Cocycles of nilpotent quotients of free groups

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Abstract

We focus on the cohomology of the k-th nilpotent quotient of a free group. We describe all the group 2-, 3-cocycles in terms of the Massey product and give expressions for some of the 3-cocycles. We also give simple proofs of some of the results on the Milnor invariant and Johnson-Morita homomorphisms.

Keywords: nilpotent group, higher Massey product, group cohomology, mapping class group, link

1 Introduction

Let F be the free group of rank q. We define F_1 to be F and F_k to be the commutator subgroup $[F_{k-1}, F]$ by induction. Accordingly, we have the central extension

$$0 \longrightarrow F_k/F_{k+1} \longrightarrow F/F_{k+1} \xrightarrow{p_k} F/F_k \longrightarrow 1. \tag{1}$$

The abelian kernel F_k/F_{k+1} is classically known to be free and of finite rank (see, e.g., [BC, MKS] and references therein). We denote the rank as $N_k \in \mathbb{N}$.

The nilpotent quotient F/F_k has been studied with applications (see, e.g., [Mas, BC, MKS] for the relation to the free Lie algebra $\oplus_{i=1}^k F_k/F_{k+1}$). The group homology of F/F_k also plays an important role in the study of low dimensional topology, including the Milnor (link) invariant, Johnson-Morita homomorphisms, and tree parts of the quantum invariant (see [GL, Ki, IO, Heap, KN, Mas, Tu, Po]). Roughly speaking, Massey products of manifolds appear in nilpotent obstructions of manifolds, as described in [CGO, FS, Ki, GL, Heap]. Moreover, the homology groups of degrees 2 and 3 are computed as

$$H_2(F/F_k;\mathbb{Z}) \cong \mathbb{Z}^{N_k}, \qquad H_3(F/F_k;\mathbb{Z}) \cong \bigoplus_{i=k}^{2k-2} \mathbb{Z}^{qN_i - N_{i+1}}. \tag{2}$$

The former is a classical result; however, the latter was a result of Igusa and Orr [IO, Corollaries 5.5 and 6.5] from spectral sequences. In particular, it seems difficult to discuss the third homology $H_3(F/F_k;\mathbb{Z})$ quantitatively.

We describe the bases of the cohomologies $H^2(F/F_k;\mathbb{Z})$ and $H^3(F/F_k;\mathbb{Z})$ as Massey products (see Theorems 3.2 and 3.3) and explicitly express some of the cocycles. We also consider the Massey products to be an algorithm to produce many expressions of cocycles. Section 5 gives 3-cocycles of some nilpotent groups and concretely describes their expressions. Thus, it is reasonable to hope that the expressions may be useful for computing various things appearing in topology; including the Morita homomorphisms [M2] and Orr link invariants [O1].

As corollaries, Section 4 provides simple proofs of four known results of [FS, Po, Tu, Heap, Ki], which are related to Massey products. The original proofs were discussed at the cohomology level, and they constructed Massey products according to circumstances. In contrast, since the theorems 3.2 and 3.3 produces cocycles, which give a basis of $H^*(F/F_k;\mathbb{Z})$ as Massey products, we give shorter proofs in terms of cocycles.

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This paper is organized as follows. Section 2 reviews Magnus expansions and Massey products, and Section 3 states our theorem with a proof. Section 4 explains the four known results mentioned above and gives alternative proofs to them based on our theorem in Section 3. Section 5 discusses an algorithm to produce cocycles.

Conventional notations. Given a group G, we denote G as G_1 and define G_k to be the commutator subgroup $[G_{k-1}, G]$ by induction. Furthermore, let F be the free group of rank q with a basis $x_1, \dots, x_q$, and let $N_k \in \mathbb{Z}$ be the rank of F_k/F_{k+1}. In addition, we assume the basic properties of group (co)-homology, as in [Bro, Sections I, II, and VII].

2 Review: Magnus expansions and higher Massey products

Let us begin by studying unipotent Magnus expansions and higher Massey products.

First, we review the Magnus expansion modulo degree k. Let $\mathbb{Z}\langle X_1, \dots, X_q\rangle$ be the polynomial ring with non-commutative indeterminates $X_1, \dots, X_q$, and $\mathcal{J}_k$ be the two-sided ideal generated by polynomials of degree $\geq k$. Consider the multiplicative map $\mathcal{M} : F \to \mathbb{Z}\langle X_1, \dots, X_q\rangle/\mathcal{J}_k$ defined by

$$\mathcal{M}(x_i) = 1 + X_i, \qquad \mathcal{M}(x_i^{-1}) = 1 - X_i + X_i^2 + \cdots + (-1)^{k-1}X_i^{k-1}. \tag{3}$$

It is known that $\mathcal{M}(F_k) = 0$. Then, by passage to this F_k, this $\mathcal{M}$ further induces the injection

$$\mathcal{M}_k : F/F_k \longrightarrow \mathbb{Z}\langle X_1, \dots, X_k\rangle/\mathcal{J}_k,$$

which we call *the Magnus expansion (of F modulo degree k)* in this paper.

Next, we review another description of the Magnus expansion [GG], which is a faithful linear representation of F/F_k. Let Ω_k be the polynomial ring $\mathbb{Z}[\lambda_i^{(j)}]$ over commuting indeterminates $\lambda_i^{(j)}$ with $i \in \{1, 2, \dots, k-1\},\ j \in \{1, \dots, q\}$. We define the group homomorphism

$$\Upsilon_k : F \longrightarrow GL_k(\Omega_k),$$

by setting

$$\Upsilon_k(x_j) = \begin{pmatrix} 1 & \lambda_1^{(j)} & 0 & \cdots & 0 \\ 0 & 1 & \lambda_2^{(j)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & \lambda_{k-1}^{(j)} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

It is known [GG] that the image $\Upsilon_k(F_k)$ consists of the identity matrix and that the induced map $F/F_k \to GL_k(\Omega_k)$ is injective. Thus, we have an isomorphism $F/F_k \cong \mathrm{Im}(\Upsilon_k)$; the correspondence $1 + X_j \mapsto \Upsilon_k(x_j)$ gives isomorphism $\mathrm{Im}(\mathcal{M}_k) \cong \mathrm{Im}(\Upsilon_k)$. The longer the word of x, the more difficult the computation of $\mathcal{M}_k(x)$; however, that of $\Upsilon_k(x)$ is still simpler. The map Υ_k is defined over Ω_k and is therefore compatible with computer programs.

Next, we review the higher Massey products, which were first defined by Kraines [Kra]. We describe the products in the non-homogenous complex of a group G with a trivial coefficient ring A. That is, as in [Bro, Chap. III.1], we define the abelian group $C^*(G; A)$ of cochains to be $\mathrm{Map}(G^n, A)$ and the coboundary map ∂_n^* by setting

$$(\partial_n^* f)(g_1, \dots, g_n) = f(g_2, \dots, g_n) + (-1)^n f(g_1, \dots, g_{n-1})$$

$$-f(g_1g_2, g_3, \dots, g_n) + f(g_1, g_2g_3, g_4 \dots, g_n) + \cdots + (-1)^{n-1} f(g_1, \dots, g_{n-1}g_n).$$

Furthermore, the cup product on $C^n(G;A)$ can be described as a canonical product. More precisely, for $u \in C^p(G;A)$ and $v \in C^q(G;A)$, the product $u \smile v \in C^{p+q}(G;A)$ is defined by

$$(u \smile v)(g_1, \dots, g_{p+q}) := (-1)^{pq} u(g_1, \dots, g_p) \cdot v(g_{p+1}, \dots, g_{p+q}) \in A.$$

For any $1 \leq i \leq n$, take a cocycle $\gamma_i \in C^{p_i}(G;A)$. Then, a *defining system* associated with $(\gamma_1, \dots, \gamma_n)$ is a set of elements $(a_{s,t})$ for $1 \leq s \leq t \leq n$ with $(s,t) \neq (1,n)$, satisfying

(i) $a_{s,t} \in C^{p_s + p_{s+1} + \cdots + p_t - t + s}(G;A)$.

