A PROOF OF SAITOH'S CONJECTURE FOR CONJUGATE HARDY H^2 KERNELS

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Abstract. In this article, we obtain a strict inequality between the conjugate Hardy H^2 kernels and the Bergman kernels on planar regular regions with $n > 1$ boundary components, which is a conjecture of Saitoh.

1. Introduction

Let D be a planar regular region with n boundary components which are analytic Jordan curves (see [12, 15]).

As in [12], $H_2^{(c)}(D)$ denotes the analytic Hardy class on D defined as the set of all analytic functions $f(z)$ on D such that the subharmonic functions $|f(z)|^2$ have harmonic majorants $U(z)$, i.e. $|f(z)|^2 \leq U(z)$ on D.

As in [12], $\hat{R}_t(z, \bar{w})$ denotes the conjugate Hardy H^2 kernel on D if

$$f(w) = \frac{1}{2\pi} \int_{\partial D} f(z) \overline{\hat{R}_t(z, \bar{w})} (\frac{\partial G(z,t)}{\partial \nu_z})^{-1} d|z| \tag{1.1}$$

holds for any holomorphic function $f \in H_2^{(c)}(D)$ which satisfies

$$\int_{\partial D} |f(z)|^2 (\frac{\partial G(z,t)}{\partial \nu_z})^{-1} d|z| < +\infty,$$

where $G(z,t)$ is the Green function on D, $f(z)$ means Fatou's nontangential boundary value, and $\frac{\partial}{\partial \nu_z}$ denotes the derivative along the outer normal unit vector ν_z. It is well-known that $\frac{\partial G(z,t)}{\partial \nu_z}$ is positive continuous on ∂D because of the analyticity of the boundary (see [12]).

When $t = w$, $\hat{R}(z, \bar{w})$ denotes $\hat{R}_w(z, \bar{w})$ for simplicity. When $z = w$, $\hat{R}(z)$ denotes $\hat{R}(z, \bar{z})$ for simplicity.

Let $B(z, \bar{w})$ be the Bergman kernel on D. When $z = w$, $B(z)$ denotes $B(z, \bar{z})$ for simplicity.

In [15] (see also [12] and [16]), the following so-called Saitoh's conjecture was posed (backgrounds and related results could be referred to Hejhal's paper [11] and Fay's book [7]).

Conjecture 1.1. *(Saitoh's Conjecture) If $n > 1$, then $\hat{R}(z) > \pi B(z)$.*

In the present article, we give a proof of the above Conjecture.

Theorem 1.1. *Conjecture 1.1 holds.*

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One of the ingredients of the present article is using the concavity of minimal L^2 integrations in [8].

2. Preparations

In the present section, we recall some known results and present some preparations, which will be used in the proof of Theorem 1.1.

2.1. **The concavity of minimal L^2 integrations.** Let $G(z,w)$ be the Green function on $D\times D$ such that $G(z,w)-\log|z-w|$ is analytic on $D\times D$.

Let $g_w(-\log r)$ be the minimal L^2 integration of the holomorphic functions f on $\{2G(\cdot,w)<-\log r\}$ satisfying $f(w)=1$. It is clear that $g_w(-\log r)$ is also the reciprocal of the Bergman kernel for $\{2G(\cdot,w)<-\log r\}$. In [2], the following concavity of the $g_w(-\log r)$ was presented by Berndtsson and Lempert.

Proposition 2.1. *(see [2], see also Proposition 4.1 in [8] for general cases) $g_w(-\log r)$ is concave with respect to $r\in(0,1]$.*

Note that $\lim_{r\to 0+0} g_w(-\log r)=0$, then Proposition 2.1 implies that

Corollary 2.1. *The inequality*

$$\lim_{r\to 1-0}\frac{g_w(-\log r)-g_w(0)}{r-1}\le g_w(0)\le \lim_{r\to 0+0}\frac{g_w(-\log r)}{r} \tag{2.1}$$

holds for any $r\in(0,1)$, where $\lim_{r\to 0+0}\frac{g_w(-\log r)}{r}$ might be $+\infty$. Moreover, the following three statements are equivalent

(1) $\lim_{r\to 1-0}\frac{g_w(-\log r)-g_w(0)}{r-1}=g_w(0)$;
(2) $\lim_{r\to 0+0}\frac{g_w(-\log r)}{r}=g_w(0)$;
(3) $g_w(-\log r)=rg(0)$ *for any* $r\in(0,1]$.

2.2. **Green function and Bergman kernel.** Note that there exists a local coordinate z' on a neighborhood U_0 of $z_0\in\partial D$ such that $\partial D|_{U_0}=\{\Im z'=0\}$, which implies the following well-known lemma.

