Maximal regularity of the Stokes system with Navier boundary condition in general unbounded domains

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Dedicated to Professor Hideo Kozono on the occasion of his 60th birthday

Abstract

Consider the instationary Stokes system in general unbounded domains $\Omega \subset \mathbb{R}^n$, $n \geq 2$, with boundary of uniform class C^3, and Navier slip or Robin boundary condition. The main result of this article is the maximal regularity of the Stokes operator in function spaces of the type $\tilde{L}^q$ defined as $L^q \cap L^2$ when $q \geq 2$, but as $L^q + L^2$ when $1 < q < 2$, adapted to the unboundedness of the domain.

1 Introduction and main result

Given an unbounded domain $\Omega \subset \mathbb{R}^n$ and a finite time interval $(0, T)$ we consider for a prescribed external force $\boldsymbol{f} : \Omega \times (0, T) \to \mathbb{R}^n$ the instationary Stokes system with Navier boundary condition

$$\begin{aligned} \boldsymbol{u}_t - \nu\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad & \text{in } \Omega \times (0,T) \\ \operatorname{div} \boldsymbol{u} = 0 \quad & \text{in } \Omega \times (0,T) \\ \boldsymbol{u}(0) = \boldsymbol{0} \quad & \text{in } \Omega \\ \boldsymbol{u}\cdot\boldsymbol{n} = 0, \quad \alpha\boldsymbol{u} + \beta(\mathsf{T}(\boldsymbol{u},p)\boldsymbol{n})_\tau = \boldsymbol{0} \quad & \text{on } \partial\Omega \times (0,T). \end{aligned} \tag{1.1}$$

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Key Words: Stokes operator; maximal regularity; Navier boundary condition; Robin boundary condition; general unbounded domains; spaces $\tilde{L}^q(\Omega)$

Mathematics Subject Classification (2010): 35B65; 35Q30; 76D07

Here $\boldsymbol{u} : \Omega \times (0,T) \to \mathbb{R}^n$, $p : \Omega \times (0,T) \to \mathbb{R}$ are the unknown velocity field and pressure, respectively. The tensor $\mathsf{T} = \mathsf{T}(\boldsymbol{u}, p) = -p\mathsf{I} + \mathsf{S}(\boldsymbol{u}) = -p\mathsf{I} + 2\nu\mathsf{D}(\boldsymbol{u})$ is the Cauchy stress tensor where $\mathsf{D}(\boldsymbol{u}) = \frac{1}{2}(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^\top)$ denotes the symmetric part of the velocity gradient, and $\nu > 0$ is the viscosity. As usual for the Stokes system we set $\nu = 1$ and obtain the viscous stress tensor

$$\mathsf{S}(\boldsymbol{u}) = \nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^\top.$$

Let $\boldsymbol{n}$ denote the unit outer normal to $\partial\Omega$, and let the subscript τ indicate the tangential component(s) of a vector field on $\partial\Omega$; to be more precise, for $\boldsymbol{y} \in \mathbb{R}^n$ we have $\boldsymbol{y}_\tau = \boldsymbol{y} - (\boldsymbol{y} \cdot \boldsymbol{n})\boldsymbol{n}$. The constants $\alpha \in [0, 1)$ and $\beta \in (0, 1]$ satisfy $\alpha + \beta = 1$. Hence the boundary condition $\alpha\boldsymbol{u} + \beta(\mathsf{T}(\boldsymbol{u}, p)\boldsymbol{n})_\tau = \mathbf{0}$ (called Navier or Robin condition or of third type) simplifies to

$$B(\boldsymbol{u}) = B_{\alpha,\beta}(\boldsymbol{u}) := \alpha\boldsymbol{u} + \beta(\mathsf{S}(\boldsymbol{u})\boldsymbol{n})_\tau = \mathbf{0} \tag{1.2}$$

and describes two different physical cases. For $\alpha = 0$, $\beta = 1$ we obtain the so-called no-stick or perfect slip condition (Navier boundary condition), meaning that the fluid is subject to no tangential stresses at the boundary. When $0 < \alpha, \beta < 1$, tangential stresses at the boundary are proportional to the tangential velocity $\boldsymbol{u}_\tau = \boldsymbol{u}$ on $\partial\Omega$ (Robin or third type boundary condition); also recall the impermeability condition $\boldsymbol{u} \cdot \boldsymbol{n} = \mathbf{0}$ on $\partial\Omega$. For references on these boundary conditions and their physical meaning see [36].

The starting point for analytic semigroup theory applied to the instationary Stokes system is the Stokes resolvent problem. From the rich literature for the Dirichlet case ($\boldsymbol{u} = \mathbf{0}$ on $\partial\Omega$) we mention [38] for a potential theoretic approach, [25] for a method using pseudodifferential operators, and [20] for multiplier techniques for the whole and half space followed by localization methods for bounded and exterior domains. Moreover, we refer to [13, 14] for infinite cylinders, and to [1, 2] for layers. Resolvent estimates in weighted function spaces for (bent) half spaces and aperture domains are considered in [21, 22, 23]. The techniques used in most of these papers exclude many other interesting unbounded domains, e.g., domains with several exits to infinity, with infinitely many holes, with spiraling exits, wedges with smooth vertex etc.

The semigroup approach for the Navier boundary condition (1.2) was first considered by Giga in [26] for a bounded domain as a special case of a more general condition. For the case $\mathbb{R}^n_+$ Saal [31] showed that the Stokes operator generates an analytic semigroup and admits a bounded H^∞-calculus. In [33] Shibata and Shimada proved the unique solvability of the Stokes resolvent system with the Navier boundary condition for bounded and exterior domains. This is done by a cut-off technique, where - as for the Dirichlet case - existence and uniqueness are proven successively for the whole space, the half space, bent half spaces and a bounded (or exterior) domain. We note that an inhomogeneous divergence as well as non-zero boundary conditions are included; this will also be used in our analysis. Finally, Shimada [36] proved maximal $L^s(L^q)$-regularity of the instationary system with both Navier and Robin boundary condition for bounded domains.

For the case of the Neumann boundary condition where $\mathbf{T}(\boldsymbol{u}, p) \cdot \boldsymbol{n} = \boldsymbol{\varphi}$ is prescribed on $\partial\Omega$ similar results were obtained by Shibata and Shimizu, see [34] for the resolvent equation in bounded and exterior domains and [35] for the instationary system in a bounded domain. The Neumann and further boundary conditions were also treated by Shibata [32] and in several papers of Solonnikov and Grubb (e.g. in [27]) using pseudo-differential operators. Boundary conditions in terms of differential forms were considered by Miyakawa [30].

Due to counter-examples by Bogovskij and Maslennikova [4, 29] the Helmholtz decomposition of vector fields in $L^q(\Omega)$, $1 < q < \infty$, on an unbounded smooth domain may fail unless $q = 2$. By analogy, a bounded Helmholtz projection P_q with the properties required to define the Stokes operator $A_q = -P_q\Delta$ when $q \neq 2$ may not exist. Therefore, in [7, 8, 9, 10, 11] H. Kozono, H. Sohr and the first author of this article introduced the spaces

$$\tilde{L}^q(\Omega) := \begin{cases} L^q(\Omega) + L^2(\Omega), & \text{if } 1 \leq q < 2, \\ L^q(\Omega) \cap L^2(\Omega), & \text{if } 2 \leq q \leq \infty. \end{cases} \tag{1.3}$$

The corresponding norm is defined as $\|u\|_{\tilde{L}^q} = \max\{\|u\|_q, \|u\|_2\}$ when $q \geq 2$, and as $\inf\{\|u_1\|_q + \|u_2\|_2 : u = u_1 + u_2, u_1 \in L^q(\Omega), u_2 \in L^2(\Omega)\}$ when $1 \leq q < 2$. For bounded domains $\tilde{L}^q(\Omega) = L^q(\Omega)$ with equivalent norms. We note that functions in $\tilde{L}^q(\Omega)$ locally behave like L^q-functions, but globally exploit L^2-properties. By well-known results of interpolation theory, $\tilde{L}^q(\Omega)' \cong \tilde{L}^{q'}(\Omega)$ when $1 \leq q < \infty$.

By analogy, function spaces like $\tilde{L}^q_\sigma(\Omega)$ of solenoidal vector fields, $\tilde{G}^q(\Omega) = \{\nabla p \in \tilde{L}^q(\Omega)\}$ of gradient fields and $\tilde{W}^{k,q}(\Omega)$ of weakly differentiable functions will be defined. In [9] the authors showed for general uniformly smooth domain $\Omega \subset \mathbb{R}^n$ that in $\tilde{L}^q(\Omega)$, $1 < q < \infty$, the corresponding Helmholtz projection $\tilde{P}_q$ is a bounded operator yielding the algebraic and topological decomposition $\tilde{L}^q(\Omega) = \tilde{L}^q_\sigma(\Omega) \oplus \tilde{G}^q(\Omega)$. Its norm is bounded by a constant depending only on q and the type τ_Ω of the domain; for the definition of τ_Ω see Assumption 1.1 below.

The Stokes operator $\tilde{A}_q = -\tilde{P}_q\Delta$ with Dirichlet boundary condition generates an analytic semigroup ([8, 11]) on $\tilde{L}^q_\sigma(\Omega)$ and has the property of maximal regularity ([10]). Moreover, Kunstmann [28] showed that $\tilde{A}_q$ admits a bounded H^∞-calculus. These results are applied by Riechwald and the first author in

[15, 16, 17] in order to develop the theory of mild, strong and very weak solutions to the Navier-Stokes equations with Dirichlet boundary condition in uniformly smooth domains. For a recent review including proofs we refer to [12].

To work in general unbounded domains we use the exhaustion method, *i.e.*, we approximate Ω from the interior by a sequence of increasing bounded domains $\Omega_j \subset \Omega$. In the case of the Dirichlet boundary condition $\boldsymbol{u} = \boldsymbol{0}$ on $\partial\Omega$ (see [8]) the boundary condition is included in the definition of the space $W_0^{1,q}(\Omega)$ and easily transferred from the spaces $W_0^{1,q}(\Omega_j)$ to $W_0^{1,q}(\Omega)$ as $j \to \infty$. A similar approach cannot directly be applied to the Robin boundary condition. Moreover, we do not have a global trace theorem at hand for general unbounded domains. Therefore, we pose some further restrictions on the domain Ω, see Assumption 1.1 below. Actually, it is not clear whether there are uniform C^3-domains *not* fulfilling Assumption 1.1; for precise definitions we refer to Sect. 2.1 below.