(ii) When $s = t$, the diagonal map $a_{s,s}$ is equal to γ_s in $C^{p_s}(G;A)$.

(iii) $\partial^*(a_{s,t}) = \sum_{r=s}^{t-1} (-1)^{p_s + p_{t+1} + \cdots + p_t - t + s} a_{s,r} \smile a_{r+1,t}$.

Given such a defining system, we can define a cocycle of the form

$$\sum_{r:\ 1 \leq r \leq n-1} (-1)^{p_1 + p_2 + \cdots + p_r - r + 1} a_{1,r} \smile a_{r+1,n} \in C^{p_1 + p_2 + \cdots + p_n - n + 2}(G;A). \tag{4}$$

Following [Kra], the *n-fold Massey product*, $\langle \gamma_1, \gamma_2, \dots, \gamma_n \rangle$, is defined to be the set of cohomology classes of cocycles associated with all possible defining systems. While there are many interpretations of the higher Massey product, we use the Massey products as a method to yield cocycles from cocycles of lower degree.

Remark 2.1. It is known that if $p_1 = p_2 = \cdots = p_n = 1$ and every m-fold Massey product with $m < n$ chosen from $\{\gamma_1, \dots, \gamma_n\}$ is nullcohomologous, the *n-fold Massey product* $\langle \gamma_1, \gamma_2, \dots, \gamma_n \rangle$ is a singleton in $H^2(G;A)$ [FS].

3 Main theorem: generators of $H^*(F/F_k)$

We describe the bases of some cohomologies of F/F_k in terms of Massey products (Theorems 3.2 and 3.3).

Before stating the theorem, we concretely define some defining systems. For $1 \leq t \leq q$, let $\alpha_t : F/F_k \to \mathbb{Z}$ be the homomorphism that sends x_s to $\delta_{s,t}$, where $\delta_{s,t}$ is the Kronecker delta. We regard α_t as a 1-cocycle of F/F_k. Then, given a k-tuple $I = (i_1, \dots, i_k) \in \{1, 2, \dots, q\}^k$, we have 1-cocycles $\alpha_{i_1}, \dots, \alpha_{i_k}$. Furthermore, for $1 \leq s \leq t \leq k$, let us consider the evaluation of the coefficient of $X_{i_s} \cdots X_{i_t}$. That is, we set up the linear map

$$\beta_{i_s i_{s+1} \cdots i_t} : \mathbb{Z}\langle X_1, \dots, X_q \rangle / \mathcal{J}_k \longrightarrow \mathbb{Z}; \qquad \sum a_{j_1 \dots j_a} X_{j_1} \cdots X_{j_a} \longmapsto a_{i_s i_{s+1} \dots, i_t}. \tag{5}$$

Let us denote the composite $\beta_{i_s \cdots i_t} \circ \mathcal{M}_k$ by $c_{i_s \cdots i_t}$, which we will use many times.

Lemma 3.1. *Consider the case $p_1 = p_2 = \cdots = p_n = 1$ with $n = k$. Let $a_{s,t} : F \to \mathbb{Z}$ be the composite $c_{i_s \cdots i_t}$. Then, the set of $(a_{s,t})$ is a defining system associated with $(\alpha_{i_1}, \dots, \alpha_{i_k})$. In particular, the resulting 2-cocycle is represented by*

$$F/F_k \times F/F_k \longrightarrow \mathbb{Z}; \quad (x, y) \longmapsto \sum_{\ell:\ 1 \leq \ell \leq k-1} c_{i_1 i_2 \cdots i_\ell}(x) c_{i_{\ell+1} \cdots i_k}(y). \tag{6}$$

Proof. From the unipotent Magnus expansion, the right side in (iii) is equivalent to the product of upper triangular matrices. Thus, it is not so difficult to check (iii) by direct computation. Thus, the formula of the Massey products (4) readily means (6). □

Let us also review the standard sequences from [CFL]. Equip the set of all sequences $\bigcup_{s=1}^{\infty}\{1,2,\dots,q\}^s$ with the lexicographical order. Then, a sequence $I = i_1 i_2 \cdots i_k$ is said to be *standard* if $I < i_s i_{s+1} \cdots i_k$ for any $2 \le s \le k$. For example, $\{123\}$ and $\{1223\}$ are standard, but $\{213\}$ and $\{3142\}$ are not standard. Let $\mathfrak{U}_k$ be the set of standard sequences of length k. It is known that the order of $\mathfrak{U}_k$ is equal to N_k (see, e.g., [MKS] and [CFL, Theorem 1.5]). The following is also known:

$$N_k = \mathrm{rank}(F_k/F_{k+1}) = \mathrm{rank}\big(H^2(F/F_k;\mathbb{Z})\big) = |\mathfrak{U}_k| = \frac{1}{k}\sum_{d:\ d|k} \mu(\frac{k}{d})q^d \in \mathbb{N}, \tag{7}$$

where μ is the Möbius function (see, e.g., [Witt] or [CFL, Theorem 1.5]). In addition, there are studies on a basis of F_k/F_{k+1}, e.g., Hall basis from the viewpoint of a free Lie algebra (see, e.g., [Hall, MKS]), and "standard commutators" obtained from standard sequences (see [CFL]).

Since this paper focuses on applications to the homology, we give a basis of $H^2(F/F_k;\mathbb{Z})$ in terms of cocycles. Indeed, the second and third cohomologies of F/F_k are generated by Massey products. First, we observe the second one, although it is classically known (see [FS, O2, Tu]).

Theorem 3.2 (see [FS, O2, Tu]). *(I) Every j-fold Massey product with $j < k$ is zero. In particular, for any standard index $i_1 \cdots i_k \in \mathfrak{U}_k$, the k-fold one $\langle \alpha_{i_1}, \dots, \alpha_{i_k}\rangle$ is uniquely defined in $H^2(F/F_k;\mathbb{Z})$ and represented by a 2-cocycle in* (6).

(II) The second cohomology $H^2(F/F_k;\mathbb{Z}) \cong \mathbb{Z}^{N_k}$ is spanned by the k-fold Massey products $\langle \alpha_{i_1}, \dots, \alpha_{i_k}\rangle$ running over all the standard sequences $(i_1 \cdots i_k) \in \mathfrak{U}_k$.

