# BIFURCATION SETS OF REAL POLYNOMIAL FUNCTIONS OF TWO VARIABLES AND NEWTON POLYGONS

MASAHARU ISHIKAWA†, TAT-THANG NGUYEN‡, AND TIẾN-SƠN PHẠM*

Abstract. In this paper, we determine the bifurcation set of a real polynomial function of two variables for non-degenerate case in the sense of Newton polygons by using a toric compactification. We also count the number of singular phenomena at infinity, called "cleaving" and "vanishing", in the same setting. Finally, we give an upper bound of the number of atypical values at infinity in terms of its Newton polygon. To obtain the upper bound, we apply toric modifications to the singularities at infinity successively.

## 1. Introduction

Let $f\colon \mathbb{K}^2 \to \mathbb{K}$ be a polynomial function, where $\mathbb{K}$ is either $\mathbb{C}$ or $\mathbb{R}$. It is well-known that there exists a finite set $B \subset \mathbb{K}$ such that $f\colon \mathbb{K}^2 \setminus f^{-1}(B) \to \mathbb{K} \setminus B$ is a $C^\infty$ locally trivial fibration. The smallest one among such finite sets is called the *bifurcation set*, which we denote by $B_f$. Let $\Sigma_f$ denote the set of critical values of $f$. A singular phenomenon at infinity can be formulated as follows. We say that $f$ is *trivial at infinity* at $c \in \mathbb{K}$ if there exist a small open disc $D$ centered at $c$ and a compact set $K \subset \mathbb{K}^2$ such that the restriction of $f$ to $f^{-1}(D) \setminus K$ is a $C^\infty$ trivial fibration. Otherwise, we say that $f$ has a singularity at infinity at $c \in \mathbb{K}$ and $c$ is called an *atypical value of $f$ at infinity*. We denote the set of atypical values at infinity by $B_{\infty,f}$. Obviously, $B_f = B_{\infty,f} \cup \Sigma_f$.

There are many studies aiming to determine atypical values of polynomial maps at infinity. In the case $\mathbb{K} = \mathbb{C}$, the results of Suzuki [26], Hà and Lê [9] and Hà and Nguyên [10] are known to be pioneering works in these studies, where geometrical and topological characterizations of the atypical values at infinity are given. For more

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details, we refer the reader to the survey [13] and the recent papers [11, 18] with the references therein.

We now assume that $\mathbb{K} = \mathbb{R}$. Tibăr and Zaharia [27] characterized atypical non-singular fibers of real polynomial functions from real smooth algebraic surfaces due to their asymptotic behaviour at infinity. For real polynomial functions of two variables, in [5] Coste and de la Puente gave another characterization of the atypical values at infinity by using "clusters" and, in particular, they provided an algorithmic method to determine them effectively. See [6, 12, 17] for further studies related to this topic.

In this paper, we study atypical values at infinity of real polynomial functions of two variables using Newton polygons, associated toric compactifications and successive toric modifications. These techniques were used by the first author in [14] for determining the bifurcation sets of complex polynomial functions algorithmically.

To state our results, we prepare some terminologies. Set $f(x,y) = \sum_{(m,n)} a_{m,n} x^m y^n$, where $m, n \geq 0$. Let $\Delta(f)$ be the convex hull of the integral points $(m,n) \in \mathbb{R}^2$ with $a_{m,n} \neq 0$. Remark that, following [19], it is usual to study the convex hull of these integral points and the origin $(0,0)$. In this paper, we need to study the above different polytope to get a necessary and sufficient condition in Theorem 1.1 below. A vector $P = {}^t(p,q) \neq (0,0)$ consisting of coprime integers $p$ and $q$ is called a *primitive covector*. For a given $P$, let $d(P;f)$ denote the minimal value of the linear function $pX + qY$ for $(X,Y) \in \Delta(f)$. Set $\Delta(P;f) := \{(X,Y) \in \Delta(f) \mid pX + qY = d(P;f)\}$, which is called a *face* of $\Delta(f)$ if $\dim \Delta(P;f) = 1$. The partial sum $f_P(x,y) := \sum_{(m,n)\in\Delta(P;f)} a_{m,n} x^m y^n$ is called the *boundary function* for the covector $P$. If $\Delta(P;f)$ is a face then it is called the *face function*. Let $\Gamma_\infty^+(f)$ (resp. $\Gamma_\infty^0(f)$, $\Gamma_\infty^-(f)$) denote the set of faces $\Delta(P;f)$ of $f$ such that $P = {}^t(p,q)$ satisfies either $p < 0$ or $q < 0$ and satisfies $d(P;f) > 0$ (resp. $d(P;f) = 0$, $d(P;f) < 0$). For a set $\Gamma(f)$ of faces of $\Delta(f)$, we say that $f$ is *non-degenerate* on $\Gamma(f)$ if the system of equations $\frac{\partial f_P}{\partial x} = \frac{\partial f_P}{\partial y} = 0$ has no solutions in $(\mathbb{R} \setminus \{0\})^2$ for any face $\Delta(P;f)$ in $\Gamma(f)$.

A face $\Delta(P;f)$ in $\Gamma_\infty^0(f)$ is called a *bad face*. For a bad face $\Delta(P;f)$, let $b_P(t)$ denote the polynomial of one variable $t$ defined by

$$f_P(x,y) = b_P(t(x,y)), \quad t(x,y) = x^{|q|} y^{|p|},$$

where $P = {}^t(p,q)$. We say that $f_P$ is *Morse* if $b_P(t)$ is a Morse function on $\mathbb{R} \setminus \{0\}$ (i.e., it has only non-degenerate critical points on $\mathbb{R} \setminus \{0\}$) and $f$ is *Morse on* $\Gamma_\infty^0(f)$ if $f_P$ is Morse for any bad face $\Delta(P;f) \in \Gamma_\infty^0(f)$.

The following theorem can be seen as a real counterpart of the complex result derived by Némethi and Zaharia [22, Proposition 6] (see also [2, Lemma 8] and [28]).

**Theorem 1.1.** *Let $f(x,y)$ be a real polynomial map with $f(0,0)=0$ that is non-degenerate on $\Gamma^+_\infty(f)\cup\Gamma^-_\infty(f)$ and Morse on $\Gamma^0_\infty(f)$. Then, $c\in B_f$ if and only if one of the following holds:*

(i) $c\in\Sigma_f$;

(ii) $c=0$ *and there exists* $\mathit{\Delta}(P;f)\in\Gamma^+_\infty(f)$ *such that* $f_P(x,y)=0$ *has a solution in* $(\mathbb{R}\setminus\{0\})^2$;

(iii) $c$ *is a critical value of* $b_P|_{\mathbb{R}\setminus\{0\}}$ *for a bad face* $\mathit{\Delta}(P;f)$.

Remark that the assumptions in Theorem 1.1 are satisfied for generic choice of coefficients of $f$.

It is worth mentioning that in the setting of (real, complex or mixed) polynomial maps $f$ of more than two variables, the results established by Némethi and Zaharia [22, Theorem 1], Chen and Tibăr [3, Theorem 1.1], and Chen, Dias, Takeuchi and Tibăr [4, Theorem 1.1 and Corollary 1.2] only give necessary conditions (in terms of the Newton polyhedron of $f$) for a value $c$ to be an element in the bifurcation set of $f$.

Next we count the number of atypical values at infinity under the assumptions in Theorem 1.1. It is known in [27, 5] that an atypical value $c\in B_{\infty,f}$ is characterized by the existence of a cleaving or vanishing family whose limit is $f=c$. The precise definitions of these families are given in Section 2. In the next theorem, we determine the number of cleaving and vanishing families. A face function $f_P(x,y)$ has a factorization of the form

$$f_P(x,y)=Ax^\alpha y^\beta\prod_{j=1}^{r}(x^q-A_jy^p)^{\nu_j},$$

where $P={}^t(p,q)$, $\alpha,\beta\in\mathbb{Z}$ and $A,A_j\in\mathbb{C}\setminus\{0\}$ with $A_i\neq A_j$ for $i\neq j$. If $x^q-A_jy^p=0$ has a solution in $(\mathbb{R}\setminus\{0\})^2$ then we say that $(x^q-A_jy^p)^{\nu_j}$ is a *factor with real solution of multiplicity $\nu_j$. We denote the number of such factors by $r(P;f)$, that is, we set $r(P;f)=r$.* For each bad face $\mathit{\Delta}(P;f)\in\Gamma^0_\infty(f)$, let $r'(P;f)$ denote the number of non-zero real roots of $\frac{db_P}{dt}(t)=0$. Let $\mathrm{cleav}(f)$ and $\mathrm{vanish}(f)$ denote the numbers of cleaving families and vanishing families of $f$, respectively.

**Theorem 1.2.** *Suppose that $f$ satisfies the conditions in Theorem 1.1. Suppose further that $f$ has only isolated singularities. Then*

$$\mathrm{cleav}(f)+\mathrm{vanish}(f)=2(R^++R^0)\quad\textit{and}\quad 0\le\mathrm{vanish}(f)\le 2R^0,$$

*where*

$$R^+=\sum_{\mathit{\Delta}(P;f)\in\Gamma^+_\infty(f)}r(P;f),\quad R^0=\sum_{\mathit{\Delta}(P;f)\in\Gamma^0_\infty(f)}r'(P;f).$$

*In particular, if there is no bad face then there is no vanishing family.*

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Note that $r(P;f)$ is less than the number of lattice points on $\Delta(P;f)$ if $\Delta(P;f) \in \Gamma_\infty^+(f)$ and $r'(P;f)$ is also if $\Delta(P;f) \in \Gamma_\infty^0(f)$. We remark that we cannot expect the equality in Theorem 1.2 if we relax the conditions as degenerate case or non-Morse case.

Even if $f$ does not satisfy the assumptions in Theorem 1.1, by applying toric modifications successively, we can obtain an upper bound of the number of elements in $B_{\infty,f}$. For each face $\Delta(R_i;f) \in \Gamma_\infty^-(f)$, set $\mu(R_i;f) = \sum_{j=1}^{\eta}(\mu_j - 1)$, where $\mu_1, \dots, \mu_\eta$ are the multiplicities of factors of $f_{R_i}(x,y)$ with real solution. Note that $\mu(R_i;f)$ is less than the number of lattice points on $\Delta(R_i;f)$. Let $R^+$ and $R^0$ be the integers defined in Theorem 1.2. Let $|B_{\infty,f}|$ and $|\Sigma_f|$ denote the numbers of elements in $B_{\infty,f}$ and $\Sigma_f$, respectively.