Assumption 1.1. A uniform C^3-domain $\Omega \subset \mathbb{R}^n$ of type $\tau_\Omega = (\tilde{\alpha}, \tilde{\beta}, K)$ is assumed to have the following representation: There exists a sequence $\{\Omega_j\}_{j\in\mathbb{N}}$ of bounded uniform C^3-domains of the same type τ_Ω such that $\Omega_j \subset \Omega$ and

- $\Omega_j \subset \Omega_{j+1}$ for all $j \in \mathbb{N}$ and $\Omega = \bigcup_{j=1}^{\infty} \Omega_j$,
- $\Gamma_j := \partial\Omega_j \cap \partial\Omega \neq \emptyset$ for all $j \in \mathbb{N}$,
- $\Gamma_j \subset \Gamma_{j+1}$ for all $j \in \mathbb{N}$ and $\partial\Omega = \bigcup_{j=1}^{\infty} \Gamma_j$.

To define the Stokes operator with Navier boundary condition $B_{\alpha,\beta}(\boldsymbol{u}) = \boldsymbol{0}$ we introduce for $1 < q < \infty$ the Sobolev space

$$W_B^{2,q}(\Omega) = \big\{\boldsymbol{u} \in W^{2,q}(\Omega) : B(\boldsymbol{u}) = \boldsymbol{0} \text{ on } \partial\Omega\big\}.$$

The boundary condition for the space $W_B^{2,q}(\Omega)$ is understood locally in the sense of usual traces. Then for a bounded domain Ω the domain of the Stokes operator $A_{q,B} = -P_q\Delta$ is given by $\mathcal{D}(A_{q,B}) = L^q_\sigma(\Omega) \cap W_B^{2,q}(\Omega)$. However, this definition is not suitable for general unbounded domains. For this reason, let

$$\tilde{\mathcal{D}}^q_B(\Omega) = \mathcal{D}(\tilde{A}_{q,B}) = \begin{cases} \mathcal{D}(A_{q,B}) \cap \mathcal{D}(A_{2,B}), & 2 \le q < \infty, \\ \mathcal{D}(A_{q,B}) + \mathcal{D}(A_{2,B}), & 1 < q < 2. \end{cases} \tag{1.4}$$

Using the Helmholtz projection $\tilde{P}_q$ we define the Stokes operator with Navier boundary condition for a general uniformly smooth domain as

$$\tilde{A}_{q,B} = -\tilde{P}_q\Delta : \tilde{\mathcal{D}}^q_B(\Omega) \subset \tilde{L}^q_\sigma(\Omega) \to \tilde{L}^q_\sigma(\Omega). \tag{1.5}$$

Concerning the Stokes resolvent system $\lambda\boldsymbol{u} + \tilde{A}_{q,B}\boldsymbol{u} = \tilde{P}_q\boldsymbol{f}$ related to (1.1) we cite the following results [18]:

Theorem 1.2 (Resolvent problem for $\tilde{A}_{q,B}$). *Let $1 < q < \infty$, $0 < \varepsilon < \frac{\pi}{2}$, $\delta > 0$. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a uniform C^3-domain of type τ_Ω and let Assumption 1.1 be satisfied. Then the following assertions hold:*

(i) *The sector $\mathcal{S}_\varepsilon = \{\lambda \in \mathbb{C} : \lambda \neq 0, |\arg \lambda| < \pi - \varepsilon\}$ is contained in the resolvent set of $-\tilde{A}_{q,B}$, and the resolvent $(\lambda + \tilde{A}_{q,B})^{-1} : \tilde{L}^q_\sigma(\Omega) \to \tilde{L}^q_\sigma(\Omega)$ satisfies the resolvent estimate*

$$\|\lambda \boldsymbol{u}\|_{\tilde{L}^q(\Omega)} + \|\boldsymbol{u}\|_{\tilde{W}^{2,q}(\Omega)} \leq C\|\boldsymbol{f}\|_{\tilde{L}^q(\Omega)} \tag{1.6}$$

for $\boldsymbol{f} \in \tilde{L}^q_\sigma(\Omega)$, $\boldsymbol{u} = (\lambda + \tilde{A}_{q,B})^{-1}\boldsymbol{f}$, $\lambda \in \mathcal{S}_\varepsilon$ with $|\lambda| \geq \delta$, where $C = C(q, \varepsilon, \delta, \tau_\Omega) > 0$.

(ii) *The Stokes operator $\tilde{A}_{q,B} : \tilde{\mathcal{D}}^q_B(\Omega) \to \tilde{L}^q_\sigma(\Omega)$ is a densely defined closed operator, and $-\tilde{A}_{q,B}$ generates an analytic semigroup $\{e^{-t\tilde{A}_{q,B}}\}_{t\geq 0}$ in $\tilde{L}^q_\sigma(\Omega)$ satisfying the estimate*

$$\|e^{-t\tilde{A}_{q,B}}\boldsymbol{f}\|_{\tilde{L}^q(\Omega)} \leq Ce^{\delta t}\|\boldsymbol{f}\|_{\tilde{L}^q(\Omega)} \tag{1.7}$$

for $\boldsymbol{f} \in \tilde{L}^q_\sigma(\Omega)$, $t \geq 0$, where $C = C(q, \delta, \tau_\Omega) > 0$.

(iii) *The norms*

$$\|\cdot\|_{\tilde{W}^{2,q}}, \quad \|\cdot\|_{\tilde{L}^q} + \|\tilde{A}_{q,B}\cdot\|_{\tilde{L}^q}, \quad \|\cdot\|_{\tilde{L}^q} + \|(1+\tilde{A}_{q,B})\cdot\|_{\tilde{L}^q}, \quad \|(1+\tilde{A}_{q,B})\cdot\|_{\tilde{L}^q}$$

are equivalent on $\mathcal{D}(1 + \tilde{A}_{q,B}) := \mathcal{D}(\tilde{A}_{q,B})$ with a constant depending on Ω only through τ_Ω. Moreover, the adjoint operator satisfies $\langle \tilde{A}_{q,B}\boldsymbol{u}, \boldsymbol{v}\rangle = \langle \boldsymbol{u}, \tilde{A}_{q',B}\boldsymbol{v}\rangle$ for all $\boldsymbol{u} \in \mathcal{D}(\tilde{A}_{q,B})$, $\boldsymbol{v} \in \mathcal{D}(\tilde{A}_{q',B})$, and $(\tilde{A}_{q,B})' = \tilde{A}_{q',B}$.

Here and in the following we will frequently omit the symbol Ω in norms like $\|\cdot\|_{L^q(\Omega)}$ when the domain is clear from the context.

The main result of this article concerns the maximal regularity of the Stokes operator $\tilde{A}_{q,B}$.

Theorem 1.3 (Maximal regularity for $\tilde{A}_{q,B}$). *Let $1 < q, s < \infty$, $0 < T < \infty$. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a uniform C^3-domain of type τ_Ω and let Assumption 1.1 be satisfied. Then the following assertions hold:*

(i) *For every $\boldsymbol{f} \in L^s(0, T; \tilde{L}^q_\sigma(\Omega))$ and every $\boldsymbol{u}_0 \in \tilde{\mathcal{D}}^q_B(\Omega)$ there exists a unique solution $\boldsymbol{u} \in L^s(0, T; \tilde{\mathcal{D}}^q_B(\Omega))$ with $\boldsymbol{u}_t \in L^s(0, T; \tilde{L}^q_\sigma(\Omega))$ of the Cauchy problem*

$$\boldsymbol{u}_t + \tilde{A}_{q,B}\boldsymbol{u} = \boldsymbol{f}, \quad \boldsymbol{u}(0) = \boldsymbol{u}_0,$$

satisfying the estimate

$$\|\boldsymbol{u}_t\|_{L^s(0,T;\tilde{L}^q(\Omega))} + \|\boldsymbol{u}\|_{L^s(0,T;\tilde{L}^q(\Omega))} + \|\tilde{A}_{q,B}\boldsymbol{u}\|_{L^s(0,T;\tilde{L}^q(\Omega))} \\ \le C\big(\|\boldsymbol{f}\|_{L^s(0,T;\tilde{L}^q(\Omega))} + \|\boldsymbol{u}_0\|_{\tilde{\mathcal{D}}^q_B(\Omega)}\big) \qquad (1.8)$$

with a constant $C = C(\tau_\Omega, T, q, s) > 0$. *By Theorem 1.2 (iii) a similar estimate holds for the term* $\|\boldsymbol{u}\|_{L^s(0,T;\tilde{W}^{2,q}(\Omega))}$.

(ii) *For every* $\boldsymbol{f} \in L^s(0,T;\tilde{L}^q(\Omega))$ *and every* $\boldsymbol{u}_0 \in \tilde{\mathcal{D}}^q_B(\Omega)$ *the instationary Stokes system* (1.1) *has a unique solution* $(\boldsymbol{u}, \nabla p) \in L^s(0,T;\tilde{\mathcal{D}}^q_B(\Omega)) \times L^s(0,T;\tilde{G}^q(\Omega))$ *with* $\boldsymbol{u}_t \in L^s(0,T;\tilde{L}^q_\sigma(\Omega))$, *defined by* $\boldsymbol{u}_t + \tilde{A}_{q,B}\boldsymbol{u} = \tilde{P}_q\boldsymbol{f}$, $\boldsymbol{u}(0) = \boldsymbol{u}_0$, *as well as* $\nabla p(t) = (I - \tilde{P}_q)(\boldsymbol{f} + \Delta\boldsymbol{u})(t)$ *and satisfying*

$$\|\boldsymbol{u}_t\|_{L^s(0,T;\tilde{L}^q(\Omega))} + \|\boldsymbol{u}\|_{L^s(0,T;\tilde{W}^{2,q}(\Omega))} + \|\nabla p\|_{L^s(0,T;\tilde{L}^q(\Omega))} \\ \le C\big(\|\boldsymbol{f}\|_{L^s(0,T;\tilde{L}^q(\Omega))} + \|\boldsymbol{u}_0\|_{\tilde{\mathcal{D}}^q_B(\Omega)}\big) \qquad (1.9)$$

with a positive constant $C = C(\tau_\Omega, T, q, s)$.