Proof. (I) Recall the definitions of (ii) and of (5). Thus, every lower Massey product is nullcohomologous by using $a_{s,t}$ for some (s,t) by induction on k.
(II) Denote the sum of $c_{i_1\cdots i_k}$ on F_k/F_{k+1} as $\mathcal{S}$, where $i_1 \cdots i_k \in \mathfrak{U}_k$. Namely, $\mathcal{S} := \oplus_{I\in\mathfrak{U}_k} c_I : F_k/F_{k+1} \to \mathbb{Z}^{N_k}$. It is known [CFL, Theorems 3.5 and 3.9] that the sum $\mathcal{S}$ is surjective; hence, it is bijective. Accordingly, the centrally extended group operation on $F/F_k \times F_k/F_{k+1}$ from the 2-cocycles $\oplus_{I\in\mathfrak{U}_k}\langle \alpha_{i_1}, \dots, \alpha_{i_k}\rangle$ is, by definition, formulated as

$$(g,\alpha)\cdot(h,\beta) = \big(gh, \mathcal{S}^{-1}\big(\bigoplus_{i_1\cdots i_k\in\mathfrak{U}_k} c_{i_1 i_2\cdots i_k}(\alpha+\beta) + \sum_{\ell:\ 1\le\ell<k} c_{i_1 i_2\cdots i_\ell}(g)c_{i_{\ell+1}\cdots i_k}(h)\big)\big) \tag{8}$$

for $g,h \in F/F_k$ and $\alpha,\beta \in F_k/F_{k+1}$. It is worth noting that the subsequences $i_1 i_2 \dots i_\ell$, $i_{\ell+1}\cdots i_k$ are also standard. Therefore, via the Magnus expansion, this group on $F/F_k \times F_k/F_{k+1}$ is isomorphic to the nature F/F_{k+1} as central extensions over F/F_k (cf. matrix multiplications). Hence, the second cohomology $H^2(F/F_k) \cong \mathbb{Z}^{N_k}$ is generated by the sum $\oplus_{I\in\mathfrak{U}_k}\langle \alpha_{i_1}, \dots, \alpha_{i_k}\rangle$, which provides a basis of $H^2(F/F_k)$, as required. □

Further, we give a basis of the third cohomology $H^3(F/F_k;\mathbb{Z})$.

Theorem 3.3. *For any $k \le \ell \le 2k-2$, consider the projection $p_\ell : F/F_\ell \to F/F_k$. Then, there are homomorphisms $\mathfrak{s}_\ell : Z^3(F/F_\ell;\mathbb{Z}) \to Z^3(F/F_k;\mathbb{Z})$ such that*

$$\alpha_s \smile \langle \alpha_{i_1}, \dots, \alpha_{i_\ell}\rangle = p_\ell^* \circ \mathfrak{s}_\ell(\alpha_s \smile \langle \alpha_{i_1}, \dots, \alpha_{i_\ell}\rangle) \in Z^3(F/F_\ell;\mathbb{Z}) \tag{9}$$

for any $(i_1 \cdots i_\ell) \in \mathfrak{U}_\ell, 1 \le s \le q$, and the following set of 3-cocycles is a basis of the third cohomology $H^3(F/F_k;\mathbb{Z}) \cong \oplus_{\ell=k}^{2k-2}\mathbb{Z}^{qN_\ell - N_{\ell+1}}$.

$$\bigcup_{k\le\ell\le 2k-2} \big\{ \ \mathfrak{s}_\ell\big(\alpha_s \smile \langle \alpha_{i_1}, \dots, \alpha_{i_\ell}\rangle\big) \ \big| \ (i_1\cdots i_\ell) \in \mathfrak{U}_\ell, \ \ 1 \le s \le q, \ \ (i_1\cdots i_\ell s) \notin \mathfrak{U}_{\ell+1} \ \big\}. \tag{10}$$

Proof. First, we discuss some of the results from Igusa and Orr [IO]. They constructed a certain filtration on the homology $\mathcal{F}^s H_3(F/F_k;\mathbb{Z})$ [IO, Theorem 6.7] such that two isomorphisms

$$\frac{\mathcal{F}^\ell H_3(F/F_k;\mathbb{Z})}{\mathcal{F}^{\ell+1} H_3(F/F_k;\mathbb{Z})} \cong \mathbb{Z}^{qN_{\ell-1}-N_\ell}, \tag{11}$$

$$\mathcal{F}^\ell H_3(F/F_k;\mathbb{Z}) \cong \mathrm{Im}((p_{\ell-1})_* : H_3(F/F_{\ell-1};\mathbb{Z}) \to H_3(F/F_k;\mathbb{Z})),$$

hold for $k < \ell < 2k$, and the left quotients in (11) are zero otherwise. In particular, $\mathcal{F}^\ell H_3(F/F_k;\mathbb{Z})$ is a direct summand of $H_3(F/F_k;\mathbb{Z})$, and $\mathrm{Ker}((p_{\ell-1})_*)$ is a direct summand of $H_3(F/F_{\ell-1})$, if $k < \ell < 2k$. Thus, for $\ell \geq k$, we readily have the composite

$$H_3(F/F_k;\mathbb{Z}) \xrightarrow{\text{projection}} \mathcal{F}^{\ell+1} H_3(F/F_k) \cong H_3(F/F_\ell)/\,\mathrm{Ker}((p_\ell)_*) \hookrightarrow H_3(F/F_\ell;\mathbb{Z}).$$

Applying this composite to $\mathrm{Hom}(\bullet,\mathbb{Z})$ is regarded as a map $\iota_\ell : H^3(F/F_\ell;\mathbb{Z}) \to H^3(F/F_k;\mathbb{Z})$. Accordingly, let us set up the composite of the cup product on F/F_ℓ and ι_ℓ:

$$\Theta_\ell : H^1(F/F_\ell;\mathbb{Z}) \otimes H^2(F/F_\ell;\mathbb{Z}) \stackrel{\smile}{\longrightarrow} H^3(F/F_\ell;\mathbb{Z}) \stackrel{\iota_\ell}{\longrightarrow} H^3(F/F_k;\mathbb{Z}).$$

We show that the direct sum $\oplus_{\ell=k}^{2k-2}\Theta_\ell$ surjects onto $H^3(F/F_k;\mathbb{Z})$. Consider the Lyndon-Hochschild spectral sequence from the central extension $F_k/F_{k+1} \to F/F_{k+1} \to F/F_k$. Then, as shown in the proof of [IO, Lemma 5.8], we find an exact sequence,

$$H_3(F/F_{k+1};\mathbb{Z}) \longrightarrow H_3(F/F_k;\mathbb{Z}) = E^2_{3,0} \stackrel{d^2_{3,0}}{\longrightarrow} E^2_{1,1} \stackrel{\delta_*}{\longrightarrow} H_2(F/F_k;\mathbb{Z}) \to 0, \tag{12}$$

satisfying $d^2_{3,0}(\mathcal{F}^{k+1}H_3(F/F_k;\mathbb{Z})) = \mathrm{Ker}(\delta_*)$ and $d^2_{3,0}(\mathcal{F}^k H_3(F/F_k;\mathbb{Z})) = 0$. Keep in mind that each term is free. Furthermore, consider the spectral sequence on the cohomology level. Noting the identity

$$E_2^{1,1} = H^1(F/F_k; F_k/F_{k+1}) \cong H^1(F/F_k;\mathbb{Z}) \otimes H^2(F/F_k;\mathbb{Z}),$$

we dually obtain the following exact sequence from (12):

$$0 \longrightarrow H^2(F/F_k) \longrightarrow H^1(F/F_k) \otimes H^2(F/F_{k-1}) \stackrel{d_2^{3,0}}{\longrightarrow} \mathcal{F}^{k+1} H^3(F/F_k).$$

As is usual with the cup$_1$-product, the differential map $d_2^{3,0}$ is equal to the cup product. To summarize, this sequence and the isomorphisms (11) imply the surjectivity of $\oplus_{\ell=k}^{2k-2}\Theta_\ell$, as required. Moreover, the construction of Θ_ℓ admits the desired section $\mathfrak{s}_\ell$ satisfying (9).

