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Some modules over Lie algebras related to the Virasoro algebra

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Abstract. In this paper, we study restricted modules over a class of $\frac{1}{2}\mathbb{Z}$-graded Lie algebras $\mathfrak{g}$ related to the Virasoro algebra. We in fact give the classification of certain irreducible restricted $\mathfrak{g}$-modules in the sense of determining each irreducible restricted module up to an irreducible module over a subalgebra of $\mathfrak{g}$ which contains its positive part. Several characterizations of these irreducible $\mathfrak{g}$-modules are given. By the correspondence between restricted modules over $\mathfrak{g}$ and modules over the vertex algebra associated to $\mathfrak{g}$, we get the classification of certain irreducible modules over vertex algebras associated to these $\mathfrak{g}$.

1. Introduction

For a vertex operator algebra V, there are three kinds of modules, i.e., weak, admissible and ordinary V-modules, and the notion of weak modules for vertex operator algebras just corresponds to the notion of modules for vertex algebras. One of the fundamental tasks in the representation theory of vertex operator algebras is to classify all irreducible admissible and ordinary modules. But it is also interesting to classify irreducible modules for vertex algebras. In fact, it is even challenging to do such classification for vertex operator algebras which are not rational or C_2-cofinite (see [10]). A rough classification of irreducible modules for vertex algebras related to the Virasoro algebra is obtained in this paper. This is not achieved directly in the theory of vertex algebras, but with the help of the theory of Lie algebras. The strategy we used is to view these modules as modules over Lie algebras.

We call a Lie algebra $\mathfrak{g}$ G-graded if $\mathfrak{g} = \oplus_{g\in G}\mathfrak{g}_g$ and $[\mathfrak{g}_g, \mathfrak{g}_h] \subseteq \mathfrak{g}_{g+h}$ for any $g, h \in G$, where G is an abelian group. And we call a $\mathfrak{g}$-module M G-graded if $M = \oplus_{g\in G}M_g$ and $\mathfrak{g}_g M_h \subseteq M_{g+h}$ for any $g, h \in G$. Among infinite dimensional $\mathbb{Z}$-graded Lie algebras, the most important one is the Virasoro algebra $\mathfrak{V}$, which has a basis $\{L_n, C \mid n \in \mathbb{Z}\}$ subject to the following bracket relations

$$[C, L_m] = 0,\ [L_m, L_n] = (n-m)L_{m+n} + \frac{m^3-m}{12}\delta_{m+n,0}C, \quad \forall m, n \in \mathbb{Z}.$$

Recently, certain irreducible modules over the Virasoro, Heisenberg-Virasoro and Schrödinger-Virasoro algebra, W-algebra $W(2,2)$ were respectively classified in [7], [1] and [2], whose constructions are slight generalizations of highest weight modules. In the present paper, we consider a class of $\frac{1}{2}\mathbb{Z}$-graded Lie algebras $\mathfrak{g}$ which includes all these algebras except for the Virasoro algebra (see Remark 3.3). It is easily observed from [7, Theorem 2] that some of the irreducible $\mathfrak{V}$-modules studied there, are in fact irreducible

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restricted $\mathfrak{V}$-modules. $\mathfrak{g}$-modules of this kind are studied in this paper. To be more precise, we determine each irreducible restricted $\mathfrak{g}$-module up to an irreducible module over a subalgebra which contains the positive part of $\mathfrak{g}$ (see Theorem 2.2 and Proposition 3.2); and we give several characterizations of these modules (see Theorem 3.1). Finally, we prove that these modules also exhaust (inequivalent) irreducible modules over the vertex algebra associated to $\mathfrak{g}$ (see Theorem 3.5). Indeed, this is our motivation to study restricted $\mathfrak{g}$-modules.

The organization of this paper is as follows. In Section 2, we first give an explicit form of Lie algebras under investigation. Then similar as the construction of Verma modules we construct the induced module $\mathrm{Ind}(M)$, the main object of this paper. The main result of this section is to show the irreducibility of $\mathrm{Ind}(M)$ under certain conditions. Several characterizations of these irreducible $\mathfrak{g}$-modules $\mathrm{Ind}(M)$ are given in Section 3. This result can also be viewed as the classification of certain irreducible $\mathfrak{g}$-modules. Using this classification and the known relation between restricted $\mathfrak{g}$-modules and modules over the vertex algebra $V_{\mathfrak{g}}$ associated to $\mathfrak{g}$ we classify certain irreducible modules over $V_{\mathfrak{g}}$.

Throughout this paper, we denote by $\mathbb{C}$, $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Z}_+$ the sets of complex numbers, integers, positive integers, and nonnegative integers, respectively, and denote by $U(L)$ the universal enveloping algebra of a Lie algebra L. And all vector spaces are assumed to be over $\mathbb{C}$.

2. Preliminary and Irreducibility

To be more precise, we study Lie algebras $\mathfrak{g} = V^0 + V^1 + V^2 + \cdots + V^n$ ($n \in \mathbb{N}$) with $V^0 = \mathfrak{V}$ satisfying the following conditions

(1) each V^i is a (nonzero) $\mathbb{Z}$ or $\frac{1}{2}+\mathbb{Z}$-graded $\mathfrak{V}$-module such that $\dim(V^i)_l \le 1$ for all $0 \ne l \in \frac{1}{2}\mathbb{Z}$ and $\dim(V^i)_0 < \infty$, and $[V^i, V^j] = 0$ for any $1 \le i \ne j \le n$;

(2) $\mathfrak{g}$ is $\frac{1}{2}\mathbb{Z}$-graded: $\mathfrak{g} = \oplus_{l\in\frac{1}{2}\mathbb{Z}}\mathfrak{g}_l$ with $\mathfrak{g}_l = \sum_{i=0}^n (V^i)_l$ for all $l \in \frac{1}{2}\mathbb{Z}$;

(3) there exists $\rho \in \{1, \cdots, n\}$ for which V^ρ is $\mathbb{Z}$-graded, $[V^\rho, V^\rho] = 0$, $[(V^0)_{l+q}, (V^\rho)_{-q}] \ne 0$ and $[(V^\rho)_{l+q}, (V^0)_{-q}] \ne 0$ for any $l \in \mathbb{Z}_+$, $q \in \mathbb{N}$;

(4) $[(V^i)_{l+q}, (V^i)_{-q}] \subseteq (V^\rho)_l$ for any $1 \le i \le n$, $q \in \frac{1}{2}\mathbb{Z}$, $l \in \mathbb{Z}$ and $[(V^i)_{l+q}, (V^i)_{-q}] \ne 0$ for any $1 \le i \ne \rho \le n$, $l \in \mathbb{Z}_+$, $q \in \frac{1}{2}\mathbb{N}$ such that $(V^i)_{-q} \ne 0$.

