# OBTUSE CONSTANTS OF ALEXANDROV SPACES

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Abstract. We introduce a new geometric invariant called the obtuse constant of spaces with curvature bounded below. We first find relations between this invariant and the normalized volume. We also discuss the case of maximal obtuse constant equal to $\pi/2$, where we prove some rigidity for spaces. Although we consider Alexandrov spaces with curvature bounded below, the results are new even in the Riemannian case.

## 1. Introduction

In the present paper, we introduce a new geometric invariant called the obtuse constant of a space with curvature bounded below, and investigate the relations between this invariant and the normalized volume.

For this invariant, there are some historical backgrounds. For a positive integer $n$ and $D, v > 0$, let $\mathcal{M}(n, D, v)$ denote the family of $n$-dimensional closed Riemannian manifolds with sectional curvature $\geq -1$, diameter $\leq D$ and volume $\geq v$. In [2], Cheeger proved that for every $M \in \mathcal{M}(n, D, v)$, the length of every periodic closed geodesic has length $\geq \ell_{n,D}(v) > 0$ for some uniform constant $\ell_{n,D}(v)$. In [4], Grove and Petersen extended Cheeger's argument as follows: There are positive constants $\delta = \delta_{n,D}(v)$ and $\epsilon = \epsilon_{n,D}(v)$ such that for every $M \in \mathcal{M}(n, D, v)$ and for every distinct $p, q \in M$ with distance $|p, q| < \delta$, either $q$ is $\epsilon$-regular to $p$, or $p$ is $\epsilon$-regular to $q$. Those results were keys to control local geometry of the space, and brought significant results, topological finiteness of Riemannian manifolds (see [2], [4], [5]).

In this paper, we do not need to restrict ourselves to Riemannian manifolds. Let $M$ be a complete Alexandrov space with curvature $\geq \kappa$. For two points $p, q$ of $M$, $\Uparrow_p^q$ denotes the set of all directions at $p$ of all minimal geodesics from $p$ to $q$. We also use the symbol $\uparrow_p^q$ to indicate $\uparrow_p^q \in \Uparrow_p^q$.

Let $\mathcal{A}(n, D, v)$ denote the family of $n$-dimensional compact Alexandrov spaces with curvature $\geq -1$, diameter $\leq D$ and volume $\geq v$.

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Grove and Petersen's result mentioned above still holds for Alexandrov spaces (see [10]). This implies that for every $M \in \mathcal{A}(n, D, v)$ and for every distinct $p, q \in M$ with $|p, q| < \delta$ there exists a point $x \in M$ such that either $\angle(\Uparrow_p^q, \uparrow_p^x) > \pi/2 + \epsilon$ for some $\uparrow_p^x \in \Uparrow_p^x$ or $\angle(\Uparrow_q^p, \uparrow_q^x) > \pi/2 + \epsilon$ for some $\uparrow_q^x \in \Uparrow_q^x$. However such a point $x$ was assumed to be close to those points $p$ or $q$ in general. As we see later, if one can take such a point $x$ relatively far away from $p$ or $q$, it will be useful in some situations.

The above is a motivation to our invariants, which we are going to define in detail. First we suppose that $M$ is compact. Let $R = R_M = \mathrm{rad}(M)$ be the radius of $M$:

$$R_p := \sup_{q \in M} |p, q|, \qquad R := \inf_{p \in M} R_p.$$

For $p \neq q \in M$, set

$$\mathrm{ob}(p; q) := \sup_{x \in B(p, R/2)^c} \left( \max_{\uparrow_p^x \in \Uparrow_p^x} \angle(\Uparrow_p^q, \uparrow_p^x) - \pi/2 \right),$$

which we call the *obtuse constant of* $\{p, q\}$ *at* $p$, and define the *obtuse constant at* $p$ *and* $q$ by

$$\mathrm{ob}(p, q) := \max\{\mathrm{ob}(p; q), \mathrm{ob}(q; p)\}.$$

Finally we define the *obtuse constant* $\mathrm{ob}(M)$ of $M$ as

$$\mathrm{ob}(M) := \liminf_{|p,q| \to 0} \mathrm{ob}(p, q) \in [0, \pi/2].$$

**Theorem 1.1.** *There exists a uniform positive constant* $\epsilon_{n,D}(v)$ *such that*

$$\mathrm{ob}(M) > \epsilon_{n,D}(v)$$

*for every* $M \in \mathcal{A}(n, D, v)$.

*More precisely, there exists also a positive constant* $\delta_{n,D}(v)$ *such that if* $M \in \mathcal{A}(n, D, v)$ *and* $p, q$ *are distinct points of* $M$ *with* $|p, q| < \delta_{n,D}(v)$, *then* $\mathrm{ob}(p, q) > \epsilon_{n,D}(v)$.

This generalizes the result of Grove and Petersen as stated before.

The converse to Theorem 1.1 is also true. Let $\mathcal{A}(n, D)$ denote the family of $n$-dimensional compact Alexandrov spaces with curvature $\geq -1$ and diameter $\leq D$. Notice that the obtuse constant is rescaling invariant. Therefore for $M \in \mathcal{A}(n, D)$ it is natural to compare $\mathrm{ob}(M)$ with the *normalized volume* by the diameter defined as

$$\tilde{v}(M) := \frac{\mathcal{H}^n(M)}{(\mathrm{diam}\,(M))^n},$$

where $\mathcal{H}^n(M)$ denotes the $n$-dimensional Hausdorff measure of $M$.

**Theorem 1.2.** *There exists a positive continuous function* $C_{n,D}(\epsilon)$ *with* $\lim_{\epsilon \to 0} C_{n,D}(\epsilon) = 0$ *such that for every* $M \in \mathcal{A}(n, D)$, *we have*

$$\mathrm{ob}(M) < C_{n,D}(\tilde{v}(M)).$$

In the case of nonnegative curvature, as an immediate consequence of Theorems 1.1 and 1.2, we have

**Corollary 1.3.** *There exist positive continuous functions $\epsilon_n(t)$ and $C_n(t)$ with $\lim_{t\to 0}\epsilon_n(t) = \lim_{t\to 0} C_n(t) = 0$ such that for every compact Alexandrov $n$-space $M$ of nonnegative curvature, we have*

$$\epsilon_n(\tilde{v}(M)) \le \mathrm{ob}(M) \le C_n(\tilde{v}(M)).$$

From Theorems 1.1, 1.2 and Corollary 1.3, we conclude that there is a strong relation between the obtuse constant and the normalized volume.

Next we discuss the noncompact case. Suppose that $M$ is noncompact complete Alexandrov space with curvature $\ge \kappa$ $(\kappa \le 0)$. Set

$$\mathrm{ob}_\infty(p,q) := \limsup_{x\to\infty} \max\left\{ \max_{\uparrow_p^x \in \Uparrow_p^x} \angle(\Uparrow_p^q, \uparrow_p^x), \max_{\uparrow_q^x \in \Uparrow_q^x} \angle(\Uparrow_q^p, \uparrow_q^x) \right\} - \pi/2,$$

which we call the *obtuse constant at $p$ and $q$ from infinity.* We define the *obtuse constant* $\mathrm{ob}_\infty(M)$ *of $M$ from infinity* as

$$\mathrm{ob}_\infty(M) := \liminf_{|p,q|\to 0} \mathrm{ob}_\infty(p,q) \in [0, \pi/2]$$

In the geometry of complete noncompact spaces with nonnegative curvature, the notion of asymptotic cone or volume growth rate plays an important role. For instance, any complete noncompact Riemannian manifold with nonnegative curvature having maximal volume growth is known to be diffeomorphic to an Euclidean space.

Let $M$ be an $n$-dimensional complete noncompact Alexandrov space with curvature $\ge 0$, and for any fixed $p \in M$, let

$$v_\infty(M) := \lim_{R\to\infty} \frac{\mathcal{H}^n(B(p,R))}{R^n}$$

be the volume growth rate of $M$.

As a noncompact version of Theorems 1.1 and 1.2, we have the following:

**Theorem 1.4.** *There exist continuous increasing functions $\epsilon_n$ and $C_n$ with $\epsilon_n(0) = C_n(0) = 0$ such that for every complete noncompact Alexandrov $n$-space with nonnegative curvature, we have*

$$\epsilon_n(v_\infty(M)) \le \mathrm{ob}_\infty(M) \le C_n(v_\infty(M)).$$

*In particular, $v_\infty(M) = 0$ if and only if $\mathrm{ob}_\infty(M) = 0$.*

Finally we consider the maximal case of the obtuse constants equal to $\pi/2$. We need to define a variant of the notion on the injectivity radius. For an Alexandrov space $M$ with curvature bounded below, let us denote by 1-inj$(M)$ the supremum of $r \ge 0$ such that for every $p \in M$ and every direction $\xi \in \Sigma_p$ at $p$ there exists a minimal geodesic $\gamma$ starting from $p$ in the direction of at least one of $\xi$ or the opposite $-\xi$

(if any) of length $\geq r$. We call $1\text{-inj}(M)$ the *one-side injectivity radius of* $M$. It should be noted that if $p \in \partial M$ and $\xi \in \Sigma_p \setminus \partial\Sigma_p$, then the opposite $-\xi$ does not exist, and therefore there always exists a minimal geodesic in the direction $\xi$ of length $\geq r$ if $1\text{-inj}(M) \geq r$. We have the following rigidity:

**Theorem 1.5.** *If a compact Alexandrov space $M$ with curvature $\geq \kappa$ and radius $R$ has* $\mathrm{ob}(M) = \pi/2$, *then* $1\text{-inj}(M) \geq R/2$.