**Theorem 1.3.** *The following inequality holds:*

$$|B_{\infty,f}| \leq \epsilon(R^+) + R^0 + \sum_{\Delta(P;f)\in\Gamma_\infty^-(f)} \mu(P;f),$$

*where* $\epsilon(R^+) = 0$ *if* $R^+ = 0$ *and* $\epsilon(R^+) = 1$ *if* $R^+ > 0$.

A similar result for complex polynomial functions had been obtained in [21]; see also [7, 8, 14, 15, 16, 20] for related results.

The paper is organized as follows. In Section 2, we introduce the definition of an admissible toric compactification with respect to primitive covectors, and give the definitions of cleaving and vanishing families and their equivalence relations. In the subsequent three sections, we give the proofs of Theorem 1.1, Theorem 1.2 and Theorem 1.3. Two examples are given in the end of Section 3. The definition of an admissible toric modification is given in the beginning of Section 5, before giving the proof of Theorem 1.3.

## 2. Preliminaries

2.1. **Toric compactification.** We first recall some definitions given in [19] which will be used in this work. Set $f(x,y) = \sum_{(m,n)} a_{m,n}x^m y^n$, where $m,n \geq 0$. A boundary function $f_P(x,y)$ is said to be *non-degenerate* if the system of equations $\frac{\partial f_P}{\partial x} = \frac{\partial f_P}{\partial y} = 0$ has no solutions in $(\mathbb{R}\setminus\{0\})^2$. Otherwise it is said to be *degenerate*. The polynomial $f$ is called *convenient* if $\Delta(f)$ intersects both positive axes.

Let $\Gamma_\infty^+(f)$ (resp. $\Gamma_\infty^0(f)$, $\Gamma_\infty^-(f)$) denote the set of faces $\Delta(P;f)$ of $f$ such that $P = {}^t(p,q)$ satisfies either $p < 0$ or $q < 0$ and satisfies $d(P;f) > 0$ (resp. $d(P;f) = 0$, $d(P;f) < 0$). For a set $\Gamma(f)$ of faces of $\Delta(f)$, we say that $f$ is *non-degenerate* on $\Gamma(f)$ if $f_P$ is non-degenerate for any face $\Delta(P;f)$ in $\Gamma(f)$. Note that the non-degeneracy condition in [19] corresponds to the non-degeneracy on $\Gamma_\infty^-(f)$ in this paper.

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Let $f:\mathbb{R}^2\to\mathbb{R}$ be a polynomial function. We give the definition of an admissible toric compactification with respect to the Newton polygon $\Delta(f)$. Let $Q_i={}^t(p_i,q_i)$, $i=1,2,\dots,n$, be primitive covectors that satisfy the following:

(1) either $p_i$ or $q_i$ is negative;
(2) $\Delta(Q_i;f)$ is a face of $\Delta(f)$; and
(3) the indices are assigned in the counter-clockwise orientation.

Let $R_i={}^t(r_i,s_i)$, $i=1,2,\dots,m$, be primitive covectors that satisfy the following:

(1) $R_1={}^t(1,0)$, $R_2={}^t(0,1)$;
(2) either $r_i$ or $s_i$ is negative for each $R_i$, $i=3,\dots,m$;
(3) $\{Q_i\}$ is contained in $\{R_3,\dots,R_m\}$;
(4) the indices are assigned in the counter-clockwise orientation; and
(5) the determinants of the matrices $(R_i,R_{i+1})$, $i=1,\dots,m-1$, and $(R_m,R_1)$ are 1.

For convenience, we set $R_{m+1}=R_1$. For each $\mathrm{Cone}(R_i,R_{i+1})$, $i=2,\dots,m$, an affine coordinate chart $(u_i,v_i)\in\mathbb{R}^2$ is defined by the coordinate transformation

$$x=u_i^{r_i}v_i^{r_{i+1}},\quad y=u_i^{s_i}v_i^{s_{i+1}}.$$

Then a smooth toric variety $X$ is obtained by gluing these coordinate charts, which is described as

$$X=(\mathbb{R}\setminus\{0\})^2\cup\left(\bigcup_{i=1}^{m}E(R_i)\right)=\mathbb{R}^2\cup\left(\bigcup_{i=3}^{m}E(R_i)\right),$$

where $E(R_i)$ is the exceptional divisor corresponding to the covector $R_i$. The real variety $X$ is called the *admissible toric compactification* of $\mathbb{R}^2$ associated with $\{R_1,\dots,R_m\}$.

Let $U_i$ denote the local chart with coordinates $(u_i,v_i)$ corresponding to $\mathrm{Cone}(R_i,R_{i+1})$ for $i=2,\dots,m$. On $U_i$, the function $f$ has the form

$$f(u_i,v_i)=u_i^{d(R_i;f)}v_i^{d(R_{i+1};f)}(g_i(v_i)+u_ih_i(u_i,v_i)),\tag{2.1}$$

where $g_i$ is a polynomial of one variable $v_i$ and $h_i$ is a polynomial of two variables $(u_i,v_i)$. The divisor $E(R_i)$ in this chart is given by $u_i=0$.

For an algebraic curve $C$ in $\mathbb{R}^2$, its closure in $X$ is called the *strict transform* of $C$. Set $f(x,y)=x^\alpha y^\beta F(x,y)$, where $\alpha$ (resp. $\beta$) is a non-negative integer such that $x$ (resp. $y$) does not divide $F$. Let $V_f$, $V_F$, $V_1$ and $V_2$ denote the strict transforms of $f(x,y)=0$, $F(x,y)=0$, $x=0$ and $y=0$ in $X$, respectively. In particular, $V_f=V_F\cup V_1\cup V_2$. Note that $V_1$ (resp. $V_2$) is empty if $\alpha$ (resp. $\beta$) is 0.

**Lemma 2.1.** (1) *For $i=3,\dots,m$, $V_F$ does not intersect $E(R_i)$ unless $R_i\in\{Q_1,\dots,Q_n\}$.*
(2) *For $i=3,\dots,m-1$, $V_F$ does not intersect $E(R_i)\cap E(R_{i+1})$.*

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(3) *If* $\alpha \geq 1$ *(resp.* $\beta \geq 1$*) then* $V_1$ *(resp.* $V_2$*) intersects* $E(R_m)$ *(resp.* $E(R_3)$*) transversely.*

*Proof.* All the assertions in this lemma are well-known. For instance, the explanation in [24] restricted to the two variable case works for real polynomial maps also. We only check the assertion (3) to confirm the usage of indices. The curves $V_1$ and $E(R_m)$ are given on $U_m$ as $\{(u_m, v_m) \in U_m \mid v_m = 0\}$ and $\{(u_m, v_m) \in U_m \mid u_m = 0\}$, respectively. Hence they intersect transversely. Similarly, $V_2$ and $E(R_3)$ are given on $U_2$ as $\{(u_2, v_2) \in U_2 \mid u_2 = 0\}$ and $\{(u_2, v_2) \in U_2 \mid v_2 = 0\}$, respectively. Hence they intersect transversely. $\square$

**Lemma 2.2.** *Let $i$ be an index in $\{3, \ldots, m\}$. Suppose that $R_i$ satisfies one of the following:*

(i) $\Delta(R_i; f)$ *is not a bad face and* $f_{R_i}$ *is non-degenerate.*
(ii) $\Delta(R_i; f)$ *is a bad face and* $b_{R_i}(t) = 0$ *has no non-zero real multiple root.*

*Then there is a one-to-one correspondence between the intersection points of* $E(R_i)$ *and* $V_F$ *and the non-zero real roots of* $g_i(v_i) = 0$*. Moreover, they intersect transversely at these points.*

*Proof.* The divisor $E(R_i)$ is given on $U_i$ as $\{(u_i, v_i) \in U_i \mid u_i = 0\}$. On the other hand, $V_F \cap U_i$ is the set $\{(u_i, v_i) \in U_i \mid g_i(v_i) + u_i h_i(u_i, v_i) = 0\}$ with excluding isolated points on $u_i = 0$. Let $(0, s)$ be an intersection point of $u_i = 0$ and $g_i(v_i) + u_i h_i(u_i, v_i) = 0$, where $s \in \mathbb{R} \setminus \{0\}$. Since $s$ is a single root of $g_i(v_i) = 0$ in both of cases (i) and (ii), $\frac{\partial(g_i(v_i)+u_i h_i(u_i,v_i))}{\partial v_i}(0, s) \neq 0$. Hence $g_i(v_i) + u_i h_i(u_i, v_i) = 0$ is smooth at $(0, s)$, i.e, $(0, s)$ is not an isolated point, and $V_F$ intersects $u_i = 0$ transversely at $(0, s)$. $\square$

*Remark* 2.3. The assumption of non-degeneracy of $f_{R_i}$ is necessary. For example if $g_i(v_i) + u_i h_i(u_i, v_i) = (v_i - 1)^2 + u_i^2$ then the intersection point $(0, 1)$ with $u_i = 0$ is isolated.

2.2. **Cleaving and vanishing at infinity.** Let $N$ be a small, compact tubular neighborhood of $\cup_{i=3}^m E(R_i)$ in $X$.

**Definition 2.4.** A continuous family $\{(\gamma_t, \delta_t, c_t)\}_{t \in (0,1)}$ of triples of a proper arc $\gamma_t$ in $N \setminus \cup_{i=3}^m E(R_i)$ whose endpoints lie on the boundary $\partial N$, a closed, connected subset $\delta_t \subset \gamma_t$, which is either a closed arc or a point, and a real number $c_t$ is called a *cleaving family* of $f$ if it satisfies the following:

(1) $\gamma_t \subset f^{-1}(c_t)$; and
(2) $c := \lim_{t \to 0} c_t$ satisfies $|c| < \infty$ and $\delta := \lim_{t \to 0} \delta_t \subset \cup_{i=3}^m E(R_i)$.