Remark 2.1. *Without loss of generality, we may assume $\rho = n$ from now on. As a consequence of the conditions* (1) *and* (3) *we have $0 \ne [(V^0)_{l+q}, (V^n)_{-q}] \subseteq (V^n)_l$ for any $l \in \mathbb{Z}_+$, $q \in \mathbb{N}$. In particular, $(V^n)_l \ne 0$ for any $l \in \mathbb{Z}_+$.*

To avoid any ambiguity, we write $\mathfrak{g}$ as $\mathfrak{g}^{(n)}$ if necessary. Here we give some examples of infinite dimensional Lie algebras satisfying the above conditions:

- $\mathfrak{g}^{(1)} = \mathcal{V}ir(a,b) = \oplus_{i\in\mathbb{Z}}(\mathbb{C}L_i \oplus \mathbb{C}I_i) \oplus \mathbb{C}C_1 \oplus \sum_{i\in\mathbb{Z}}\mathbb{C}\mathcal{C}_{2,i}$ (cf. [3]), the universal central extension of $\mathcal{W}(a,b)$ (see [4]) except for the case $(a,b) \ne (0,1)$, which satisfies the following (nontrivial) relations:

$$[L_i, L_j] = (j-i)L_{i+j} + \delta_{i+j,0}\frac{i^3-i}{12}C_1,$$
$$[L_i, I_j] = (a+j+bi)I_{i+j} + \delta_{i+j,0}\mathcal{C}_{2,i},$$

where $\{L_i, I_i, C_k \mid i \in \mathbb{Z}, k = 1, 2\}$ is linearly independent, $a, b \in \mathbb{C}$ such that $\pm a + l \notin b\mathbb{N}$ for any $l \in \mathbb{N}$ and

$$\mathcal{C}_{2,i} = \begin{cases} (i^2+i)C_2 & \text{if (a,b)=(0,0),} \\ \frac{i^3-i}{12}C_2 & \text{if (a,b)=(0,}-1\text{),} \\ iC_2 & \text{if (a,b)=(0,1),} \\ 0 & \text{otherwise.} \end{cases}$$

- $\mathfrak{g}^{(1)}$ = the W-algebra $W(2,2)$ (see [9]) which has a basis $\{L_i, I_i, C \mid n \in \mathbb{Z}\}$ subject to the following (nontrivial) relations:

$$[L_i, L_j] = (j-i)L_{i+j} + \frac{i^3-i}{12}\delta_{i+j,0}C,$$
$$[L_i, I_j] = (j-i)I_{i+j} + \frac{i^3-i}{12}\delta_{i+j,0}C.$$

- $\mathfrak{g}^{(2)}$ = the deformative Schrödinger-Virasoro algebra (see [8]) which has a basis $\{L_i, I_i, Y_{i+s}, C \mid i \in \mathbb{Z}, s = 0 \text{ or } \frac{1}{2}\}$ subject to the following (nontrivial) relations:

$$[L_i, L_j] = (j-i)L_{i+j} + \frac{i^3-i}{12}\delta_{i+j,0}C,$$
$$[L_i, I_j] = (j-\lambda i+2\mu)I_{i+j},$$
$$[L_i, Y_{j+s}] = (j+s-\frac{\lambda+1}{2}i+\mu)Y_{i+j+s},$$
$$[Y_{i+s}, Y_{j+s}] = (j-i)I_{i+j+2s},$$

  where $\lambda, \mu \in \mathbb{C}$ such that $\pm 2\mu - l \notin \lambda\mathbb{N}$ for any $l \in \mathbb{N}$.
- $\mathfrak{g}^{(n)} (n \geq 2)$ = a Lie algebra with basis $\{L_i, I_i, Y_{i+s}^{(j)}, C \mid i \in \mathbb{Z}, 1 \leq j \leq n-1, s = 0 \text{ or } \frac{1}{2}\}$ subject to the same nontrivial relations as the deformative Schrödinger-Virasoro algebra with Y_{i+s} replaced by $Y_{i+s}^{(j)}$ for all j.

Let M be a module over a Lie algebra L and X be a subspace of L. For any $v \in M$ and $n \in \mathbb{Z}_+$, denote $X^n v = \text{span}\{x_1 x_2 \cdots x_n v \mid x_i \in X \text{ for } i = 1, 2, \cdots, n\}$. The action of X on M is called *locally nilpotent* if for any $v \in M$ there exists $n \in \mathbb{Z}_+$ such that $X^n v = 0$ and *locally finite* if $\dim\big(\sum_{n\in\mathbb{Z}_+} \mathbb{C}X^n v\big) < +\infty$ for any $v \in M$. A $\mathfrak{g}$-module M is called *restricted* if for any $v \in M$ there exists $n \in \frac{1}{2}\mathbb{Z}_+$ such that $\mathfrak{g}_m v = 0$ for all $m \geq n$.

Before presenting our results, we first need to do some preparations. Let $\mathbb{M}$ be the set of elements of form $\mathbf{i} = (\cdots, i_2, i_1)$ with each $i_k \in \mathbb{Z}_+$ such that $\sum_{k\geq 1} i_k < \infty$. Let ϵ_i denote the element of $\mathbb{M}$ such that the i-th entry from the right is 1 and all the other entries being zero and $\mathbf{0}$ denote the element of $\mathbb{M}$ with all its entries being zero. For any $\mathbf{i} \in \mathbb{M}$, write $\mathbf{w}(\mathbf{i}) = \sum_{k\geq 1} k i_k$. For any $\mathbf{0} \neq \mathbf{i} \in \mathbb{M}$, let p and q be the maximal and minimal integers such that $i_p \neq 0$ and $i_q \neq 0$ respectively, and set $\mathbf{i}' = \mathbf{i} - \epsilon_p$ and $\mathbf{i}'' = \mathbf{i} - \epsilon_q$.

For any $\mathbf{i}, \mathbf{j} \in \mathbb{M}$, define

$$\mathbf{j} > \mathbf{i} \iff \text{there exists } 1 \leq r \in \mathbb{Z}_+ \text{ such that } j_r > i_r \text{ and } j_s = i_s \text{ for all } s > r$$

and

$$\mathbf{j} \succ \mathbf{i} \Longleftrightarrow \text{there exists } 1 \le r \in \mathbb{Z}_+ \text{ such that } j_r > i_r \text{ and } j_s = i_s \text{ for all } 1 \le s < r.$$

And define a total order " $\succ$ " on $\underbrace{\mathbb{M} \times \cdots \times \mathbb{M}}_{n+1}$ by decreeing

$(\mathbf{i}^{(n)}, \mathbf{i}^{(n-1)}, \cdots, \mathbf{i}^{(0)}) \succ (\mathbf{j}^{(n)}, \mathbf{j}^{(n-1)}, \cdots, \mathbf{j}^{(0)}) \Longleftrightarrow$
$\exists 1 \le r \le n-1$ such that $\mathbf{j}^{(k)} = \mathbf{i}^{(k)}$ for $0 \le k \le r-1$ and $\big(\mathbf{i}^{(r)}, \mathbf{w}(\mathbf{i}^{(r)})\big) \succ \big(\mathbf{j}^{(r)}, \mathbf{w}(\mathbf{j}^{(r)})\big)$
or $(\mathbf{i}^{(n-1)}, \cdots, \mathbf{i}^{(0)}) = (\mathbf{j}^{(n-1)}, \cdots, \mathbf{j}^{(0)})$ and $\mathbf{i}^{(n)} > \mathbf{j}^{(n)}$.