In the noncompact case, we have

**Theorem 1.6.** *If a complete noncompact $n$-dimensional Alexandrov space $M$ with curvature $\geq \kappa$ has* $\mathrm{ob}_\infty(M) = \pi/2$, *then* $1\text{-inj}(M) = \infty$.

*Suppose in addition that $M$ has nonempty boundary. Then $M$ is homeomorphic to the Euclidean half space $\mathbb{R}^n_+$, and any distinct two points of $\partial M$ are on a line of $M$ which is contained in $\partial M$.*

In the case of nonnegative curvature, we have the following result.

**Theorem 1.7.** *Let $M$ be a complete noncompact $n$-dimensional Alexandrov space with nonnegative curvature. Suppose that* $\mathrm{ob}_\infty(M) = \pi/2$. *Then we have the following.*

(1) *If $M$ has no boundary, then* $1\text{-inj}(M(\infty)) \geq \pi/2$;
(2) *If $M$ has nonempty boundary, then $M$ is isometric to $\mathbb{R}^n_+$.*

Note that the estimate $1\text{-inj}(M(\infty)) \geq \pi/2$ in Theorem 1.7 (1) is sharp, because there is a surface of revolution of nonnegative curvature satisfying $\mathrm{ob}_\infty(M) = \pi/2$ and $M(\infty)$ is a circle of length $\pi$ (see Example 6.3). We also cannot expect that $M(\infty)$ has no singular points in Theorem 1.7 (1) (see Remark 6.6 and Conjecture 6.7).

The organization of the present paper is as follows: After preliminaries about Alexandrov spaces in Section 2, we prove Theorem 1.1 in Section 3, where we apply an argument in [3] to our setting. For the proof of Theorem 1.2, we apply the Lipschitz submersion theorem in [16], which is carried out in Section 4. To prove Theorem 1.4, we consider the convergence to the asymptotic cone, and apply ideas of the proof of Theorems 1.1 and 1.2. This is done in Section 5. In Section 6, we discuss the case when the obtuse constants attain the maximum value $\pi/2$, where we obtain the rigidity results, Theorems 1.5, 1.6 and 1.7, together with the example showing that Theorem 1.7 is sharp (see Theorem 6.4). In Section 7, we introduce the notions of *comparison obtuse constant* in terms of comparison angles and discuss the rigidity cases. This invariant does not depend on the choice of the lower curvature bound. In Section 8, we consider the notions of $\kappa$-obtuse constant from infinity, which does depend on the choice of the lower curvature bound $\kappa$ of a noncompact space. These new invariants give more restriction on the space in the maximal case, and we have a strong rigidity in the case of nonnegative curvature, which might be of independent interest.

## 2. Preliminaries

In this paper, $|x, y|$ denotes the distance between two points $x, y$ in a metric space. An isometric embedding from an interval to a metric space is called a *minimal geodesic*. Furthermore, a fixed minimal geodesic between two points $x$ and $y$ is sometimes denoted by $xy$. For $\kappa \in \mathbb{R}$, we denote by $\mathbb{M}_\kappa$ the simply-connected complete surface of constant curvature $\kappa$, which is called the $\kappa$-plane. For distinct three points $x, y, z$ in a metric space, we denote by $\tilde{\triangle}_\kappa xyz$ a geodesic triangle in $\mathbb{M}_\kappa$ with the length of three sides $|x, y|$, $|y, z|$ and $|z, x|$, where $|x, y| + |y, z| + |z, x| < 2\pi/\sqrt{\kappa}$ if $\kappa > 0$. Vertices of $\tilde{\triangle}_\kappa xyz$ will be denoted by $\tilde{x}, \tilde{y}, \tilde{z}$. Furthermore, the angle of $\tilde{\triangle}_\kappa xyz$ at $\tilde{x}$ is denoted by $\tilde{\angle}_\kappa yxz$ and is called the *$\kappa$-comparison angle* of $x, y, z$ at $x$. We also write $\tilde{\angle} yxz$ by omitting $\kappa$ depending on the context.

2.1. **Basics of Alexandrov spaces.** Let us recall the definition of Alexandrov spaces, following [1]. An *Alexandrov space $M$ of curvature $\geq \kappa$* is a locally complete metric space satisfying the following:

(1) for any two points in $M$, there exists a minimal geodesic joining them;
(2) every point has a neighborhood $U$ such that for any two minimal geodesics $xy, xz$ contained in $U$ with the same starting point $x$, and for any $s \in xy$ and $t \in xz$, we have
$$|s, t| \geq |\tilde{s}, \tilde{t}|.$$
Here, $\tilde{s} \in \tilde{x}\tilde{y}$ and $\tilde{t} \in \tilde{x}\tilde{z}$ are taken in the comparison triangle $\tilde{\triangle}_\kappa xyz = \triangle \tilde{x}\tilde{y}\tilde{z}$ with $|x, s| = |\tilde{x}, \tilde{s}|$ and $|x, t| = |\tilde{x}, \tilde{t}|$.

When an Alexandrov space is complete as a metric space, due to [1], the property (2) holds globally.

From the definition, the monotonicity of comparison angle holds for an Alexandrov space, that is, for two geodesics $xy$ and $xz$ in an Alexandrov space $M$ of curvature $\geq \kappa$ as above, and $s \in xy - \{x\}$, $t \in xz - \{x\}$, we have
$$\tilde{\angle} yxz \leq \tilde{\angle} sxt. \tag{2.1}$$
Here, $\tilde{\angle}$ means $\tilde{\angle}_\kappa$. In particular, the limit
$$\angle(xy, xz) := \lim_{xy \ni s \to x,\, xz \ni t \to x} \tilde{\angle} sxt$$
always exists. It is called the *angle* between $xy$ and $xz$. When the geodesics $xy$ and $xz$ are fixed, we write $\angle yxz = \angle(xy, xz)$. By the definition of the angle, we obtain
$$\angle yxz \geq \tilde{\angle} yxz. \tag{2.2}$$

When an Alexandrov space is complete, (2.1) and (2.2) are also true for any geodesics.

From now on, $M$ denotes an Alexandrov space of curvature $\geq \kappa$. Furthermore, we assume that $M$ has at least two points. For a point $x \in M$, let us set $\Gamma_x$ the set of all non-trivial geodesics starting from $x$. It is known that the angle $\angle$ is a pseudo-distance function on $\Gamma_x$. The completion of the metric space $\Sigma_x^0$ induced from $(\Gamma_x, \angle)$ is called the *space of directions at $p$ (in $M$)* which is denoted by $\Sigma_x = \Sigma_x M$. The distance function on $\Sigma_x$ is written as $\angle$, the same symbol as the angle. An element of $\Sigma_x$ is called a direction at $x$. Furthermore, for geodesics $xy, xz \in \Gamma_x$, $\angle yxz = 0$ if and only if $xy \subset xz$ or $xz \subset xy$ as the images of geodesics. In particular, any Alexandrov space does not admit a branching geodesic. The equivalent class of $xy$ is denoted by $\uparrow_x^y$. Let $\Uparrow_x^y$ denote the set of all directions of geodesics from $x$ to $y$.

It is known that the Lebesgue covering dimension of $M$ is the same as the Hausdorff dimension of it, which is called the *dimension of $M$* and is written as $\dim M$ ([1], [13]). From now on, we assume that $\dim M < \infty$. This assumption implies that, the space of directions $\Sigma_x$ at $x \in M$ is compact and becomes an Alexandrov space of curvature $\geq 1$ and of dimension equal to $\dim M - 1$. Here, we used a convention that the metric space of two points with distance $\pi$ is regarded as an Alexandrov space of curvature $\geq 1$ and dimension zero.

For $\delta > 0$, $p \in M$ is *$\delta$-strained* if there exists a collection $\{(a_i, b_i)\}_{1 \leq i \leq n}$ of pairs of points, where $n = \dim M$, such that

$$\tilde{\angle} a_i p b_i > \pi - \delta, \qquad \tilde{\angle} a_i p a_j > \pi/2 - \delta,$$
$$\tilde{\angle} b_i p b_j > \pi/2 - \delta, \qquad \tilde{\angle} a_i p b_j > \pi/2 - \delta,$$

hold for all $i \neq j$. Such a collection $\{(a_i, b_i)\}$ is called a *$\delta$-strainer at $p$*. Let us denote by $\delta$-str.rad$(p)$ the supremum of $\min\{|p, a_i|, |p, b_i|\}_i$, where the supremum runs over all $\delta$-strainers at $p$, which is called the *$\delta$-strained radius at $p$*.

The set $\mathcal{R}_\delta(M)$ of all $\delta$-strained points in $M$ is known to have full measure in the $n$-dimensional Hausdorff measure $\mathcal{H}^n$ ([1], [9]). In particular, $\mathcal{R}(M) = \bigcap_{\delta > 0} \mathcal{R}_\delta(M)$ also has full measure and is dense in $M$. A point in $\mathcal{R}(M)$ is said to be *regular*. It is known that $p$ is regular if and only if $\Sigma_p$ is isometric to the sphere of constant curvature one. A point in $M \setminus \mathcal{R}_\delta(M)$ (resp. in $M \setminus \mathcal{R}(M)$) is said to be $\delta$-singular (resp. singular). The set of all singular points is denoted by $\mathcal{S}(M)$.