If there exists a cleaving family with limit $f = c$, then we say that the curve $f = c$ is *cleaving at infinity*.

Note that the definition of a cleaving family depends on the compactification $X$ of $\mathbb{R}^2$, though the existence of a cleaving family and its value $c$ do not. In this sense, the statement "$f = c$ is cleaving at infinity" does not depend on the choice of $X$. This definition coincides with that in [5, p.30] if we state it without compactification. Obviously, if $f = c$ is cleaving at infinity then $c \in B_f$.

**Definition 2.5.** A continuous family $\{(C_t, c_t)\}_{t\in(0,1)}$ of pairs of a real number $c_t$ and a connected component $C_t$ of $f = c_t$ in $\mathbb{R}^2$ is called a *vanishing family* if it satisfies the following:

(1) $C_t \subset N \setminus \cup_{i=3}^m E(R_i)$; and
(2) $c := \lim_{t\to 0} c_t$ satisfies $|c| < \infty$ and $C := \lim_{t\to 0} C_t \subset \cup_{i=3}^m E(R_i)$.

If there exists a vanishing family with limit $f = c$, then we say that the curve $f = c$ is *vanishing at infinity*.

Note that the definition of a vanishing family depends on the compactification $X$ of $\mathbb{R}^2$, though the existence of a vanishing family and its value $c$ do not. In this sense, the statement "$f = c$ is vanishing at infinity" does not depend on the choice of $X$. In [27], the value $c \in B_f$ is characterized by the first Betti numbers and Euler characteristics of fibers and "vanishing" and "splitting" phenomena. The definition of a vanishing family coincides with the "vanishing" in [27] if we state it without compactification. Obviously, if $f = c$ is vanishing at infinity then $c \in B_f$.

**Lemma 2.6** ([27, 5], see p.31 in [5]). *Suppose that $c \in B_f$. Then one of the following holds:*

(i) $c \in \Sigma_f$;
(ii) *$f = c$ is cleaving at infinity;*
(iii) *$f = c$ is vanishing at infinity.*

Since the definitions of these families depend on the choice of the compact neighborhoods $N$, the parameter $t$ and the subsets $\{\delta_t\}_{t\in(0,1)}$, we need to introduce an equivalence relation to remove these ambiguities. The equivalence relation is defined as follows.

**Definition 2.7.** (1) Two cleaving families $\{(\gamma_t, \delta_t, c_t)\}_{t\in(0,1)}$ and $\{(\gamma'_t, \delta'_t, c'_t)\}_{t'\in(0,1)}$, defined in compact tubular neighborhoods $N$ and $N'$ of $\cup_{i=3}^m E(R_i)$ respectively, are *equivalent* if there exists $\varepsilon > 0$ such that for any $s \in (0, \varepsilon)$ there exists $s' \in (0, 1)$ such that $c_s = c'_{s'}$ and $\gamma_s \cap \gamma'_{s'} \neq \emptyset$.
(2) Two vanishing families $\{(C_t, c_t)\}_{t\in(0,1)}$ and $\{(C'_t, c'_t)\}_{t'\in(0,1)}$, defined in compact tubular neighborhoods $N$ and $N'$ of $\cup_{i=3}^m E(R_i)$ respectively, are *equivalent* if there exists $\varepsilon > 0$ such that for any $s \in (0, \varepsilon)$ there exists $s' \in (0, 1)$ such that $c_s = c'_{s'}$ and $C_s = C'_{s'}$.

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Later, we will count the numbers of cleaving and vanishing families up to these equivalence relations.

## 3. Proof of Theorem 1.1 and examples

Theorem 1.1 will follow from the next proposition.

**Proposition 3.1.** *Suppose that $f(x,y)$ is non-degenerate on $\Gamma^+_\infty(f) \cup \Gamma^-_\infty(f)$ and that $b_P(t) = 0$ has no non-zero real multiple root for any bad face $\Delta(P;f)$. Then, $0 \in B_f$ if and only if either* (i) $0 \in \Sigma_f$ *or* (ii) *there exists $\Delta(P;f) \in \Gamma^+_\infty(f)$ such that $f_P(x,y) = 0$ has a solution in $(\mathbb{R} \setminus \{0\})^2$. Moreover, if it is in case* (ii) *then $f = 0$ is cleaving at infinity.*

We divide the proof into three lemmas. Note that the proofs of the first two lemmas for complex polynomials are written, for example, in [23, 25], which are based on the Curve Selection Lemma at infinity, and their arguments work in real case also. We here give different proofs based on the toric compactification $X$. Let $N$ denote a small, compact tubular neighborhood of $\cup_{i=3}^m E(R_i)$ in $X$.

**Lemma 3.2.** *Suppose that $f$ is convenient and non-degenerate on $\Gamma^-_\infty(f)$. Then $0 \in B_f$ if and only if $0 \in \Sigma_f$.*

*Proof.* Since $\Sigma_f \subset B_f$, it is enough to show that if $0 \in B_f$ then $0 \in \Sigma_f$. Assume that $0 \notin \Sigma_f$. By Lemma 2.6, it is enough to check that $f = 0$ is not cleaving and not vanishing at infinity. Note that there is no bad face since $f$ is convenient, and $V_f$ intersects $\cup_{i=3}^m E(R_i)$ transversely by Lemma 2.1 and 2.2.

We first prove that $f = 0$ is not cleaving at infinity. Assume that $f = 0$ is cleaving at infinity. Then $V_f$ must intersect $E(R_i)$ for some $i = 3, \dots, m$. Let $p$ be an intersection point of $V_f$ and $E(R_i)$. Note that $p \in \delta$, where $\delta$ is the limit of closed, connected sets $\{\delta_t\}_{t \in (0,1)}$ in Definition 2.4. Since $f$ is convenient, we have $d(R_i; f) < 0$. Then any nearby fiber of $V_f$ in $N$ near $p$ is a simple arc connecting a point near $V_f \cap \partial N$ and the point $p$, see Figure 1. On $N \setminus \bigcup_{i=3}^m E(R_i) \subset \mathbb{R}^2$, we can construct a non-zero smooth vector field that gives a $C^\infty$ locally trivial fibration at infinity with central fiber $V_f \cap (N \setminus \bigcup_{i=3}^m E(R_i))$ by choosing a family of smooth circles going to the infinity and being transverse to the nearby fibers suitably. Hence the fibration at infinity near $p$ is trivial, which contradicts the assumption that $f = 0$ is cleaving at $p$.

Next we check that $f = 0$ is not vanishing at infinity. Since $d(R_i; f) < 0$ for any $i = 3, \dots, m$, if a vanishing family exists then there exists a sequence $\{(t_j, s)\}_{j \in \mathbb{N}}$ of points on $U_i$ for some $i \in \{3, \dots, m\}$ such that $\lim_{j \to \infty} t_j = 0$, $\lim_{j \to \infty} f|_{U_i}(t_j, s) = 0$ and $g_i(s) \neq 0$. However, from (2.1), we see that $|\lim_{j \to \infty} f|_{U_i}(t_j, s)| = \infty$, which is a contradiction. Thus $0 \notin B_f$ by Lemma 2.6. □

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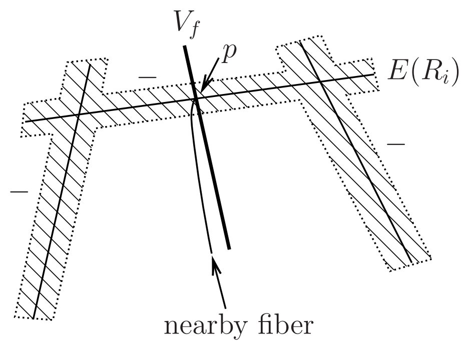


FIGURE 1. The triviality of the fibration at infinity in the case $d(R_i; f) < 0$. The sign $-$ means that $E(R_i)$ satisfies $d(R_i; f) < 0$. The painted region is the neighborhood $N$.

**Lemma 3.3.** *Suppose that $f(0,0) \neq 0$ and $f$ is not convenient. Suppose further that $f$ is non-degenerate on $\Gamma^-_\infty(f)$ and $b_P(t) = 0$ has no non-zero real multiple root for any bad face $\Delta(P; f)$. Then $0 \in B_f$ if and only if $0 \in \Sigma_f$.*

*Proof.* We assume $0 \notin \Sigma_f$ and prove $0 \notin B_f$. By Lemma 2.6, it is enough to check that $f = 0$ is not cleaving and not vanishing at infinity.

First we show that $f = 0$ is not cleaving at infinity. Let $R_i$ be a covector such that $E(R_i)$ intersects $V_f$, where $i = 3, \ldots, m$. If $d(R_i; f) = 0$ then the condition (ii) in Lemma 2.2 holds by the assumption. By Lemma 2.2, all nearby fibers of $f = 0$ near $E(R_i)$ are transverse to $E(R_i)$, see Figure 2. Hence the fibration at the infinity is trivial. The triviality also holds if $d(R_i; f) < 0$ as we had seen in the proof of Lemma 3.2.

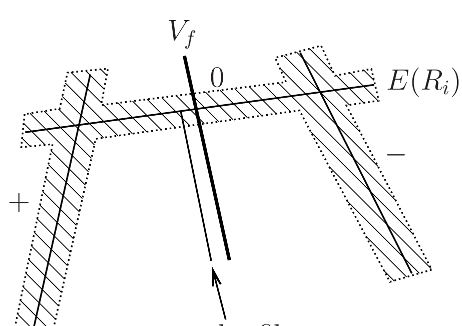


FIGURE 2. The triviality of the fibration at infinity in the case $d(R_i; f) = 0$. The sign 0 means that $E(R_i)$ satisfies $d(R_i; f) = 0$.

Next we check that $f = 0$ is not vanishing at infinity. If there is a vanishing family whose limit intersects $E(R_i)$ with $d(R_i; f) = 0$ then, since $f(u_i, v_i)$ in (2.1) has no factor $u_i^{-1}$, the limit in $U_i$ should be given by $f(u_i, v_i) = 0$, which is nothing but $V_f \cap U_i$. If $g_i(v_i) = 0$ in (2.1) has a non-zero real solution then, since it is not a multiple root, the limit cannot be contained in $E(R_i)$ with $d(R_i; f) = 0$. If $g_i(v_i) = 0$ has no non-zero real solution then $V_f \cap E(R_i) \cap U_i = \emptyset$. Hence, in either case, there is no vanishing family. The limit cannot intersect $E(R_i)$ with $d(R_i; f) < 0$ by the same reason as we had seen in the proof of Lemma 3.2. This completes the proof. □

Finally, we study the case where either $x|f$ or $y|f$.