Set

$$\mathcal{S} = \left\{ \underline{d} = (d_0, d_1, \cdots, d_n) \in \mathbb{Z}_+^{n+1} \mid d_0 = 0, d_n \ge 2d_i \text{ for } 1 \le i \le n-1 \right\},$$

$$\mathfrak{g}_{\underline{d}} = \sum\nolimits_{i \in \frac{1}{2}\mathbb{Z}_+} \Big((V^0)_{i-d_0} \oplus (V^1)_{i-d_1} \oplus \cdots \oplus (V^n)_{i-d_n} \Big)$$

and

$$\mathfrak{g}_{+\infty} = \sum\nolimits_{i \in \mathbb{Z}_+} (V^0)_i + V^1 + \cdots + V^n.$$

It is easy to check that for any $\underline{d} \in \mathcal{S} \cup \{+\infty\}$, $\mathfrak{g}_{\underline{d}}$ is a subalgebra of $\mathfrak{g}$ by using the assumptions on $\mathfrak{g}$. Let L be a Lie algebra, $\mathfrak{a}$ a subalgebra of L, M an $\mathfrak{a}$-module and Y a subset of $\mathfrak{a}$. Set $\mathrm{Ann}_M(Y) = \{v \in M \mid yv = 0 \text{ for } y \in Y\}$ and form the induced module $\mathrm{Ind}_{\mathfrak{a}}^L(M) := U(L) \otimes_{U(\mathfrak{a})} M$, which is simply written as $\mathrm{Ind}(M)$ if the context is clear. For any vector space V, define

$$\delta_{V,0} = \begin{cases} 1 & \text{if } V = 0, \\ 0 & \text{if } V \ne 0. \end{cases}$$

For a $\mathfrak{g}_{\underline{d}}$-module M, it is hard to give a sufficient and necessary condition for $\mathrm{Ind}(M)$ to be irreducible. The following result provides a sufficient conditon.

Theorem 2.2. *Let $\underline{d} \in \mathcal{S} \cup \{+\infty\}$ and M be an irreducible $\mathfrak{g}_{\underline{d}}$-module for which there exists $k \in \mathbb{Z}_+$ such that*

(1) $\mathrm{Ann}_M(V^n)_k = 0$ *if* $k \ne 0$; $\sum_{q \in \mathbb{Z}} (1 - \delta_{[(V^0)_{-q}, (V^n)_q], 0}) \mathrm{Ann}_M [(V^0)_{-q}, (V^n)_q] = 0$ *and* $\sum_{1 \le i \le n-1} \sum_{q \in \frac{1}{2}\mathbb{Z}_+} (1 - \delta_{[(V^i)_{-q}, (V^i)_q], 0}) \mathrm{Ann}_M [(V^i)_{-q}, (V^i)_q] = 0$ *if* $k = 0$;
(2) $(V^0)_{k+d_n+p} M = (V^n)_{k+p} M = (V^i)_{k+d_i+p} M = 0$ *for all* $1 \le i \le n-1$ *and* $p \in \frac{1}{2}\mathbb{N}$.

Then

(i) $\mathrm{Ind}(M)$ *is an irreducible* $\mathfrak{g}$*-module*;
(ii) *the actions of* $(V^0)_{k+d_n+p}$, $(V^n)_{k+p}$ *and* $(V^i)_{k+d_i+p}$ *on* $\mathrm{Ind}(M)$ *for all* $1 \le i \le n-1$ *and* $p \in \frac{1}{2}\mathbb{N}$ *are locally nilpotent.*

We denote by $u_k^{(i)}$ a basis of nonzero $\left(V^i\right)_k$ with $u_k^{(0)} = L_k$ for $0 \ne k \in \frac{1}{2}\mathbb{Z}$ and $i \in \{0, 1, \cdots, n\}$. Note that for any i, the set consisting of all indexes $k \le -\frac{1}{2} - d_i$

such that $(V^i)_k \neq 0$ is denumerable, say, $\cdots < -I_2^{(i)} < -I_1^{(i)} \leq -\frac{1}{2} - d_i$. Set $U_i^{\mathbf{j}} = \cdots (u^{(i)}_{-I_2^{(i)}})^{j_2} (u^{(i)}_{-I_1^{(i)}})^{j_1}$ for any $\mathbf{j} = (\cdots, j_2, j_1) \in \mathbb{M}$. Take $0 \neq u \in \mathrm{Ind}(M)$. Then u can be (uniquely) written as the following form

$$\sum_{\mathbf{j}^{(n)},\cdots,\mathbf{j}^{(0)} \in \mathbb{M}} U_n^{\mathbf{j}^{(n)}} \cdots U_1^{\mathbf{j}^{(1)}} U_0^{\mathbf{j}^{(0)}} u_{\mathbf{j}^{(n)},\cdots,\mathbf{j}^{(0)}} \text{ (finite sum)}, \tag{2.1}$$

where all $(\mathbf{j}^{(n)}, \cdots, \mathbf{j}^{(0)}) \in \mathbb{M}$ and $u_{\mathbf{j}^{(n)},\cdots,\mathbf{j}^{(0)}} \in M$. Set

$$\mathrm{supp}(u) = \Big\{ \big(\mathbf{j}^{(n)}, \cdots, \mathbf{j}^{(0)}\big) \in \mathbb{M} \times \mathbb{M} \cdots \times \mathbb{M} \mid u_{\mathbf{j}^{(n)},\cdots,\mathbf{j}^{(0)}} \neq 0 \Big\}.$$

Let $m(u) := \big(\mathbf{k}^{(n)}, \cdots, \mathbf{k}^{(0)}\big)$ be the maximum in $\mathrm{supp}(u)$ with respect to the total order $\succ$ on $\underbrace{\mathbb{M} \times \cdots \times \mathbb{M}}_{n+1}$.