2.2. **Tangent cones.** Let $M$ be an $n$-dimensional complete Alexandrov space and $p \in M$. Let $\kappa$ denote a lower curvature bound of $M$. For simplicity, we assume $\kappa < 0$. We consider the function

$$f_\kappa(s) := \frac{\sinh(\sqrt{-\kappa}\, s)}{\sqrt{-\kappa}}.$$

We define the $\kappa$-tangent cone $T_p^\kappa M$ as follows. Let us consider the product $\Sigma_p \times [0, \infty)$. For $(\xi, s), (\eta, t) \in \Sigma_p \times [0, \infty)$, we define a distance between them as

$$f'_\kappa(|(\xi,s),(\eta,t)|) = f'_\kappa(s)f'_\kappa(t) + \kappa f_\kappa(s)f_\kappa(t)\cos\angle(\xi,\eta),$$

where $f'_\kappa(s) = \frac{d}{ds}f_\kappa(s)$. This formula comes from the cosine formula on the $\kappa$-plane. Clearly, $|(\xi,s),(\eta,t)| = 0$ if and only if $(\xi,s) = (\eta,t)$ or $s = t = 0$. So, we obtain a metric space by smashing $\Sigma_p \times \{0\}$ in $\Sigma_p \times [0,\infty)$ and set

$$T^\kappa_p M := \Sigma_p \times [0,\infty)/\Sigma_p \times \{0\}.$$

This is called the $\kappa$-tangent cone in this paper. A point in $T^\kappa_p M$ corresponding to $\Sigma_p \times \{0\}$ is denoted by $o$ and is called the origin. Then, $T^\kappa_p M$ becomes a complete $n$-dimensional Alexandrov space of curvature $\geq \kappa$. A logarithmic map $\log^\kappa_p : M \to T^\kappa_p M$ is defined by

$$\log^\kappa_p(x) := (\uparrow^x_p, |p,x|)$$

where, if $x = p$, we set $\log^\kappa_p(p) := o$. Then, by the definition of Alexandrov spaces, $\log^\kappa_p$ is a distance noncontracting map. An important point is that $\log^\kappa_p$ has a nice left inverse as follows.

**Theorem 2.1** ([11], [12])**.** *Let $M$ and $p$ be be as above, and $\kappa < 0$. Then, there exists a 1-Lipschitz map*

$$\mathrm{gexp}^\kappa_p : T^\kappa_p M \to M$$

*satisfying* $\mathrm{gexp}^\kappa_p \circ \log^\kappa_p = \mathrm{id}$.

*In particular,* $\mathrm{gexp}^\kappa_p$ *is surjective and*

$$\mathrm{gexp}^\kappa_p(B(o,r)) = B(p,r)$$

*for any $r \geq 0$ with $r \leq R_p$.*

## 3. Proof of Theorem 1.1

In this section, we are going to prove an equivalent formulation (Theorem 3.1) to Theorem 1.1 as follows. To state it, let us consider a moduli space $\mathcal{A}$ consisting of all $n$-dimensional compact Alexandrov spaces of curvature $\geq -D^2$ with diameter one and normalized volume $\geq v$. Since the obtuse constant $\mathrm{ob}(M)$ is a scale-invariant, Theorem 1.1 is equivalent to

**Theorem 3.1.** *There exist positive numbers $\epsilon_{n,D}(v)$ and $\delta_{n,D}(v)$ depending only on $n, D, v$ such that for any $M \in \mathcal{A}$ and $p \neq q \in M$ with $|p,q| \leq \delta_{n,D}(v)$, we have*

$$\mathrm{ob}(p,q) \geq \epsilon_{n,D}(v).$$

To prove Theorem 3.1, we prepare several numerical constants.

**Definition 3.2** (A constant $V_1$)**.** For each $M \in \mathcal{A}$, we consider

$$V_1(M) := \min_{p \in M} \mathcal{H}^n(M - B(p, R_M/2))$$

and define a universal constant

$$V_1 := V_1(n, D, v) := \min_{M \in \mathcal{A}} V_1(M) = \min_{M \in \mathcal{A}} \min_{p \in M} \mathcal{H}^n(M - B(p, R_M/2)).$$

Here, the existence of those minima in the above definitions follows from the fact that both the radius $R_M$ and the $n$-dimensional Hausdorff measure of balls (of a fixed radius) in $M \in \mathcal{A}$ are continuous in the Gromov-Hausdorff topology.

**Definition 3.3** (A constant $R_{\min}$)**.** Let

$$R_{\min} := R_{\min}(n, D, v) := \min_{M \in \mathcal{A}} R_M.$$

Since each $M \in \mathcal{A}$ has diameter one, $R_{\min} \geq 1/2$.

**Definition 3.4** (A constant $C_1$)**.** Let

$$C_1 := C_1(n, D, v) := \int_{R_{\min}/2}^{1} \left( \frac{1}{D} \sinh(Ds) \right)^{n-1} ds.$$

Due to Bishop-Gromov's type inequality for Alexandrov spaces, we have

**Lemma 3.5** ([3])**.** *Let $\Sigma$ be an $(n-1)$-dimensional Alexandrov space of curvature $\geq 1$, and $A \subset M$ a closed set. Then, we have*

$$\frac{\mathcal{H}^{n-1}(B_{\pi/2+\epsilon}(A)) - \mathcal{H}^{n-1}(B_{\pi/2-\epsilon}(A))}{\mathcal{H}^{n-1}(B_{\pi/2+\epsilon}(A))} \leq \theta_n(\epsilon)$$

*Here, $\theta_n(\epsilon)$ is a positive continuous function so that* $\lim_{\epsilon \to 0} \theta_n(\epsilon) = 0$.

Explicitly, $\theta_n$ is given as

$$\theta_n(\epsilon) = \frac{\mathcal{H}^{n-1}(B(o, \pi/2+\epsilon)) - \mathcal{H}^{n-1}(B(o, \pi/2-\epsilon))}{\mathcal{H}^{n-1}(B(o, \pi/2+\epsilon))}$$

where $o \in \mathbb{S}^{n-1}$ is a base point.

**Definition 3.6** (A constant $\epsilon$)**.** Let us fix $\epsilon = \epsilon_{n,D}(v) > 0$ satisfying

$$C_1(n, D, v)\mathcal{H}^{n-1}(\mathbb{S}^{n-1})\theta_n(\epsilon) \leq V_1(n, D, v)/2.$$

**Definition 3.7** (A constant $\delta$)**.** Let us fix a positive number $\delta = \delta(n, D, v) \ll R_{\min}$ satisfying the following conditions.

Recall that $\mathbb{M}_{-D^2}$ denotes a simply-connected complete surface of constant curvature $-D^2$. For $a, b, c \in \mathbb{M}_{-D^2}$ with $|a, b| \leq \delta$, $1 \geq |b, c| \geq R_{\min}/3$, we have

$$\begin{aligned} &\angle acb \leq \epsilon/10; \\ &|\angle bac + \angle abc - \pi| \leq \epsilon/10. \end{aligned} \tag{3.3}$$

Furthermore, for any $p \in M \in \mathcal{A}$, we have

$$\sup_{0<r<1} \mathcal{H}^n(B(p, r+\delta)) - \mathcal{H}^n(B(p, r)) \leq V_1(n, D, v)/3. \tag{3.4}$$

Note that the condition (3.3) means that the triangle $abc$ given in the definition is thin. The inequality (3.4) is obtained by Bishop-Gromov's inequality.

*Proof of Theorem 3.1 (and 1.1).* Let $\epsilon$ and $\delta$ be given as above. Suppose that the statement of Theorem 3.1 fails (for constants $\epsilon$ and $\delta$). Then, there exist $M \in \mathcal{A}$ and $p \neq q \in M$ such that

$$|p, q| \leq \delta$$

but

$$\mathrm{ob}(p; q) \leq \epsilon/2 \quad \text{and} \quad \mathrm{ob}(q; p) \leq \epsilon/2. \tag{3.5}$$

The above condition (3.5) implies that if $x \in M$ satisfies $|x, p| > R_M/2 + \delta$, then

$$\tilde{\angle} pqx \leq \pi/2 + \epsilon/2 \quad \text{and} \quad \tilde{\angle} qpx \leq \pi/2 + \epsilon/2.$$

Hence, we have

$$\tilde{\angle} pqx \geq \pi/2 - \frac{7}{10}\epsilon \quad \text{and} \quad \tilde{\angle} qpx \geq \pi/2 - \frac{7}{10}\epsilon.$$

In particular,

$$\angle(\Uparrow_q^p, \uparrow_q^x) \geq \pi/2 - \frac{7}{10}\epsilon \quad \text{and} \quad \angle(\Uparrow_p^q, \uparrow_p^x) \geq \pi/2 - \frac{7}{10}\epsilon$$

for arbitrary $\uparrow_q^x \in \Uparrow_q^x$ and $\uparrow_p^x \in \Uparrow_p^x$. Therefore, if $|x, p| > R_M/2 + \delta$, then

$$\left|\angle(\Uparrow_p^q, \uparrow_p^x) - \pi/2\right| \leq 7\epsilon/10 \quad \text{and} \quad \left|\angle(\Uparrow_q^p, \uparrow_q^x) - \pi/2\right| \leq 7\epsilon/10.$$

By the definition of the constant $V_1$ and (3.4), we obtain

$$\begin{aligned} V_1(n, D, v) &\leq \mathcal{H}^n(M - B(p, R_M/2)) \\ &\leq \mathcal{H}^n(M - B(p, R_M/2 + \delta)) + V_1(n, D, v)/3. \end{aligned}$$

Now we set

$$A := \left\{ x \in M \;\middle|\; \begin{array}{l} |xp| > R_M/2 \text{ and} \\ |\angle(\Uparrow_p^q, \uparrow_p^x) - \pi/2| \leq \epsilon \\ \text{for some } \uparrow_p^x \in \Uparrow_p^x \end{array} \right\};$$

$$A_T := \left\{ v \in T_p^{-D^2} M \;\middle|\; \begin{array}{l} 1 \geq |v| > R_{\min}/2 \text{ and} \\ |\angle(\Uparrow_p^q, v) - \pi/2| \leq \epsilon \end{array} \right\}.$$

Here, $T_p^{-D^2} M$ is the $\kappa$-tangent cone at $p$. Due to Theorem 2.1, there exists a surjective 1-Lipschitz map

$$\mathrm{gexp}_p^{-D^2} : T_p^{-D^2} M \to M$$

satisfying $B(p, R') = \mathrm{gexp}_p^{-D^2}(B(o, R'))$ for every $0 < R' \leq \max_{y \in M} |p, y|$. Hence, we have

$$M - B(p, R_M/2 + \delta) \subset A \subset \mathrm{gexp}_p^{-D^2}(A_T).$$

Therefore, by Lemma 3.5

$$\begin{aligned}
2V_1/3 &\le \mathcal{H}^n(M - B(p, R_M/2+\delta)) \\
&\le \mathcal{H}^n(A_T) \\
&\le C_1(n,D,v)(\mathcal{H}^{n-1}(B(\Uparrow_p^q, \pi/2+\epsilon)) - \mathcal{H}^{n-1}(B(\Uparrow_p^q, \pi/2-\epsilon))) \\
&\le C_1(n,D,v)\mathcal{H}^{n-1}(\mathbb{S}^{n-1})\theta_n(\epsilon) \\
&\le V_1/2.
\end{aligned}$$

This is a contradiction. □

## 4. Collapsing case

In this section, we prove Theorem 1.2 by contradiction.