This article is organized as follows. In Sect. 2, we describe several preliminaries and recall necessary results for the bounded domain case. Sect. 3 contains the proof of Theorem 1.3 for bounded domains when $1 < s = q < \infty$ and $\boldsymbol{u}_0 = \boldsymbol{0}$. We solve the instationary equation in a bounded domain, focusing on the maximal regularity estimate in $\tilde{L}^q(\Omega)$ with a constant depending on Ω only through its type τ_Ω. For the case $2 \le s = q < \infty$, we use the localization procedure and local estimates in L^q as well as the global L^2-estimate. The case $1 < s = q < 2$ is treated by duality arguments.

The unbounded domain, see Sect. 4, is represented by a sequence of bounded domains, see Assumption 1.1. Extending the solutions of the instationary system in each of these bounded domains to the unbounded domain Ω we obtain a sequence with a uniform maximal regularity estimate in Ω. This uniformity is achieved since *a priori* constants in Theorem 1.3 do depend on the domain only through the type τ_Ω. Finally, weak-limit procedures yield a solution to the instationary system in the unbounded domain. The uniqueness of solutions is shown separately in Subsect. 4.3, together with the proof of the remaining cases $1 < s \ne q < \infty$, $\boldsymbol{u}(0) = \boldsymbol{u}_0 \ne \boldsymbol{0}$.

2 Preliminaries

2.1 Basic notation

Let us recall the definition of a uniform C^k-domain and its essential properties.

Definition 2.1. *A domain* $\Omega \subset \mathbb{R}^n$, $n \geq 2$, *is called a uniform* C^k*-domain of type* $\tau_\Omega = (\tilde{\alpha}, \tilde{\beta}, K)$, *where* $k \in \mathbb{N}$, $k \geq 2$, $\tilde{\alpha} > 0$, $\tilde{\beta} > 0$ *and* $K > 0$, *if for each* $x_0 \in \partial\Omega$ *there exist - after a translation and rotation - a Cartesian coordinate system with origin at* x_0 *and coordinates* $y = (y', y_n)$, $y' = (y_1, \dots, y_{n-1})$, *and a* C^k*-function* $h(y')$, $|y'| \leq \tilde{\alpha}$, *with* $\|h\|_{C^k} \leq K$ *such that the neighborhood*

$$U_{\tilde{\alpha},\tilde{\beta},h}(x_0) := \{y \in \mathbb{R}^n : h(y') - \tilde{\beta} < y_n < h(y') + \tilde{\beta},\ |y'| < \tilde{\alpha}\}$$

of x_0 *satisfies* $U_{\tilde{\alpha},\tilde{\beta},h}(x_0) \cap \partial\Omega = \{y = (y', y_n) \in \mathbb{R}^n : h(y') = y_n,\ |y'| < \tilde{\alpha}\}$ *and*

$$U^-_{\tilde{\alpha},\tilde{\beta},h}(x_0) := \{y \in \mathbb{R}^n : h(y') - \tilde{\beta} < y_n < h(y'),\ |y'| < \tilde{\alpha}\} = U_{\tilde{\alpha},\tilde{\beta},h}(x_0) \cap \Omega.$$

Notice that the constants $\tilde{\alpha}, \tilde{\beta}, K$ do not depend on $x_0 \in \partial\Omega$; moreover, the parameter τ_Ω is only related to Ω but is not a function of Ω. We may choose the new coordinate system $y = (y', y_n)$ such that the axes of y' are tangential to $\partial\Omega$ in x_0. Thus we have $h(0) = 0$, $\nabla' h(0) = 0$, and due to a continuity argument for each given constant $M > 0$ we can choose $\tilde{\alpha} > 0$ sufficiently small such that

$$\|h\|_{C^1} \leq M. \tag{2.1}$$

Considering a uniform C^3-domain of type $\tau_\Omega = (\tilde{\alpha}, \tilde{\beta}, K)$ there exists a covering of $\overline{\Omega}$ by open balls $B_j = B_r(x_j)$ where $x_j \in \overline{\Omega}$ and $r = r(\tau_\Omega) > 0$, *i.e.* $\overline{\Omega} \subset \bigcup_j B_j$, such that with appropriate functions $h_j \in C^3$

$$\overline{B}_j \subset U_{\tilde{\alpha},\tilde{\beta},h_j}(x_j) \quad \text{if} \quad x_j \in \partial\Omega, \qquad \overline{B}_j \subset \Omega \quad \text{if} \quad x_j \in \Omega.$$

The index j runs from 1 to some finite number $N \in \mathbb{N}$ if Ω is bounded and $j \in \mathbb{N}$ for Ω unbounded. The covering $\{B_j\}$ can be chosen in such a way that no more than some fixed number $N_0 = N_0(\tau_\Omega)$ of the balls have a nonempty intersection. Moreover, there exists a partition of unity $\{\varphi_j\}$, $\varphi_j \in C_0^\infty(\mathbb{R}^n)$, related to this covering, such that

$$0 \leq \varphi_j \leq 1, \quad \operatorname{supp} \varphi_j \subset B_j, \quad \sum_j \varphi_j = 1 \text{ on } \overline{\Omega} \tag{2.2}$$

$$\|\nabla \varphi_j\|_\infty,\ \|\nabla^2 \varphi_j\|_\infty,\ \|\nabla^3 \varphi_j\|_\infty \leq C = C(\tau_\Omega) \tag{2.3}$$

uniformly in j. For $x_j \in \Omega$ let us assume that $\operatorname{supp} \varphi_j \subset B_j^-$, where B_j^- denotes the lower half-ball of B_j.

Now we are able to localize our problem along $\partial\Omega$ to domains of the form

$$H' := H'_{\tilde{\alpha},\tilde{\beta},r,h} = \{y \in \mathbb{R}^n : h(y') - \tilde{\beta} < y_n < h(y'),\ |y'| < \tilde{\alpha}\} \cap B_r(0), \tag{2.4}$$

where we assume $\overline{B_r(0)} \subset \{y \in \mathbb{R}^n : h(y') - \tilde{\beta} < y_n < h(y') + \tilde{\beta},\ |y'| < \tilde{\alpha}\}$, and the function $h \in C_0^3(B_r'(0))$ satisfies $h(0) = 0$, $\nabla' h(0) = 0$, and the smallness asumption $\|h\|_{C^1} \le M$ is satisfied for some given $M > 0$. Here $B_r'(0)$, $0 < r = r(\tau_\Omega) < \tilde{\alpha}$, denotes an $(n-1)$-dimensional ball.

Thanks to the properties of the support of φ_j we can even work in domains H the boundary of which decomposes into two disjoint parts $\partial_1 H$, $\partial_2 H$ such that

$$H \subset H' \quad \text{and} \quad \partial H = \partial_1 H \dot{\cup} \partial_2 H, \text{ where } \partial_1 H \subset \{y \in \mathbb{R}^n : y_n = h(y')\}. \tag{2.5}$$

We choose H so that $\left(\operatorname{supp} \varphi_j \cap \overline{H'}\right) \subset \overline{H}$ and $\operatorname{dist}(\operatorname{supp} \varphi_j, \partial_2 H) > 0$, see Figure 1 below, and that H is a uniform C^3-domain of type τ_Ω. Such a domain H is bounded and uniformly star-shaped with respect to some ball

$$B_{r'}(x_0) \subset H \quad \text{where } 0 < \tilde{r} = \tilde{r}(\tau_\Omega) \le r' \le r = r(\tau_\Omega). \tag{2.6}$$

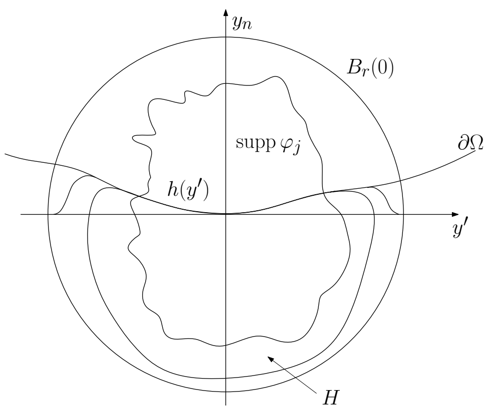

Figure 1: Illustration of a local domain H.

If Ω is unbounded, then it can be expressed as a union of countably many bounded uniform C^3-domains $\Omega_j \subset \Omega$, $j \in \mathbb{N}$, such that $\Omega_j \subset \Omega_{j+1}$ for all $j \in \mathbb{N}$ and $\Omega = \bigcup_{j=1}^\infty \Omega_j$. Each of these subdomains is of the same type $(\tilde{\alpha}', \tilde{\beta}', K')$ and we may assume that $\tilde{\alpha} = \tilde{\alpha}', \tilde{\beta} = \tilde{\beta}', K = K'$, *i.e.* $\tau_{\Omega_j} = \tau_\Omega$. Under Assumption 1.1 to hold in Theorem 1.3 we even suppose that $\Gamma_j := \partial\Omega_j \cap \partial\Omega \neq \emptyset$, $\Gamma_j \subset \Gamma_{j+1}$ for all $j \in \mathbb{N}$, and $\partial\Omega = \bigcup_{j=1}^\infty \Gamma_j$.

Let us introduce the following spaces of Sobolev type. Given $1 < q, q' < \infty$ such that $1 = \frac{1}{q} + \frac{1}{q'}$, let $W^{1,q}(\Omega)$ and $W_0^{1,q}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{W^{1,q}}}$ with norm $\|\cdot\|_{W^{1,q}(\Omega)}$ denote the usual L^q-Sobolev spaces. Then

$$\begin{aligned} W^{-1,q}(\Omega) &= \left(W_0^{1,q'}(\Omega)\right)' \\ \hat{W}^{1,q}(\Omega) &= \left\{u \in L^q_{\mathrm{loc}}(\overline{\Omega}) : \nabla u \in L^q(\Omega)\right\} \\ \hat{W}^{-1,q}(\Omega) &= \left(\hat{W}^{1,q'}(\Omega)\right)'. \end{aligned}$$

In the space $\hat{W}^{1,q}(\Omega)$ we identify two elements differing by a constant and equip it with the norm $\|\nabla\cdot\|_{L^q(\Omega)}$. If Ω is bounded, we may identify

$$\hat{W}^{1,q}(\Omega) = W^{1,q}(\Omega) \cap L^q_0(\Omega), \quad L^q_0(\Omega) := \Big\{u \in L^q(\Omega) : \int_\Omega u = 0\Big\}.$$

Note that we will often omit the symbol Ω for spaces of functions defined on Ω to keep notation short; by analogy, $\|u,\, v,\, \dots\| := \|u\| + \|v\| + \|\dots\|$ for some norm $\|\cdot\|$ even when u and v etc. may be functions, vector fields etc. with different number of components.