Lemma 2.3. *Let M be a $\mathfrak{g}_{\underline{d}}$-module satisfying the conditions in Theorem 2.2. For any $u \in \mathrm{Ind}(M) \setminus M$, let $\big(\mathbf{k}^{(n)}, \cdots, \mathbf{k}^{(0)}\big)$ be its maximum in* $\mathrm{supp}(u)$ *and r minimal such that $\mathbf{k}^{(r)} \neq \mathbf{0}$. Set $\hat{i} = \max\{s \mid k_s^{(r)} \neq 0\}$ and $\hat{j} = \min\{s \mid k_s^{(r)} \neq 0\}$.*

(a) *If $r = 0$, then $m(u^{(n)}_{I_{\hat{j}}^{(0)}+k} u) = \big(\mathbf{k}^{(n)}, \cdots, \mathbf{k}^{(1)}, (\mathbf{k}^{(0)})''\big)$.*

(b) *If $0 < r < n$, then $m(u^{(r)}_{I_{\hat{j}}^{(r)}+k} u) = \big(\mathbf{k}^{(n)}, \cdots, \mathbf{k}^{(r+1)}, (\mathbf{k}^{(r)})'', \mathbf{0}, \cdots, \mathbf{0}\big)$.*

(c) *If $r = n$, then $m(L_{I_{\hat{i}}^{(n)}+k} u) = \big((\mathbf{k}^{(n)})', \mathbf{0}, \cdots, \mathbf{0}\big)$.*

Proof. The idea of this proof comes essentially from [7, 1] (see also [2]). Let k be the nonnegative integer satisfying conditions (1) and (2) in Theorem 2.2. Assume u has the form (2.1).

(a) Note by the condition (2) in Theorem 2.2 that $u^{(n)}_{I_{\hat{j}}^{(0)}+k} v = 0$ for any $v \in M$. It follows this and the conditions $[V^n, V^n] = [V^i, V^j] = 0, \forall 1 \leq i \neq j \leq n$ that

$$\begin{aligned} & u^{(n)}_{I_{\hat{j}}^{(0)}+k} U_n^{\mathbf{k}^{(n)}} \cdots U_0^{\mathbf{k}^{(0)}} u_{\mathbf{k}^{(n)},\cdots,\mathbf{k}^{(0)}} \\ &= U_n^{\mathbf{k}^{(n)}} \cdots U_1^{\mathbf{k}^{(1)}} [u^{(n)}_{I_{\hat{j}}^{(0)}+k}, u^{(0)}_{-I_{\hat{i}}^{(0)}}] U_0^{(\mathbf{k}^{(0)})'} u_{\mathbf{k}^{(n)},\cdots,\mathbf{k}^{(0)}} + \cdots + \\ & \quad U_n^{\mathbf{k}^{(n)}} \cdots U_1^{\mathbf{k}^{(1)}} U_0^{(\mathbf{k}^{(0)})''} [u^{(n)}_{I_{\hat{j}}^{(0)}+k}, u^{(0)}_{-I_{\hat{j}}^{(0)}}] u_{\mathbf{k}^{(n)},\cdots,\mathbf{k}^{(0)}}. \end{aligned} \tag{2.2}$$

Note that by the condition (3) on $\mathfrak{g}$ we see that $0 \neq [u^{(n)}_{I_{\hat{j}}^{(0)}+k}, u^{(0)}_{-I_{\hat{j}}^{(0)}}] \in (V^n)_k$. Then by the assumption on k, $\mathrm{Ann}_M [u^{(n)}_{I_{\hat{j}}^{(0)}+k}, u^{(0)}_{-I_{\hat{j}}^{(0)}}] = 0$. Thus in particular, $[u^{(n)}_{I_{\hat{j}}^{(0)}+k}, u^{(0)}_{-I_{\hat{j}}^{(0)}}] u_{\mathbf{k}^{(n)},\cdots,\mathbf{k}^{(0)}} \neq 0$ and therefore

$$\big(\mathbf{k}^{(n)}, \cdots, \mathbf{k}^{(1)}, (\mathbf{k}^{(0)})''\big) = m\big(u^{(n)}_{I_{\hat{j}}^{(0)}+k} U_n^{\mathbf{k}^{(n)}} \cdots U_0^{\mathbf{k}^{(0)}} u_{\mathbf{k}^{(n)},\cdots,\mathbf{k}^{(0)}}\big).$$

Denote

$$(\mathbf{j}_1^{(n)},\cdots,\mathbf{j}_1^{(0)})=m(u^{(n)}_{I^{(0)}_{\hat{j}}+k}U_n^{\mathbf{j}^{(n)}}\cdots U_0^{\mathbf{j}^{(0)}}u_{\mathbf{j}^{(n)},\cdots,\mathbf{j}^{(0)}})\text{ for }(\mathbf{j}^{(n)},\cdots,\mathbf{j}^{(0)})\in\operatorname{supp}(u).$$

Case 1. $\mathbf{j}^{(0)}=\mathbf{k}^{(0)}$.

Note that in this case,

$$(\mathbf{j}_1^{(n)},\cdots,\mathbf{j}_1^{(1)},\mathbf{j}_1^{(0)})=\big(\mathbf{j}^{(n)},\cdots,\mathbf{j}^{(1)},(\mathbf{k}^{(0)})''\big)\preceq\big(\mathbf{k}^{(n)},\cdots,\mathbf{k}^{(1)},(\mathbf{k}^{(0)})''\big)$$

and the equality holds if and only if $\mathbf{j}^{(l)}=\mathbf{k}^{(l)}$ for all $l=1,2,\cdots,n$, by a similar formula (2.2) for $(\mathbf{j}^{(n)},\cdots,\mathbf{j}^{(1)},\mathbf{j}^{(0)})$ and the fact that $(\mathbf{j}^{(n)},\cdots,\mathbf{j}^{(1)},\mathbf{j}^{(0)})\preceq(\mathbf{k}^{(n)},\cdots,\mathbf{k}^{(1)},\mathbf{k}^{(0)})$.

Case 2. $\mathbf{j}^{(0)}\neq\mathbf{k}^{(0)}$.

In this case we have $\big(\mathbf{j}^{(0)},\mathbf{w}(\mathbf{j}^{(0)})\big)\prec\big(\mathbf{k}^{(0)},\mathbf{w}(\mathbf{k}^{(0)})\big)$. Then either $\mathbf{w}(\mathbf{j}^{(0)})<\mathbf{w}(\mathbf{k}^{(0)})$ or $\mathbf{w}(\mathbf{j}^{(0)})=\mathbf{w}(\mathbf{k}^{(0)})$ and $\mathbf{j}^{(0)}\prec\mathbf{k}^{(0)}$. If $\mathbf{w}(\mathbf{j}^{(0)})<\mathbf{w}(\mathbf{k}^{(0)})$, then $\mathbf{w}(\mathbf{j}_1^{(0)})\le\mathbf{w}(\mathbf{j}^{(0)})-\hat{j}<\mathbf{w}(\mathbf{k}^{(0)})-\hat{j}=\mathbf{w}(\mathbf{k}^{(0)})''$ and therefore $(\mathbf{j}_1^{(n)},\cdots,\mathbf{j}_1^{(0)})\preceq\big(\mathbf{k}^{(n)},\cdots,\mathbf{k}^{(1)},(\mathbf{k}^{(0)})''\big)$.