The proof will be completed by showing the linear independence of (10), as follows. Note from the definition of $(i_1 \cdots i_\ell) \in \mathfrak{U}_\ell$ that $(i_1 \cdots i_\ell s) \in \mathfrak{U}_{\ell+1}$ if and only if $s > i_1$. Thus, if $s > i_1$, then $i_\ell i_1 \cdots i_{\ell-1} s$ is not standard because of $i_k \geq s$. We now discuss [GL, Proposition 4.5] showing the equality

$$\alpha_s \smile \langle \alpha_{i_1}, \ldots, \alpha_{i_\ell} \rangle = \alpha_{i_\ell} \smile \langle \alpha_{i_1}, \ldots, \alpha_{i_{\ell-1}}, \alpha_s \rangle \in H^3(F/F_\ell;\mathbb{Z}).$$

Hence, $\alpha_s \smile \langle \alpha_{i_1}, \ldots, \alpha_{i_\ell} \rangle$ with $(i_1 \cdots i_\ell) \in \mathfrak{U}_\ell$ and $(i_1 \cdots i_\ell s) \in \mathfrak{U}_{\ell+1}$ is cohomologous to a cocycle in (10). By (9) and functoriality, a similar conclusion can be obtained even in the case $k < \ell \leq 2k-2$. Hence, from the surjectivity of $\oplus_{\ell=k}^{2k-2}\Theta_\ell$, comparing the rank of $H^3(F/F_k)$ with the order of (10) leads to the linear independence, as required. □

We immediately obtain a corollary from the above proof.

Corollary 3.4. *For any $s \leq q$, the map $H^2(F/F_k;\mathbb{Z}) \to H^3(F/F_k;\mathbb{Z})$, which sends β to $\alpha_s \smile \beta$ is injective.*

Proof. By Theorem 3.2 (II), the 2-cocycles $\langle \alpha_{i_1}, \dots, \alpha_{i_k} \rangle$ gives a basis of $H^2(F/F_k;\mathbb{Z})$. By (10), the 3-cocycles $\alpha_s \smile \langle \alpha_{i_1}, \dots, \alpha_{i_k} \rangle$ are linearly independent. Hence, we obtain the desired injectivity. □

4 Applications: simple proofs

There are some topological invariants using $H^*(F/F_k)$, and some results on the Milnor link-invariants and mapping class groups of surfaces. In this section, we give simple proofs of the results from Fenn-Sjerve [FS], Turaev [Tu], Porter [Po], Kitano [Ki], and Heap [Heap].

4.1 Theorem of Fenn-Sjerve on Massey product

We state the Fenn-Sjerve theorem [FS] and give an alternative proof using Theorem 3.2. Assume $k \geq 3$. Let $W_1, \dots, W_t$ be words in F_k and $R \subset F$ be the normal closure of $W_1, \dots, W_t$ and F_{k+1}. Let $\mathcal{G}$ be the quotient group F/R. Since $H_1(F) \cong H_1(\mathcal{G})$, the 1-cocycle $\alpha_j : F \to \mathbb{Z}$ induces the 1-cocycle $\alpha_j : \mathcal{G} \to \mathbb{Z}$ for $j \leq q$.

Let $p : \mathcal{G} \to F/F_k$ be the projection. Recall Hopf's theorem, which claims the isomorphisms

$$H_2(\mathcal{G}) \cong (R \cap [F,F])/[F,R], \qquad H_2(F/F_k) \cong (F_k \cap F_2)/[F,F_k] = F_k/F_{k+1}.$$

Noticing $W_j \in R \cap [F,F]$, we denote the pushforward $p_*(W_j)$ by $\mathcal{W}_j$ and regard it as a 2-cycle of F/F_k.

Theorem 4.1 ([FS])**.** *Suppose that all of $W_1, \dots, W_t$ lie in F_k. For any $\ell < k$, every ℓ-fold Massey product $\langle \alpha_{i_1}, \dots, \alpha_{i_\ell} \rangle$ vanishes. On the other hand, k-fold Massey products are defined and evaluated on $\{W_j\}$ according to the formula*

$$\sum_{j_1,\dots,j_k} [\langle \alpha_{j_1}, \dots, \alpha_{j_k} \rangle, W_j\ ] X_{j_1} \cdots X_{j_k} = \mathcal{M}_{k+1}(\mathcal{W}_j) \in \mathcal{M}_{k+1}(F_k/F_{k+1}). \tag{13}$$

The outer $[,]$ is the pairing of $H^2(\mathcal{G};\mathbb{Z})$ and $H_2(\mathcal{G};\mathbb{Z})$.

To prove this theorem, we give a lemma:

Lemma 4.2. *Take a standard index $I = i_1 \cdots i_k \in \mathfrak{U}_k$ and the associated Massey product $\langle \alpha_{i_1}, \dots, \alpha_{i_k} \rangle$. Via the isomorphism $H_2(F/F_k) \cong F_k/F_{k+1}$ above, the Kronecker product $[\langle \alpha_{j_1}, \dots, \alpha_{j_k} \rangle, \bullet] : H_2(F/F_k) \to \mathbb{Z}$ coincides with the map $c_{i_1 \cdots i_k} : F_k/F_{k+1} \to \mathbb{Z}$.*

Proof. For any $a \in F/F_k$, we choose a representative $\bar{a} \in F$. It follows from [Bro, Exercise 4 in II.5] that the correspondence $(g,h) \mapsto \overline{g}\overline{h}(\overline{gh})^{-1}$ induces the isomorphism $H_2(F/F_k) \to F_k/F_{k+1}$. By the cocycle expression of $\langle \alpha_{j_1}, \dots, \alpha_{j_k} \rangle$ in Lemma 3.1, a similar discussion to (8) readily deduces the required coincidence. □

Proof of Theorem 4.1. Given a standard index $I = i_1 \cdots i_k \in \mathfrak{U}_k$, let $W_I \in F_k/F_{k+1}$ be a generator of the $i_1 \cdots i_k$-th summand of $\mathbb{Z}^{N_k} \cong F_k/F_{k+1}$. A paper [CFL, §2] explicitly describes the word W_I and refers to it as *the standard commutator*. Another paper [CFL, Lemma 3.4] showed that, for any standard index $j_1 \cdots j_k$, the coefficient of $X_{j_1} \cdots X_{j_k}$ in $\mathcal{M}_{k+1}(W_I)$ is $\delta_{i_1,j_1} \cdots \delta_{i_k,j_k} \in \{0,1\}$. Therefore, by Lemma 4.2, this is equal to $[\langle \alpha_{j_1}, \dots, \alpha_{j_k} \rangle, \mathcal{W}_I]$. In summary, if $\mathcal{G}$ is presented by $F/\langle F_{k+1}, W_I \rangle$, the equality (13) holds.