Lemma 2.2 (Poincaré and Friedrichs inequalities)**.** *Let* $1 < q < \infty$ *and* H *be a bounded domain as in* (2.5)*. Let either* $u \in W^{1,q}_0(H)$ *or* $u \in W^{1,q}(H)$*,* $\int_H u = 0$*. Then*

$$\|u\|_{L^q(H)} \le C(q, \tau_\Omega)\|\nabla u\|_{L^q(H)}. \tag{2.7}$$

In the case of vector fields $\boldsymbol{u} \in W^{1,q}(H)$ *satisfying* $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ *on* ∂H *a similar estimate holds.*

Proof. The result for $u \in W^{1,q}_0(H)$ is well known. For $u \in W^{1,q}(H)$, $\int_H u = 0$, the inequality holds with a constant $C = C(q, \Omega)$, see [24, Theorem II.5.4]. The more concrete dependence $C = C(q, \tau_\Omega)$ uses the uniform star-shapedness (2.6) and is proved in [19].

Concerning $\boldsymbol{u} \in W^{1,q}(H)$ satisfying $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on ∂H we apply [24, Exercise II.4.5] where the inequality is proven with a constant $C \le \operatorname{diam} H(|q-2|+n+1)$. Here $\operatorname{diam} H \le 2r(\tau_\Omega)$. □

Lemma 2.3 (Divergence equation, [8], Lemma 2.1 in [11])**.** *Let* $1 < q < \infty$*.*

(i) *There is a bounded linear operator* $R : L^q_0(H) \to W^{1,q}_0(H)$ *such that* $\operatorname{div} Rf = f$ *for all* $f \in L^q_0(H)$*. Moreover,*

$$\|Rf\|_{W^{1,q}(H)} \le C(q, \tau_\Omega)\|f\|_{L^q(H)} \quad \textit{for all } f \in L^q_0(H).$$

(ii) *There exists a constant* $C = C(q, \tau_\Omega) > 0$ *such that for every* $p \in L^q_0(H)$

$$\|p\|_{L^q(H)} \;\le\; C\|\nabla p\|_{W^{-1,q}(H)} \;=\; C \sup_{\boldsymbol{0}\neq\boldsymbol{v}\in W^{1,q'}_0(H)} \frac{|\langle p, \operatorname{div}\boldsymbol{v}\rangle|}{\|\nabla\boldsymbol{v}\|_{L^{q'}(H)}}. \tag{2.8}$$

Finally, we mention some interpolation inequalites for functions from $W^{2,q}(\Omega)$.

Lemma 2.4 (Interpolation estimates, [8], Lemma 2.3 in [11])**.** *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded* C^2*-domain of type* τ_Ω*.*

(i) *Let* $1 < q < \infty$. *Then for every* $0 < M < 1$ *there exists a positive constant* $C = C(M, q, \tau_\Omega)$ *such that for all* $u \in W^{2,q}(\Omega)$

$$\|\nabla u\|_{L^q(\Omega)} \leq M\|\nabla^2 u\|_{L^q(\Omega)} + C\|u\|_{L^q(\Omega)}. \tag{2.9}$$

(ii) *Let* $2 \leq q < \infty$. *Then for every* $0 < M < 1$ *there exists a positive constant* $C = C(M, q, \tau_\Omega)$ *such that for all* $u \in W^{2,q}(\Omega)$

$$\|u\|_{L^q(\Omega)} \leq M\|\nabla^2 u\|_{L^q(\Omega)} + C\big(\|\nabla^2 u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\big). \tag{2.10}$$

2.2 Maximal regularity for half spaces

Following [36] we introduce several function spaces used only in Lemmata 2.5 and 2.7 as well as in Proposition 2.6 below. Let $D \subset \mathbb{R}^n$ be a domain, $I \subset \mathbb{R}$ a time interval and X a Banach space. Given $1 < s, q < \infty$, $0 < T \leq \infty$ we define the spaces

$$\begin{aligned}
\dot{L}^s(I; X) &= \{u \in L^s(\mathbb{R}; X) : u = 0 \text{ for } t \notin I\}\\
\dot{W}^{1,s}([0,T); X) &= \{u \in W^{1,s}((-\infty, T); X) : u = 0 \text{ for } t < 0\}\\
W^{2,1}_{q,s}(D \times I) &= L^s(I; W^{2,q}(D)) \cap W^{1,s}(I; L^q(D))\\
\dot{W}^{2,1}_{q,s}(D \times [0,T)) &= \{u \in W^{2,1}_{q,s}(D \times (-\infty, T)) : u = 0 \text{ for } t < 0\},
\end{aligned}$$

where the second last one is equipped with the norm $\|u\|_{L^s(I;W^{2,q})} + \|u\|_{W^{1,s}(I;L^q)}$. Let $\mathcal{F}$ denote the Fourier transform with respect to time, and let $\theta \in \mathbb{R}$. We set $\langle D_t\rangle^\theta u(t) = \mathcal{F}^{-1}\big[(1+\xi^2)^{\theta/2}(\mathcal{F}u)(\xi)\big](t)$ and define the Bessel potential spaces

$$\begin{aligned}
H^{\theta,s}(\mathbb{R}; X) &= \{u \in L^s(\mathbb{R}; X) : \langle D_t\rangle^\theta u \in L^s(\mathbb{R}; X)\}\\
H^{1,1/2}_{q,s}(D \times \mathbb{R}) &= H^{1/2,s}(\mathbb{R}; L^q(D)) \cap L^s(\mathbb{R}; W^{1,q}(D))\\
\dot{H}^{1,1/2}_{q,s}(D \times \mathbb{R}_+) &= \{u \in H^{1,1/2}_{q,s}(D \times \mathbb{R}) : u = 0 \text{ for } t < 0\}\\
H^{1,1/2}_{q,s}(D \times (0,T)) &= \{u : \exists\, v \in H^{1,1/2}_{q,s}(D \times \mathbb{R}) \text{ with } u = v \text{ on } D \times (0,T)\}\\
\dot{H}^{1,1/2}_{q,s}(D \times (0,T)) &= \{u : \exists\, v \in \dot{H}^{1,1/2}_{q,s}(D \times \mathbb{R}_+) \text{ with } u = v \text{ on } D \times (0,T)\}.
\end{aligned}$$

These spaces are equipped with the following norms:

$$\begin{aligned}
\|u\|_{H^{\theta,s}(\mathbb{R};X)} &= \|u\|_{L^s(\mathbb{R};X)} + \|\langle D_t\rangle^\theta u\|_{L^s(\mathbb{R};X)}\\
\|u\|_{H^{1,1/2}_{q,s}(D\times\mathbb{R})} &= \|u\|_{H^{1/2,s}(\mathbb{R};L^q(D))} + \|u\|_{L^s(\mathbb{R};W^{1,q}(D))}\\
\|u\|_{H^{1,1/2}_{q,s}(D\times(0,T))} &= \inf\{\|v\|_{H^{1,1/2}_{q,s}(D\times\mathbb{R})} : v \in H^{1,1/2}_{q,s}(D\times\mathbb{R}),\ u = v \text{ on } D\times(0,T)\}\\
\|u\|_{\dot{H}^{1,1/2}_{q,s}(D\times(0,T))} &= \inf\{\|v\|_{H^{1,1/2}_{q,s}(D\times\mathbb{R})} : v \in \dot{H}^{1,1/2}_{q,s}(D\times\mathbb{R}_+),\ u = v \text{ on } D\times(0,T)\}.
\end{aligned}$$

The first step is the maximal regularity of the Stokes operator $\tilde{A}_{q,B}$ for the exact half space $\mathbb{R}^n_+$. We write $\boldsymbol{u} = (\boldsymbol{u}', u_n)$ with $\boldsymbol{u}' = (u_1, \dots u_{n-1})$, and similarly, $\boldsymbol{h}' = (h_1, \dots h_{n-1})$ for functions with $n-1$ variables.

Lemma 2.5 (The instationary system in $\mathbb{R}^n_+$)**.** *Let $1 < s, q < \infty$, $0 < T < \infty$ and let functions*

$$\boldsymbol{f} \in L^s(0,T; L^q(\mathbb{R}^n_+)), \quad g \in L^s(0,T; W^{1,q}(\mathbb{R}^n_+)) \cap W^{1,s}(0,T; \hat{W}^{-1,q}(\mathbb{R}^n_+))$$

with $g(t=0) = 0$, $\operatorname{supp} g(t) \subset B_R$ for $t \in [0,T]$ and some $R > 0$ be given. Moreover, let $\boldsymbol{h}' \in \dot{H}^{1,1/2}_{q,s}(\mathbb{R}^n_+ \times (0,T))$. Then there is a unique solution $\boldsymbol{u} \in W^{2,1}_{q,s}(\mathbb{R}^n_+ \times (0,T))$ and $p \in L^s(0,T; \hat{W}^{1,q}(\mathbb{R}^n_+))$ of the system

$$\begin{aligned} \boldsymbol{u}_t + \boldsymbol{u} - \operatorname{div} \mathsf{S}(\boldsymbol{u}) + \nabla p = \boldsymbol{f} \qquad & \text{in } \mathbb{R}^n_+ \times (0,T) \\ \operatorname{div} \boldsymbol{u} = g \qquad & \text{in } \mathbb{R}^n_+ \times (0,T) \\ u_n = 0, \ \frac{\partial \boldsymbol{u}'}{\partial x_n} = -\boldsymbol{h}' \qquad & \text{on } \partial\mathbb{R}^n_+ \times (0,T), \\ \boldsymbol{u}(0) = \boldsymbol{0} \qquad & \text{in } \mathbb{R}^n_+, \end{aligned} \tag{2.11}$$

which satisfies the estimate

$$\begin{aligned} \|\boldsymbol{u}_t, \nabla^2 \boldsymbol{u}, \nabla p\|_{L^s(0,T;L^q(\mathbb{R}^n_+))} \leq C(s,q)\big(&\|\boldsymbol{f}, g, \nabla g\|_{L^s(0,T;L^q(\mathbb{R}^n_+))} \\ &+ \|g_t, g\|_{L^s(0,T;\hat{W}^{-1,q}(\mathbb{R}^n_+))} + \|\boldsymbol{h}'\|_{H^{1,1/2}_{q,s}(\mathbb{R}^n_+ \times (0,T))}\big). \end{aligned} \tag{2.12}$$