**Lemma 3.4.** *Suppose that $f$ is non-degenerate on $\Gamma^+_\infty(f) \cup \Gamma^-_\infty(f)$ and that $b_P(t) = 0$ has no non-zero real multiple root for any bad face $\Delta(P; f)$. Suppose further that either $x|f$ or $y|f$. Then, $0 \in B_f$ if and only if either* (i) *$0 \in \Sigma_f$ or* (ii) *there exists $\Delta(P; f) \in \Gamma^+_\infty(f)$ such that $f_P(x, y) = 0$ has a solution in $(\mathbb{R} \setminus \{0\})^2$. Moreover, if it is in case* (ii) *then $f = 0$ is cleaving at infinity.*

*Proof.* Set $f(x, y) = x^\alpha y^\beta F(x, y)$ with either $\alpha > 0$ or $\beta > 0$. For convenience, we choose $R_3, \ldots, R_m$ such that $d(R_m; f) > 0$ and $f_{R_m}$ is a monomial if $\alpha > 0$ and that $d(R_3; f) > 0$ and $f_{R_3}$ is a monomial if $\beta > 0$. Since the indices of the covectors $\{R_1, \ldots, R_m\}$ are assigned in the counter-clockwise orientation, if $\alpha > 0$ then there exists an index $k$ such that $d(R_k; f) \le 0$ and $d(R_i; f) > 0$ for $i > k$. Similarly, if $\beta > 0$ then there exists an index $k'$ such that $d(R_i; f) > 0$ for $3 \le i \le k'$ and $d(R_{k'+1}; f) \le 0$. Note that $E(R_k) \cap E(R_{k'}) = \emptyset$ when $\alpha > 0$ and $\beta > 0$.

Suppose that there exists $\Delta(R_i; f) \in \Gamma^+_\infty(f)$ with $i \ge k$ such that $f_{R_i}(x, y) = 0$ has a solution in $(\mathbb{R} \setminus \{0\})^2$. Let $i_0$ be the largest index such that $d(R_{i_0}; f) > 0$ and $f_{R_i}(x, y)$ has a solution in $(\mathbb{R} \setminus \{0\})^2$. We consider the real toric variety $X$ obtained by the admissible toric compactification of $\mathbb{R}^2$ associated with $\{R_1, \ldots, R_m\}$. Let $\gamma$ be a branch of $V_f$ in $N$ intersecting $E(R_{i_0})$ and being nearest to $E(R_{i_0+1})$. By Lemma 2.1 (3), $V_1 \cap N$ is a short arc in $N$ intersecting $E(R_m)$ transversely, see Figure 3. Thus we can find a cleaving family between $V_1$ and $\gamma$ in $N$. If there exists $\Delta(R_i; f) \in \Gamma^+_\infty(f)$ with $3 \le i \le k'$ such that $f_{R_i}(x, y) = 0$ has a solution in $(\mathbb{R} \setminus \{0\})^2$ then there exists a cleaving family by the same reason. Thus, in either case, we have $0 \in B_f$.

Now we prove the converse. Suppose that there does not exist $\Delta(P; f) \in \Gamma^+_\infty(f)$ such that $f_P(x, y) = 0$ has a solution in $(\mathbb{R} \setminus \{0\})^2$. By Lemma 2.6, it is enough to check that $f = 0$ is not cleaving and not vanishing at infinity. We first check that $f = 0$ is not cleaving. For any $\Delta(R_i; f) \in \Gamma^+_\infty(f)$, since $f_{R_i}(x, y) = 0$ has no solution in $(\mathbb{R} \setminus \{0\})^2$, $V_F$ does not intersect $E(R_i)$ by Lemma 2.2. If $V_F$ intersects $E(R_i)$ with $d(R_i; f) = 0$ then by the same argument as in the proof of Lemma 3.3, the fibration at infinity near $E(R_i)$ is trivial, see Figure 2. The triviality also holds in the case where $V_F$ intersects $E(R_i)$ with $d(R_i; f) < 0$ as we have seen in the proof of Lemma 3.2, see Figure 1. This shows that there is no cleaving family near the intersection of $V_F$ with $\cup_{i=3}^m E(R_i)$.

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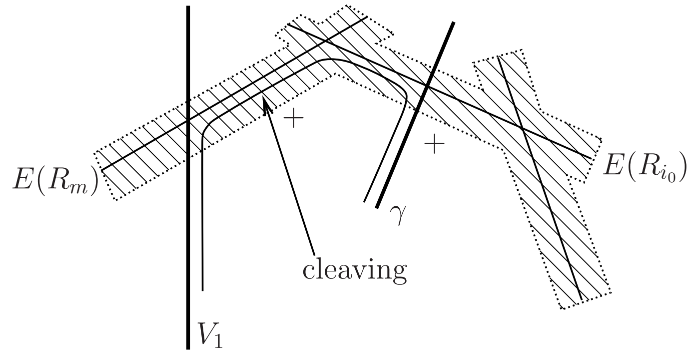


FIGURE 3. A cleaving family between $V_1$ and $\gamma$. The sign $+$ means that $E(R_i)$ satisfies $d(R_i; f) > 0$.

By Lemma 2.1, it remains to show that there is no cleaving family near the intersection of $V_1$ with $E(R_m)$ and near the intersection of $V_2$ with $E(R_3)$. We only check the former case. The latter case is proved similarly. On $U_m$, $f$ has the form

$$f(u_m, v_m) = u_m^{d(R_m;f)} v_m^{d(R_1;f)} (g_m(v_m) + u_m h_m(u_m, v_m)),$$

where $V_f \cap U_m$ corresponds to the curve $v_m^{d(R_1;f)}(g_m(v_m) + u_m h_m(u_m, v_m)) = 0$. Since the covectors $R_3, \dots, R_m$ are chosen such that $f_{R_m}$ is a monomial, $V_f$ intersects $E(R_m)$ only at $E(R_m) \cap V_1$ by Lemma 2.1. If $\Gamma^0_\infty(f) \neq \emptyset$ then a nearby fiber of $f$ passing near $V_1$ intersects $\cup_{i=3}^m E(R_i)$ at a point near $E(R_{k-1}) \cap E(R_k)$ as shown on the left in Figure 4. We can show that the fibration is trivial at infinity by choosing a family of smooth circles going to the infinity and being transverse to the nearby fibers suitably as in the proof of Lemma 3.2. If $\Gamma^0_\infty(f) = \emptyset$ then nearby fibers intersect $E(R_{k-1}) \cap E(R_k)$ as shown on the right in Figure 4 since $\Delta(R_{k-1}; f) \in \Gamma^-_\infty(f)$ and $\Delta(R_k; f) \in \Gamma^+_\infty(f)$. Therefore, the fibration is again trivial at infinity. Thus $f = 0$ is not cleaving at infinity.

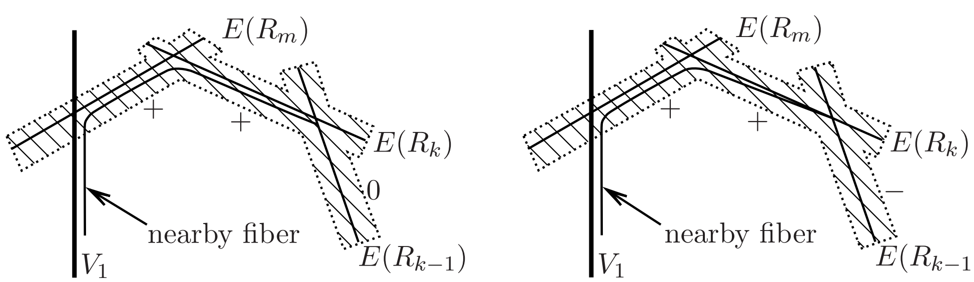


FIGURE 4. The triviality of the fibration at infinity for nearby fibers passing near $V_1$.

Next we check that $f = 0$ is not vanishing at infinity. As we explained in the proof of Lemma 3.3, a vanishing family does not exist in a neighborhood of $E(R_i)$ with $d(R_i; f) = 0$. It does not exist near $E(R_i)$ with $d(R_i; f) > 0$ also since a nearby fiber cannot stay in $N$ as we had seen in Figure 4. A vanishing family does not exist near $E(R_i)$ with $d(R_i; f) < 0$ by the same reason as we had seen in the proof of Lemma 3.2. This completes the proof. □

*Proof of Proposition 3.1.* The assertion follows from Lemmas 3.2, 3.3 and 3.4. □

*Proof of Theorem 1.1.* If it is in case (ii), we have $c = 0 \in B_f$ by Proposition 3.1. If it is in case (iii) then $f - c$ has the form

$$f(u_i, v_i) - c = v_i^{d(R_{i+1};f)}(\tilde{g}_i(v_i) + u_i h_i(u_i, v_i)),$$

where $\tilde{g}_i(v_i) = g_i(v_i) - cv_i^{-d(R_{i+1};f)}$, and there exists a non-zero real root $s$ of $\tilde{g}_i(v_i) = 0$ with multiplicity 2. That is, the strict transform $V_{f-c}$ of $f = c$ intersects $E(R_i)$ at $(u_i, v_i) = (0, s)$ with multiplicity 2. Let $U$ be a small neighborhood of $(0, s)$ in $N$ such that $U \setminus E(R_i)$ consists of two connected components, say $U'$ and $U''$. There are three cases: (1) $V_{f-c}$ intersects both of $U'$ and $U''$; (2) $V_{f-c}$ intersects one of them and does not intersect the other; (3) $V_{f-c}$ does not intersect both of $U'$ and $U''$. In case (2), there is a cleaving family and a vanishing family as shown in Figure 5. Thus we have $c \in B_f$.