We complete the proof for $\mathcal{G} = F/R$. Since F_k/F_{k+1} is abelian, we can expand $p_*(W_j)$ as $\prod_{I\in\mathfrak{U}_k}(W_I)^{a_I}$ for some $a_I \in \mathbb{Z}$. Then, from the discussion in the preceding paragraph, the equality (13) holds in the coefficient of $X_{j_1}\cdots X_{j_k}$ with respect to every standard index $j_1\cdots j_k$. Since any other coefficient is a linear sum of such coefficients of $X_{j_1}\cdots X_{j_k}$, we conclude that the equality (13) in $\mathcal{M}_{k+1}(F_k/F_{k+1})$ is satisfied. □

Remark 4.3. Finally, we should give remarks regarding the following subsections. For a group G, let us write BG for the Eilenberg-MacLane space of type $(G,1)$. Then, we can choose a connected CW complex X such that $\pi_1(X)\cong G$ and fix a classifying map $c: X\to BG$, which is obtained by killing the higher homotopy groups of X. By construction, we notice that $c_*: H_2(X)\to H_2(BG;\mathbb{Z}) = H_2(G)$ is surjective. In particular, if G is $\mathcal{G} = F/R$ as above, there are 2-cycles $\mathcal{X}_j \in H_2(X)$ with $c_*(\mathcal{X}_j) = [W_j]$. By $\pi_1(X)\cong\mathcal{G}$, the 1-cocycle α_j may be regarded as an element of $H^1(X;\mathbb{Z})$, and the pairing $[\langle\alpha_{j_1},\ldots,\alpha_{j_k}\rangle, \mathcal{X}_j\ ]$ is equal to the equality (13). To conclude, we can interpret some Massey products in $H^2(X;\mathbb{Z})$ from the viewpoints of Theorem 4.1.

4.2 Milnor invariant and Massey product

Porter [Po] and Turaev [Tu] independently showed that the Milnor link invariant is equivalent to some Massey products of the link complement space. For simplicity, this paper focuses on only the k-th leading terms of the Milnor invariant and gives an alternative proof of their result.

To state the theorems, we begin by reviewing the Milnor invariant according to [Mil, Tu]. We assume that the reader has elementary knowledge of knot theory, as in [Po, Tu]. Let M be an integral homology 3-sphere, that is, a closed 3-manifold such that $H_*(M;\mathbb{Z})\cong H_*(S^3;\mathbb{Z})$. Choose a link $L\subset M$ with q components and a meridian-longitude pair $(\mathfrak{m}_\ell, \mathfrak{l}_\ell)$ for $\ell\le q$, where $\mathfrak{l}_\ell$ may be the preferred longitude. Fix an open tubular neighborhood of L, and denote it by νL. Then, it is known ([Mil]; see also [Tu, Lemma 1.2]) that the m-th nilpotent quotient $\pi_1(M\setminus\nu L)/\pi_1(M\setminus\nu L)_m$ has the group presentation

$$\Big\langle\ x_1,\ldots,x_q\ \Big|\ [x_\ell, w_\ell^{(m)}] = 1 \text{ for } \ell\le q,\quad F_m\ \Big\rangle, \tag{14}$$

where x_j is represented by the j-th meridian, and $w_j^{(m)}$ is defined by the j-th longitude in $\pi_1(M\setminus\nu L)$.

For brevity, let us assume the existence of $k\in\mathbb{Z}$ such that $w_j^{(m)}$ is trivial in the k-th quotient for any $m\le k$, i.e., $w_j^{(k)}\in F_k$. We call the existence *Assumption $\mathcal{A}_k$*. By considering $w_\ell^{(k)}$ to be a word in F_k/F_{k+1}, we focus on the value $\mathcal{M}_{k+1}(w_\ell^{(k)})\in\mathbb{Z}\langle X_1,\ldots,X_q\rangle/\mathcal{J}_{k+1}$. The coefficient of $X_{i_1}\cdots X_{i_k}$ in $\mathcal{M}_{k+1}(w_\ell^{(k)})$ is called *the k-th Milnor μ-invariant* of L, and it is denoted as $\mu(i_1\cdots i_k;\ell)$. Let $\alpha_j:\pi_1(M\setminus\nu L)\to\mathbb{Z}$ be the homomorphism that sends $\mathfrak{m}_\ell$ to $\delta_{\ell,j}$.

Theorem 4.4 (The minimal non-vanishing case of [Tu][Po]). *Suppose the assumption $\mathcal{A}_k$. Let $[\mathfrak{l}_\ell]\in H_2(M\setminus\nu L;\mathbb{Z})$ be a 2-cycle corresponding to the ℓ-th longitude $\mathfrak{l}_\ell$. For any index $I = i_1 i_2\cdots i_k$, the $(k+1)$-fold Massey product $\langle\alpha_{i_1},\ldots,\alpha_{i_{k-1}},\alpha_{i_k},\alpha_\ell\rangle$ is uniquely defined, and the following equality holds:*

$$\mu(i_1\cdots i_k;\ell) = \ [\langle\alpha_{i_1},\ldots,\alpha_{i_{k-1}},\alpha_{i_k},\alpha_\ell\rangle, [\mathfrak{l}_\ell]]\in\mathbb{Z}. \tag{15}$$

Proof. Take the classifying map $\mathcal{C} : M \setminus \nu L \to B\pi_1(M \setminus \nu L)$ explained in Remark 4.3. We first claim that the pushforward $\mathcal{C}_*([\mathfrak{l}_\ell]) \in H_2(\pi_1(M \setminus \nu L); \mathbb{Z})$ corresponds to the relation $\mathfrak{m}_\ell \mathfrak{l}_\ell \mathfrak{m}_\ell^{-1} \mathfrak{l}_\ell^{-1}$. Consider the ℓ-th torus boundary $\partial_\ell(M \setminus \nu L)$ as a $K(\mathbb{Z}^2, 1)$-space, where this $\mathbb{Z}$ is generated by the pair $\mathfrak{m}_\ell, \mathfrak{l}_\ell$. The group $\pi_1(\partial_\ell(M \setminus \nu L)) \cong \mathbb{Z}^2$ is presented by $\langle \mathfrak{m}_\ell, \mathfrak{l}_\ell \mid \mathfrak{m}_\ell \mathfrak{l}_\ell \mathfrak{m}_\ell^{-1} \mathfrak{l}_\ell^{-1} \rangle$. Following Hopf's theorem, the generator κ of $H_2(\partial_\ell(M \setminus \nu L)) \cong \mathbb{Z}$ corresponds to $\mathfrak{m}_\ell \mathfrak{l}_\ell \mathfrak{m}_\ell^{-1} \mathfrak{l}_\ell^{-1}$. Putting the inclusion $\iota_\ell : \partial_\ell(M \setminus \nu L) \to M \setminus \nu L$, the pushforward $(\iota_\ell)_*(\kappa)$ is identified with $[\mathfrak{l}_\ell]$ by definitions. Thus, the pushforward $\mathcal{C}_*([\mathfrak{l}_\ell]) = (\mathcal{C} \circ \iota_\ell)_*(\kappa)$ corresponds to the relation $\mathfrak{m}_\ell \mathfrak{l}_\ell \mathfrak{m}_\ell^{-1} \mathfrak{l}_\ell^{-1}$, as required.