Proof. The proof is based on [36, Theorem 5.1] where data $\boldsymbol{f} \in \dot{L}^s(\mathbb{R}_+; L^q(\mathbb{R}^n_+))$, $\boldsymbol{h}' \in \dot{H}^{1,1/2}_{q,s}(\mathbb{R}^n_+ \times \mathbb{R}_+)$, and $g \in \dot{L}^s(\mathbb{R}_+; W^{1,q}(\mathbb{R}^n_+)) \cap \dot{W}^{1,s}([0,\infty); \hat{W}^{-1,q}(\mathbb{R}^n_+))$ on the time interval $\mathbb{R}_+$ such that $\operatorname{supp} g(t) \subset B_R$ for all $t \in \mathbb{R}$ are considered. In this situation there is a unique solution $(\boldsymbol{u}, p) \in \dot{W}^{2,1}_{q,s}(\mathbb{R}^n_+ \times [0,\infty)) \times \dot{L}^s(\mathbb{R}_+; \hat{W}^{1,q}(\mathbb{R}^n_+))$ of (2.11) in $\mathbb{R}^n_+ \times \mathbb{R}_+$, satisfying the estimate (2.12) with time interval $(0,T)$ replaced by $\mathbb{R}_+$.

To prove (2.12) with the interval $[0,T]$ we define extensions $\boldsymbol{F}, G, \boldsymbol{H}'$ of $\boldsymbol{f}, g, \boldsymbol{h}'$, respectively, from $[0,T]$ to $\mathbb{R}$ as follows: Let $\boldsymbol{F}(t) = \boldsymbol{0}$ when $t \notin [0,T]$, and

$$G(t) = 0 \text{ when } t < 0 \text{ or } t > 2T, \text{ but } G(t) = g(2T - t) \text{ for } t \in [T, 2T].$$

Obviously, $\|\boldsymbol{F}\|_{\dot{L}^s(\mathbb{R};L^q)} = \|\boldsymbol{f}\|_{L^s(0,T;L^q)}$, and $\|G\|_{L^s(\mathbb{R};W^{1,q})} = 2^{1/s}\|g\|_{L^s(\mathbb{R};W^{1,q})}$; an analogous identity holds for $\|G\|_{W^{1,s}(\mathbb{R};\hat{W}^{-1,q})}$. Moreover, $\operatorname{supp} G(t) \subset B_R$ for all $t \in \mathbb{R}$. For $\boldsymbol{h}' \in \dot{H}^{1,1/2}_{q,s}(\mathbb{R}^n_+ \times (0,T))$, we choose an extension $\boldsymbol{H}' \in \dot{H}^{1,1/2}_{q,s}(\mathbb{R}^n_+ \times \mathbb{R}_+)$ such that $\boldsymbol{H}' = \boldsymbol{h}'$ on $\mathbb{R}^n_+ \times (0,T)$ and

$$\|\boldsymbol{H}'\|_{H^{1,1/2}_{q,s}(\mathbb{R}^n_+ \times \mathbb{R})} \leq 2\|\boldsymbol{h}'\|_{H^{1,1/2}_{q,s}(\mathbb{R}^n_+ \times (0,T))}. \tag{2.13}$$

Now we look for a solution of the system

$$\begin{aligned} \boldsymbol{z}_t + \boldsymbol{z} - \operatorname{div} \mathbf{S}(\boldsymbol{z}) + \nabla\theta &= \boldsymbol{F} && \text{in } \mathbb{R}^n_+ \times \mathbb{R}_+ \\ \operatorname{div} \boldsymbol{z} &= G && \text{in } \mathbb{R}^n_+ \times \mathbb{R}_+ \\ z_n = 0, \quad \frac{\partial \boldsymbol{z}'}{\partial x_n} &= -\boldsymbol{H}' && \text{on } \partial\mathbb{R}^n_+ \times \mathbb{R}_+, \\ \boldsymbol{z}(0) &= \boldsymbol{0} && \text{in } \mathbb{R}^n_+. \end{aligned}$$

According to [36, Theorem 5.1] there is a unique solution $\boldsymbol{z} \in \dot{W}^{2,1}_{q,s}(\mathbb{R}^n_+ \times [0,\infty))$, $\theta \in \dot{L}^s(\mathbb{R}_+; \hat{W}^{1,q}(\mathbb{R}^n_+))$ satisfying the estimate

$$\begin{aligned} &\|\boldsymbol{z}_t,\, \nabla^2\boldsymbol{z},\, \nabla\theta\|_{L^s(0,T;L^q)} \le \|\boldsymbol{z}_t,\, \nabla^2\boldsymbol{z},\, \nabla\theta\|_{L^s(\mathbb{R};L^q)} \\ &\quad\le C\big(\|\boldsymbol{F},\, G,\, \nabla G\|_{L^s(\mathbb{R};L^q)} + \|G_t,\, G\|_{L^s(\mathbb{R};\hat{W}^{-1,q})} + \|\boldsymbol{H}'\|_{H^{1,1/2}_{q,s}(\mathbb{R}^n_+\times\mathbb{R})}\big) \\ &\quad\le C\big(\|\boldsymbol{f},\, g,\, \nabla g\|_{L^s(0,T;L^q)} + \|g_t,\, g\|_{L^s(0,T;\hat{W}^{-1,q})} + \|\boldsymbol{h}'\|_{H^{1,1/2}_{q,s}(\mathbb{R}^n_+\times(0,T))}\big) \end{aligned}$$

with constants $C = C(s,q)$. Then $(\boldsymbol{u},p) = (\boldsymbol{z},\theta)\big|_{(0,T)}$ solves the system (2.11) and satisfies the desired estimate.

Uniqueness follows from the existence of solutions to the dual problem. □

In the next step we consider the case of bent half spaces. Given $\omega \in C^3(\mathbb{R}^{n-1})$ we define the bent half space

$$\tilde{H}_\omega = \{y \in \mathbb{R}^n : y_n > \omega(y')\}.$$

For the control of ω we use the definition $\|\nabla'\omega\|_{C^k} = \sum_{|\alpha'|\le k} \|\partial^{\alpha'}_{x'}\nabla'\omega\|_{L^\infty}$.

As auxiliary tools we need two estimates from [35, Propositions 2.6 and 2.8]:

Proposition 2.6. *Let $1 < s,q < \infty$.*

(i) Let $1 \le R < \infty$ and $D \subset \mathbb{R}^n$ be a domain. Then for any $v \in W^{1,s}(\mathbb{R}; L^q(D))$ there holds

$$\|v\|_{H^{1/2,s}(\mathbb{R};L^q(D))} \le C(s,q)\big(R^{-1/2}\|v_t\|_{L^s(\mathbb{R};L^q(D))} + R^{1/2}\|v\|_{L^s(\mathbb{R};L^q(D))}\big).$$

(ii) For $\boldsymbol{v} \in L^s(\mathbb{R}; W^{2,q}(\mathbb{R}^n_+)) \cap W^{1,s}(\mathbb{R}; L^q(\mathbb{R}^n_+))$ there holds

$$\|\langle D_t\rangle^{1/2}\nabla\boldsymbol{v}\|_{L^s(\mathbb{R};L^q(\mathbb{R}^n_+))} \le C(s,q)\big(\|\boldsymbol{v}_t,\, \boldsymbol{v},\, \nabla\boldsymbol{v},\, \nabla^2\boldsymbol{v}\|_{L^s(\mathbb{R};L^q(\mathbb{R}^n_+))}\big).$$

Proof. (i) is proved in [35, Proposition 2.6] for a fixed domain $D \subset \mathbb{R}^n$ with constant $C = C(s,q,D)$. To show that C can be chosen independent of the domain we use the trivial extension operator $E_D : L^q(D) \to L^q(\mathbb{R}^n)$ such that $E_D u(x) = 0$ for $x \notin D$. Since $\langle D_t\rangle^{1/2}$ commutes with E_D and thus $\|\langle D_t\rangle^{1/2}u\|_{L^s(\mathbb{R};L^q(D))} \le \|\langle D_t\rangle^{1/2}E_D u\|_{L^s(\mathbb{R};L^q(\mathbb{R}^n))}$, [35, Proposition 2.6] yields the assertion.

