigma_{12} \circ \sigma_{01} = \sigma_{02}$, and it suffices to show that $\#_c(\tau_n) \neq 1$. Now we claim that $\#_c(\tau_{i+1})$ is either $\#_c(\tau_i) + 1$ or $\#_c(\tau_i) - 1$. Notice that $\#_c(\tau_1) = 2$ due to Fact 3.5(1). There are now two cases: If both 0 and $k_{i+1}$ are contained in the same cycle $\mu$ of the decomposition of $\tau_i$, then by Fact 3.5(1), $(0 k_{i+1}) \circ \mu$ splits into two cycles which have 0 and $k_{i+1}$, respectively. Therefore, $\#_c(\tau_{i+1}) = \#_c(\tau_i) + 1$. If 0 and $k_{i+1}$ are contained in different cycles $\mu_0$ and $\mu_1$, respectively, then by Fact 3.5(3), $(0 k_{i+1}) \circ \mu_0 \circ \mu_1$ becomes a single cycle of size $|\mu_0| + |\mu_1|$, so we have $\#_c(\tau_{i+1}) = \#_c(\tau_i) - 1$. Inductively, we get

$$\#_c(\tau_i) = \begin{cases} 0 \pmod{2} & \text{if } i \equiv 1 \pmod{2} \\ 1 \pmod{2} & \text{if } i \equiv 0 \pmod{2}. \end{cases}$$

So for odd $n$, $\#_c(\tau_n)$ must be even. Therefore the cycle decomposition of $\sigma_{02}$ cannot be a single cycle, and $\alpha$ must be of RN-type.

**Remark 3.7.** Let us summarize the previous theorem as follows: Let $\beta$ be a minimal 2-chain with a 1-shell boundary. Then,

- if its length is 1 modulo 4, it may be of NR-type; or
- if its length is 3 modulo 4, it must be of RN-type.

For the first case, indeed we can find an NR-type 2-chain with a 1-shell boundary: Let $\alpha$ be a 2-chain with $\text{supp}(\alpha) = \{0, 1, 2\}$, which is a chain-walk with a representation
\[ \sum_{j=0}^{4k} (-1)^j a_j \text{ such that } \partial^0 a_{4k} = f_{12}, \partial^j a_{4k} = f_{02}, \text{ and } \partial^2 a_0 = f_{01}; \text{ for each } 0 \leq j \neq j_1 < 2k, \text{ and } i \in \{0, 1, 2\}, \partial^i a_{2j+1} \neq \partial^i a_{2j+1}; f_{01} \neq \partial^2 a_{2j+1}; f_{02} \neq \partial^1 a_{2j+1}; \text{ for } 0 \leq j < 2k, \partial^0 a_j = \partial^0 a_{2k+1+j} \neq f_{12}; \text{ and no other relations between the boundaries of each } 2\text{-simplex terms. In this case, we obtain the unique matrix expression} \\
M = \begin{pmatrix} k k+1 \cdots 2k & 0 & 1 & \cdots & k-1 \\
2k & 0 & \cdots & k-1 & k & \cdots & 2k-1 \\
0 & 1 & \cdots & k & k+1 & \cdots & 2k \end{pmatrix}.
\]

In Example 3.2, \( \alpha \) has two 2-simplex terms which have the same sign and the same image under the 0-th boundary operator \( \partial^0 \), for example \( \partial^0 a_0 = \partial^0 a_4 \), and then \( \alpha \) is of RN-type. The following theorem says this does not happen by accident.

**Theorem 3.8.** Let \( \alpha \) be a minimal 2-chain of length \( 2n+1 \) having the 1-shell boundary \( f_{12} - f_{02} + f_{01} \) with \( \text{supp}(\alpha) = \{0, 1, 2\} \), which is a chain-walk with a representation \( \sum_{j=0}^{2n} (-1)^j a_j \). Suppose one of the following holds:

1. \( \partial^j a_{2j-1} = \partial^j a_{2j-1} \) for some \( 0 < j_0 < j_1 \leq n \) and \( 0 \leq \ell \leq 2 \);
2. \( \partial^j a_{2j} = \partial^j a_{2j} \) for some \( 0 \leq j_0 < j_1 \leq n \) and \( 0 \leq \ell \leq 2 \).