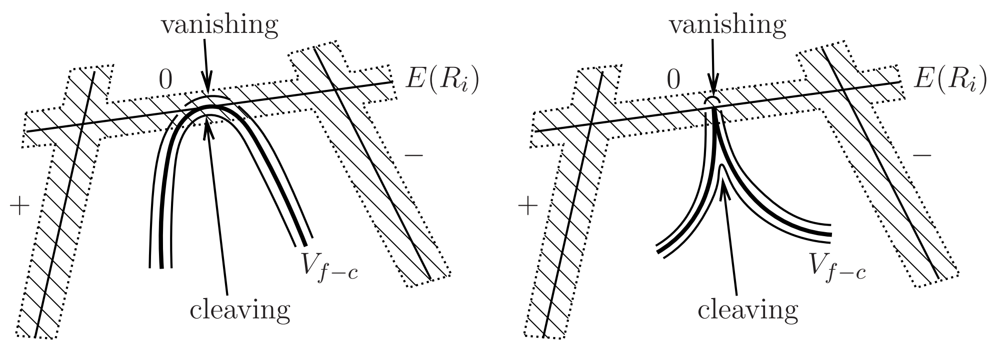


FIGURE 5. Cleaving and vanishing at infinity on $E(R_i)$ with $\Delta(R_i; f) \in \Gamma_\infty^0(f)$ (Case where $V_{f-c}$ is in one side).

In case (1), since the multiplicity is 2, $V_{f-c}$ in $U$ consists of either two curves intersecting each other transversely and also intersecting $E(R_i)$ transversely, see on the left in Figure 6, or one curve with multiplicity 2 intersecting $E(R_i)$ transversely. In the former case, $f = c$ is cleaving at infinity from both sides as shown in the figure. In the latter case, $c \in \Sigma_f$. In case (3), $f = c$ has vanishing families from both sides, see on the right in Figure 6. In any case, we have $c \in B_f$.

Conversely, if both of (ii) and (iii) are not satisfied then applying Proposition 3.1 to $f(x, y) - c$, we can conclude that $c \notin B_f$ unless $c \in \Sigma_f$. □

*Remark* 3.5. The "Morse condition" on bad faces is crucial especially in case (1) in the above proof. For a non-zero real root $s$ of $\tilde{g}_i(v_i) = 0$, let $\mu$ denote its multiplicity.

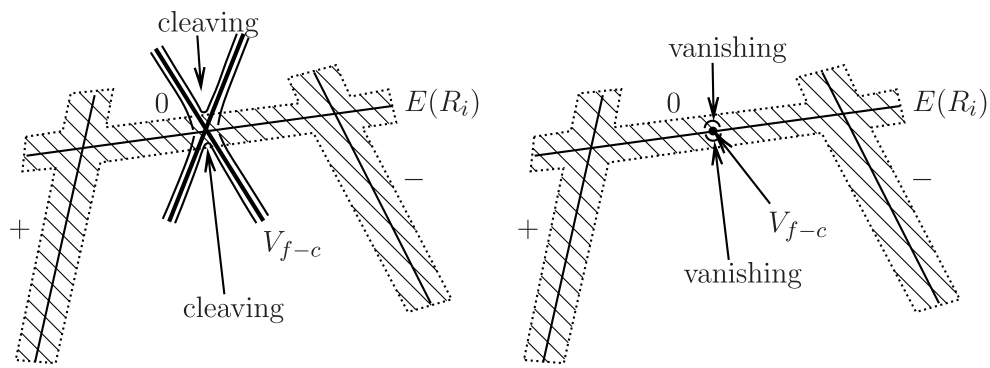


FIGURE 6. Cleaving and vanishing at infinity on $E(R_i)$ with $\Delta(R_i; f) \in \Gamma^0_\infty(f)$ (Case where $V_{f-c}$ is either in both sides or isolated).

If $\mu$ is odd then $V_{f-c}$ can be one curve being tangent to $E(R_i)$ with multiplicity $\mu$ and intersects both of $U'$ and $U''$. In this case, the fibration is trivial in this neighborhood. Hence we cannot generalize the assertion in the case where $\mu$ is odd. On the other hand, if $\mu$ is even then the assertion in Theorem 1.1 still holds by the same argument as in the proof.

**Example 3.6.** Consider the polynomial function $f(x, y) = x(1 + x^m y^{2n})$, where $m, n \geq 1$. From the Newton polygon $\Delta(f)$, there are two covectors orthogonal to the face of $\Delta(f)$, i.e.,

$$Q_1 = \begin{pmatrix} -2n \\ m \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 2n \\ -m \end{pmatrix}.$$

We can easily check that the conditions (i) and (iii) in Theorem 1.1 are not satisfied. If $c \neq 0 = f(0, 0)$ then (ii) is also not satisfied. When $c = 0$, only the covector $Q_2$ satisfies $\Delta(Q_2; \tilde{f}) \in \Gamma^+_\infty(\tilde{f})$. We can easily check that (ii) is satisfied if and only if $m$ is odd. Thus $B_f = \{0\}$ if $m$ is odd and $B_f = \emptyset$ if $m$ is even.

We here explain how the cleaving and vanishing families appear in a real toric variety in the case where $f(x, y) = x(1 + xy^2)$. Set

$$R_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad R_3 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad R_4 = Q_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix},$$
$$R_5 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad R_6 = Q_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad R_7 = R_1.$$

These primitive covectors satisfy the conditions in Section 2 and the associated admissible toric compactification $X$ becomes as shown in Figure 7. We can see from the figure that there are two cleaving families up to equivalence relation defined in Definition 2.7 and there is no vanishing family.

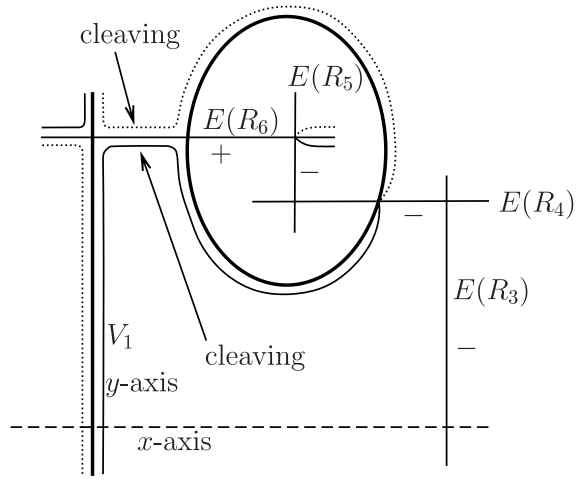


FIGURE 7. A connected component of $f = \varepsilon$ and a connected component of $f = -\varepsilon$, with sufficiently small $\varepsilon > 0$, are described. Both of them are cleaving as $\varepsilon \to 0$.

**Example 3.7.** Consider the polynomial function

$$f(x, y) = x + \frac{1}{m}x^m y^m + \frac{2a}{m+1}x^{m+1}y^{m+1} + \frac{1}{m+2}x^{m+2}y^{m+2}$$

with $m \geq 2$. It has no singular point and hence $\Sigma_f = \emptyset$.

From the Newton polygon $\Delta(f)$, the covectors orthogonal to the faces are

$$Q_1 = \begin{pmatrix} -m-2 \\ m+1 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad Q_3 = \begin{pmatrix} m \\ 1-m \end{pmatrix}.$$

Only the face $\Delta(Q_2; f)$ is a bad face. To apply Theorem 1.1, we need to assume that $b_{Q_2}(t) = t^m(\frac{1}{m} + \frac{2a}{m+1}t + \frac{1}{m+2}t^2)$ is a Morse function on $\mathbb{R} \setminus \{0\}$. The critical points are the roots of $\frac{db_{Q_2}}{dt}(t) = t^{m-1}(1 + 2at + t^2) = 0$. Thus $b_{Q_2}$ is Morse on $\mathbb{R} \setminus \{0\}$ if and only if $a \neq \pm 1$. If $-1 < a < 1$ then $b_{Q_2}$ has no critical point on $\mathbb{R} \setminus \{0\}$. If $|a| > 1$ then $t_0 = -a - \sqrt{a^2 - 1}$ and $t_1 = -a + \sqrt{a^2 - 1}$ are the critical points of $b_{Q_2}$. The face $\Delta(Q_3; f)$ is in $\Gamma^+_\infty(f)$ and $f_{Q_3}(x, y) = x + x^m y^m = 0$ has a solution $(-1, -1)$ in $(\mathbb{R} \setminus \{0\})^2$. Hence $0 \in B_f$. The bifurcation set $B_f$ is now determined for $|a| \neq 1$: $B_f = \{0, b_Q(t_0), b_Q(t_1)\}$ if $|a| > 1$ and $B_f = \{0\}$ if $|a| < 1$.

Now we explain how the cleaving and vanishing families appear in a real toric variety. Set

$$R_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; R_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \; R_3 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \; R_4 = Q_1 = \begin{pmatrix} -m-2 \\ m+1 \end{pmatrix},$$
$$R_5 = Q_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \; R_6 = Q_3 = \begin{pmatrix} m \\ 1-m \end{pmatrix},$$
$$R_{6+k} = \begin{pmatrix} m-k \\ 1-m+k \end{pmatrix} (k = 1, \dots, m-3), \; R_{m+4} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \; R_{m+5} = R_1.$$

These primitive covectors satisfy the conditions in Section 2 and the associated admissible toric compactification $X$ becomes as shown in Figure 8, which is in the case where $m = 8$ and $|a| > 1$. If $|a| < 1$ then $V_f$ does not intersect $E(R_5)$.

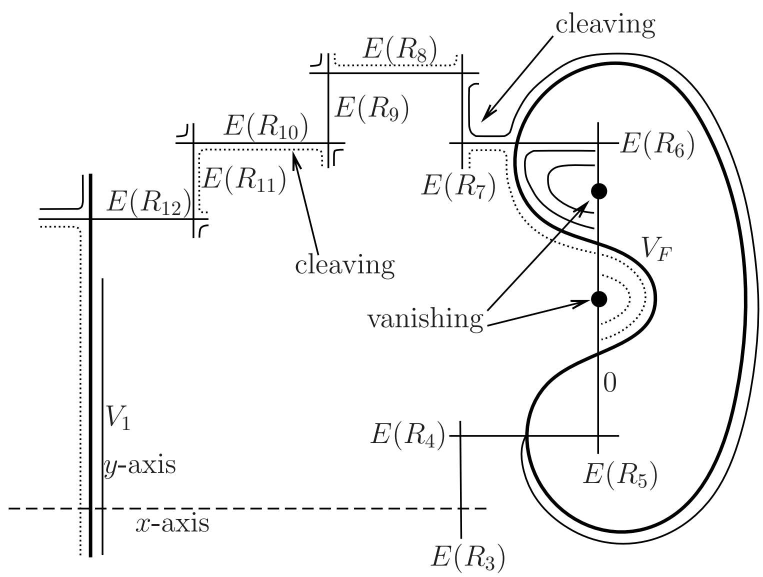


FIGURE 8. A part of a connected component of $f = \varepsilon$ and a part of a connected component of $f = -\varepsilon$, with sufficiently small $\varepsilon > 0$, are described. Both of them are cleaving as $\varepsilon \to 0$.