Assume that $\mathbf{w}(\mathbf{j}^{(0)})=\mathbf{w}(\mathbf{k}^{(0)})$ and $\mathbf{j}^{(0)}\prec\mathbf{k}^{(0)}$. Let $s=\min\{s\in\mathbb{N}\mid j_s^{(0)}\neq 0\}$. Since $\mathbf{j}^{(0)}\prec\mathbf{k}^{(0)}$, $s\ge\hat{j}$. If $s>\hat{j}$, then $\mathbf{w}(\mathbf{j}_1^{(0)})\le\mathbf{w}(\mathbf{j}^{(0)})-s<\mathbf{w}(\mathbf{j}^{(0)})-\hat{j}=\mathbf{w}(\mathbf{k}^{(0)})-\hat{j}=\mathbf{w}(\mathbf{k}^{(0)})''$ and therefore $(\mathbf{j}_1^{(n)},\cdots,\mathbf{j}_1^{(0)})\preceq\big(\mathbf{k}^{(n)},\cdots,\mathbf{k}^{(1)},(\mathbf{k}^{(0)})''\big)$. If $s=\hat{j}$, then

$$\begin{aligned}(\mathbf{j}_1^{(n)},\cdots,\mathbf{j}_1^{(0)})&=m(u^{(n)}_{I^{(0)}_{\hat{j}}+k}U_n^{\mathbf{j}^{(n)}}\cdots U_0^{\mathbf{j}^{(0)}}u_{\mathbf{j}^{(n)},\cdots,\mathbf{j}^{(0)}})=m(u^{(n)}_{I^{(0)}_{s}+k}U_n^{\mathbf{j}^{(n)}}\cdots U_0^{\mathbf{j}^{(0)}}u_{\mathbf{j}^{(n)},\cdots,\mathbf{j}^{(0)}})\\&=(\mathbf{j}^{(n)},\cdots,\mathbf{j}^{(1)},(\mathbf{j}^{(0)})'')\prec(\mathbf{j}^{(n)},\cdots,\mathbf{j}^{(1)},(\mathbf{k}^{(0)})'')\\&\preceq(\mathbf{k}^{(n)},\cdots,\mathbf{k}^{(1)},(\mathbf{k}^{(0)})'').\end{aligned}$$

So in either case, we obtain $(\mathbf{j}_1^{(n)},\cdots,\mathbf{j}_1^{(0)})\preceq\big(\mathbf{k}^{(n)},\cdots,\mathbf{k}^{(1)},(\mathbf{k}^{(0)})''\big)$ and the equality holds only when $(\mathbf{j}^{(n)},\cdots,\mathbf{j}^{(0)})=(\mathbf{k}^{(n)},\cdots,\mathbf{k}^{(0)})$, proving (a).

(b) follows similar arguments as for (a). Here we need to point out that $u^{(r)}_{I^{(r)}_{\hat{j}}+k}\neq 0$. Note by the condition (4) on $\mathfrak{g}$ that $[(V^r)_{I^{(r)}_{\hat{j}}+k},(V^r)_{-I^{(r)}_{\hat{j}}}]\neq 0$, since $(V^r)_{-I^{(r)}_{\hat{j}}}\neq 0$. This, in particular, implies $(V^r)_{I^{(r)}_{\hat{j}}+k}\neq 0$. Then as a basis element of $(V^r)_{I^{(r)}_{\hat{j}}+k}$, $u^{(r)}_{I^{(r)}_{\hat{j}}+k}\neq 0$.

(c) For any $(\mathbf{j}^{(n)},\mathbf{0},\cdots,\mathbf{0})\in\operatorname{supp}(u)$, we have

$$\begin{aligned}&L_{I^{(n)}_{\hat{i}}+k}U_n^{\mathbf{j}^{(n)}}u_{\mathbf{j}^{(n)},\mathbf{0},\cdots,\mathbf{0}}\\&=L_{I^{(n)}_{\hat{i}}+k}\big(\cdots(u^{(n)}_{-I^{(n)}_2})^{j_2^{(n)}}(u^{(n)}_{-I^{(n)}_1})^{j_1^{(n)}}u_{\mathbf{j}^{(n)},\mathbf{0},\cdots,\mathbf{0}}\big)\quad(\text{by the conditon }[V^n,V^n]=0)\\&=j_{\hat{i}}^{(n)}U_n^{\mathbf{j}^{(n)}-\epsilon_{\hat{i}}}[L_{I^{(n)}_{\hat{i}}+k},u^{(n)}_{-I^{(n)}_{\hat{i}}}]u_{\mathbf{j}^{(n)},\mathbf{0},\cdots,\mathbf{0}}+\sum_{l<\hat{i}}j_l^{(n)}U_n^{\mathbf{j}^{(n)}-\epsilon_l}[L_{I^{(n)}_{\hat{i}}+k},u^{(n)}_{-I^{(n)}_l}]u_{\mathbf{j}^{(n)},\mathbf{0},\cdots,\mathbf{0}}\\&=j_{\hat{i}}^{(n)}U_n^{(\mathbf{j}^{(n)})'}[L_{I^{(n)}_{\hat{i}}+k},u^{(n)}_{-I^{(n)}_{\hat{i}}}]u_{\mathbf{j}^{(n)},\mathbf{0},\cdots,\mathbf{0}}\quad(\text{since }[L_{I^{(n)}_{\hat{i}}+k},u^{(n)}_{-I^{(n)}_l}]u_{\mathbf{j}^{(n)},\mathbf{0},\cdots,\mathbf{0}}=0)\end{aligned}$$

and $[L_{I^{(n)}_{\hat{i}}+k}, u^{(n)}_{-I^{(n)}_{\hat{i}}}]u_{\mathbf{j}^{(n)},\mathbf{0},\cdots,\mathbf{0}} \neq 0$. That is,

$$\operatorname{supp}\big(L_{I^{(n)}_{\hat{i}}+k}U_n^{\mathbf{j}^{(n)}}u_{\mathbf{j}^{(n)},\mathbf{0},\cdots,\mathbf{0}}\big) = \begin{cases}\big\{\big((\mathbf{j}^{(n)})',\mathbf{0},\cdots,\mathbf{0}\big)\big\} & \text{if } j^{(n)}_{\hat{i}} \neq 0,\\ \big\{(\mathbf{0},\cdots,\mathbf{0},\mathbf{0})\big\} & \text{if } j^{(n)}_{\hat{i}} = 0.\end{cases}$$