Then \( \alpha \) is of RN-type.

**Proof.** Assume (1) holds. Let \( M \) be a matrix expression of \( \alpha \). Consider the triple \( (\sigma_{01}, \sigma_{12}, \sigma_{02}) \) with respect to \( M \). If one of the permutations can be decomposed into (more than) two disjoint cycles, then we are done. Therefore we can assume that all the three permutations are not properly decomposed. Now let \( p := M(\ell, j_0) \neq q := M(\ell, j_1) \).

Due to (1), we can swap the entries \( p \) and \( q \) from \( M \) to obtain a new matrix \( M' \). Thus \( M'(\ell, j_0) = M(\ell, j_1) = q \) and \( M'(\ell, j_1) = M(\ell, j_0) = p \). Notice that \( M'(i, j) = M(i, j) \) except for \( (i, j) = (\ell, j_0), (\ell, j_1) \). Now we have three cases:

Case 1) \( \ell = 2 \) : Since \( \ell = 2 \), only the third row is changed. So, \((\tau_{01}, \tau_{12}, \tau_{02}) := (\sigma_{01}, (p \circ \sigma_{12}, (p \circ \sigma_{02}))\) is a triple of permutations induced from \( M' \), and \( \tau_{12} \) (or \( \sigma_{02} \)) has two disjoint cycles due to Fact 3.5(1).

Case 2) \( \ell = 1 \) : Similarly to Case 1), \( ((p \circ \sigma_{01}, \sigma_{12} \circ (p \circ \sigma_{02}) \circ (p \circ \sigma_{02}) \circ (p \circ \sigma_{02})) \) is a triple induced from \( M' \) and again due to Fact 3.5(1), \( (p \circ \sigma_{01}) \circ (p \circ \sigma_{02}) \) is decomposed into two disjoint cycles.

Case 3) \( \ell = 0 \) : Similarly, \( (\sigma_{01} \circ (p \circ q), \sigma_{12}, \sigma_{02} \circ (p \circ q)) \) is a new triple and \( \sigma_{01} \circ (p \circ q) \) splits into two disjoint cycles by Fact 3.5(2).

In conclusion, for any case, new matrix expression \( M' \) witnesses \( \alpha \) being of RN-type.

For the second condition, the theorem holds by the same argument. \( \square \)

4. Lascar 2-chains

In this section, we look closely at RN-type 2-chains. We present the notions of planar type, Lascar type, and tower type 2-chains, which are all of RN-type unless the length of any given 2-chain is 1. We shall show these three properties of RN-type 2-chains are all
equivalent: Given a minimal 2-chain, if it is equivalent to a 2-chain satisfying one of the three properties, then it is also equivalent to 2-chains which satisfies the others. This is an interesting result as the notion of planar type comes from geometry while that of Lascar type comes from model theory (in particular, found in the proof of Fact 1.9).

**Remark/Notation 4.1.** Let \( \Delta \subseteq \mathcal{P}(\omega) \). We say \( \Delta \) is an abstract simplicial complex if for \( u \in \Delta \) and \( v \subseteq u \), \( v \) is again in \( \Delta \). The vertex set of \( \Delta \) is the set \( \bigcup \Delta \). For a fixed finite set \( X = \{ x_0 < x_1 < \cdots < x_n \} \subseteq \omega \), we say the power set of \( X \), \( \mathcal{P}(X) \) is called an abstract \( n \)-simplex. Let \( S_n \) denote the set of abstract \( n \)-simplices and let \( C_n \) the free abelian group generated by \( S_n \); its elements are called abstract \( n \)-chains. Next, we define abstract \( n \)-th boundary maps \( \partial_n : C_n \rightarrow C_{n-1} \) as follows: if \( \mathcal{P}(X) \) is an \( n \)-simplex, the boundary of \( \mathcal{P}(X) \) is defined as \( \partial_n(\mathcal{P}(X)) = \sum_{i=0}^{n} (-1)^i \mathcal{P}(X \setminus \{x_i\}) \) and we extend linearly to all \( n \)-chains in \( C_n \).