(ii) is a consequence of [35, Proposition 2.8]. □

Lemma 2.7 (The instationary system in bent half spaces)**.** *Let* $0 < T < \infty$ *and* $\tilde{H}_\omega$ *denote a bent half space with* $\omega \in C^3(\mathbb{R}^{n-1})$. *Let*

$$\boldsymbol{f} \in L^s(0,T;L^q(\tilde{H}_\omega)), \quad g \in L^s(0,T;W^{1,q}(\tilde{H}_\omega)) \cap W^{1,s}(0,T;\hat{W}^{-1,q}(\tilde{H}_\omega)),$$

with $\operatorname{supp} g(t) \subset B_R$ *for any* $t \in [0,T]$ *and some* $R > 0$, $g(0) = 0$ *be given. Moreover, let* $\boldsymbol{h} \in \dot{H}^{1,1/2}_{q,s}(\tilde{H}_\omega \times (0,T))$. *Assume that* $\boldsymbol{u} \in W^{2,1}_{q,s}(\tilde{H}_\omega \times (0,T))$ *and* $p \in L^s(0,T;\hat{W}^{1,q}(\tilde{H}_\omega))$ *solve*

$$\begin{aligned} \boldsymbol{u}_t + \boldsymbol{u} - \operatorname{div} \mathbf{S}(\boldsymbol{u}) + \nabla p &= \boldsymbol{f} && \text{in } \tilde{H}_\omega \times (0,T) \\ \operatorname{div} \boldsymbol{u} &= g && \text{in } \tilde{H}_\omega \times (0,T) \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0, \quad B_{\alpha,\beta}(\boldsymbol{u}) &= \boldsymbol{h} && \text{on } \partial\tilde{H}_\omega \times (0,T) \\ \boldsymbol{u}(0) &= \boldsymbol{0} && \text{in } \tilde{H}_\omega. \end{aligned} \tag{2.14}$$

Then there is a constant $0 < K_0 = K_0(q) < 1$ *such that if* $\|\nabla'\omega\|_{C^0} \le K_0$ *then the solution* $(\boldsymbol{u}, \nabla p)$ *satisfies the estimate*

$$\begin{aligned} &\|\boldsymbol{u}_t\|_{L^s(0,T;L^q(\tilde{H}_\omega))} + \|\boldsymbol{u}\|_{L^s(0,T;W^{2,q}(\tilde{H}_\omega))} + \|\nabla p\|_{L^s(0,T;L^q(\tilde{H}_\omega))} \\ &\qquad \le C\big(\|\boldsymbol{f}\|_{L^s(0,T;L^q(\tilde{H}_\omega))} + \|\boldsymbol{u},\, g\|_{L^s(0,T;W^{1,q}(\tilde{H}_\omega))} \\ &\qquad\quad + \|g_t,\, g\|_{L^s(0,T;\hat{W}^{-1,q}(\tilde{H}_\omega))} + \|\boldsymbol{h}\|_{H^{1,1/2}_{q,s}(\tilde{H}_\omega \times (0,T))}\big) \end{aligned}$$

with a constant $C = C(q, s, \|\nabla'\omega\|_{C^2}, T) > 0$.

Proof. Let $\boldsymbol{u} \in W^{2,1}_{q,s}(\tilde{H}_\omega \times (0,T))$ and $p \in L^s(0,T;\hat{W}^{1,q}(\tilde{H}_\omega))$ solve (2.14) in $\tilde{H}_\omega \times (0,T)$. In order to show the estimate we follow [36] and reduce the problem to the half-space.

We consider the bijection $\varphi : \tilde{H}_\omega \to \mathbb{R}^n_+$, $x = (x', x_n) \mapsto (x', x_n - \omega(x')) =: y$, with Jacobian $J_\varphi = 1$. For a function v defined on $\tilde{H}_\omega$ we set $\tilde{v}(y) = v(x)$. For derivatives we have $\frac{\partial}{\partial x_n} = \frac{\partial}{\partial y_n}$ and $\frac{\partial}{\partial x_j} = \frac{\partial}{\partial y_j} - \frac{\partial \omega}{\partial y_j}\frac{\partial}{\partial y_n}$, $j = 1, \dots, n-1$.

Let us denote $K_i = \|\nabla'\omega\|_{C^i}$, $i \ge 0$, and assume that $K_0 \le 1$. Setting $\boldsymbol{z}'(y) = \tilde{\boldsymbol{u}}'(y)$, $z_n(y) = \tilde{u}_n(y) - \nabla'\omega(y') \cdot \tilde{\boldsymbol{u}}'(y)$ and $\theta(y) = \tilde{p}(y)$ we see that $(\boldsymbol{z}, \theta)$ satisfies the following problem in the half-space

$$\begin{aligned} \boldsymbol{z}_t + \boldsymbol{z} - \operatorname{div} \mathbf{S}(\boldsymbol{z}) + \nabla\theta &= \tilde{\boldsymbol{f}} + \boldsymbol{R}(\boldsymbol{z}, \theta) && \text{in } \mathbb{R}^n_+ \times (0,T) \\ \operatorname{div} \boldsymbol{z} &= \tilde{g} && \text{in } \mathbb{R}^n_+ \times (0,T) \\ z_n = 0, \quad -\beta \frac{\partial \boldsymbol{z}'}{\partial y_n} &= \frac{\tilde{\boldsymbol{h}}}{\sqrt{1 + |\nabla'\omega|^2}} + \boldsymbol{R}_\partial(\boldsymbol{z}) && \text{on } \partial\mathbb{R}^n_+ \times (0,T) \\ \boldsymbol{z}(0) &= \boldsymbol{0} && \text{in } \mathbb{R}^n_+, \end{aligned} \tag{2.15}$$

where $\tilde{\boldsymbol{f}}, \tilde{g}$ are defined by $\boldsymbol{f}, g$ as described above and $\operatorname{supp} \tilde{g}(t)$ is contained in a fixed compact set for all t. The remainder terms $\boldsymbol{R}(\boldsymbol{z}, \theta)$ and $\boldsymbol{R}_\partial(\boldsymbol{z})$ are linear with respect to $\boldsymbol{z}_t, \boldsymbol{z}, \nabla\boldsymbol{z}, \nabla^2\boldsymbol{z}, \nabla\theta$. To be more precise, $\boldsymbol{R}$ depends on $\boldsymbol{z}_t, \boldsymbol{z}, \nabla\boldsymbol{z}, \nabla^2\boldsymbol{z}$ and $\nabla\theta$, but at each instance $\boldsymbol{z}_t$, $\nabla^2\boldsymbol{z}$ and $\nabla\theta$ are multiplied by components of $\nabla'\omega$, whereas $\boldsymbol{R}_\partial$ depends on $\boldsymbol{z}, \nabla\boldsymbol{z}$ such that $\nabla\boldsymbol{z}$ is multiplied by $\nabla'\omega$. It is important to note that $\boldsymbol{R}_\partial(\boldsymbol{z})$ also contains the term $\alpha\boldsymbol{z}'$ from the boundary condition $B_{\alpha,\beta}(\boldsymbol{u}) = 0$.

Hence, assuming $K_0 \le 1$, there are constants C and C_{K_1,K_2} such that

$$\begin{aligned}&\|\boldsymbol{R}(\boldsymbol{z},\theta)\|_{L^s(0,T;L^q(\mathbb{R}^n_+))} \\ &\quad \le C\big(K_0\|\boldsymbol{z}_t,\ \nabla^2\boldsymbol{z},\ \nabla\theta\|_{L^s(0,T;L^q(\mathbb{R}^n_+))} + C_{K_1,K_2}\|\boldsymbol{z}\|_{L^s(0,T;W^{1,q}(\mathbb{R}^n_+))}\big).\end{aligned} \tag{2.16}$$

To get an estimate of $\|\boldsymbol{R}_\partial(\boldsymbol{z})\|_{H^{1,1/2}_{q,s}(\mathbb{R}^n_+\times(0,T))}$ we define the extension $\boldsymbol{Z}$ of $\boldsymbol{z}$ from $[0,T]$ to $\mathbb{R}$ as the extension G of g in the proof of Lemma 2.5 above, *i.e.*, $\boldsymbol{Z}(t) = \boldsymbol{0}$ when $t < 0$ or $t > 2T$, but $\boldsymbol{Z}(t) = \boldsymbol{z}(2T-t)$ for $t \in [T,2T]$. Hence $\boldsymbol{Z} \in W^{2,1}_{q,s}(\mathbb{R}^n_+\times\mathbb{R})$ and $\|\boldsymbol{Z}\|_{L^s(\mathbb{R};L^q(\mathbb{R}^n_+))} = 2^{1/s}\|\boldsymbol{z}\|_{L^s(0,T;L^q(\mathbb{R}^n_+))}$; similar results hold for $\|\boldsymbol{Z}_t\|_{L^s(\mathbb{R};L^q(\mathbb{R}^n_+))}$ and $\|\boldsymbol{Z}\|_{L^s(\mathbb{R};W^{2,q}(\mathbb{R}^n_+))}$. Then for $\boldsymbol{R}_\partial(\boldsymbol{z})$ we have the estimate

$$\begin{aligned}\|\boldsymbol{R}_\partial(\boldsymbol{z})\|_{H^{1,1/2}_{q,s}(\mathbb{R}^n_+\times(0,T))} &\le \|\boldsymbol{R}_\partial(\boldsymbol{Z})\|_{H^{1,1/2}_{q,s}(\mathbb{R}^n_+\times\mathbb{R})} \\ &\le CK_0\|\nabla\boldsymbol{Z}\|_{H^{1,1/2}_{q,s}(\mathbb{R}^n_+\times\mathbb{R})} + C_{K_1,K_2}\|\boldsymbol{Z}\|_{H^{1,1/2}_{q,s}(\mathbb{R}^n_+\times\mathbb{R})}.\end{aligned} \tag{2.17}$$

The first term on the right-hand side is analyzed using Proposition 2.6 (ii) as follows:

$$\begin{aligned}\|\nabla\boldsymbol{Z}\|_{H^{1,1/2}_{q,s}(\mathbb{R}^n_+\times\mathbb{R})} &= \|\nabla\boldsymbol{Z}\|_{L^s(\mathbb{R};W^{1,q}(\mathbb{R}^n_+))} + \|\nabla\boldsymbol{Z}\|_{H^{1/2,s}(\mathbb{R};L^q(\mathbb{R}^n_+))} \\ &= 2\|\nabla\boldsymbol{Z}\|_{L^s(\mathbb{R};W^{1,q}(\mathbb{R}^n_+))} + \|\langle D_t\rangle^{1/2}\nabla\boldsymbol{Z}\|_{L^s(\mathbb{R};L^q(\mathbb{R}^n_+))} \\ &\le c(s,q)\|\boldsymbol{Z}_t,\ \boldsymbol{Z},\ \nabla\boldsymbol{Z},\ \nabla^2\boldsymbol{Z}\|_{L^s(\mathbb{R};L^q(\mathbb{R}^n_+))}.\end{aligned}$$

For the second term we use Proposition 2.6 (i) and get with $\varepsilon > 0$ that

$$\begin{aligned}\|\boldsymbol{Z}\|_{H^{1,1/2}_{q,s}(\mathbb{R}^n_+\times\mathbb{R})} &= \|\boldsymbol{Z}\|_{L^s(\mathbb{R};W^{1,q}(\mathbb{R}^n_+))} + \|\boldsymbol{Z}\|_{H^{1/2,s}(\mathbb{R};L^q(\mathbb{R}^n_+))} \\ &\le C_\varepsilon\|\boldsymbol{Z}\|_{L^s(\mathbb{R};W^{1,q}(\mathbb{R}^n_+))} + \varepsilon\|\boldsymbol{Z}_t\|_{L^s(\mathbb{R};L^q(\mathbb{R}^n_+))}.\end{aligned} \tag{2.18}$$

Summarizing the last two estimates with (2.17) we see that

$$\begin{aligned}&\|\boldsymbol{R}_\partial(\boldsymbol{z})\|_{H^{1,1/2}_{q,s}(\mathbb{R}^n_+\times(0,T))} \\ &\quad \le C(K_0+\varepsilon)\|\boldsymbol{Z}_t,\ \nabla^2\boldsymbol{Z}\|_{L^s(\mathbb{R};L^q(\mathbb{R}^n_+))} + C_\varepsilon\|\boldsymbol{Z}\|_{L^s(\mathbb{R};W^{1,q}(\mathbb{R}^n_+))}\end{aligned} \tag{2.19}$$

where $\boldsymbol{Z}$ may be replaced by $\boldsymbol{z}$ with a minor change of C, C_ε.