Lemma 2.1. *The Green function $G(z,w)$ has an analytic extension on $(U\times V)\setminus\{z=w\}$, where U is a neighborhood of $\bar{D}$ and $V\subset\subset D$.*

Note that the Bergman kernel $B(z,\bar{w})$ on $D\times D$ equals $\frac{\partial}{\partial z}\frac{\partial}{\partial\bar{w}}G(z,w)$ (see [1]), then it follows from Lemma 2.1 that $B(\cdot,\bar{w})$ is smooth on a neighbohood of $\bar{D}$ for any given $w\in D$. Note that $\frac{B(\cdot,\bar{w})}{B(w,\bar{w})}$ is the (unique) holomorphic function satisfying $\int_D|\frac{B(\cdot,\bar{w})}{B(w,\bar{w})}|^2=g_w(0)$ and $\frac{B(\cdot,\bar{w})}{B(w,\bar{w})}(w)=1$ (see [1]), then it follows that

Remark 2.1. *There exists a (unique) holomorphic function f $(=\frac{B(\cdot,\bar{w})}{B(w,\bar{w})})$, which is smooth on a neighhorhood of $\bar{D}$ such that $f(w)=1$ and $\int_D|f|^2=g_w(0)$.*

2.3. **A solution of a conjecture of Suita.** We recall the following solution of a conjecture posed by Suita [13].

Theorem 2.1. *([10]) Let $c_\beta(z_0)=\lim_{z\to z_0}\exp(G(z,z_0)-\log|z-z_0|)$. Then $(c_\beta(z_0))^2=\pi B(z_0)$ holds for some $z_0\in D$ if and only if D conformally equivalent to the unit disc, i.e. $n=1$.*

We would like to recall that $(c_\beta(z_0))^2 \leq \pi B(z_0)$ was proved by Blocki in [3] for planar domains D (the referee kindly mentions that a new approach of considering the properties of the Bergman kernel for sublevel sets of the Green function for planar domains D was first introduced in [4]), and Guan-Zhou in [9] for open Riemann surfaces, i.e. the original form in [13].

Note that $2G(z,z_0) - 2\log|z-z_0|$ is harmonic on D (continuous near z_0), then it follows that

Lemma 2.2. $\frac{(c_\beta(z_0))^2}{\pi} = \lim_{r\to 0+0} \frac{r}{g_{z_0}(-\log r)}$.

Proof. Note that $c_\beta(z_0) = \lim_{z\to z_0} \exp(G(z,z_0) - \log|z-z_0|)$ implies that for any $\varepsilon > 0$, then there exists a neighborhood U_0 of z_0 such that

$$|G(z,z_0) - \log(c_\beta(z_0)|z-z_0|)| < \varepsilon. \tag{2.2}$$

As $G(z,z_0)$ and $\log(c_\beta(z_0)|z-z_0|)$ both go to $-\infty$ $(z \to z_0)$, then there exists $\delta_0 > 0$, such that $\{G(z,z_0) < \frac{1}{2}\log r\} \subset U_0$ and $\{\log(c_\beta(z_0)|z-z_0|) < \frac{1}{2}\log r\} \subset U_0$ for any $r \in (0,\delta_0)$. It follows from (2.2) that for any $r \in (0, \delta_0 e^{-2\varepsilon})$,

$$\begin{aligned}\{\log(c_\beta(z_0)|z-z_0|) + \varepsilon < \frac{1}{2}\log r\} &\subset \{G(z,z_0) < \frac{1}{2}\log r\} \\ &\subset \{\log(c_\beta(z_0)|z-z_0|) - \varepsilon < \frac{1}{2}\log r\} \subset U_0\end{aligned} \tag{2.3}$$

which implies

$$\begin{aligned}\mu\{\log(c_\beta(z_0)|z-z_0|) + \varepsilon < \frac{1}{2}\log r\} &\leq g_{z_0}(-\log r) \\ &\leq \mu\{\log(c_\beta(z_0)|z-z_0|) - \varepsilon < \frac{1}{2}\log r\},\end{aligned} \tag{2.4}$$

i.e.

$$c_\beta^{-2}(z_0)\pi r e^{-2\varepsilon} \leq g_{z_0}(-\log r) \leq c_\beta^{-2}(z_0)\pi r e^{2\varepsilon}, \tag{2.5}$$

where μ is the Lebesgue measure on $\mathbb{C}$. Then Lemma 2.2 has been proved by the arbitrariness of $\varepsilon > 0$. $\square$

Note that

$$B(z_0) = \frac{1}{g_{z_0}(0)}, \tag{2.6}$$

then it follows from Theorem 2.1 and Lemma 2.2 that

Remark 2.2. *Statement (2) in Corollary 2.1 holds if and only if D is the unit disc.*

2.4. Conjugate analytic Hardy space on D.

Lemma 2.3. *For any given $w_0 \in D$, and holomorphic function f on D which is continuous on $\bar{D}$,*

$$\lim_{r\to 1-0} \frac{\int_{\{e^{2G(z,w_0)}\geq r\}} |f(z)|^2}{1-r} = \int_{\partial D} |f(z)|^2 \left(\frac{\partial 2G(z,w_0)}{\partial \nu_z}\right)^{-1} d|z| \tag{2.7}$$

holds, where $\frac{\partial}{\partial \nu_z}$ is the derivative along the outer normal unit vector ν_z.