We say that a surjective map $f : M \to X$ between Alexandrov spaces is an $\epsilon$-*almost Lipschitz submersion* if

1. it is an $\epsilon$-approximation;
2. for every $p, q \in M$, we have
$$\left| \frac{|f(p), f(q)|}{|p,q|} - \sin\theta \right| < \epsilon,$$
where $\theta$ denotes the infimum of $\angle qpx$ when $x$ runs over the fiber $f^{-1}(f(p))$.

We recall the following result from [16, Theorem 0.2 and Lemma 4.19].

**Theorem 4.1.** *For given positive integer $m$ and $\mu_0 > 0$ there are $\delta = \delta_m > 0$ and $\epsilon = \epsilon_m(\mu_0) > 0$ satisfying the following: Let $X$ be an $m$-dimensional complete Alexandrov space with curvature $\ge -1$ and with $\delta$-*str-rad$(X) > \mu_0$. *Then if the Gromov-Hausdorff distance between $X$ and a complete Alexandrov space $M$ with curvature $\ge -1$ is less than $\epsilon$, then there exists a map $f : M \to X$ such that*

1. *it is a $\tau(\delta,\epsilon)$-almost Lipschitz submersion;*
2. *it is $(1-\tau(\delta,\epsilon))$-open in the sense that for every $p \in M$ and $x \in X$ there exists a point $q \in f^{-1}(x)$ such that $|f(p), f(q)| \ge (1-\tau(\delta,\epsilon))|p,q|$.*

*Here $\tau(\delta,\epsilon)$ is a positive constant depending only on $m$, $\mu_0$ and $\delta$, $\epsilon$ satisfying* $\lim_{\delta,\epsilon\to 0}\tau(\delta,\epsilon) = 0$.

*Proof of Theorem 1.2.* Suppose it is not true. Then there would exist a sequence $M_i$ in $\mathcal{A}(n,D)$ with $\tilde{v}(M_i) \to 0$ and $\mathrm{ob}(M_i) > c > 0$ for some uniform constant $c$. When $\mathrm{diam}\,(M_i) \to 0$, we rescale the metric so that $\mathrm{diam}\,(M_i) = 1$ with respect to the new metric. Then since $\mathcal{H}^n(M_i) \to 0$, passing to a subsequence, we may assume that $M_i$ collapses to a lower dimensional Alexandrov space $X$ with $\dim X \ge 1$. Let $m = \dim X$, and take a regular point $x_0$ of $X$ and small $r_0 > 0$ such that the $B(x_0, r_0) \subset \mathcal{R}_\delta(X)$ with $\delta < \delta_m$ and that the $\delta$-strain radius of $B(x_0, r_0)$ is greater than a constant $\mu_0 > 0$. Applying Theorem 4.1 to $B := B(x_0, r_0)$, we have a $\tau(\delta,\epsilon)$-almost Lipschitz submersion $f_i : U_i \to B$. By the coarea formula (see [6] for instance), we obtain

$$\int_{U_i} C_n(f_i, p) d\mathcal{H}^n(p) = \int_B \mathcal{H}^{n-m}(f_i^{-1}(x))\, d\mathcal{H}^m(x),$$

where $C_n(f_i, p)$ denotes the coarea factor of $f_i$ at $p$. Since $f_i$ is a $\tau(\delta,\epsilon)$-almost Lipschitz submersion, we see that $|C_n(f_i, p) - 1| < \tau(\delta,\epsilon)$. Let $B_0$ be the set of points $y \in B$ such that $\mathcal{H}^{n-m}(f_i^{-1}(y)) > 0$. It follows that $B_0$ is dense in $B$. For $y_0 \in B_0$, one can take distinct points $p$ and $q$ in $f_i^{-1}(y_0)$ which are sufficiently close to each other. Lemma 4.11 of [16] shows that $|\angle(\uparrow_p^q, H_p) - \pi/2| < \tau(\delta,\epsilon)$ for every $\uparrow_p^q \in \Uparrow_p^q$, where $H_p \subset \Sigma_p$ denotes the horizontal directions at $p$ defined as

$$H_p = \{\, \uparrow_p^x \;|\, |p, x| \geq \mu_0 \,\}$$

(see [16]). It follows that for every $x \in B(p, R/2)^c$ and every $\uparrow_p^x \in \Uparrow_p^x$, $|\angle(\uparrow_p^x, \Uparrow_p^q) - \pi/2| < \tau(\delta,\epsilon)$. Similarly we have $|\angle(\uparrow_q^x, \Uparrow_q^p) - \pi/2| < \tau(\delta,\epsilon)$ for all $x \in B(q, R/2)^c$ and $\uparrow_q^x \in \Uparrow_q^x$, and therefore $\mathrm{ob}(p, q) < \tau(\delta,\epsilon)$. This completes the proof of Theorem 1.2. □

**Problem 4.2.** *Probably, the fiber $f^{-1}(x)$ has positive $(n-m)$-dimensional Hausdorff measure for all $x \in X$ in the situation of Theorem 4.1.*

*Proof of Corollary 1.3.* The conclusion follows from Theorems 1.1 and 1.2. The desired functions $\epsilon_n$ and $C_n$ in the conclusion are defined as follows, for instance. We construct only $\epsilon_n$. Let

$$\mathcal{A} := \left\{ M \;\middle|\; \begin{array}{l} M \text{ is an } n\text{-dimensional compact} \\ \text{Alexandrov space of nonnegative curvature} \end{array} \right\}$$

and set

$$\epsilon'_n(\tilde{v}) := \inf \{\mathrm{ob}(M) \mid M \in \mathcal{A} \text{ with } \tilde{v}(M) \geq \tilde{v}\}$$

for $\tilde{v} > 0$. Then, $\epsilon'_n$ satisfies

$$\epsilon'_n(\tilde{v}(M)) \leq \mathrm{ob}(M)$$

for every $M \in \mathcal{A}$. Furthermore, by Theorem 1.1, $\epsilon'_n(\tilde{v}) > 0$ for any $\tilde{v} > 0$. From Theorem 1.2, we have

$$\lim_{\tilde{v}\to 0} \epsilon'_n(\tilde{v}) = 0.$$

Note that the problem of maximizing $\tilde{v}(M)$ in $\mathcal{A}$ is equivalent to the problem of maximizing the usual volume in the restricted class of $M$'s whose diameter is one, because $\mathrm{ob}(M)$ and $\tilde{v}(M)$ are scale invariants. Since a maximizing sequence in the latter class has a convergent subsequence, there is a maximal value of $\tilde{v}(M)$ in $\mathcal{A}$, say $\tilde{v}_{n,\max}$.

Let us define a step function $\epsilon''_n : (0, \tilde{v}_{n,\max}] \to [0, \pi/2]$ by

$$\epsilon''_n(\tilde{v}) := \epsilon'_n(\tilde{v}_{n,\max}/k) \text{ if } \tilde{v} \in (\tilde{v}_{n,\max}/k, \tilde{v}_{n,\max}/(k-1)]$$

which bounds $\epsilon'_n$ from below. Furthermore, we consider the piecewise linear function connecting points $(\tilde{v}_{n,\max}/(k-1), \epsilon''_n(\tilde{v}_{n,\max}/k))$'s. Then, the function $\epsilon_n$ satisfies the desired condition of the conclusion of Corollary 1.3. $\square$

**Remark 4.3.** Clearly, Theorems 1.1 and 1.2 and Corollary 1.3 hold for balls in Alexandrov spaces. For a ball $B(p,r)$ in an $n$-dimensional (possibly noncompact) complete Alexandrov space centered at $p$ and radius $r$ with $r \le R_p$, we set $\mathrm{ob}(B(p,r))$ as follows. For $x \neq y \in B(p,r/2)$, we set

$$\mathrm{ob}_{B(p,r)}(x;y) := \sup_{z \in B(p,r)-B(p,r/2)} \left( \max_{\uparrow_x^z \in \Uparrow_x^z} \angle(\Uparrow_x^y, \uparrow_x^z) - \pi/2 \right)$$

and we define

$$\mathrm{ob}(B(p,r)) := \liminf_{x,y \in B(p,r/2) \text{ and } |x,y| \to 0} \max \left\{ \mathrm{ob}_{B(p,r)}(x;y), \mathrm{ob}_{B(p,r)}(y;x) \right\}.$$

Then, for instance, corresponding to Corollary 1.3, we have

$$\epsilon_n(\tilde{v}(B(p,r))) \le \mathrm{ob}(B(p,r)) \le C_n(\tilde{v}(B(p,r)))$$

for any point $p$ in an $n$-dimensional Alexandrov space $M$ of nonnegative curvature and any $r > 0$ with $r \le R_p$.