Then it is easy to see that $m(L_{I^{(n)}_{\hat{i}}+k}u) = \big((\mathbf{k}^{(n)})',\mathbf{0},\cdots,\mathbf{0}\big)$. □

Proof of Theorem 2.2: Let W be any nonzero $\mathfrak{g}$-submodule of $\mathrm{Ind}(M)$. Take $0 \neq u \in W$ such that $m(u)$ is minimal among $m(u')$ for all $0 \neq u' \in W$. If $m(u) \neq (\mathbf{0},\cdots,\mathbf{0})$, then by the lemma above there exists $0 \neq w \in W$ such that $m(w) \prec m(u)$, contradicting the choice of u. Thus, $m(u) = (\mathbf{0},\cdots,\mathbf{0})$ and therefore $u \in M$. Now the irreducibility of $\mathrm{Ind}(M)$ follows from that of M.

Remark 2.4. *Let $\underline{d}$ and M be as in Theorem* 2.2 *except that M may not be irreducible. Then*

$$M = \left\{u \in \mathrm{Ind(M)} \;\middle|\; \begin{matrix}(V^0)_{k+d_n+p}u = (V^n)_{k+p}u = (V^i)_{k+d_i+p}u = 0,\\ \forall 1 \leq i \leq n-1,\ \tfrac{1}{2} \leq p\end{matrix}\right\}$$

and Lemma 2.3 still holds.

3. Characterization and VA-Module

Define $\mathfrak{g}^{\underline{x}}$ for $\underline{x} = (x_0, x_1, \cdots, x_n) \in (\frac{1}{2}\mathbb{Z})^{n+1}$ to be the subalgebra of $\mathfrak{g}$ generated by $(V^i)_j$ with $x_i \leq j \in \frac{1}{2}\mathbb{Z}$ for $i = 0, 1, \cdots, n$. As in [1, Section 3] we have the following characterizations of certain irreducible $\mathfrak{g}$-modules.

Theorem 3.1. *Let S be an irreducible $\mathfrak{g}$-module such that $[(V^0)_p, (V^i)_q] = (V^i)_{p+q}$ on S for any $p \in \mathbb{Z}_+$, $q \in \frac{1}{2}\mathbb{N}$, $i = 1, \cdots, n$ and that*

$$\sum\nolimits_{q\in\mathbb{Z}}(1-\delta_{[(V^0)_{-q},(V^n)_q],0})\mathrm{Ann}_S[(V^0)_{-q},(V^n)_q] = 0$$

$$\text{and}\quad \sum\nolimits_{1\leq i\leq n-1}\sum\nolimits_{q\in\frac{1}{2}\mathbb{Z}_+}(1-\delta_{[(V^i)_{-q},(V^i)_q],0})\mathrm{Ann}_S[(V^i)_{-q},(V^i)_q] = 0.$$

Then the following conditions are equivalent.

1. *There exists $t \in \mathbb{Z}$ such that the actions of $(V^i)_r, i = 0, 1, \cdots, n$ on S are locally finite for all $t \leq r \in \frac{1}{2}\mathbb{Z}$.*
2. *There exists $t \in \mathbb{Z}$ such that the actions of $(V^i)_r, i = 0, 1, \cdots, n$ on S are locally nilpotent for all $t \leq r \in \frac{1}{2}\mathbb{Z}$.*
3. *There exists $\underline{t} \in \mathbb{Z}^{n+1}$ such that S is a locally finite $\mathfrak{g}^{\underline{t}}$-module.*
4. *There exists $\underline{t} \in \mathbb{Z}^{n+1}$ such that S is a locally nilpotent $\mathfrak{g}^{\underline{t}}$-module.*
5. *There exist $\underline{d} \in \mathbb{Z}^{n+1}$ and an irreducible $\mathfrak{g}_{\underline{d}}$-module M satisfying the conditions in Theorem* 2.2 *such that $S \cong \mathrm{Ind}(M)$.*

Proof. The following implications (5) ⇒ (3) ⇒ (1), (5) ⇒ (4) ⇒ (2) and (2) ⇒ (1) are clear. So we only need to show (1) ⇒ (5).

By (1) we know that there exists $t \in \mathbb{Z}_+$ such that the actions of $(V^i)_r, i = 0, 1, \cdots, n$ on S are locally finite for all $t \le r \in \frac{1}{2}\mathbb{Z}$. In particular, there exists a nonzero $v \in S$ such that $L_t v = \lambda v$ for some $\lambda \in \mathbb{C}$.

Choose any $\frac{1}{2} < a_i \in \frac{1}{2}\mathbb{Z}$ such that $(V^i)_{t+a_i} \ne 0$ and denote

$$N_{a_i} = \sum\nolimits_{m \in \mathbb{Z}_+} \mathbb{C} L_t^m (V^i)_{t+a_i} v,\ i = 0, 1, \cdots, n,$$

which are all finite dimensional. Note by the assumption $[(V^0)_p, (V^i)_q] = (V^i)_{p+q}$ $(p \ge 0, q > 0)$ on S that

$$(V^i)_{t+a_i+(m+1)t} v = [L_t, (V^i)_{t+a_i+mt}] v = (L_t - \lambda)(V^i)_{t+a_i+mt} v$$

for $m \in \mathbb{Z}_+$ and $i = 0, 1, \cdots, n$, from which we know that

$$(V^i)_{t+a_i+mt} v \subseteq N_{a_i} \text{ implies } (V^i)_{t+a_i+(m+1)t} v \subseteq N_{a_i} \text{ for } m \in \mathbb{Z}_+ \text{ and } i = 0, 1, \cdots, n.$$

So induction on m shows

$$(V^i)_{t+a_i+mt} v \subseteq N_{a_i} \text{ for } m \in \mathbb{Z}_+ \text{ and } i = 0, 1, \cdots, n.$$

In particular, $\sum_{m \in \mathbb{Z}_+} \mathbb{C}(V^i)_{t+a_i+mt} v$ for $i = 0, 1, \cdots, n$ are all finite dimensional and so are

$$\sum\nolimits_{p \in \mathbb{Z}_+} \mathbb{C}(V^i)_{t+a_i+p} v = \mathbb{C}(V^i)_{t+a_i} v + \sum\nolimits_{j=t+1}^{2t} \sum\nolimits_{m \in \mathbb{Z}_+} (V^i)_{j+a_i+mt} v,\ i = 0, 1, \cdots, n.$$