It is kindly pointed out by the referee that the above lemma is an immediate consequence of the coarea formula. For the convinience of the reader, we retain the following elementary proof.

Proof. As $\frac{\partial}{\partial\nu_z}G(z,w_0)$ is positive on ∂D, it is clear that $\frac{\partial}{\partial y}G(z_b,w_0)\neq 0$ or $\frac{\partial}{\partial x}G(z_b,w_0)\neq 0$, where $z_b\in\partial D$. Then there exists a neighborhood U_b of z_b with coordinates $(u,v)=(x,2G(x+\sqrt{-1}y,w_0))$ or $(2G(x+\sqrt{-1}y,w_0),y)$ on U_b. Note that ∂D is compact, then there exist finite U_b covering ∂D. It is clear that one can choose finite unitary decomposition $\{\rho_\lambda\}_\lambda$ such that $\sum\rho_\lambda=1$ near ∂D, and for any λ, $Supp(\rho_\lambda)\subset U_b$ for some z_b.

Without loss of generality, we assume that $\frac{\partial}{\partial y}G(z_b,w_0)\neq 0$, where $z_b\in\partial D$. Then there exists a neighborhood U_b of z_b with coordinates $(u,v)=(x,2G(x+\sqrt{-1}y,w_0))$, where $u\in(a_1,a_2), v\in(\log r_b,-\log r_b)$ and $r_b\in(0,1)$.

It suffices to consider $|f|^2\rho$ instead of $|f|^2$ in (2.7), where $Supp\{\rho\}\subset\subset U_b$ and ρ is smooth. It is clear that $\frac{\partial u}{\partial x}=1$, $\frac{\partial u}{\partial y}=0$, $\frac{\partial v}{\partial x}=\frac{\partial}{\partial x}2G(z,w_0)$, and $\frac{\partial v}{\partial y}=\frac{\partial}{\partial y}2G(z,w_0)$, which implies that $\frac{\partial x}{\partial u}=1$, $\frac{\partial y}{\partial u}=-\frac{\frac{\partial}{\partial x}2G(z,w_0)}{\frac{\partial}{\partial y}2G(z,w_0)}$, $\frac{\partial x}{\partial v}=0$, and $\frac{\partial y}{\partial v}=(\frac{\partial}{\partial y}2G(z,w_0))^{-1}$. It is clear that equalities

$$\nu_z=\frac{(\frac{\partial}{\partial x}2G(z,w_0),\frac{\partial}{\partial y}2G(z,w_0))}{((\frac{\partial}{\partial x}2G(z,w_0))^2+(\frac{\partial}{\partial y}2G(z,w_0))^2)^{1/2}}$$

and

$$\frac{\partial}{\partial\nu_z}2G(z,w_0)=((\frac{\partial}{\partial x}2G(z,w_0))^2+(\frac{\partial}{\partial y}2G(z,w_0))^2)^{1/2}$$

hold, which imply

$$((\frac{\partial x}{\partial u})^2+(\frac{\partial y}{\partial u})^2)^{1/2}=\frac{\frac{\partial}{\partial\nu_z}2G(z,w_0)}{|\frac{\partial}{\partial y}2G(z,w_0)|}. \tag{2.8}$$

Replacing the integral variables, one can obtain

$$\begin{aligned}&\int_{\{e^{2G(z,w)}\geq r\}}|f(z)|^2\rho\\ =&\int_{\{a_1<u<a_2,\log r\leq v\leq 0\}}|f(z(u,v))|^2\rho(z(u,v))(|\frac{\partial}{\partial y}2G(z(u,v),w_0)|)^{-1},\end{aligned}\tag{2.9}$$