On the local chart $U_5$ with coordinates $(u_5, v_5)$, for each $j = 0, 1$, we have

$$f(u_5, v_5) - f(t_j) = (v_5 - t_j)^2 \hat{g}_j(v_5) + u_5 v_5^8$$

with $\hat{g}_j(t_j) \neq 0$. Thus $V_{f-f(t_j)}$ is tangent to $E(R_5)$ at $(u_5, v_5) = (0, t_j)$ with multiplicity 2. This is in case (2) in the proof of Theorem 1.1. Hence we see that there are a cleaving family and a vanishing family for each $j = 0, 1$. Since a vanishing family does not appear in the settings in Lemma 3.3 and 3.4, we see that there is no other vanishing family. There are two cleaving families with limit $f = 0$ as shown in Figure 8. Here we count the numbers of cleaving and vanishing families up to equivalence relations in

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Definition 2.7. In summary, this example has four cleaving families and two vanishing families. For other $m$'s more than 1, we can easily check that $f$ also has the same numbers of cleaving and vanishing families.

## 4. Proof of Theorem 1.2

In this section, we give the proof of Theorem 1.2, which determines the number of cleaving and vanishing families counted up to equivalence relations in Definition 2.7.

*Proof of Theorem 1.2.* Let $E^+$ and $E^0$ denote the union of $E(R_i)$'s with $\Delta(R_i; f) \in \Gamma^+_\infty(f)$ and with $\Delta(R_i; f) \in \Gamma^0_\infty(f)$, and let $N^+$ and $N^0$ denote a small, compact neighborhood of $E^+$ and $E^0$ in $X$, respectively. A cleaving family appears either in $N^+$ or $N^0$, see Lemma 3.4 and the proof of Theorem 1.1. A vanishing family appears only in $N^0$, see the proof of Theorem 1.1.

First we observe it in $N^+$. Suppose that $E^+ \neq \emptyset$. We may assume that $f$ has the form $f(x, y) = x^\alpha y^\beta F(x, y)$ with either $\alpha > 0$ or $\beta > 0$. We set $E^+ = E^+_x \cup E^+_y$, where $E^+_x = \cup^m_{i=k} E(R_i)$ and $E^+_y = \cup^{k'}_{i=3} E(R_i)$. Here $k$ is the index such that $d(R_{k-1}; f) \leq 0$ and $d(R_k; f) > 0$ (cf. Figure 4) and $k'$ is the index such that $d(R_{k'}; f) > 0$ and $d(R_{k'+1}; f) \leq 0$. Note that if $\alpha = 0$ (resp. $\beta = 0$) then $E^+_x$ (resp. $E^+_y$) is empty and that $E^+_x \cap E^+_y = \emptyset$. Let $R^+_x$ (resp. $R^+_y$) be the sum of $r(R_i; f)$'s for $i \geq k$ (resp. $3 \leq i \leq k'$) with $\Delta(R_i; f) \in \Gamma^+_\infty(f)$. Note that $R^+ = R^+_x + R^+_y$.

We first count the number of cleaving families in a compact neighborhood $N^+_x$ of $E^+_x$. Remark that $R^+_x$ is equal to the number of intersection points $V_F \cap E^+_x$. A nearby fiber in $N^+_x$ yields a "cleaving" if and only if both of the endpoints of the fiber in $N^+_x$ is on the boundary $\partial N^+_x$. If an endpoint is not on $\partial N^+_x$ then it is on $E(R_{k-1})$ if $d(R_{k-1}; f) = 0$ and on the intersection $E(R_{k-1}) \cap E(R_k)$ if $d(R_{k-1}; f) < 0$. Such an endpoint can appear on all of the four quadrants on the chart $U_{k-1}$ corresponding to $\mathrm{Cone}(R_{k-1}, R_k)$. Since $V_1$ intersects $E(R_m)$ at one point, $V_f$ intersects $E^+_x$ at $R^+_x + 1$ points. Adding the 4 endpoints on $U_{k-1}$, there are totally $4R^+_x + 8$ endpoints. Hence the curve $\{f = \varepsilon\} \cup \{f = -\varepsilon\}$ has $2R^+_x + 4$ connected components in $N^+_x$. However, the endpoints lying on $E(R_{k-1})$ do not contribute to "cleavings". Moreover, there is no connected component both of whose endpoints are on $E(R_{k-1})$. This can be checked as follows: Try to describe a curve in $N^+_x$ starting at one of the endpoints on $E(R_{k-1})$. Then we meet $V_f$ before coming back near $E(R_{k-1})$. Thus the curve must go out from $\partial N^+_x$. There are four connected components which do not contribute to "cleavings". Hence the number of cleaving families in $N^+_x$ is $2R^+_x$. This is true even if $E^+_x = \emptyset$ since there is no cleaving family in this case.

The number of cleaving families in a compact neighborhood $N^+_y$ of $E^+_y$ can be counted by the same way and it becomes $2R^+_y$. Since $E^+_x \cap E^+_y = \emptyset$, the countings in $N_x^+$ and $N_y^+$ do not conflict. Hence the total number of cleaving families in $N^+$ is $2R_x^+ + 2R_y^+ = 2R^+$.

Next we observe it in $N^0$. Since $f$ has only isolated singularities, each critical point of $b_P(t)|_{\mathbb{R}\setminus\{0\}}$ corresponds to an intersection point of $V_F$ and $E^0$ as shown in Figures 5 and 6. In either case, for each intersection point, the sum of the number of cleaving families and that of vanishing families is 2. Hence the total number of cleaving and vanishing families in $N^0$ is $2R^0$. This completes the proof. $\square$

## 5. An upperbound of $B_{\infty,f}$

In this section, we do not assume that $f$ is non-degenerate on $\Gamma^+_\infty(f) \cup \Gamma^-_\infty(f)$ and also do not assume that $f$ is Morse on $\Gamma^0_\infty(f)$.

We will prove Theorem 1.3 by applying successive admissible toric modifications for each singularity on $\cup_{i=3}^m E(R_i)$ appearing due to degeneracies. We first introduce an admissible toric modification. Though an admissible toric modification is usually defined for a polynomial function or a locally analytic function, we define it for rational functions given as in (2.1). Note that such a modification had been used in [14] for studying singularities at infinity of complex polynomial functions.

Let $U \subset \mathbb{R}^2$ be a small neighborhood of the origin and let $\hat{f} : U \to \mathbb{R}$ be a real rational function on $U$ whose expansion is given by $\hat{f}(x,y) = \sum_{(m,n)} a_{m,n} x^m y^n$, where $(m,n) \in \mathbb{Z}$ with $m > -M$ for some non-negative integer $M$ and $n \geq 0$. We define the Newton polygon $\Delta^{\mathrm{loc}}(\hat{f})$ of $\hat{f}$ by the convex hull of $\cup_{(m,n)}((m,n) + \mathbb{R}^2_{\geq 0})$, where $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} \mid x \geq 0\}$ and the union is taken for all $(m,n)$ such that $a_{m,n} \neq 0$. For a given primitive covector $P = {}^t(p,q)$ with $p, q > 0$, let $d(P;\hat{f})$ denote the minimal value of the linear function $pX + qY$, where $(X,Y) \in \Delta^{\mathrm{loc}}(\hat{f})$. Set $\Delta(P;\hat{f}) := \{(X,Y) \in \Delta^{\mathrm{loc}}(\hat{f}) \mid pX + qY = d(P;\hat{f})\}$, which is called a *face* if $\dim \Delta(P;\hat{f}) = 1$. The partial sum $\hat{f}_P(x,y) = \sum_{(m,n)\in\Delta(P;\hat{f})} a_{m,n} x^m y^n$ is called the *boundary function* for the covector $P$. If $\Delta(P;\hat{f})$ is a face then it is called the *face function*. A boundary function $\hat{f}_P$ is said to be *degenerate* if $\frac{\partial \hat{f}_P}{\partial x} = \frac{\partial \hat{f}_P}{\partial y} = 0$ has a solution in $(\mathbb{R}\setminus\{0\})^2$. Otherwise it is said to be *non-degenerate*.

Let $\hat{f}$ be a rational function given as above and let $\hat{Q}_i = {}^t(\hat{p}_i, \hat{q}_i)$, $i = 1, \ldots, \hat{n}$, be primitive covectors such that

(1) both $\hat{p}_i$ and $\hat{q}_i$ are positive;
(2) $\Delta(\hat{Q}_i;\hat{f})$ is a compact face;
(3) the indices are assigned in the counter-clockwise orientation.

Let $\hat{R}_i = {}^t(\hat{r}_i, \hat{s}_i)$, $i = 1, \ldots, \hat{m}$, be primitive covectors which satisfy the following:

(1) $\hat{R}_1 = {}^t(1,0)$ and $\hat{R}_{\hat{m}} = {}^t(0,1)$;
(2) both $\hat{r}_i$ and $\hat{s}_i$ are positive for each $\hat{R}_i$, $i = 2, \ldots, \hat{m}-1$;
(3) $\hat{Q}_i$ is contained in $\{\hat{R}_2, \ldots, \hat{R}_{\hat{m}-1}\}$;

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(4) the indices are assigned in the counter-clockwise orientation;
(5) the determinants of the matrices $(\hat{R}_i, \hat{R}_{i+1})$, $i = 1, \dots, \hat{m} - 1$, are 1.