We are now in a position to complete the proof. By the assumption, $w_\ell^{(k)}$ lies in F_k. Therefore, $[x_\ell, w_\ell^{(k)}] \in F_{k+1}$. Let W_ℓ be $[x_\ell, w_\ell^{(k)}]$, and let $\mathcal{G}$ be $F/\langle F_{k+1}, W_1, \dots, W_q \rangle$. It immediately follows from Theorem 4.1 of Fenn-Sjerve that the $(k+1)$-fold $\langle \alpha_{i_1}, \dots, \alpha_{i_k}, \alpha_\ell \rangle$ is uniquely defined (see Remark 2.1), and that the left side of (15) equals the right side of (13), and the right one is the coefficient of $X_{i_1} \cdots X_{i_k} X_\ell$ in $\mathcal{M}_{k+1}([x_\ell, w_\ell^{(k+1)}])$. Then, we can verify, by directly computing $\mathcal{M}_{k+1}(x_\ell w_\ell^{(k+1)} x_\ell^{-1} (w_\ell^{(k+1)})^{-1})$, that it is equal to the coefficient of $X_{i_1} \cdots X_{i_k}$ in $\mathcal{M}_{k+1}(w_\ell^{(k)})$, that is, $\mu(i_1 \cdots i_k; \ell)$, as required. □

Incidentally, Milnor defined the μ-invariant of higher degree [Mil], and Porter [Po] and Turaev [Tu] described the higher invariant in terms of the Massey product (see also [KN], which provides a refinement of the higher invariant). Although we omit the details, the higher degree relation can be proven in the same manner as Theorem 4.4.

4.3 Johnson homomorphisms and Massey products

Now let us focus on the Johnson homomorphisms of the mapping class groups of surfaces (see, e.g., [Joh1, M1, Day, GL, Heap, M2] for the significance of these homomorphisms). This section paraphrases [Ki], which describes the relation between the Johnson homomorphisms and Massey products.

First, we should establish the notations. Let $\Sigma_{g,r}$ be a compact oriented surface of genus g with r boundaries. Let $\Gamma_{g,1}$ be the group of isotopy classes of orientation-preserving homeomorphisms of $\Sigma_{g,1}$ which are the identity on the boundary. Suppose $q = 2g$ and let F be $\pi_1(\Sigma_{g,1})$. Then, the action of $\Gamma_{g,1}$ on $F = \pi_1(\Sigma_{g,1})$ can be regarded as a homomorphism, $\Gamma_{g,1} \to \mathrm{Aut}(F)$. Subject to F_k, we have $\rho_k : \Gamma_{g,1} \to \mathrm{Aut}(F/F_k)$. We commonly denote the kernel $\mathrm{Ker}(\rho_k)$ by $\mathcal{T}(k)$. We have the filtration $\Gamma_{g,1} \supset \mathcal{T}(2) \supset \mathcal{T}(3) \supset \cdots$.

Next, let us review the homomorphism (16) below. Fix any $f \in \mathcal{T}(k)$. Given $[\gamma] \in H_1(\Sigma_{g,1}) = F/F_2$, choose a representative $\gamma \in \pi_1(\Sigma_{g,1})$. Then, $f_*(\gamma)\gamma^{-1}$ lies in F_k since the action of $f \in \mathcal{T}(k)$ on F/F_k is trivial. Then, *the k-th Johnson homomorphism* is defined as the map,

$$\tau_k : \mathcal{T}(k) \longrightarrow \mathrm{Hom}(H_1(\Sigma_{g,1}), \quad F_k/F_{k+1}), \tag{16}$$

which sends $[f]$ to the homomorphism $[\gamma] \to [f_*(\gamma)\gamma^{-1}]$. It is known that τ_k is well-defined and a homomorphism. The following is also well-known (see [Joh2, M2]): for $f \in \mathcal{T}(k)$, f lies in $\mathcal{T}(k+1)$ if and only if $\tau_m(f) = 0$ for any $m \leq k$.

Next, let us examine the mapping torus $T_{f,1}$ for a fixed $f \in \mathcal{T}(k)$ with $k \geq 2$. Here, $T_{f,1}$ is the quotient space of $\Sigma_{g,1} \times [0,1]$ subject to the relation $(y, 0) \sim (f(y), 1)$ for any $y \in \Sigma_{g,1}$. Since $f \in \mathcal{T}(k)$ with $k \geq 2$, we have $H_*(T_{f,1}) \cong H_*(\Sigma_{g,1} \times S^1)$. Furthermore, we fix a basis $\{x_1, \dots, x_{2g}\}$ of the free group $F = \pi_1(\Sigma_{g,1})$. Then, following the van Kampen argument, we can verify the presentation

$$\pi_1(T_{f,1}) \cong \langle x_1,\dots,x_{2g},\gamma \mid [x_1,\gamma]f_*(x_1)x_1^{-1},\dots,[x_{2g},\gamma]f_*(x_{2g})x_{2g}^{-1} \rangle. \tag{17}$$

Here, γ represents a generator of $\pi_1(S^1)$. Since $T_{f,1}$ is a $\Sigma_{g,1}$-bundle over S^1 by definition, it is a $K(\pi,1)$-space. Hence, $H_*(\pi_1(T_{f,1})) \cong H_*(T_{f,1}) \cong H_*(\Sigma_{g,1}\times S^1)$. Moreover, Hopf's theorem implies that the relations $[x_1,\gamma]f_*(x_1)x_1^{-1},\dots,[x_{2g},\gamma]f_*(x_{2g})x_{2g}^{-1}$ represent a basis $\{\mathcal{X}_1,\dots,\mathcal{X}_{2g}\}$ of $H_2(T_{f,1})\cong\mathbb{Z}^{2g}$.

Now let us state and prove Proposition 4.5. Since the boundary $\partial T_{f,1}$ is the torus $\partial\Sigma_{g,1}\times S^1$, we can define T_f^γ to be the resulting space obtained by filling in the torus $\partial T_{f,1}\simeq\partial\Sigma_{g,1}\times S^1$ with the solid torus $\partial\Sigma_{g,1}\times D^2$. The space T_f^γ is called *the Dehn filling along a curve on* $\partial T_{f,1}$. Denoting the inclusion $T_{f,1}\to T_f^\gamma$ as ι^γ, we see that the homology $H_2(T_f^\gamma;\mathbb{Z})\cong H^1(T_f^\gamma;\mathbb{Z})\cong\mathbb{Z}^{2g}$ is spanned by the pushforwards $\{\iota_*^\gamma(\mathcal{X}_1),\dots,\iota_*^\gamma(\mathcal{X}_{2g})\}$. Moreover, we should note that the presentation of $\pi_1(T_f^\gamma)$ is

$$\pi_1(T_f^\gamma)\cong\langle\, x_1,\dots,x_{2g} \mid f_*(x_1)x_1^{-1},\dots,f_*(x_{2g})x_{2g}^{-1}\,\rangle. \tag{18}$$

Proposition 4.5 (cf. [Ki] and [GL, Corollary 4.1])**.** *Let $\{\mathcal{X}_1,\dots,\mathcal{X}_{2g}\}$ be the basis of $H_2(T_{f,1})$, and $\iota^\gamma: T_{f,1}\to T_f^\gamma$ be the inclusion as above. Let $x_j^*:\pi_1(T_f^\gamma)\to\mathbb{Z}$ be the 1-cocycle that sends x_i to $\delta_{j,i}$. Define the map $\tau_k':\mathcal{T}(k)\to\mathrm{Hom}(H_1(\Sigma_{g,1}),\ \mathcal{M}_{k+1}(F_k/F_{k+1}))$ by letting $\tau_k'(f)$ be the homomorphism*

$$x_i \longmapsto \sum_{j_1,\dots,j_k}[\langle x_{j_1}^*,\dots,x_{j_k}^*\rangle,\iota_*^\gamma(\mathcal{X}_i)\ ]X_{j_1}\cdots X_{j_k}. \tag{19}$$

The outer $[,]$ is the pairing of $H^2(T_f^\gamma)$ and $H_2(T_f^\gamma)$. Then, $\tau_k'(f)(x_i)$ is equal to $\mathcal{M}_{k+1}\big(\tau_k(f)(x_i)\big)$.