which implies

$$\begin{aligned}&\lim_{r\to 1-0}\frac{\int_{\{e^{2G(z,w)}\geq r\}}|f(z)|^2\rho}{1-r}\\ =&\lim_{r\to 1-0}\frac{\int_{\{a_1<u<a_2,\log r\leq v\leq 0\}}|f(z(u,v))|^2\rho(z(u,v))(|\frac{\partial}{\partial y}2G(z(u,v),w_0)|)^{-1}}{1-r}\\ =&\lim_{r\to 1-0}\frac{\int_{\{\{a_1<u<a_2,\log r\leq v\leq 0\}}|f(z(u,v))|^2\rho(z(u,v))(|\frac{\partial}{\partial y}2G(z(u,v),w_0)|)^{-1}}{-\log r}\\ =&\int_{\{a_1<u<a_2\}}|f(z(u,0))|^2\rho(z(u,0))(|\frac{\partial}{\partial y}2G(z(u,0),w_0)|)^{-1}du\\ =&\int_{\partial D}|f(z)|^2\rho(z)(\frac{\partial 2G(z,w_0)}{\partial\nu_z})^{-1}d|z|,\end{aligned}\tag{2.10}$$

where the last equality follows from (2.8). Then Lemma 2.3 has been proved. □

3. Proof of Theorem 1.1

We prove Theorem 1.1 in two steps: firstly we prove that $''\geq''$ holds, secondly we prove that $''=''$ does not hold.

Step 1. Let $f(z)=\frac{B(\cdot,\bar{w})}{B(w,\bar{w})}$, which implies that

$$\int_D |f|^2 = g_w(0). \tag{3.1}$$

It follows from Remark 2.1 that f is continuous on $\bar{D}$, which implies that $1=f(w)=\frac{1}{2\pi}\int_{\partial D} f(z)\overline{\hat{R}(z,\bar{w})}(\frac{\partial G(z,w)}{\partial \nu_z})^{-1}d|z|$. By Cauchy-Schwartz Lemma, it follows that

$$\begin{aligned}1 \leq&\frac{1}{(2\pi)^2}(\int_{\partial D}|f(z)|^2(\frac{\partial G(z,w)}{\partial \nu_z})^{-1}d|z|)\\ &\times(\int_{\partial D}|\hat{R}(z,\bar{w})|^2(\frac{\partial G(z,w)}{\partial \nu_z})^{-1}d|z|).\end{aligned} \tag{3.2}$$

As f is continuous on $\bar{D}$, it follows from Lemma 2.3 that

$$\begin{aligned}\lim_{r\to 1-0}\frac{1-r}{\int_{\{e^{2G(z,w)}\geq r\}}|f(z)|^2} =&(\int_{\partial D}|f(z)|^2(\frac{\partial 2G(z,w)}{\partial \nu_z})^{-1}d|z|)^{-1}\\ \leq&\frac{1}{2\pi^2}(\int_{\partial D}|\hat{R}(z,\bar{w})|^2(\frac{\partial G(z,w)}{\partial \nu_z})^{-1}d|z|)\\ =&\frac{1}{2\pi^2}(\int_{\partial D}\hat{R}(z,\bar{w})\overline{\hat{R}(z,\bar{w})}(\frac{\partial G(z,w)}{\partial \nu_z})^{-1}d|z|)\\ =&\frac{1}{\pi}\hat{R}(w,\bar{w})=\frac{1}{\pi}\hat{R}(w).\end{aligned} \tag{3.3}$$

Note that

$$g_w(-\log r)\leq\int_{\{e^{2G(z,w)}<r\}}|f(z)|^2,$$

then it follows from (3.1) that

$$g_w(0)-g_w(-\log r)\geq\int_D|f(z)|^2-\int_{\{e^{2G(z,w)}<r\}}|f(z)|^2=\int_{\{e^{2G(z,w)}\geq r\}}|f(z)|^2,$$

which implies

$$\lim_{r\to 1-0}\frac{r-1}{g_w(-\log r)-g_w(0)}\leq\lim_{r\to 1-0}\frac{1-r}{\int_{\{e^{2G(z,w)}\geq r\}}|f(z)|^2}. \tag{3.4}$$

It follows from (2.1), (2.6), (3.3) and (3.4) that

$$\begin{aligned}B(w)=\frac{1}{g_w(0)}\leq&\lim_{r\to 1-0}\frac{r-1}{g_w(-\log r)-g_w(0)}\\ \leq&\lim_{r\to 1-0}\frac{1-r}{\int_{\{e^{2G(z,w)}\geq r\}}|f(z)|^2}\leq\frac{1}{\pi}\hat{R}(w).\end{aligned} \tag{3.5}$$

Then we obtain that $''\geq''$ holds.

Step 2. It suffices to prove $B(w)\neq\frac{1}{\pi}\hat{R}(w)$. We prove by contradiction: if not, then $B(w)=\frac{1}{\pi}\hat{R}(w)$ holds. It follows from (3.5) that $\frac{1}{g_w(0)}=\lim_{r\to 1-0}\frac{r-1}{g_w(-\log r)-g_w(0)}$ (statement (1) of Corollary 2.1). Combining with Corollary 2.1 and Remark 2.2, we obtain that $n=1$ which contradicts $n>1$.

Then Theorem 1.1 has been proved.

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