## 5. Volume growth and obtuse constant from infinity

This section is devoted to prove Theorem 1.4.

In this section, let $M$ denote noncompact complete Alexandrov $n$-space of nonnegative curvature. As written in the introduction, we discuss about a relation between the volume growth rate

$$v_\infty(M) = \lim_{R \to \infty} \frac{\mathcal{H}^n(B(x,R))}{R^n}$$

and the obtuse constant from infinity.

*Proof of Theorem 1.4.* We first prove that $\mathrm{ob}_\infty(M)$ has a lower bound $\epsilon_n(v_\infty(M))$. Assuming $v_\infty(M) \ge v > 0$, we prove

$$\inf_{p \neq q} \mathrm{ob}_\infty(p,q) \ge \epsilon_n(v) > 0. \tag{5.6}$$

Note that this claim is stronger than the first inequality in Theorem 1.4. Fix a base point $p \in M$. From the assumption, there exists $R_0 > 0$ such that if $R \ge R_0$, then

$$\mathcal{H}^n(B(p,R)) \ge vR^n/2.$$

That is, the unit ball $B_R := \frac{1}{R}B(p,R)$ centered at $p$ in $\frac{1}{R}M$ has volume not less than $v/2$. Hence, by Corollary 1.3 and Remark 4.3, there exists $\delta = \delta_n(v) > 0$ such that for $R \ge R_0$ and for $x, y \in B(p, R/2)$

with $0 < |x, y| \le \delta R$, we obtain a point $z \in B(p, R) - B(p, R/2)$ and directions $\uparrow_z^x$ and $\uparrow_y^z$ satisfying

$$\max\left\{\angle(\Uparrow_x^y, \uparrow_x^z), \angle(\Uparrow_y^x, \uparrow_y^z)\right\} \ge \pi/2 + \epsilon_n(v).$$

Here, $\epsilon_n(v)$ is given by Corollary 1.3. Now we prove the claim (5.6). Let us take arbitrary $x \neq y \in M$. Taking $R$ to be large, we have $x, y \in B(p, \delta R/2)$. Hence, we obtain

$$\begin{aligned}
\mathrm{ob}_\infty(x, y) &= \inf_{R > R_0} \sup_{|z,p| > R/2} \left( \max_{\uparrow_x^z, \uparrow_y^z} \left\{\angle(\Uparrow_x^y, \uparrow_x^z), \angle(\Uparrow_y^x, \uparrow_y^z)\right\} - \pi/2 \right) \\
&\ge \inf_{R > R_0} \sup_{R/2 < |z,p| \le R} \left( \max_{\uparrow_x^z, \uparrow_y^z} \left\{\angle(\Uparrow_x^y, \uparrow_x^z), \angle(\Uparrow_y^x, \uparrow_y^z)\right\} - \pi/2 \right) \\
&\ge \epsilon_n(v).
\end{aligned}$$

This completes the proof of the claim (5.6). In particular, we obtain

$$\epsilon_n(v) \le \inf_{x \neq y} \mathrm{ob}_\infty(x, y) \le \mathrm{ob}_\infty(M).$$

We shall prove that $\mathrm{ob}_\infty(M)$ has an upper bound as $C_n(v_\infty(M))$. We assume that there is no bound as $C_n$. So, there exists a sequence $\{M_i\}_i$ of noncompact complete $n$-dimensional Alexandrov spaces of nonnegative curvature satisfying $v_\infty(M_i) \to 0$ and $\mathrm{ob}_\infty(M_i) \ge \epsilon > 0$. Here, $\epsilon$ is independent of $i$. Let us take base points $p_i \in M_i$. Then, there exist $R_i \to \infty$ such that

$$0 \le \frac{\mathcal{H}^n(B(p_i, R_i))}{R_i^n} - v_\infty(M_i) < i^{-1}.$$

Therefore, by subtracting a subsequence, we may assume that $\left(\frac{1}{R_i} M_i, p_i\right)$ collapses to a pointed noncompact Alexandrov space $(X, p)$. Now, using Theorem 4.1 and the coarea formula ([6]) as in the proof of Theorem 1.2 we have

$$\mathrm{ob}_\infty(M_i) < \epsilon/2$$

for large $i$. This is a contradiction. □

**Remark 5.1.** In the proof of Theorem 1.4, we have proven a more detailed assertion as

$$\epsilon_n(v_\infty(M)) \le \inf_{p \neq q \in M} \mathrm{ob}_\infty(p, q) \le \mathrm{ob}_\infty(M) \le C_n(v_\infty(M)).$$

## 6. Maximal cases

In this section, let us discuss the maximal case of the obtuse constants equal to $\pi/2$. Let $M$ be an Alexandrov space with curvature bounded below. For every $p \in M$ and $\xi \in \Sigma_p(M)$, let 1-inj$(p; \xi)$ denote the supremum of $r \ge 0$ such that there exists a minimal geodesic $\gamma$

starting from $p$ in the direction $\xi$ or $-\xi$ (if it exists) of length $r$. Then the one-side injectivity radius of $M$ is defined as

$$\text{1-inj}(M) = \inf\{\text{1-inj}(p;\xi) \,|\, p \in M, \xi \in \Sigma_p\}.$$

One of main results of this section is to prove the following, which is a detailed version of Theorem 1.5.

**Theorem 6.1.** *If a compact Alexandrov space $M$ with curvature $\geq \kappa$ and radius $R$ has* $\mathrm{ob}(M) = \pi/2$*, then* $\text{1-inj}(M) \geq R/2$*.*

*Suppose that $M$ has nonempty boundary in addition. Then for each $p \in \partial M$ and $\xi \in \Sigma_p$ there exists a minimal geodesic in the direction $\xi$ of length $\geq R/2$. In addition, if $\xi \in \partial\Sigma_p$, then it has the opposite direction $-\xi$. In particular, $\partial M$ is totally geodesic and a $C^0$-Riemannian manifold.*

*Proof.* For any $p \in M$ and $\xi \in \Sigma_p^0$, let $\gamma$ be the geodesic from $p$ in the direction $\xi$. Take $q_i \in \gamma$ with $|p, q_i| \to 0$. By the assumption $\mathrm{ob}(M) = \pi/2$, there exists a sequence $x_i \in M$ such that one of the following holds:

(1) $\angle(\Uparrow_p^{q_i}, \uparrow_p^{x_i}) > \pi - \epsilon_i$ and $|x_i, p| \geq R/2$ for some $\uparrow_p^{x_i}$;
(2) $\angle(\Uparrow_{q_i}^{p}, \uparrow_{q_i}^{x_i}) > \pi - \epsilon_i$ and $|x_i, q_i| \geq R/2$ for some $\uparrow_{q_i}^{x_i}$,

where $\epsilon_i \to 0$. Suppose (1). Then $\{\uparrow_p^{x_i}\}$ is a Cauchy sequence and the geodesic $px_i$ converges to a geodesic $\sigma$ such that $\angle(\gamma, \sigma) = \pi$. Next suppose (2), and take $y_i \in q_ix_i$ such that $|q_i, y_i| = |q_i, q_1|$. Since $\angle y_iq_iq_1 < \epsilon_i$, $q_ix_i$ converges to a geodesic of legth $\geq R/2$ which extends $\gamma$. If $\xi$ is any element of $\Sigma_p$, we use a standard limiting argument to obtain the conclusion $\text{1-inj}(p;\xi) \geq R/2$. Thus we conclude that $\text{1-inj}(M) \geq R/2$.

Next we suppose $M$ has non-empty boundary, and take an arbitarary $p \in \partial M$ and $\xi \in \Sigma_p$. If $\xi$ is an interior direction, then there is no opposite direction to $\xi$. Take $p_i$ with $|p, p_i| \to 0$ and $\uparrow_p^{p_i} \to \xi$. It follows that there exists $x_i$ such that $\angle pp_ix_i > \pi - \epsilon_i$ and $|p_i, x_i| \geq R/2$ with $\lim_{i\to\infty} \epsilon_i = 0$. Therefore the broken geodesic $pp_ix_i$ converges to a minimal geodesic in the direction $\xi$ of length $\geq R/2$. Next assume $\xi \in \partial\Sigma_p$ and take a sequence of interior directions $\xi_i \in \Sigma_p \setminus \partial\Sigma_p$ converging to $\xi$. Then the sequence of minimal geodesics of length $R/2$ in the dirction $\xi_i$ converges to a minimal geodesic, say $\gamma$, in the dirction $\xi$ of length $R/2$. Note that $\gamma$ is contained in $\partial M$. Considering the point $q = \gamma(R/4)$ and the direction $\uparrow_q^p$, from the above assersion, we obtain a minimal geodesic $\sigma : [0, R/2] \to \partial M$ in the direction $\uparrow_q^p$. In particular, there is the opposite direction $-\xi = \dot\sigma(R/4)$. Reversing the orientation of $\sigma$ with a proper reparameterization, we see that

$$\left\{\begin{array}{l}\text{there exists a minimal geodesic } \mu : [-R/4, R/4] \to \partial M \\ \text{such that } \dot\mu(0) = \xi.\end{array}\right. \tag{6.7}$$

In particular, $\partial M$ is totally geodesic. It also follows that $\Sigma_p = \mathbb{S}^{n-2}$, where $n := \dim M$. Since $\partial M$ is an Alexandrov space with curvature $\geq \kappa$ having no singular points, a result in [7] shows that $\partial M$ is a $C^0$-Riemannian manifold. This completes the proof. □

In the noncompact case, by an argument similar to the proof of Theorem 6.1, we get the following.