Let \( T \) be a closed triangular plane region. Let \( \Delta(T) \) be a triangulation, which is a triangular subdivision, of \( T \) in the plane with only three exterior vertices assigned \( \{0, 1, 2\} \), and consider an abstract simplicial complex \( \Delta_a(T) \) whose geometric realization is homeomorphic to \( \Delta(T) \). Then \( \Delta_a(T) = \bigcup_i \mathcal{P}(s_i) \) for finite subsets \( s_i \subset \omega \) and \( |s_i| = 3 \), and this induces an abstract 2-chain \( \alpha : \Delta_a(T) \rightarrow \mathcal{P}(\{0, 1, 2\}) - \mathcal{P}(\{0, 2\}) + \mathcal{P}(\{0, 1\}), \) and \( \epsilon_i = \pm 1 \) uniquely determined. For a 2-chain \( \alpha = \sum_i \epsilon_i a_i \) in \( p \) with a 1-shell boundary, where \( a_i : \mathcal{P}(s_i) \rightarrow C_A \) is 2-simplex in \( p \), we shall write \( \alpha \) as \( \alpha : \Delta_a(T) \rightarrow C_A \), and we say \( \alpha \) has the domain of a triangulation of \( T \).

Of course we can consider a 2-chain having the domain of any triangular subdivision. But if such a 2-chain has a 1-shell boundary, then its triangulation must have only three exterior vertices.

And by a simplicial map between simplicial complexes \( L \) and \( K \), we mean a map such that whenever the vertices of \( L \) span a simplex of \( L \), their image span a simplex of \( K \). We say two triangulations are isomorphic if there is a bijective simplicial map between two triangulations.

**Remark 4.2.** Let \( T \) be a closed triangular plane region. If two triangulations of \( T \) are isomorphic, then they have the same abstract simplicial complex.

**Definition 4.3.** Let \( \alpha \) be a minimal 2-chain having the 1-shell boundary \( f_{12} - f_{02} + f_{01} \).

1. We call \( \alpha \) planar type (or simply planar) if \( \alpha : \Delta_a(T) \rightarrow C_A \) for some closed triangular plane region \( T \), where \( \Delta_a(T) \) is a planar triangulation of \( T \).
2. We call \( \alpha \) Lascar type (or Lascar) if \( |\alpha| = 1 \) or \( \alpha \) is an RN-type 2-chain of length \( 2n + 1 \), which is a chain-walk with a representation \( \sum_{i=0}^{2n} (-1)^ia_i \) (see Fact 1.18(2)) such that for each \( j < n \), \( \partial^0(a_{2j}) - \partial^0(a_{2j+1}) = 0 \). (Hence \( 1 \in \text{supp}(a_i) \) for all \( i \leq 2n \))
3. We call \( \alpha \) tower type if \( \alpha = \sum_{i=-n}^{n} \epsilon_i a_i \ (0 \leq n) \) with \( \epsilon_0 = 1 \), \( \epsilon_i = \epsilon_{-i} \ (i > 0) \), such that
(a) $\partial^2 a_0 = f_{01}$, $(\partial \epsilon_n a_n)^{0,2} = -f_{02}$ and $(\partial \epsilon_{-n} a_{-n})^{1,2} = f_{12}$;
(b) $\text{supp}(a_0) = \{0, 1, k_0\}$; for $1 \leq i \leq n$, $\text{supp}(a_i) = \{0, k_{i-1}, k_i\}$, $\text{supp}(a_{-i}) = \{1, k_{i-1}, k_i\}$; $k_n = 2$;
(c) for $0 \leq i \leq n - 1$, $(\partial \epsilon_i a_i)^{0,k_i} + (\partial \epsilon_{i+1} a_{i+1})^{0,k_i} = 0$ and $(\partial \epsilon_{-i} a_{-i})^{1,k_i} + (\partial \epsilon_{-(i+1)} a_{-(i+1)})^{1,k_i} = 0$; and
(d) $\epsilon_i (\partial^0 a_i) + \epsilon_{-i} (\partial^0 a_{-i}) = 0$ for $1 \leq i \leq n$.