For each $\mathrm{Cone}(\hat{R}_i, \hat{R}_{i+1})$, $i = 1, \dots, \hat{m}-1$, an affine coordinate chart $(u_i, v_i)$ is defined by the coordinate transformation

$$x = u_i^{\hat{r}_i} v_i^{\hat{r}_{i+1}}, \quad y = u_i^{\hat{s}_i} v_i^{\hat{s}_{i+1}}.$$

Then a real variety $Y$ is obtained by gluing these coordinate charts, which is described as

$$Y = U \cup \left( \bigcup_{i=2}^{\hat{m}-1} E(\hat{R}_i) \right),$$

where $E(\hat{R}_i)$ is the exceptional divisor corresponding to the covector $\hat{R}_i$. Let $\pi : Y \to U$ be the associated proper mapping, which is called the *admissible toric modification associated with* $\{\hat{R}_1, \dots, \hat{R}_{\hat{m}}\}$. For further information about toric modifications, see [24].

Suppose that $\hat{f}$ has the form

$$\hat{f}(x, y) = x^d (y + c)^{d'} (y^\mu \hat{g}(y) + x\hat{h}(x, y)),$$

where $d, d' \in \mathbb{Z}$, $c \neq 0$, $\hat{g}(y)$ is the expansion of a rational function of one variable $y$ with $\hat{g}(0) \neq 0$, and $h(x, y)$ is the expansion of a rational function of two variables $(x, y)$ with $|h(0, 0)| < \infty$. Let $\Delta^-(\hat{f})$, $\Delta^0(\hat{f})$ and $\Delta^+(\hat{f})$ denote the union of the compact faces $\Delta(P; \hat{f})$ of $\Delta^{\mathrm{loc}}(\hat{f})$ with $d(P; \hat{f}) < 0$, $d(P; \hat{f}) = 0$ and $d(P; \hat{f}) > 0$, respectively. Set $\ell^-(\hat{f})$ and $\ell^0(\hat{f})$ to be $-1$ plus the number of lattice points in the segment obtained by projecting $\Delta^-(\hat{f})$ and $\Delta^0(\hat{f})$ to the second axis of $\mathbb{R}^2$ on which $\Delta^{\mathrm{loc}}(\hat{f})$ is described, respectively. Set $\ell^+(\hat{f}) = \mu - \ell^-(\hat{f}) - \ell^0(\hat{f})$.

**Definition 5.1.** The integers $\ell^+(\hat{f})$, $\ell^0(\hat{f})$ and $\ell^-(\hat{f})$ are called the (+)-, (0)- and (−)-*height* of $\Delta^{\mathrm{loc}}(\hat{f})$, respectively.

These heights will be used in the proof of Theorem 1.3.

Let $f$ be a polynomial function. We first apply an admissible toric compactification $Y_1 \supset \mathbb{R}^2$ associated with primitive covectors $\{R_1, \dots, R_m\}$ with respect to $\Delta(f)$. Suppose that $f_{R_i}$ is degenerate for a face $\Delta(R_i; f)$ in $\Gamma_\infty^-(f)$. On $U_i$, $f$ is given as (2.1). Let $s_1, \dots, s_\eta$ be non-zero real roots of $g_i(v_i) = 0$ and $\mu_1, \dots, \mu_\eta$ their multiplicities. For some $\xi \in \{1, \dots, \eta\}$ with $\mu_\xi \geq 2$, which exists since $f_{R_i}$ is degenerate, we apply the change of coordinates

$$(x_1, y_1) = (u_i, v_i - s_\xi).$$

We call $(x_1, y_1)$ *translated coordinates*. The polynomial function $f$ can be extended to $Y_1$ as a rational function, and is given on the chart $(x_1, y_1)$ as

$$f^1(x_1, y_1) = x_1^{d(R_i;f)}(y_1 + s_\xi)^{d(R_{i+1};f)}(y_1^{\mu_\xi} g^1(y_1) + x_1 h^1(x_1, y_1)),$$

where $g^1(0) \neq 0$.

Assume that we have applied admissible toric modifications $\pi_i : Y_i \to Y_{i-1}$ for $i = 2, \ldots, \sigma$ successively. Let $U^\sigma$ be a neighborhood of the origin on the coordinate chart $(u_\sigma, v_\sigma)$ in $Y_\sigma$ obtained after the successive toric modifications and translations of coordinates. We call $(u_\sigma, v_\sigma)$ *translated coordinates* also. Let $f^\sigma$ be the restriction of the pull-back of $f$ to $U^\sigma$, which is given as

$$f^\sigma(x_\sigma, y_\sigma) = x_\sigma^{d_\sigma}(y_\sigma + s_\xi^\sigma)^{d'_\sigma}(y_\sigma^{\mu_\sigma} g^\sigma(y_\sigma) + x_\sigma h^\sigma(x_\sigma, y_\sigma)),$$

where $d_\sigma, d'_\sigma \in \mathbb{Z}$ with $d_\sigma < 0$, $s_\xi^\sigma \neq 0$, $\mu_\sigma \geq 2$ and $g^\sigma(0) \neq 0$. Applying an admissible toric modification $\pi_{\sigma+1} : Y_{\sigma+1} \to Y_\sigma$ on $U^\sigma$ with respect to $\varDelta^{\mathrm{loc}}(f^\sigma)$, we obtain a sequence of admissible toric modifications inductively.

We say that a sequence $Y_\tau \to \cdots \to Y_1 \supset \mathbb{R}^2$ of successive toric modifications is *terminated* if there are no translated coordinates for further toric modifications. Note that a sequence of successive toric modifications is terminated in finite steps. The finiteness is proved in [14, Lemma 4.3] for complex polynomial case and the same proof works for real case also.

Let $Y_\tau \to \cdots \to Y_\sigma \to \cdots \to Y_1 \supset \mathbb{R}^2$ be a sequence of admissible toric modifications, which is not necessary to be terminated. Let $\{R_1^\sigma, \ldots, R_{m_\sigma}^\sigma\}$ be the primitive covectors for the toric modification $\pi_{\sigma+1} : Y_{\sigma+1} \to Y_\sigma$ with respect to $\varDelta^{\mathrm{loc}}(f^\sigma)$, containing primitive covectors $\{Q_1^\sigma, \ldots, Q_{n_\sigma}^\sigma\}$ orthogonal to the compact faces of $\varDelta^{\mathrm{loc}}(f^\sigma)$. For each $j = 2, \ldots, m_\sigma - 1$, on the local chart $U_j^\sigma$ in $Y_{\sigma+1}$ corresponding to $\mathrm{Cone}(R_j^\sigma, R_{j+1}^\sigma)$, the pull-back $f_j^\sigma$ of $f$ is given as

$$f_j^\sigma(u_{\sigma,j}, v_{\sigma,j}) = u_{\sigma,j}^{d(Q_j^\sigma;f^\sigma)} v_{\sigma,j}^{d(Q_{j+1}^\sigma;f^\sigma)}(g_j^\sigma(v_{\sigma,j}) + u_{\sigma,j} h_j^\sigma(u_{\sigma,j}, v_{\sigma,j})).$$

Now we define an integer $\lambda(Q_j^\sigma; f^\sigma)$ by

$$\lambda(Q_j^\sigma; f^\sigma) = \begin{cases} 0 & d(Q_j^\sigma; f^\sigma) > 0 \\ r'(Q_j^\sigma; f^\sigma) & d(Q_j^\sigma; f^\sigma) = 0 \\ \sum_{\xi \in \Xi_{\sigma,j}} (\mu_\xi - 1) & d(Q_j^\sigma; f^\sigma) < 0, \end{cases} \tag{5.1}$$

where $r'(Q_j^\sigma; f^\sigma)$ is the number of non-zero real roots of $\frac{\partial b_{Q_j^\sigma}}{dt}(t) = 0$, $\Xi_{\sigma,j}$ is the set of indices of non-zero real multiple roots of $g_j^\sigma(v_{\sigma,j}) = 0$ at which we did not apply further successive toric modifications in $Y_\tau \to \cdots \to Y_1 \supset \mathbb{R}^2$, and $\mu_\xi$ is the multiplicity of the root with index $\xi \in \Xi_{\sigma,j}$. Set $\epsilon_\sigma = 1$ if the (+)-height of $\varDelta(f_j^\sigma)$ is more than or equal to 2, and set $\epsilon_\sigma = 0$ otherwise.

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Similarly, for the primitive covectors $\{Q_1, \ldots, Q_n\}$ orthogonal to the faces of $\Delta(f)$, we define

$$\lambda(Q_i; f) = \begin{cases} 0 & d(Q_i; f) > 0 \\ r'(Q_i; f) & d(Q_i; f) = 0 \\ \sum_{\xi \in \Xi_i} (\mu_\xi - 1) & d(Q_i; f) < 0, \end{cases}$$

where $r'(Q_i; f)$ is the number of non-zero real roots of $\frac{\partial b_{Q_i}}{dt}(t) = 0$, $\Xi_i$ is the set of indices of real multiple roots of $g_i(v_i) = 0$ in (2.1) at which we did not apply further successive toric modifications, and $\mu_\xi$ is the multiplicity of the root with index $\xi \in \Xi_i$. Set $\epsilon(R^+) = 0$ if $R^+ = 0$ and $\epsilon(R^+) = 1$ if $R^+ > 0$ as in Theorem 1.3.

**Proposition 5.2.** *Let $Y_\tau \to \cdots \to Y_1 \supset \mathbb{R}^2$ be a sequence of successive toric modifications that is terminated. Then*

$$|B_{\infty,f}| \leq \epsilon(R^+) + \sum_{i=1}^{n} \lambda(Q_i; f) + \sum_{\sigma} \left( \epsilon_\sigma + \sum_{j=1}^{n_\sigma} \lambda(Q_j^\sigma; f^\sigma) \right),$$

*where $\sigma$ runs over all indices of translated coordinates appearing in the successive toric modifications.*

*Proof.* Since the sequence of successive toric modifications is terminated, $\Xi_{\sigma,j} = \emptyset$ for any $(\sigma, j)$.