**Theorem 6.2.** *If a complete noncompact Alexandrov $n$-space $M$ of curvature $\geq \kappa$ has $\mathrm{ob}_\infty(M) = \pi/2$, then $1$-$\mathrm{inj}(M) = \infty$.*

*Suppose additionally that $M$ has nonempty boundary. Then for every $p \in \partial M$ and every $\xi \in \Sigma_p$ there exists a geodesic ray in the direction $\xi$. In particular, $M$ is a $C^0$-Riemannian manifold homeomorphic to $\mathbb{R}^n_+$, and any distinct two points of $\partial M$ are on a line of $M$ which is contained in $\partial M$.*

*Proof.* The proofs of the conclusions except the last statement are similar to those of Theorem 6.1, and hence omitted. We prove only the last statement. Suppose that $M$ has nonempty boundary, and let $p, q \in \partial M$ be distinct points. By the first statement of the theorem, there exist geodesic rays $\gamma : [0, \infty) \to M$, $\sigma : [0, \infty) \to M$ such that $\gamma(0) = p = \sigma(|p, q|)$ and $\sigma(0) = q = \gamma(|p, q|)$. Note that $\gamma$ and $\sigma$ are contained in $\partial M$. For any large $R > 0$, considering the point $x = \gamma(R)$ and the direction $\uparrow_x^p$, we obtain a geodesic ray in the direction $\uparrow_x^p$. Letting $R \to \infty$ together with an argument similar to (6.7), we see that $\gamma$ and $\sigma$ define a line in $\partial M$ containing $p$ and $q$. From the first statement of the theorem, for any $p \in \partial M$, the exponential map $\exp_p : T_pM \to M$ is defined and provides a homeomorphism between $M$ and $\mathbb{R}^n_+$. This completes the proof. □

Let $M$ be a surface of revolution with vertex $p_0$ homeomorphic to $\mathbb{R}^2$ having Riemannian metric

$$g = dr^2 + m(r)^2\, d\theta^2,$$

with respect to a polar coordinates $(r, \theta)$ around $p_0$. Note that

$$m(0) = 0, \quad m'(0) = 1, \quad m'' + Km = 0.$$

We assume that

(1) the Gaussian curvature $K$ of $M$ is nonnegative;
(2) the total curvature of $M$ is at most $\pi$:

$$\int_M K\, dM \leq \pi.$$

Note that the ideal boundary $M(\infty)$ of $M$ is a circle of length $2\pi - \displaystyle\int_M K\, dM \geq \pi$ (see [15]).

**Example 6.3.** As an example, consider the hyperboloid $M_a$ defined by
$$z = a\sqrt{x^2+y^2+1}.$$
Then its asymptotic cone $(M_a)_\infty$ is written as
$$(M_a)_\infty = \{z = a\sqrt{x^2+y^2}\}.$$
Therefore $M_a$ satisfies all the above assumptions when $0 \le a \le \sqrt{3}$.

The following Theorem 6.4 shows that Theorem 1.7 is sharp.

**Theorem 6.4.** *Le $M$ be a complete open surface of revolution having nonnegative Gaussian curvature such that*
$$\int_M K\, dM \le \pi.$$
*Then* $\mathrm{ob}_\infty(M) = \pi/2$.

*Proof.* First we recall the description of geodesics in $M$. Let $(r(s), \theta(s))$ be the coordinates of a unit speed geodesic $\gamma(s)$ on $M$, and $\zeta = \zeta(s)$ be the angle, $0 \le \zeta \le \pi$, between $\gamma$ and the positive direction of the parallel circle $r =$ constant. The Clairaut relation states that
$$m(r(s))\cos\zeta(s) = \text{constant} = \nu, \tag{6.8}$$
where $\nu$ is called the Clairaut constant of $\gamma$. Moreover we have
$$\frac{d\theta}{dr} = \frac{\theta'}{r'} = \epsilon\frac{\nu}{m(r)\sqrt{m^2(r)-\nu^2}}, \tag{6.9}$$
where $\epsilon = \pm 1$ is determined by the sign of $r'$ (see [14, Proposition 7.1.3]).

Let $L(t)$ denote the length of geodesic sphere $S(p_0, t) := \partial B(p_0, t)$. Since
$$\lim_{t\to\infty}\frac{L(t)}{t} = L(M(\infty)) > 0,$$
we have $\int_1^\infty \frac{dt}{L^2(t)} < \infty$. It follows from [14, Theorem 7.2.1] that the set of poles of $M$ coincides with a closed ball around $p_0$ of positive radius $r(M) > 0$. Therefore for every $p, q \in M$ if one of $p, q$ is contained in $B(p_0, r(M))$, then obviously we have $\mathrm{ob}_\infty(p, q) = \pi/2$.

Therefore in the below, we assume that $p, q \in M \setminus B(p_0, r(M))$. Let $A_p$ denote the set of velocity vectors $v \in \Sigma_p$ of the geodesic rays emanating from $p$. Let us first show that $A_p$ contains a closed arc of length $2\pi - \int_M K\, dM \ge \pi$. Let $m(A_p)$ denote the measure of $A_p$. By the result due to Maeda [8], we know that
$$\inf_{p\in M} m(A_p) = 2\pi - \int_M K\, dM \ge \pi.$$
From this point of view, the claim is likely to be true. In the argument below, we confirm this.

We may assume that $(r(p), \theta(p)) = (r_0, 0)$ and $r_0 > r(M)$. Let $\xi_0 \in \Sigma_p$ (resp. $\eta_0 \in \Sigma_p$) denote the positive direction of the meridian through $p$ (resp. the positive direction of the parallel circle through $p$). For each $t \in [-\pi, \pi]$, we let

$$\xi_t = \cos t \cdot \xi_0 + \sin t \cdot \eta_0$$

Denote by $\gamma_t$ the geodesic from $p$ such that $\gamma_t'(0) = \xi_t$. For each $s \in [-\pi, \pi]$, we let $\sigma_s$ be the geodesic ray from $p_0$ that is equal to the meridian with $\theta(\sigma_s) = s$, and take a sequence $t_i \to \infty$ and a minimal geodesic $\mu_{s,i}$ joining $p$ to $\sigma_s(t_i)$. When $s = \pi$, we choose $\mu_{\pi,i}$ in such a way that $0 \le \theta(\mu_{\pi,i}(t)) \le \pi$ for all $t \ge 0$. Then a subsequence of $\mu_{s,i}$ converges to a geodesic ray $\mu_s$ from $p$ satisfying

(1) $0 \le \theta(\mu_s(t_1)) < \theta(\mu_s(t_2)) < s$ for all $0 \le t_1 < t_2 < \infty$;
(2) $\lim_{t\to\infty} \theta(\mu_s(t)) = s$.

Take $t_* \in (0, \pi]$ such that $\xi_{t_*} = \mu_\pi'(0)$. We claim that

$$t_* \ge \pi - \frac{1}{2}\int_M K\, dM \ge \pi/2. \tag{6.10}$$

Let $D$ denote the domain bounded by the two geodesic rays $\gamma_{t_*}$ and $\gamma_{-t_*}$ such that $p_0 \in D$. Let $\lambda_s : [0, d_s] \to M$ be a minimal geodesic from $\gamma_{t_*}(s)$ to $\gamma_{-t_*}(s)$. Note that both $\gamma_{t_*}$ and $\gamma_{-t_*}$ are asymptotic to $\sigma_\pi$ by symmetry, and hence

$$\lim_{s\to\infty} |\gamma_{\pm t_*}(s), \sigma_\pi(s)|/s = 0. \tag{6.11}$$

It follows that $\lambda_s$ is contained in $D$ for large enough $s > 0$. Let

$$\alpha_+(s) := \angle\gamma_{-t_*}(s)\gamma_{t_*}(s)p, \quad \alpha_-(s) := \angle\gamma_{t_*}(s)\gamma_{-t_*}(s)p,$$
$$\tilde\alpha_+(s) := \tilde\angle\gamma_{-t_*}(s)\gamma_{t_*}(s)p, \quad \tilde\alpha_-(s) := \tilde\angle\gamma_{t_*}(s)\gamma_{-t_*}(s)p.$$

In view of (6.11), considering 1-strainers $(p, \gamma_{t_*}(2s))$ at $\gamma_{t_*}(s)$ and $(p, \gamma_{-t_*}(2s))$ at $\gamma_{-t_*}(s)$, we have

$$\lim_{s\to\infty} |\alpha_\pm(s) - \tilde\alpha_\pm(s)| = 0.$$

Since $\lim_{s\to\infty} \tilde\alpha_\pm(s) = \pi/2$, we obtain $\lim_{s\to\infty} \alpha_\pm(s) = \pi/2$. The Gauss-Bonnet theorem then implies that

$$\begin{aligned}\int_D K\, dM &= \lim_{s\to\infty}(\alpha_+(s) + \alpha_-(s) + \angle_p(D) - \pi)\\ &= \angle_p(D) = 2(\pi - t_*) \le \int_M K\, dM,\end{aligned}$$

where $\angle_p(D)$ denotes the inner angle of $D$ at $p$. It follows that $t_* \ge \pi - \frac{1}{2}\int_M K\, dM \ge \pi/2$ as required.