To complete the proof we apply the corresponding half-space estimate (2.12) to (2.15) and obtain that

$$\begin{aligned}\|\boldsymbol{z}_t,\, \nabla^2\boldsymbol{z},\, \nabla\theta\|_{L^s(0,T;L^q(\mathbb{R}^n_+))} &\le C(s,q)\big(\|\tilde{\boldsymbol{f}},\, \boldsymbol{R}(\boldsymbol{z},\theta)\|_{L^s(0,T;L^q(\mathbb{R}^n_+))}\\ &\quad + \|\tilde{g}\|_{L^s(0,T;W^{1,q}(\mathbb{R}^n_+))} + \|\tilde{g}_t,\, \tilde{g}\|_{L^s(0,T;\hat{W}^{-1,q}(\mathbb{R}^n_+))}\\ &\quad + \|\tilde{\boldsymbol{h}},\, \boldsymbol{R}_\partial(\boldsymbol{z})\|_{H^{1,1/2}_{q,s}(\mathbb{R}^n_+\times(0,T))}\big);\end{aligned} \tag{2.20}$$

the term $(1+|\nabla'\omega|^2)^{-1/2}$ in front of $\tilde{\boldsymbol{h}}$ in (2.15) does not change the character of this inequality. Choosing in (2.16), (2.19) the constants K_0 and ε sufficiently small, the $L^s(0,T;L^q(\mathbb{R}^n_+))$-norms of $\boldsymbol{z}_t, \nabla^2\boldsymbol{z}$ and $\nabla\theta$ can be absorbed from the right hand side of (2.20). Thus we are led to the estimate

$$\begin{aligned}&\|\boldsymbol{z}_t,\, \nabla^2\boldsymbol{z},\, \nabla\theta\|_{L^s(0,T;L^q(\mathbb{R}^n_+))}\\ &\quad\le C(s,q)\big(\|\tilde{\boldsymbol{f}},\, \tilde{g},\, \nabla\tilde{g}\|_{L^s(0,T;L^q(\mathbb{R}^n_+))} + \|\tilde{g}_t,\, \tilde{g}\|_{L^s(0,T;\hat{W}^{-1,q}(\mathbb{R}^n_+))}\\ &\qquad + \|\tilde{\boldsymbol{h}}\|_{H^{1,1/2}_{q,s}(\mathbb{R}^n_+\times(0,T))} + \|\boldsymbol{z}\|_{L^s(0,T;W^{1,q}(\mathbb{R}^n_+))}\big).\end{aligned}$$

Finally, it suffices to estimate $\tilde{\boldsymbol{f}}, \tilde{g}, \tilde{g}_t$ and $\tilde{\boldsymbol{h}}$ by $\boldsymbol{f}, g, g_t$ and $\boldsymbol{h}$ in their respective norms to get the desired estimate for the solution $(\boldsymbol{u}, p)$. □

The main result of this subsection concerns maximal regularity estimates for solutions with support in bounded domains of type H, cf. (2.4), (2.5), (2.6).

Proposition 2.8 (The instationary system in H). *Let $0 < T < \infty$ and let $\boldsymbol{f} \in L^q(0,T;L^q(H))$ and $g \in L^q(0,T;W^{1,q}(H)) \cap W^{1,q}(0,T;\hat{W}^{-1,q}(H))$ satisfying $g(0) = 0$ be given, Moreover, let $\boldsymbol{h} \in L^q(0,T;W^{1,q}(H)) \cap W^{1,q}(0,T;L^q(H))$, $\boldsymbol{h}\cdot\boldsymbol{n}|_{\partial H\times(0,T)} = 0$, $\boldsymbol{h}(0) = \boldsymbol{0}$. Assume that the functions $\boldsymbol{u} \in W^{2,1}_{q,q}(H\times(0,T))$ and $p \in L^q(0,T;W^{1,q}(H))$ solve (2.14) (with $\tilde{H}_\omega$ replaced by H) and that uniformly for* a.a. $t \in [0,T]$

$$\operatorname{dist}(\operatorname{supp}\boldsymbol{u}(t) \cup \operatorname{supp} p(t), \partial_2 H) > 0. \tag{2.21}$$

Then there is a constant $C = C(\tau_\Omega, q, T) > 0$ such that

$$\begin{aligned}&\|\boldsymbol{u}_t, \boldsymbol{u},\, \nabla\boldsymbol{u},\, \nabla^2\boldsymbol{u},\, \nabla p\|_{L^q(0,T;L^q(H))}\\ &\qquad\le C\big(\|\boldsymbol{f},\, g,\, \nabla g\|_{L^q(0,T;L^q(H))} + \|g_t,\, g\|_{L^q(0,T;\hat{W}^{-1,q}(H))}\\ &\qquad\quad + \|\boldsymbol{h}\|_{H^{1,1/2}_{q,q}(H\times(0,T))} + \|\boldsymbol{u}\|_{L^q(0,T;W^{1,q}(H))}\big).\end{aligned} \tag{2.22}$$

An analogous result holds for the backward Stokes system where $(2.14)_1$ is replaced by $-\boldsymbol{u}_t + \boldsymbol{u} - \operatorname{div}\mathbf{S}(\boldsymbol{u}) + \nabla p = \boldsymbol{f}$ with the initial value $\boldsymbol{u}(T) = \boldsymbol{0}$, and $\boldsymbol{g}(T) = 0$, $\boldsymbol{h}(T) = \boldsymbol{0}$.

Proof. Due to (2.21) we extend $\boldsymbol{u}$, p by zero so that $(\boldsymbol{u}, \nabla p)$ may be considered as a solution of the Stokes system in a bent half space and use Lemma 2.7. The smallness assumption is satisfied thanks to (2.1), where we choose $M < 1$. In our case $C = C(q, \|\nabla' \omega\|_{C^2}, T)$ means that $C = C(\tau_\Omega, q, T)$.

Moreover, since $\boldsymbol{h}(0) = \boldsymbol{0}$ or $\boldsymbol{h}(T) = 0$, $\boldsymbol{h}$ satisfies the estimate (2.18), *i.e.*,

$$\|\boldsymbol{h}\|_{H^{1,1/2}_{q,q}(H\times(0,T))} \leq C\|\boldsymbol{h}\|_{L^q(0,T;W^{1,q}(H))} + \varepsilon\|\boldsymbol{h}_t\|_{L^q(0,T;L^q(H))}, \tag{2.23}$$

where $\varepsilon > 0$ can be chosen sufficiently small.

For the backward equation the transformation $\tilde{t} = T - t$ is used. □

2.3 Maximal regularity of $A_{q,B}$ for bounded domains

We consider the instationary Stokes system

$$\begin{aligned} \boldsymbol{u}_t - \Delta\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad & \text{in } \Omega\times(0,T) \\ \operatorname{div}\boldsymbol{u} = 0 \quad & \text{in } \Omega\times(0,T) \\ \boldsymbol{u}(0) = \boldsymbol{u}_0 \quad & \text{at } t = 0 \\ \boldsymbol{u}\cdot\boldsymbol{n} = 0, \quad B(\boldsymbol{u}) = 0 \quad & \text{on } \partial\Omega\times(0,T), \end{aligned} \tag{2.24}$$

for bounded domains and define in view of the variation of constants formula the operators

$$\mathcal{J}_{s,q}\boldsymbol{f}(t) = \int_0^t e^{-(t-\tau)A_{q,B}}\boldsymbol{f}(\tau)\,\mathrm{d}\tau \quad \text{and} \quad \mathcal{J}'_{s,q}\boldsymbol{g}(t) = \int_t^T e^{-(\tau-t)A_{q,B}}\boldsymbol{g}(\tau)\,\mathrm{d}\tau$$

for $\boldsymbol{f}, \boldsymbol{g} \in L^s(0,T; L^q_\sigma(\Omega))$.

Theorem 2.9 (Maximal Regularity)**.** *Let $1 < q, s < \infty$, $0 < T < \infty$, and let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded C^3-domain.*

(i) *If $\boldsymbol{f} \in L^s(0,T; L^q(\Omega))$, $\boldsymbol{u}_0 \in \mathcal{D}_{q,s} = (L^q_\sigma(\Omega), \mathcal{D}(A_{q,B}))_{1-1/s,s}$, the instationary Stokes system (2.24) admits a unique solution $\boldsymbol{u} \in W^{2,1}_{q,s}(\Omega\times(0,T))$ and $p \in L^s(0,T; W^{1,q}(\Omega))$, which enjoys the estimate*

$$\begin{aligned} \|\boldsymbol{u}_t\|_{L^s(0,T;L^q(\Omega))} + \|\boldsymbol{u}\|_{L^s(0,T;W^{2,q}(\Omega))} &+ \|p\|_{L^s(0,T;W^{1,q}(\Omega))} \\ &\leq C\big(\|\boldsymbol{f}\|_{L^s(0,T;L^q(\Omega))} + \|\boldsymbol{u}_0\|_{\mathcal{D}_{q,s}}\big), \end{aligned} \tag{2.25}$$

where $C = C(q, s, T, \Omega)$ is independent of $\boldsymbol{u}, p, \boldsymbol{f}, \boldsymbol{u}_0$.