We first check the contribution of the faces $\Delta(Q_j^\sigma; f^\sigma)$ with $d(Q_j^\sigma; f^\sigma) > 0$ to $|B_{\infty,f}|$. Consider the variety $Y_{\sigma+1}$ obtained by an admissible toric modification $\pi_{\sigma+1} : Y_{\sigma+1} \to Y_\sigma$ with respect to $\Delta^{\mathrm{loc}}(f^\sigma)$. Let $\ell^+(f^\sigma)$ be the $(+)$-height of $\Delta^{\mathrm{loc}}(f^\sigma)$. Suppose that $\ell^+(f^\sigma) \geq 2$. If there are cleaving families near $E(Q_j^\sigma)$ with $d(Q_j^\sigma; f^\sigma) > 0$ then their limits correspond to the same value in $B_{\infty,f}$. Hence its contribution is at most 1. If $\ell^+(f^\sigma) = 0$ then there is no contribution. If $\ell^+(f^\sigma) = 1$ and there is no face $\Delta(P; f^\sigma)$ with $d(P; f^\sigma) > 0$ then there is no contribution also. Suppose that $\ell^+(f^\sigma) = 1$ and there is such a face, say $\Delta(Q_{j0}^\sigma; f^\sigma)$. Then, as shown in Figure 9, we see that the fibration of nearby fibers passing near $E(Q_{j0}^\sigma)$ is trivial. Note that this can be shown by choosing a family of smooth circles going to the infinity and being transverse to the nearby fibers suitably as in the proof of Lemma 3.2. Hence the contribution of the face $\Delta(Q_{j0}^\sigma; f^\sigma)$ is 0. There is no vanishing family in a neighborhood of $E(Q_j^\sigma)$ with $d(Q_j^\sigma; f^\sigma) > 0$ in $Y_{\sigma+1}$. Thus the contribution of the faces $\Delta(Q_j^\sigma; f^\sigma)$ with $d(Q_j^\sigma; f^\sigma) > 0$ is at most $\epsilon_\sigma$.

Next we check the contribution of the faces $\Delta(Q_j^\sigma; f^\sigma)$ with $d(Q_j^\sigma; f^\sigma) = 0$. The number of values appearing as their limits is at most the number of non-zero real roots of $\frac{db_{Q_j^\sigma}}{dt}(t) = 0$. Hence the contribution is at most $r'(Q_j^\sigma; f^\sigma)$.

The same observation can be applied to neighborhoods of the divisors $E(Q_i)$, $i = 1, \ldots, n$, and we have the upper bound $\epsilon(R^+) + \sum_{i=1}^n \lambda(Q_i; f)$ of the contribution. This completes the proof. $\square$

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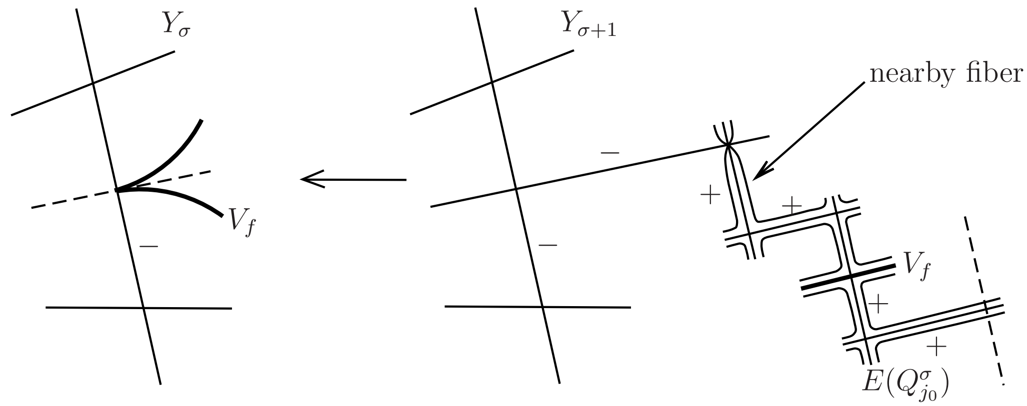


FIGURE 9. The triviality of the fibration in the case $\ell^+(f^\sigma) = 1$.

*Remark* 5.3. The phenomenon having the triviality of the fibration in the case $\ell^+ = 1$, shown in Figure 9, appears in complex polynomial case also. A face $\Delta(P; f^\sigma)$ with $d(P; f^\sigma) > 0$ in the case $\ell^+ = 1$ is called a *stable boundary face* in [14, Definition 5.5]. The exception of such a face is also pointed out in [1, Example 2.9 (3)].

*Proof of Theorem 1.3.* Let $Y_\tau \to \cdots \to Y_1 \supset \mathbb{R}^2$ be a sequence of successive toric modifications that is not terminated. Set

$$\Lambda_\tau = \epsilon(R^+) + \sum_{i=1}^{n} \lambda(Q_i; f) + \sum_{\sigma=1}^{\tau} \left( \epsilon_\sigma + \sum_{j=1}^{n_\sigma} \lambda(Q^\sigma_j; f^\sigma) \right).$$

We first prove that this sum does not increase after an admissible toric modification $\pi_{\tau+1} : Y_{\tau+1} \to Y_\tau$, i.e., prove the inequality $\Lambda_{\tau+1} \le \Lambda_\tau$.

Apply a toric modification $\pi_{\tau+1}$ at the origin of translated coordinates $(x_\tau, y_\tau)$. The pull-back $f^{\tau+1} = \pi^*_{\tau+1} f^\tau$ of $f$ has the form

$$f^{\tau+1}(x_{\tau+1}, y_{\tau+1}) = x_{\tau+1}^{d_{\tau+1}} (y_{\tau+1} + s_\xi)^{d'_{\tau+1}} (y_{\tau+1}^{\mu_\xi} g^{\tau+1}(y_{\tau+1}) + x_{\tau+1} h^{\tau+1}(x_{\tau+1}, y_{\tau+1})),$$

where $d_{\tau+1}, d'_{\tau+1} \in \mathbb{Z}$ with $d_{\tau+1} < 0$, $s_\xi \neq 0$, $\mu_\xi \ge 2$ and $g^{\tau+1}(0) \neq 0$. Let $\ell^+$, $\ell^0$ and $\ell^-$ denote the $(+)$-, $(0)$- and $(-)$-heights of $\Delta^{\rm loc}(f^{\tau+1})$, respectively. Note that $\ell^+ + \ell^0 + \ell^- \le \mu_\xi$.

We will prove that the total contribution of the faces $\Delta(Q^{\tau+1}_j; f^{\tau+1})$ to $\Lambda_{\tau+1}$ is at most $\ell := \ell^+ + \ell^0 + \ell^- - 1$. From $\Delta^{\rm loc}(f^{\tau+1})$, we see that the contribution $\sum_{\xi \in \Xi_{\tau+1,j}} (\mu_\xi - 1)$ in (5.1) in the case $d(Q^{\tau+1}_j; f^{\tau+1}) < 0$ is at most $\ell^- - 1$. The contribution in the case $d(Q^{\tau+1}_j; f^{\tau+1}) = 0$ is at most $\ell^0$ and the contribution in the case $d(Q^{\tau+1}_j; f^{\tau+1}) > 0$ is $\epsilon_{\tau+1}$. Hence if $\ell^- > 0$ then the total contribution is at most $\ell$. Suppose that $\ell^- = 0$. If $\ell^+ = 0$ then the face $\Delta(P; f^{\tau+1})$ with $d(P; f^{\tau+1}) = 0$ contains the origin $(0, 0)$ and the contribution in the case $d(Q^{\tau+1}_j; f^{\tau+1}) = 0$ becomes at most $\ell^0 - 1$. Hence the total contribution is at most $\ell$. If $\ell^+ \ge 2$ then the contribution in the case $d(Q^{\tau+1}_j; f^{\tau+1}) > 0$ becomes at most $\ell^+ - 1$, and hence the total contribution is also at most $\ell$. If $\ell^+ = 1$ then $\epsilon_{\tau+1} = 0$, i.e., the contribution in the case $d(Q_j^{\tau+1}; f^{\tau+1}) > 0$ is 0. Hence the total contribution is at most $\ell$. Since $\ell := \ell^+ + \ell^0 + \ell^- - 1 \leq \mu_\xi - 1$, we have

$$\Lambda_{\tau+1} \leq \epsilon(R^+) + \sum_{i=1}^{n} \lambda(Q_i; f) + \sum_{\sigma=1}^{\tau} \left( \epsilon_\sigma + \sum_{j=1}^{n_\sigma} \lambda(Q_j^\sigma; f^\sigma) \right) - (\mu_\xi - 1) + \ell \leq \Lambda_\tau.$$

Thus $\Lambda_\tau$ does not increase. By the same argument, we have $\Lambda_1 \leq \Lambda_0 := \epsilon(R^+) + \sum_{i=1}^n \lambda(Q_i; f)$.

Suppose that a sequence of successive toric modifications is terminated at $\tau = \tau_0$. By Proposition 5.2 we have $|B_{\infty,f}| \leq \Lambda_{\tau_0}$. We then apply the inequality $\Lambda_{\tau+1} \leq \Lambda_\tau$ inductively:

$$|B_{\infty,f}| \leq \Lambda_{\tau_0} \leq \Lambda_{\tau_0-1} \leq \cdots \leq \Lambda_1 \leq \Lambda_0 = \epsilon(R^+) + R^0 + \sum_{\Delta(P;f) \in \Gamma_\infty^-(f)} \mu(P; f).$$

This completes the proof. □

*Remark* 5.4. Let $Y_\tau \to \cdots \to Y_1 \supset \mathbb{R}^2$ be a sequence of successive toric modifications that is terminated. Then a vanishing family appears only in a neighborhood of a divisor $E(Q_j^\sigma)$ with $d(Q_j^\sigma; f^\sigma) = 0$.

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Department of Mathematics, Hiyoshi Campus, Keio University, 4-1-1 Hiroshi, Kohoku-ku, Yokohama-shi, Kanagawa, 223-8521, Japan

*Email address*: ishikawa@keio.jp

Institute of Mathematics, Vietnam Academy of Science and Technology, 18 Hoang Quoc Viet road, Cau Giay district, 10307 Hanoi, Vietnam

*Email address*: ntthang@math.ac.vn

Department of Mathematics, University of Dalat, 1 Phu Dong Thien Vuong, Dalat, Vietnam

*Email address*: sonpt@dlu.edu.vn

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