Now we show that $\gamma_t$ is a geodesic ray for each $t \in [-\pi/2, \pi/2]$. Let $\hat t$ denote the maximum of those $t \in [0, \pi/2]$ that $\gamma_s$ is a geodesic ray for all $s \in [0, t]$. It suffices to show that $\hat t = \pi/2$. Suppose that $\hat t < \pi/2$. Since $t_* \ge \pi/2 > \hat t$ and both $\gamma_{\hat t}$ and $\gamma_{t_*} = \mu_\pi$ are geodesic rays, we have $0 \le \theta(\gamma_{\hat{t}}(s)) < \pi$ for all $s \ge 0$. By (6.8), $\theta(\gamma_t(s))$ is monotone increasing in $s$, and therefore there is a unique limit

$$\theta_t(\infty) := \lim_{s\to\infty} \theta(\gamma_t(s)) \in [0,\pi]$$

for every $t \in [0,\hat{t}]$. If $\theta_{\hat{t}}(\infty) = \pi$, in a way similar to (6.10) we would have $\hat{t} \ge \pi/2$, which is a contradiction. Thus we see $\theta_{\hat{t}}(\infty) < \pi$. From continuity, there is some $\tilde{t} \in (\hat{t}, \pi/2)$ such that

$$0 \le \theta(\gamma_t(s)) < \pi, \quad 0 \le \theta_t(\infty) < \pi,$$

for any $0 \le t \le \tilde{t}$ and all $s \ge 0$. Obviously $\theta_t(\infty)$ is continuous in $t \in [0,\tilde{t}]$. For $0 \le t_1 < t_2 \le \tilde{t}$, let $\theta_i(s) := \theta(\gamma_{t_i}(s))$, and $\nu_i$ the Clairaut constants of $\gamma_{t_i}$ for $i = 1,2$. Since $\nu_1 < \nu_2$, the formula (6.9) implies that $d\theta_1/dr < d\theta_2/dr$, and hence $\theta_{t_1}(\infty) < \theta_{t_2}(\infty)$. Thus $\theta_t(\infty)$ is injective in $t \in [0,\tilde{t}]$. This yields that $\gamma_t$ coincides with the geodesic ray $\mu_{\theta_t(\infty)}$ for all $t \in [0,\tilde{t}]$, which is a contradiction to the definition of $\hat{t}$. Thus we conclude that $\hat{t} = \pi/2$ and $\gamma_t$ is a geodesic ray for every $t \in [-\pi/2, \pi/2]$ by symmetry.

Finally we show that $\mathrm{ob}_\infty(p,q) \ge \pi/2 - \tau(\delta)$ with $\delta = |p,q|$ and $\lim_{\delta\to 0}\tau(\delta) = 0$. Take a minimal geodesic $\gamma : [0,\delta] \to M$ from $p$ to $q$. First assume that $r(q) = r(p)$. Since $\zeta(0) = \zeta(\delta)$ and $\zeta(\delta/2) = 0$, we have from (6.8)

$$\cos\zeta(0) = \frac{m(r(\delta/2))}{m(r(0))}, \tag{6.12}$$

where $|m(r(0)) - m(r(\delta/2))| \le \frac{\delta}{2}m' \le \frac{\delta}{2}$ because of nonnegative curvature. It follows that

$$\left|1 - \frac{m(r(\delta/2))}{m(r(0))}\right| \le \frac{\delta}{2m(r(M))}.$$

Together with (6.12), this yields

$$\zeta(0) \le \sqrt{\frac{\delta}{m(r(M))}} =: \delta_1.$$

Let $\gamma_{\pi/2}$ be the geodesic ray from $p$ defined above. We may assume that $\angle(\gamma'_{\pi/2}(0), \gamma'(0)) = \zeta(0)$. For large enough $R > 0$, we have

$$\begin{aligned}\angle pq\gamma_{\pi/2}(R) &\ge \tilde{\angle} pq\gamma_{\pi/2}(R)\\ &\ge \pi - \tilde{\angle} qp\gamma_{\pi/2}(R) - \tilde{\angle} p\gamma_{\pi/2}(R)q\\ &\ge \pi - \zeta(0) - o_R \ge \pi - \delta_1 - o_R,\end{aligned}$$

where $\lim_{R\to\infty} o_R = 0$, and hence $\mathrm{ob}_\infty(p,q) \ge \pi/2 - \delta_1$.

Next assume $r(p) < r(q)$. If $\angle(\uparrow_p^q, \xi_0) \le \pi/2$, then $p$ and $q$ are on a geodesic ray. In the other case, taking $p_1 \in pq$ with $r(p) = r(p_1)$, one can show that $\zeta(0) \le \delta_1$ and

$$\angle pq\gamma_{\pi/2}(R) \ge \tilde{\angle} pq\gamma_{\pi/2}(R) \ge \pi - \delta_1 - o_R$$

by a similar manner. Thus we conclude that $\mathrm{ob}_\infty(M) = \pi/2$. $\square$

**Remark 6.5.** From the argument in the proof of Theorem 6.4, we directly obtain

$$\widetilde{\mathrm{ob}}_\infty(M) = \pi/2,$$

under the the same hypothesis as Theorem 6.4, where $\widetilde{\mathrm{ob}}_\infty(M)$ is the comparison obtuse constant of $M$ from infinity defined in Section 7. See also (7.15).

**Remark 6.6.** Theorem 6.4 shows that the estimate $1\text{-inj}(M(\infty)) \geq \pi/2$ in Theorem 1.7 (1) is sharp. It should also be noted that one cannot expect that $M(\infty)$ has no singular points in Theorem 1.7 (1), because if one take $N = M \times \mathbb{R}$, where $M$ is a non-flat open surface as in Theorem 6.4, then $\mathrm{ob}_\infty(N) = \pi/2$ and $N(\infty)$ is the spherical suspension over $M(\infty)$. Note that $N(\infty)$ has the two singular points at the vertices of the suspension since the length of the circle $M(\infty)$ is less than $2\pi$.

*Proof of Theorem 1.7.* Theorem 1.7 (2) immediately follows from Theorem 6.2 and the splitting theorem. Suppose $M$ has no boundary. For $\epsilon_i \to 0$ and $p \in M$, consider the asymptotic limit

$$\lim_{i\to\infty}(\epsilon_i M, p) = (M_\infty, o), \tag{6.13}$$

where the asymptotic cone $M_\infty$ is the Euclidean cone over $M(\infty)$. Identify $M(\infty) = M(\infty) \times \{1\} \subset M_\infty$. For every $\xi \in M(\infty)$ and every geodesic $\gamma : [0, \delta] \to M_\infty$ from $\xi$, fix any $0 < a < \delta$, and set $\eta := \gamma(\delta)$ and $\xi_a = \gamma(a)$. Take sequences $p_i, q_i$ and $x_i \in p_i q_i$ in $\epsilon_i M$ such that

$$p_i \to \xi, \ \ x_i \to \xi_a, \ \ q_i \to \eta,$$

as $i \to \infty$ under the convergence (6.13). On the interior of a minimal geodesic $p_i x_i$, take points $y_{i,\alpha}$ such that $y_{i,\alpha} \to x_i$ as $\alpha \to \infty$. From the assumption, for any sequences $R_i \to \infty$ and $o_i \to 0$, if $\alpha$ is large enough compared to $i$, one can find points $z_i$ with $|x_i, z_i| \geq R_i/\epsilon_i$ and either

$$\angle z_i x_i y_{i,\alpha} > \pi - o_i \ \text{ or } \ \angle z_i y_{i,\alpha} x_i > \pi - o_i.$$

Letting $i \to \infty$, we obtain a geodesic ray emanating from $\xi_a$ either in the direction $\gamma'(a)$ or in the opposite direction $-\gamma'(a)$. Then letting $a \to 0$, we conclude that there is a geodesic ray $\sigma$ starting from $\xi$ such that either $\sigma'(0) = \gamma'(0)$ or else there is the opposite direction $-\gamma'(0)$ and $\sigma'(0) = -\gamma'(0)$. Thus we have $1\text{-inj}(M_\infty) = \infty$.

Now for any direction $v \in \Sigma_\xi(M(\infty)) \subset \Sigma_\xi(M_\infty)$, there is a geodesic ray $\sigma$ of $M_\infty$ starting from $\xi$ either in the direction $v$ or else in the direction $-v$ if $-v$ exists. For each $t \geq 0$, let $\xi_t :=\uparrow_o^{\sigma(t)} \in M(\infty)$. It is easy to see that there is a unique limit $\xi' = \lim_{t\to\infty} \xi_t$ and $\angle(\xi, \xi') = \pi/2$. Thus $\xi_t$ provides a shortest segment in $M(\infty)$ from $\xi$ to $\xi'$ in the direction $v$ or $-v$ if any. This shows 1-inj$(M(\infty)) \geq \pi/2$ as required. $\square$

**Conjecture 6.7.** For a compact Alexandrov space $M$ with curvature bounded below, if $\mathrm{ob}(M) = \pi/2$, then the double $D(M)$ would have no singular points. Similarly, for a noncompact Alexandrov space $M$ with curvature bounded below, if $\mathrm{ob}_\infty(M) = \pi/2$, then the double $D(M)$ would have no singular points as well (compare Section 7).