Proof. As mentioned above, $f_*(x_i)x_i^{-1}\in F_k$ since $f\in\mathcal{T}(k)$. The statement readily follows from Theorem 4.1 and Remark 4.3 with $W_j=f_*(x_j)x_j^{-1}$ and $t=2g$. □

4.4 Vanishing condition of Morita homomorphism

As a lift of the Johnson homomorphism τ_k, Morita [M2] defined a map $\tilde\tau_k:\mathcal{T}(k)\to H_3(F/F_k)$, which is called *the Morita homomorphism*. Furthermore, Heap [Heap] showed the vanishing condition of $\tilde\tau_k$ in terms of the (relative) bordism theory. The purpose of this subsection is to give a simpler proof of the result (Theorem 4.6).

Let us review the map $\tilde\tau_k$. Fix $f\in\mathcal{T}(k)$. Thus, the relation $f_*(x_j)x_j^{-1}$ vanishes in the k-th nilpotent quotient of $\pi_1(T_f^\gamma)$. Thus, from (18), we have the canonical surjection,

$$\phi_{f,k}^\gamma:\pi_1(T_f^\gamma)\longrightarrow\pi_1(T_f^\gamma)/\big(\pi_1(T_f^\gamma)\big)_k\cong F/F_k.$$

Let $[T_f^\gamma]\in H_3(T_f^\gamma)\cong\mathbb{Z}$ be the fundamental 3-class. Then, $\tilde\tau_k(f)$ is defined to be the pushforward $(\phi_{f,k}^\gamma)_*([T_f^\gamma])\in H_3(F/F_k)$. It is known [M2, Heap] that $\tilde\tau_k(f)$ depends only on $f\in\mathcal{T}(k)$ and that $\tilde\tau_k$ is a lift of the Johnson map τ_k.

Theorem 4.6 ([Heap, Theorem 5])**.** *Let $f\in\mathcal{T}(k)$. Then, $\tilde\tau_k(f)=0$ if and only if $f\in\mathcal{T}(2k-1)$.*

Proof. First, let us make some observations. Note from Proposition 4.5 that $f\in\mathcal{T}(k+1)$ if and only if the pairing $[\langle x_{j_1}^*,\dots,x_{j_k}^*\rangle,\iota_*^\gamma(\mathcal{X}_i)\ ]$ in (19) is zero for any standard index $j_1\cdots j_k$. Furthermore, Theorem 3.3 implies that $\tilde\tau_k(f)$ is zero if and only if the pairing

$$\langle\mathfrak{s}_\ell(x_s^*\smile\langle x_{j_1}^*,\dots,x_{j_\ell}^*\rangle),\ \ (\phi_{f,k}^\gamma)_*([T_f^\gamma])\ \rangle \tag{20}$$

is zero for any sequence $(s, j_1, \cdots, j_\ell)$ in (10), where we replace α_j by x_j^*.

We now complete the proof. Suppose $f \in \mathcal{T}(2k-1)$. Then, we can define the Johnson homomorphism τ_ℓ for $\ell \leq 2k-2$ and have $\tau_\ell = 0$. Hence, all the pairings $[\langle x_{j_1}^*, \ldots, x_{j_\ell}^* \rangle, \iota_*^\gamma(\mathcal{X}_s)\ ]$ are zero. Since $H_2(T_f^\gamma; \mathbb{Z}) \cong \mathbb{Z}^g$ is generated by $\mathcal{X}_1, \ldots, \mathcal{X}_{2g}$ mentioned above, the cap product $[T_f^\gamma] \cap \alpha_s$ is a sum $\sum_{m=1}^{2g} n_{s,m} \cdot \iota_*^\gamma(\mathcal{X}_m)$ for some $n_{s,m} \in \mathbb{Z}$. Thus, the pairing in (20) is zero. Hence, $\tilde{\tau}_k(f) = 0$. Conversely, let us assume $\tilde{\tau}_k(f) = 0$. Then, for any $k \leq \ell \leq 2k-2$, the pairing (20) is zero. When $\ell = k$, Corollary 3.4 implies that the pairing (19) is zero. From Theorem 4.5, we have $\tau_k(f) = 0$, i.e., $f \in \mathcal{T}(k+1)$; thus, we can define $\tau_{k+1}(f)$. By repeating this argument, we can define $\tau_\ell(f)$ and show $\tau_\ell(f) = 0$ for any $\ell \leq 2k-2$. Hence, we conclude $f \in \mathcal{T}(2k-1)$, as required. □

Finally, we discuss some of the results on the Morita map $\tilde{\tau}_k$. Some papers [M2, Heap, Day] studied on the map $\tilde{\tau}_k$. Massuyeau [Mas] showed an equivalence between the map $\tilde{\tau}_k$ and a certain restriction of "the total Johnson homomorphism", and gave a computation of $\tilde{\tau}_k$ with rational coefficients in terms of a symplectic expansion. However, our situation is with integral coefficients; if we can determine the homeomorphism type of T_f^γ, we can hope to compute $\tilde{\tau}_k(f)$ by using the 3-cocycles of Theorem 3.3.

5 Expressions of some 3-cocycles

In general, it is practically important to give explicit expressions of group cocycles. This section focuses on quotient groups of F/F_k and gives an algorithm to describe their 3-cocycles. As mentioned in the introduction, we regard the higher Massey products as an algorithm to produce cocycles.

5.1 3-cocycles of F/F_k

First, let us focus on the group F/F_k and give presentations of the 3-cocycles $\mathfrak{s}_k(\alpha_s \smile \langle \alpha_{i_1}, \ldots, \alpha_{i_k} \rangle)$ in Theorem 3.3 when $\ell = k$ and $\ell = k+1$ with $k \geq 3$.