Then there exist $l_i{}' \in \mathbb{N}$ such that

$$\sum\nolimits_{p \in \mathbb{Z}_+} \mathbb{C}(V^i)_{t+a_i+p} v = \sum\nolimits_{p=0}^{l_i{}'} \mathbb{C}(V^i)_{t+a_i+p} v,\ i = 0, 1, \cdots, n. \tag{3.1}$$

Similarly, there exist $l_i{}'' \in \mathbb{N}$ such that

$$\sum\nolimits_{p \in \mathbb{Z}_+} \mathbb{C}(V^i)_{t+a_i+\frac{1}{2}+p} v = \sum\nolimits_{p=0}^{l_i{}''} \mathbb{C}(V^i)_{t+a_i+\frac{1}{2}+p} v,\ i = 0, 1, \cdots, n. \tag{3.2}$$

Note that for any i, the set consisting of all indexes $k \ge t + a_i$ such that $(V^i)_k \ne 0$ is denumerable, say, $I_1^{(i)} < I_2^{(i)} < \cdots$. Set $l = \max\{l_i{}', l_i{}'' \mid i = 0, 1, \cdots, n\}$ and $\hat{i} = \max\left\{k \in \mathbb{N} \mid t + a_i \le I_k^{(i)} \le t + a_i + \frac{1}{2} + l\right\}$ for all $i = 0, 1, \cdots, n$. Denote

$$V' = \sum\nolimits_{m_1^{(0)}, \cdots, m_{\hat{n}}^{(n)} \in \mathbb{Z}_+} \mathbb{C}\left(u^{(0)}_{I_1^{(0)}}\right)^{m_1^{(0)}} \cdots \left(u^{(0)}_{I_{\hat{0}}^{(0)}}\right)^{m_{\hat{0}}^{(0)}} \cdots \left(u^{(n)}_{I_1^{(n)}}\right)^{m_1^{(n)}} \cdots \left(u^{(n)}_{I_{\hat{n}}^{(n)}}\right)^{m_{\hat{n}}^{(n)}} v$$

which is finite dimensional by (1).

CLAIM. *V' is a (finite dimensional) $\mathfrak{g}^{\underline{t+a}}$-module, where $\underline{t+a} = (t+a_0, \cdots, t+a_n)$.*

Note that $u^{(i)}_{I_s^{(i)}} v'$ for all $v' \in V'$, $i = 0, 1, \cdots, n$ and $s \in \mathbb{N}$ can be written as a sum of vectors of the form:

$$\left(u^{(0)}_{I^{(0)}_1}\right)^{m^{(0)}_1}\cdots\left(u^{(0)}_{I^{(0)}_{\hat{0}}}\right)^{m^{(0)}_{\hat{0}}}\cdots\left(u^{(n)}_{I^{(n)}_1}\right)^{m^{(n)}_1}\cdots\left(u^{(n)}_{I^{(n)}_{\hat{n}}}\right)^{m^{(n)}_{\hat{n}}}u^{(i)}_{I^{(i)}_r}v\quad (r\geq 1). \tag{3.3}$$

So it suffices to show that all elements above lie in V'. By (3.1) and (3.2), we only need to show that elements in (3.3) with $t+a_i\leq I^{(i)}_r\leq t+a_i+\frac{1}{2}+l$ lie in V'. This is clear for $i=n$ in (3.3). For all $i=0,1,\cdots,n-1$, we have

$$\begin{aligned}
&\left(u^{(0)}_{I^{(0)}_1}\right)^{m^{(0)}_1}\cdots\left(u^{(0)}_{I^{(0)}_{\hat{0}}}\right)^{m^{(0)}_{\hat{0}}}\cdots\left(u^{(n)}_{I^{(n)}_1}\right)^{m^{(n)}_1}\cdots\left(u^{(n)}_{I^{(n)}_{\hat{n}}}\right)^{m^{(n)}_{\hat{n}}}u^{(i)}_{I^{(i)}_r}v\\
=&\left(u^{(0)}_{I^{(0)}_1}\right)^{m^{(0)}_1}\cdots\left(u^{(i)}_{I^{(i)}_r}\right)^{m^{(i)}_r+1}\cdots\left(u^{(n)}_{I^{(n)}_{\hat{n}}}\right)^{m^{(n)}_{\hat{n}}}v+\\
&\left(u^{(0)}_{I^{(0)}_1}\right)^{m^{(0)}_1}\cdots\left(u^{(i)}_{I^{(i)}_{r-1}}\right)^{m^{(i)}_{r-1}}\left[\left(u^{(i)}_{I^{(i)}_r}\right)^{m^{(i)}_r}\cdots\left(u^{(n)}_{I^{(n)}_{\hat{n}}}\right)^{m^{(n)}_{\hat{n}}},u^{(i)}_{I^{(i)}_r}\right]v.
\end{aligned}$$

The first term on the right hand side lies in V' and the second term can be written as a sum of elements which have the same form as (3.3) but with smaller $\sum_{i=0}^n\sum_{j=1}^{\hat{i}}m^{(i)}_j$. By induction, all elements in (3.3) lie in V', proving the claim.

It follows from the claim that there exists a minimal $l\in\mathbb{Z}_+$ such that $(L_m+\alpha_1L_{m+1}+\cdots+\alpha_lL_{m+l})V'=0$ for some $m\geq t+a_0$ and $\alpha_i\in\mathbb{C}$. Then applying L_m gives

$$(\alpha_1[L_m,L_{m+1}]+\cdots+\alpha_l[L_m,L_{m+l}])V'=0,$$

which together with the minimality of l implies $l=0$, that is, $L_mV'=0$. Therefore

$$0=L_jL_mV'=[L_j,L_m]V'+L_mL_jV'=(m-j)L_{m+j}V',\ \forall j\geq t+a_0,$$

that is, $L_{m+j}V'=0$ for all $j\geq m+a_0$. Now by again our assumption $[(V^0)_p,(V^i)_q]=(V^i)_{p+q}$ on S for any $p\in\mathbb{Z}_+$, $q\in\frac{1}{2}\mathbb{N}$, we have $(V^i)_{m+j+q}V'=[L_{m+j},(V^i)_q]V'=0$ for any $q\geq t+a_i$ and $i\geq 1$.