The Lascar type 2-chains are so named, because such type chains are crucially used in the proof of Fact 1.9, which are to do with the Lascar distance. Note that each of the three type 2-chains is of RN-type if its length is $\geq 3$.

We show all the three type 2-chains are equivalent. In above definition, if $|\alpha| = 3$ then $\alpha$ is equivalent to a Lascar 2-chain by Fact 1.18.

Figure 2: An example of a Lascar type 2-chain. The 0-th boundaries of adjacent 2-simplices are cancelled out in a pair after taking boundary map.

Figure 3: A tower type 2-chain (so planar)
**Theorem 4.4.** Let \( \alpha \) be a minimal 2-chain with a 1-shell boundary \( f_{12} - f_{02} + f_{01} \). The following are equivalent.

1. \( \alpha \) is equivalent to a Lascar type 2-chain.
2. \( \alpha \) is equivalent to a tower type 2-chain.
3. \( \alpha \) is equivalent to a planar 2-chain.

**Proof.** When \( |\alpha| = 1 \), nothing to prove, so we assume \( |\alpha| \geq 3 \).

(1) \( \Rightarrow \) (2) Assume \( \alpha = \sum_{i=0}^{n} (-1)^{i} a_i \) is a Lascar 2-chain as in Definition 4.3, with \( \text{supp}(a_i) = \{0, k_i, k_{i+1}\} \). Thus \( k_0 = k_2 = \cdots = k_{2n} = 1 \), and due to the RS-operation, we can assume that \( 2 < k_1 < k_3 < \cdots < k_{2n-1} \). Since \( k_1, k_3 \) are distinct, we can apply the CR-operation to \(-a_1 + a_2\) and obtain \(-b_{-1} + b_1\) with \( \text{supp}(b_1) = \{1, k_1, k_3\} \) and \( \text{supp}(a) = \{0, k_1, k_3\} \). Similarly we apply the CR-operation to \(-a_3 + a_4\) and obtain \(-b_{-2} + b_2\) with \( \text{supp}(b_2) = \{1, k_3, k_5\} \) and \( \text{supp}(a) = \{0, k_3, k_5\} \). Iterate this process and lastly we get \( b_{-n} - b_n \) with \( \text{supp}(b_{-n}) = \{1, 2, k_{2n-1}\} \), \( \text{supp}(b_n) = \{0, 2, k_{2n-1}\} \) by applying the CR-operation to \(-a_{2n-1} + a_{2n}\). Now put \( b_0 := a_0 \). Then it follows \( b_0 + \sum_{i=1}^{n} (-b_{-i} + b_i) + (b_{-n} - b_n) \) is a tower type 2-chain.

(2) \( \Rightarrow \) (1) This can be shown by reversely taking the process described in the proof of (1) \( \Rightarrow \) (2).

(2) \( \Rightarrow \) (3) Clear from Figure 3.