Using the notation in §3, given an index $i_1 i_2 \cdots i_\ell$ and $s \in \mathbb{N}$, let us define the map

$$\Gamma_{s i_1 i_2 \cdots i_\ell} : (F/F_\ell)^3 \longrightarrow \mathbb{Z}; \qquad (x, y, z) \longmapsto c_s(x) \Big(\sum_{j:\ 1 \leq j \leq \ell-1} c_{i_1 i_2 \cdots i_j}(y) c_{i_{j+1} \ldots i_\ell}(z) \Big).$$

The simplest case is when $\ell = k$, in which the 3-cocycle $\mathfrak{s}_k(\alpha_s \smile \langle \alpha_{i_1}, \ldots, \alpha_{i_\ell} \rangle)$ is exactly presented as $\Gamma_{s i_1 i_2 \cdots i_k}$, since $\mathfrak{s}_k = \mathrm{id}$. Next, to obtain presentations of $\mathfrak{s}_\ell(\alpha_s \smile \langle \alpha_{i_1}, \ldots, \alpha_{i_\ell} \rangle)$ with $k < \ell \leq 2k-2$, it is sufficient to explicitly give a function $\mathfrak{b} : (F/F_\ell)^2 \to \mathbb{Z}$ such that $\Gamma_{s i_1 i_2 \cdots i_\ell} - \partial_2^*(\mathfrak{b}) : (F/F_\ell)^3 \to \mathbb{Z}$ induces a map $(F/F_k)^3 \to \mathbb{Z}$.

For example, we describe the case $\ell = k+1$. Define the map $\mathfrak{b}$ by setting

$$\mathfrak{b}(x, y) = c_s(x) c_{i_1 \cdots i_{k+1}}(y) c_{i_{k+2}}(y) + c_{s i_1}(x) c_{i_2 \cdots i_{k+2}}(y) + c_{s i_{k+2}}(x) c_{i_1 \cdots i_{k+1}}(y).$$

Then, as a result of the difference $(\Gamma_{s i_1 i_2 \cdots i_{k+1}} - \partial_2^* \mathfrak{b})(x, y, z)$, we obtain

Proposition 5.1. *The cohomology 3-class $\alpha_s \smile \langle \alpha_{i_1}, \ldots, \alpha_{i_k} \rangle$ is represented by $\Gamma_{s i_1 i_2 \cdots i_k}$.*

When $\ell = k+1$, the 3-class $\mathfrak{s}_{k+1}(\alpha_s \smile \langle \alpha_{i_1}, \ldots, \alpha_{i_{k+1}} \rangle)$ is represented by the map

$$(x, y, z) \longmapsto \sum_{\ell=2}^{k} \Big(c_s(x) \big(c_{i_1 \cdots i_\ell}(y) c_{i_{\ell+1} \cdots i_{k+1}}(z) - c_{i_1 \cdots i_{\ell-1}}(y) c_{i_\ell \cdots i_{k+1}}(z) (c_{i_{k+1}}(y) + c_{i_{k+1}}(z)) \big)$$

$$-c_{si_1}(x)c_{i_2\cdots i_\ell}(y)c_{i_{\ell+1}\cdots i_{k+1}}(z) - c_{si_{k+1}}(x)c_{i_2\cdots i_{\ell-1}}(y)c_{i_\ell\cdots i_k}(z)\Big).$$

Since the length of every sequence in each term is less than k, the map $\Gamma_{si_1i_2\cdots i_{k+1}} - \partial_2^*\mathfrak{b}$ can be regarded as a map from $(F/F_k)^3$.

We should mention that while Igusa and Orr [IO, §10] express the 3-cocycles $\alpha_s \smile \langle \alpha_{i_1}, \dots, \alpha_{i_k} \rangle$ in terms of Igusa's picture, our description using Massey products is simpler and compatible with the non-homogenous complex of G.

Concerning the higher case $\ell > k+1$, the author attempted to describe the 3-cocycles but made little progress.

5.2 Quotient groups by central elements

This subsection discusses the situation in §4.1 or [FS]. Namely, we fix central elements $W_1, \dots, W_t \in F_k/F_{k+1}$ and let $\mathcal{G}$ be the quotient group of F/F_{k+1} subject to $W_1, \dots, W_t$. For simplicity, let us assume $k > 2$ and that there are standard sequences $I^{(j)} = (i_1^{(j)}, \dots, i_k^{(j)}) \in \mathfrak{U}_k$ with $j \leq t$, which are mutually distinct, and that W_j corresponds to the k-fold Massey product $\langle \alpha_{i_1^{(j)}}, \dots, \alpha_{i_k^{(j)}} \rangle$. Such an assumption appears in discussions on the higher Milnor invariant (see [Tu, KN]).

Then, as in (5), we can easily check that the following map is well-defined and a 2-cocycle:

$$\phi_{\Lambda_j} : \mathcal{G} \times \mathcal{G} \longrightarrow \mathbb{Z}; \quad ([X],[Y]) \longmapsto \sum_{\ell=1}^{k-1} c_{i_1^{(j)} i_2^{(j)} \cdots i_\ell^{(j)}}(X) c_{i_{\ell+1}^{(j)} \cdots i_k^{(j)}}(Y),$$

where we represent any element of the group $\mathcal{G}$ by the representative from F/F_k. We should mention the 5-term exact sequence from the central extension $F/F_k \to \mathcal{G}$:

$$0 \to H^1(\mathcal{G};\mathbb{Z}) \xrightarrow{\cong} H^1(F/F_k;\mathbb{Z}) \longrightarrow \mathbb{Z}^t \xrightarrow{\delta^*} H^2(\mathcal{G};\mathbb{Z}) \to H^2(F/F_k;\mathbb{Z}).$$

In fact, we can verify that the 2-cocycle ϕ_{Λ_j} corresponds to the image $\delta^*(W_j)$.

Now let us give some 3-cocycles of $\mathcal{G}$.

Proposition 5.2. *Let $\mathcal{G}$ be the group $F/\langle F_k, W_1, \dots, W_s \rangle$, as mentioned above. Fix $r, s \in \{1, \dots, q\}$ such that $(ri_1^{(j)} \dots i_{k-1}^{(j)})$ and $(i_2^{(j)} \dots i_k^{(j)} s)$ differ from other indexes $i_1^{(j')} \cdots, i_k^{(j')}$ for $j' \leq s$. Then, the Massey product $\langle \alpha_r, \phi_{\Lambda_j}, \alpha_s \rangle$ is defined and represented by the map*

$$(x,y,z) \longmapsto c_r(x)\Big(\sum_{\ell:\ 1 \leq \ell < k} c_{i_1^{(j)} \cdots i_\ell^{(j)}}(y) c_{i_{\ell+1}^{(j)} \cdots i_k^{(j)} s}(z)\Big) - \Big(\sum_{\ell:\ 1 < \ell \leq k} c_{ri_1^{(j)} \cdots i_\ell^{(j)}}(x) c_{i_{\ell+1}^{(j)} \cdots i_k^{(j)}}(y)\Big) c_s(z).$$

Proof. Note the equalities

$$\alpha_r \smile \phi_{\Lambda_j} = \partial^*\Big(\sum_{\ell:\ 1 \leq \ell < k} c_{ri_1^{(j)} \cdots i_\ell^{(j)}}(x) c_{i_{\ell+1}^{(j)} \cdots i_k^{(j)}}(y)\Big), \qquad \phi_{\Lambda_j} \smile \alpha_s = \partial^*\Big(\sum_{\ell:\ 1 < \ell \leq k} c_{i_1^{(j)} \cdots i_\ell^{(j)}}(x) c_{i_{\ell+1}^{(j)} \cdots i_k^{(j)} s}(y)\Big).$$

Then, from the definition of the triple Massey product, we have a representative. □

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