## 7. Comparison obtuse constants

Let $M$ be a compact $n$-dimensional Alexandrov space of curvature $\geq \kappa$. For $p \neq q \in M$, we set

$$\widetilde{\mathrm{ob}}_\kappa(p;q) := \sup_{x \in B(p,R_M/2)^c} \tilde{\angle}_\kappa xpq - \pi/2,$$

and

$$\widetilde{\mathrm{ob}}_\kappa(p,q) := \max\left\{\widetilde{\mathrm{ob}}_\kappa(p;q), \widetilde{\mathrm{ob}}_\kappa(q;p)\right\}.$$

Then, we define the *comparison obtuse constant* of $M$ by

$$\widetilde{\mathrm{ob}}(M) := \liminf_{|p,q| \to 0} \widetilde{\mathrm{ob}}_\kappa(p,q).$$

Note that this is independent of the choice of $\kappa$. A trivial relation

$$\widetilde{\mathrm{ob}}(M) \leq \mathrm{ob}(M) \tag{7.14}$$

follows from (2.2). Hence, when $\widetilde{\mathrm{ob}}(M)$ is maximal, it is natural to expect that $M$ has a stronger geometric property compared with the maximal case of $\mathrm{ob}(M)$. Indeed, we obtain

**Theorem 7.1.** *Let $M$ be a compact $n$-dimensional Alexandrov space. Suppose $\widetilde{\mathrm{ob}}(M) = \pi/2$. Then, $D(M)$ has no singular points, that is, for every $p \in M$, we have $\Sigma_p = \mathbb{S}^{n-1}$ or $\Sigma_p = \mathbb{S}^{n-1}_+$. Here, $\mathbb{S}^{n-1}_+$ denotes the upper half unit sphere.*

*Proof.* Let $M$ be as in the assumption. Let us take $p \in M$. Suppose that $p$ is an interior point. Take $\xi, \eta \in \Sigma_p$ with $\angle(\xi,\eta) = \mathrm{diam}\,(\Sigma_p)$, and suppose that $\angle(\xi,\eta) < \pi$. Let us take sequences $x_i, y_i \in M$ such that $|p,x_i| = |p,y_i| \to 0$, $\uparrow_p^{x_i} \to \xi$ and $\uparrow_p^{y_i} \to \eta$. Since $\widetilde{\mathrm{ob}}(M) = \pi/2$, there exists a point $z_i \in M$ such that one of the following holds:

(1) $\tilde{\angle} x_i y_i z_i \geq \pi - \epsilon_i$ and $|y_i, z_i| \geq R/2$;
(2) $\tilde{\angle} y_i x_i z_i \geq \pi - \epsilon_i$ and $|x_i, z_i| \geq R/2$,

where $\epsilon_i \to 0$. By extracting a subsequence and by replacing $x_i$ and $y_i$ if necessarily, we may assume (1) holds for all $i$. Under the convergence of $(|p,x_i|^{-1}M, p)$ to the tangent cone $(T_pM, o_p)$, the sequence of broken geodesics $x_iy_iz_i$ converges to a ray starting from $\xi$ through $\eta$. Now we can take a direction $\zeta \in \Sigma_p$ along the ray satisfying $\angle(\xi,\zeta) > \angle(\xi,\eta)$. Since this is a contradiction, we have $\mathrm{diam}\,(\Sigma_p) = \pi$.

By the splitting theorem, $T_pM$ is isometric to the product of the line $\ell$ through $\xi, \eta$ and the space $T'$ of vectors perpendicular to $\ell$. Let $\Lambda \subset \Sigma_p$ denote the set of directions tangent to $T'$. Then, we have $\mathrm{diam}\,(\Lambda) = \pi$. Indeed, if $\bar\xi, \bar\eta \in \Lambda$ attain the diameter of $\Lambda$, then taking sequences $\bar x_i, \bar y_i \to p$ with $|\bar x_i, p| = |\bar y_i, p|$ so that $\uparrow_p^{\bar x_i} \to \bar\xi$ and $\uparrow_p^{\bar y_i} \to \bar\eta$, we have a point $\bar z_i$ in a way similar to the above argument. Then, the limit ray of $\bar x_i \bar y_i \bar z_i$ (or $\bar y_i \bar x_i \bar z_i$) under the convergence $(|p, \bar x_i|^{-1} M, p) \to (T_pM, o)$, is contained in $T'$. The existence of such a ray enforces that $\mathrm{diam}\,(\Lambda) = \pi$, and $T'$ is isometric to a product $\mathbb{R} \times T''$. Repeating this argument, finally we obtain that $T_pM$ is isometric to $\mathbb{R}^n$. Therefore, $\Sigma_p = \mathbb{S}^{n-1}$.

If $p$ is a boundary point, then Theorem 6.1 directly implies that $\Sigma_p = \mathbb{S}^{n-1}_+$. This completes the proof. $\square$

For noncompact spaces, a *comparison obtuse constant from infinity* $\widetilde{\mathrm{ob}}_\infty(M)$ is also defined as follows. Let $M$ be a noncompact complete $n$-dimensional Alexandrov space of curvature $\geq \kappa'$, where $\kappa' \leq 0$. For $\kappa \leq 0$ and $p \neq q \in M$, we set

$$\widetilde{\mathrm{ob}}_{\kappa,\infty}(p, q) := \limsup_{x \to \infty} \max\{\tilde\angle_\kappa xpq, \tilde\angle_\kappa xqp\}.$$

Then, we define

$$\widetilde{\mathrm{ob}}_\infty(M) := \liminf_{|p,q| \to 0} \widetilde{\mathrm{ob}}_{\kappa,\infty}(p, q).$$

Clearly, $\widetilde{\mathrm{ob}}_\infty(M)$ is not depending on $\kappa$ and

$$\widetilde{\mathrm{ob}}_\infty(M) \leq \mathrm{ob}_\infty(M). \tag{7.15}$$

By an argument similar to the proof of Theorem 7.1, we have

**Theorem 7.2.** *Let $M$ be an $n$-dimensional noncompact complete Alexandrov space. Assume that $\widetilde{\mathrm{ob}}_\infty(M) = \pi/2$. Then $D(M)$ has no singular points.*

**Conjecture 7.3.** Theorem 1.1 and Corollary 1.3 would hold for $\widetilde{\mathrm{ob}}(M)$ instead of $\mathrm{ob}(M)$ when $M$ is compact. Theorem 1.4 would hold for $\widetilde{\mathrm{ob}}_\infty(M)$ instead of $\mathrm{ob}_\infty(M)$ when $M$ is noncompact.

Note that from the trivial inequalities (7.14) and (7.15), Theorem 1.2 and the last inequality of Theorem 1.4 are true even for $\widetilde{\mathrm{ob}}(M)$ and $\widetilde{\mathrm{ob}}_\infty(M)$.

## 8. Comparison $\kappa$-obtuse constants from infinity

We conclude the paper with some comments on another definition of "comparison obtuse constant from infinity" for noncompact spaces which does depend on the lower curvature bound.

Let $M$ be a complete noncompact Alexandrov space with curvature $\geq \kappa$, and $p \neq q \in M$. Using our previous definition of $\widetilde{\mathrm{ob}}_{\kappa,\infty}(p,q)$, set

$$\widetilde{\mathrm{ob}}_{\kappa,\infty}(M) := \inf_{p\neq q} \widetilde{\mathrm{ob}}_{\kappa,\infty}(p,q).$$

which we call the *comparison $\kappa$-obtuse constant of $M$ from infinity.* Note that

$$\widetilde{\mathrm{ob}}_{\kappa,\infty}(M) \leq \widetilde{\mathrm{ob}}_{\infty}(M) \leq \mathrm{ob}_{\infty}(M).$$

Clearly the $\kappa$-obtuse constant from infinity does depend on the choice of the lower curvature bound $\kappa$, and $\widetilde{\mathrm{ob}}_{0,\infty}(M) \geq 0$ for $\kappa = 0$. However if $\kappa < 0$, the $\kappa$-obtuse constant from infinity could be negative. For instance, if $M$ is the domain bounded by an ideal triangle all of whose vertexes are on the ideal boundary of the hyperbolic plane $\mathbb{H}^2(-1)$. Then $\widetilde{\mathrm{ob}}_{-1,\infty}(M) = -\pi/2$.

This invariant seems interesting in itself. For instance, we have the following strong rigidity.

**Theorem 8.1.** *Let $M$ be a complete noncompact Alexandrov $n$-space with nonnegative curvature satisfying $\widetilde{\mathrm{ob}}_{0,\infty}(M) = \pi/2$. If $M$ has no boundary, then $M$ is isometric to the Euclidean space $\mathbb{R}^n$.*

*Proof.* Take $r_i \to 0$ and consider the pointed Gromov-Hausdorff convergence $(r_iM, p) \to (M_\infty, o)$, where $M_\infty$ is the asymptotic cone, which is isometric to the the Euclidean cone $K(M(\infty))$ over the ideal boundary $M(\infty)$. By Theorem 1.4, $v_\infty(M) > 0$ and hence $\dim M_\infty = n$. It suffices to show that $M_\infty$ is isometric to $\mathbb{R}^n$. First we show that $\mathrm{diam}\,(M(\infty)) = \pi$. Suppose $\mathrm{diam}\,(M(\infty)) < \pi$ and take $\xi, \eta \in M(\infty)$ with $|\xi, \eta| = \mathrm{diam}\,(M(\infty))$. We identify $M(\infty)$ as $M(\infty) \times \{1\} \subset M_\infty$, and take $x_i, y_i \in r_iM$ such that $x_i \to \xi$, $y_i \to \eta$ under the convergence $(r_iM, p) \to (M_\infty, o)$. From the assumption, we may assume that there is a geodesic ray $\gamma_i$ emanating from $x_i$ through $y_i$. Passing to a subsequence, we may also assume that $\gamma_i$ converges to a geodesic ray $\gamma_\infty$ in $M_\infty$ emanating from $\xi$ through $\eta$. Obviously we can find a point $z$ on $\gamma_\infty$ such that the direction $\zeta = \uparrow_o^z$ satisfies $|\xi, \zeta| > |\xi, \eta|$. Since this is a contradiction, we have $\mathrm{diam}\,(M(\infty)) = \pi$. By the splitting theorem, $M_\infty$ is isometric to a product $M'_\infty \times \mathbb{R}$. Repeating the argument to $M'_\infty$, we see that $M'_\infty$ is isometric to a product $M''_\infty \times \mathbb{R}$. In this way, we conclude that $M_\infty$ is isometric to $\mathbb{R}^n$. $\square$

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