(3) \( \Rightarrow \) (2) Let the 2-chain \( \alpha : \Delta_\alpha(\mathcal{T}) \to \mathcal{C} \) be planar, where \( \Delta_\alpha(\mathcal{T}) \) is an abstract simplicial complex of a closed triangular plane region \( \mathcal{T} \) with exterior vertices 0, 1, 2; and \( \Delta(\mathcal{T}) \) is a planar triangulation of \( \mathcal{T} \). Now we prove using induction on \( |\alpha| \). If \( |\alpha| = 3 \) then due to the comment before this theorem, we are done. So let us prove (2) when \( |\alpha| = 2n + 1 \) (*), with the induction hypothesis when \( 3 \leq |\alpha| \leq 2n - 1 \) (**). We take a chain-walk \( \beta \) in \( \alpha \) from \(-f_{02}\) to \( f_{12} \). (Since \( \alpha \) is planar such \( \beta \) is unique.) We prove (*) using induction on \( |\beta| \).

We have \( |\beta| \geq 2 \), since \( \alpha \) is planar and \( |\alpha| \geq 3 \). If \( |\beta| = 2 \), \( \gamma = \alpha - \beta \) is again a planar 2-chain with 1-shell boundary and by (** \( \gamma \) is equivalent to a tower type 2-chain \( \gamma' \). So \( \alpha \) is equivalent to \( \gamma' + \beta \), which is a tower type 2-chain and we are done. Let us prove (*) when \( |\beta| = m + 1 \) with the induction hypothesis for \( m \geq 2 \) (**). Let \( \beta = \sum_{i=0}^{m} \epsilon_i b_i \) with its representation.

Let \( \{2, k_0, k_1, \ldots, k_{m+1}\} \) be the support of \( \beta \) such that \( k_0 = 0 \), \( k_{m+1} = 1 \) and \( \text{supp}(b_i) = \{2, k_i, k_{i+1}\} \) for \( 0 \leq i \leq m \) (moreover 2 and \( k_i \)'s are all distinct). Then \( \beta'_{k_0, k_1, \ldots, k_{m+1}} \) is corresponding to a piecewise-linear graph \( \Gamma \) connecting two vertices 0 and 1 in \( \mathcal{T} \). And we regard this graph as a graph on an interval corresponding to \( \alpha'^{0,1} \).

Case 1) \( \Gamma \) is locally concave upward: Let \( (\epsilon_j b_j + \epsilon_{j+1} b_{j+1})^{k_j, k_{j+1}, k_{j+2}} \) be corresponding to a concave upward piece in \( \Gamma \). Obtain \( \epsilon_j b'_j + \epsilon_{j+1} b'_{j+1} \) by applying the CR-operation to \( \epsilon_j b_j + \epsilon_{j+1} b_{j+1} \), so that \( \beta' = \beta - (\epsilon_j b_j + \epsilon_{j+1} b_{j+1}) + (\epsilon_j b'_j + \epsilon_{j+1} b'_{j+1}) \) forms a chain-walk in a planar \( \alpha'' := \alpha - (\epsilon_j b_j + \epsilon_{j+1} b_{j+1}) + (\epsilon_j b'_j + \epsilon_{j+1} b'_{j+1}) \sim \alpha \). Now \( |\beta'| = m \). Hence due to (**), there is a tower type \( \alpha'' \sim \alpha' \sim \alpha_0 \).
Figure 4: Two isomorphic triangulations. \((b_2 + b_3)^{1,4,5}\) corresponds to a concave upward piece in \(\Delta'(T)\) but not in \(\Delta(T)\).

Case 2) \(\Gamma\) is concave downward: There is a triangulation \(\Delta'(T)\) isomorphic to \(\Delta(T)\), where the corresponding graph \(\Gamma'\) is locally concave upward. By Remark 4.2 we may replace \(\Delta(T)\) by \(\Delta'(T)\) and apply the same process in the proof of Case 1 to \(\Gamma'\).

\[\square\]

**Question 4.5.** In an amenable category, is there an RN-type 2-chain not equivalent to a Lascar 2-chain?

Due to Theorem 4.4, it easily follows that any RN-type 2-chain of length 3 or 5 is equivalent to a Lascar type. We guess that there is an RN-type 2-chain of length 7 which is not equivalent to a Lascar 2-chain.

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