For any $\underline{r}=(r_0,r_1,\cdots,r_n)\in\mathbb{Z}^{n+1}$, consider the vector space

$$N_{\underline{r}}=\left\{v\in S\mid (V^i)_{r_i+p}v=0\text{ for all }i=0,1,\cdots,n\text{ and }p\in\frac{1}{2}\mathbb{N}\right\}.$$

By the above discussion, $N_{\underline{r}}\neq 0$ for sufficiently large $r_i\in\mathbb{Z},i=0,1,\cdots,n$. Note by Remark 2.1 and the assumption $\mathrm{Ann}_S[(V^0)_{-q},(V^n)_q]=0$ for any $q\in\mathbb{N}$ that $\mathrm{Ann}_S(V^n)_0=0$. Thus, $N_{\underline{r}}=0$ for all $r_n<0$. Choose a $\underline{k}=(k_0,k_1,\cdots,k_n)$ from the set $\{\underline{r}\in\mathbb{Z}^{n+1}\mid N_{\underline{r}}\neq 0\}$ such that the n-th component k_n is minimal. Moreover, we may assume $k_i\in\mathbb{Z}_+$ with $k_i\geq k_n$ and $k_0-k_n\geq 2(k_j-k_n)$, where $i=0,1,\cdots,n-1$ and $j=1,2,\cdots,n-1$. Denote $M=N_{\underline{k}}$ and $\underline{d}=(d_0,d_1,\cdots,d_n)$, where $d_0=0,\ d_i=k_i-k_n,\ d_n=k_0-k_n,\ i=1,2,\cdots,n-1$. It is easy to check that M is a $\mathfrak{g}_{\underline{d}}$-module. Note also that M automatically satisfies the conditions in Theorem 2.2 with $k=k_n$.

We are going to show that $S\cong\mathrm{Ind}(M)$. Since S can be generated by M, there exists a canonical surjective map

$$\pi : \mathrm{Ind}(M) \to S \text{ such that } \pi(1 \otimes v) = v, \ \forall v \in M.$$

Now it is enough to show that π is also injective. We only focus on the case $k_n \geq 1$, since similar arguments can be applied to the case $k_n = 0$ by using the assumptions

$$\sum\nolimits_{q\in\mathbb{Z}} (1 - \delta_{[(V^0)_{-q},(V^n)_q],0}) \mathrm{Ann}_S[(V^0)_{-q}, (V^n)_q] = 0$$

$$\text{and} \quad \sum\nolimits_{1\leq i\leq n-1} \sum\nolimits_{q\in\frac{1}{2}\mathbb{Z}_+} (1 - \delta_{[(V^i)_{-q},(V^i)_q],0}) \mathrm{Ann}_S[(V^i)_{-q}, (V^i)_q] = 0.$$

Let $K = \mathrm{Ker}(\pi)$ and it is clear that $K \cap M = 0$. If $K \neq 0$, choose a vector $u \in K\backslash M$ such that $m(u)$ (see the remarks before Lemma 2.3) is minimal. Then Lemma 2.3 and Remark 2.4 would lead to a contradiction: there exists $0 \neq w \in K$ with $m(w) \prec m(u)$.

At last, we remark that M automatically satisfies the conditions in Theorem 2.2. □

If in addition S as a $\mathfrak{g}$-module is restricted, then

$$N_{\underline{r}} = \left\{ v \in S \mid (V^i)_{r_i+p} v = 0 \text{ for all } i = 0, 1, \cdots, n \text{ and } p \in \frac{1}{2}\mathbb{N} \right\}$$

is nonzero whenever each entry r_i is large enough. It follows from the last part of the proof of Theorem 3.1 we see that $S \cong \mathrm{Ind}(M)$ for some $\underline{d} \in \mathcal{S}$ and $\mathfrak{g}_{\underline{d}}$-module M. That is, we derive the following result.

Proposition 3.2. *Let S be an irreducible restricted $\mathfrak{g}$-module such that*

$$\sum\nolimits_{q\in\mathbb{Z}} (1 - \delta_{[(V^0)_{-q},(V^n)_q],0}) \mathrm{Ann}_S[(V^0)_{-q}, (V^n)_q] = 0$$

$$\text{and} \quad \sum\nolimits_{1\leq i\leq n-1} \sum\nolimits_{q\in\frac{1}{2}\mathbb{Z}_+} (1 - \delta_{[(V^i)_{-q},(V^i)_q],0}) \mathrm{Ann}_S[(V^i)_{-q}, (V^i)_q] = 0.$$

Then there exist $\underline{d} \in \mathcal{S}$ and an irreducible $\mathfrak{g}_{\underline{d}}$-module M satisfying the conditions in Theorem 2.2 such that $S \cong \mathrm{Ind}(M)$.

Remark 3.3. *Let V be a module over L and $E = L \oplus L'$ be a central extension of L (that is, $[x,y]_E = [x,y]_L$ for any $x, y \in L$ and L' lies in the center of E). Then V can be naturally viewed as an L/T-module, where $T = \{x \in L \mid xv = 0 \text{ for all } v \in V\}$ and also an E-module if the action of L' is trivial on V. So in this sense we can extend all above results for $\mathfrak{g}$ to their quotients and central extensions.*

Now we turn to the study of vertex algebras and modules over vertex algebras; we refer the reader to [6] for relevant background. Associate each V^i for $i = 0, 1, \cdots, n$ a formal series $V^i(z)$. Suppose that these $V^i(z)$ are local, i.e., $(z_1 - z_2)^k [V^i(z_1), V^j(z_2)] = 0$ for some fixed positive integer k and $i, j = 0, 1, \cdots, n$. Then there is a vertex algebra $V_{\mathfrak{g}}$ (might not be a vertex operator algebra) associated to $\mathfrak{g}$ (see [5]). And in what follows we only consider Lie algebras $\mathfrak{g}$ of this case and we identify $(V^i)_k$ for $i = 0, 1, \cdots, n$ and $k \in \frac{1}{2}\mathbb{Z}$ with subspaces of $V_{\mathfrak{g}}$ in an obvious way.

Proposition 3.4. [6] *There is one-to-one correspondence between the set of irreducible modules over the vertex algebra $V_{\mathfrak{g}}$ and the set of irreducible restricted $\mathfrak{g}$-modules.*

As a consequence of Propositions 3.2 and 3.4 we have the following result.

THEOREM 3.5. *Let S be an irreducible module over the vertex algebra $V_{\mathfrak{g}}$ such that*

$$\sum\nolimits_{q\in\mathbb{Z}}(1-\delta_{[(V^0)_{-q},(V^n)_q],0})\mathrm{Ann}_S[(V^0)_{-q},(V^n)_q]=0$$

$$\text{and}\quad \sum\nolimits_{1\le i\le n-1}\sum\nolimits_{q\in\frac{1}{2}\mathbb{Z}_+}(1-\delta_{[(V^i)_{-q},(V^i)_q],0})\mathrm{Ann}_S[(V^i)_{-q},(V^i)_q]=0.$$

Then there exist $\underline{d}\in\mathcal{S}$ and an irreducible $\mathfrak{g}_{\underline{d}}$-module M satisfying the conditions in Theorem 2.2 *such that $S\cong\mathrm{Ind}(M)$.*

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