(ii) *In particular, for $\boldsymbol{f} \in L^s(0,T; L^q_\sigma(\Omega))$ and an initial value $\boldsymbol{u}_0 \in \mathcal{D}(A_{q,B})$ the nonstationary Stokes system*

$$\boldsymbol{u}_t + A_{q,B}\boldsymbol{u} = \boldsymbol{f}, \quad \boldsymbol{u}(0) = \boldsymbol{u}_0, \tag{2.26}$$

has a unique solution $\boldsymbol{u} \in L^s(0,T;\mathcal{D}(A_{q,B}))$ *given by*

$$\boldsymbol{u}(t) = e^{-tA_{q,B}}\boldsymbol{u}_0 + \mathcal{J}_{s,q}\boldsymbol{f}(t)$$

and satisfying the estimate

$$\begin{aligned} &\|\boldsymbol{u}_t\|_{L^s(0,T;L^q(\Omega))} + \|\boldsymbol{u}\|_{L^s(0,T;L^q(\Omega))} + \|A_{q,B}\boldsymbol{u}\|_{L^s(0,T;L^q(\Omega))} \\ &\qquad \le C(q,s,T,\Omega)\big(\|\boldsymbol{f}\|_{L^s(0,T;L^q(\Omega))} + \|\boldsymbol{u}_0\|_{\mathcal{D}(A_{q,B})}\big). \end{aligned} \tag{2.27}$$

(iii) *For the same data, the backward Stokes system* $-\boldsymbol{u}_t + A_{q,B}\boldsymbol{u} = \boldsymbol{f}$ *with* $\boldsymbol{u}(T) = \boldsymbol{u}_0$, *has a unique solution* $\boldsymbol{u} \in L^s(0,T;\mathcal{D}(A_{q,B}))$ *given by* $\boldsymbol{u}(t) = e^{-(T-t)A_{q,B}}\boldsymbol{u}_0 + \mathcal{J}'_{s,q}\boldsymbol{f}(t)$, *satisfying the estimate* (2.27).

(iv) *There holds the duality relation* $(\mathcal{J}_{s,q})' = \mathcal{J}'_{s',q'}$.

(v) *In the case* $q = 2$ *the constant* $C = C(2,s,T,\Omega)$ *in* (2.27) *does* not *depend on the domain* Ω.

Proof. (i) This assertion is based on [36, Theorem 1.2]. Applying the Helmholtz projection P_q to $(2.24)_1$, we obtain (ii). Moreover, the unique solution of (2.26) can be represented by the integral formula $\boldsymbol{u}(t) = e^{-tA_q}\boldsymbol{u}_0 + \mathcal{J}_{s,q}\boldsymbol{f}(t)$.

(iii) The statements for the backward equation follow from (i), (ii).

(iv) Recalling $(A_{q,B})' = A_{q',B}$, for $\boldsymbol{f} \in L^s(0,T;L^q_\sigma(\Omega))$ and $\boldsymbol{g} \in L^{s'}(0,T;L^{q'}_\sigma(\Omega))$, we easily compute that $\langle \mathcal{J}_{s,q}\boldsymbol{f}, \boldsymbol{g}\rangle_{T,\Omega} = \langle \boldsymbol{f}, \mathcal{J}'_{s',q'}\boldsymbol{g}\rangle_{T,\Omega}$.

(v) Since $A_{2,B}$ generates an analytic semigroup the assertion follows from a general result of de Simon [6], see Lemma 2.10 below and also [37, Lemma IV.1.6.2]. Let us first consider the equation $\boldsymbol{u}_t + A_{2,B}\boldsymbol{u} = \boldsymbol{f}$, $\boldsymbol{u}(0) = \boldsymbol{0}$.

Lemma 2.10 (de Simon)**.** *Let* $1 < T \le \infty$, $1 < s < \infty$. *Let* $\boldsymbol{f} \in L^s(0,T;L^2_\sigma(\Omega))$ *and* $\boldsymbol{u} = \mathcal{J}_{s,2}\boldsymbol{f}$. *Then* $A_{2,B}\boldsymbol{u} \in L^s(0,T;L^2_\sigma(\Omega))$ *and*

$$\|\boldsymbol{u}_t,\ A_{2,B}\boldsymbol{u}\|_{L^s(0,T;L^2(\Omega))} \le C\|\boldsymbol{f}\|_{L^s(0,T;L^2(\Omega))} \tag{2.28}$$

with a constant $C = C(s)$ *independent of* T.

A thorough inspection of the arguments in [6] shows that the constant in (2.28) principally depends on the constant appearing in the resolvent estimate for A_2, which is independent of the domain Ω, see [18, Proposition 2.6 (ii)]. Thus C in (2.28) is independent of Ω. On the other hand, the dependence of the constant $C(q,s,T,\Omega)$, $q \neq 2$, in the maximal regularity estimates in Theorem 2.9 on Ω remains yet unclear.

Recall that in Theorem 2.9 $T < \infty$. If $\boldsymbol{u}_0 = \boldsymbol{0}$, the equation $\boldsymbol{u}_t = \boldsymbol{f} - A_2\boldsymbol{u}$ and (2.28) lead to estimates of $\boldsymbol{u}_t$ and $\boldsymbol{u}$ in $L^s(0,T;L^2(\Omega))$ by $\boldsymbol{f}$.

The non-homogeneous case with initial velocity $\mathbf{0} \neq \boldsymbol{u}_0 \in \mathcal{D}(A_{2,B})$ is easily reduced to the homogeneous case by considering $\boldsymbol{v}(t) = \boldsymbol{u}(t) - \boldsymbol{u}_0$.

Now the proof of Theorem 2.9 is complete. □

3 Maximal regularity of $\tilde{A}_{q,B}$ for bounded domains

Let $1 < q, s < \infty$ and $0 < T < \infty$. Similarly as for a bounded domain and the Stokes operator $A_{q,B}$, see Subsect. 2.3, we define the operators

$$\tilde{\mathcal{J}}_{s,q}\boldsymbol{f}(t) = \int_0^t e^{-(t-\tau)\tilde{A}_{q,B}}\boldsymbol{f}(\tau)\,\mathrm{d}\tau \quad \text{and} \quad \tilde{\mathcal{J}}'_{s,q}\boldsymbol{g}(t) = \int_t^T e^{-(\tau-t)\tilde{A}_{q,B}}\boldsymbol{g}(\tau)\,\mathrm{d}\tau$$

for $\boldsymbol{f} \in L^s(0,T;\tilde{L}^q_\sigma(\Omega))$ and $\boldsymbol{g} \in L^s(0,T;\tilde{L}^q_\sigma(\Omega))$, respectively. Since $(\tilde{L}^q_\sigma(\Omega))' = \tilde{L}^{q'}_\sigma(\Omega)$ as well as $(\tilde{A}_{q,B})' = \tilde{A}_{q',B}$, see Theorem 1.2 (iii), we get the duality relation

$$\langle \tilde{\mathcal{J}}_{s,q}\boldsymbol{f}, \boldsymbol{g}\rangle_{T,\Omega} = \langle \boldsymbol{f}, \tilde{\mathcal{J}}'_{s',q'}\boldsymbol{g}\rangle_{T,\Omega}.$$

Recall that for a bounded domain the spaces L^q and $\tilde{L}^q$ coincide. According to Theorem 2.9, we already know that the instationary system has a unique solution satisfiying the maximal regularity estimate with a constant $C = C(q,s,T,\Omega) > 0$. Hence, in order to apply the exhaustion method to a general unbounded domain, see Assumption 1.1, it suffices to show the estimate in the $L^s(0,T;\tilde{L}^q(\Omega))$-norm with a constant depending on Ω only through the parameter $\tau_\Omega = (\tilde{\alpha}, \tilde{\beta}, K)$.

3.1 The case $2 \leq s = q < \infty$

Let $2 \leq s = q < \infty$ so that $\tilde{L}^q(\Omega) = L^q(\Omega) \cap L^2(\Omega)$, and let $\boldsymbol{f} \in L^q(0,T;\tilde{L}^q_\sigma(\Omega))$. By Theorem 2.9 the function $\boldsymbol{u} = \tilde{\mathcal{J}}_{q,q}\boldsymbol{f}$ solves the equation

$$\boldsymbol{u}_t + \tilde{A}_{q,B}\boldsymbol{u} = \boldsymbol{u}_t - \Delta\boldsymbol{u} + \nabla p = \boldsymbol{f}, \quad \boldsymbol{u}(0) = \mathbf{0}, \tag{3.1}$$

with $\nabla p = (I - \tilde{P}_q)\Delta\boldsymbol{u}$. Our aim is to prove the estimate

$$\|\boldsymbol{u}_t,\, \boldsymbol{u},\, \nabla\boldsymbol{u},\, \nabla^2\boldsymbol{u},\, \nabla p\|_{L^q(0,T;\tilde{L}^q(\Omega))} \leq C\|\boldsymbol{f}\|_{L^q(0,T;\tilde{L}^q(\Omega))} \tag{3.2}$$

with $C = C(\tau_\Omega, T, q) > 0$.

Consider a parametrization $\{h_j\}$ of $\partial\Omega$, the covering of Ω with balls $\{B_j\}$ and the corresponding partition of unity $\{\varphi_j\}$, $1 \leq j \leq N$, as described in Subsect. 2.1. We define

$$U'_j := U^-_{\tilde{\alpha},\tilde{\beta},h_j}(x_j) \cap B_j \text{ for } x_j \in \partial\Omega, \quad U'_j := B_j \text{ for } x_j \in \Omega,\ 1 \leq j \leq N.$$

Hence we may work in domains $U_j \subset U_j'$, assume that each U_j has the form as the set H in (2.4), (2.5), (2.6), and apply the results of Proposition 2.8 for H.

We use the localization procedure as in [18]. Let $M_j = M_j(p)(t)$, $t \in (0,T)$, be the constant such that $p - M_j \in L^q(0,T; L^q_0(U_j))$. Multiplying the instationary equation by φ_j, $j = 1, \dots, N$, and adding the term $\boldsymbol{u}\varphi_j$ to both sides of the momentum equation we obtain the local system

$$
\begin{aligned}
(\boldsymbol{u}\varphi_j)_t + \boldsymbol{u}\varphi_j - \operatorname{div} \mathbf{S}(\boldsymbol{u}\varphi_j) + \nabla(\varphi_j(p - M_j)) & \\
= \boldsymbol{f}\varphi_j + (p - M_j)\nabla\varphi_j - 2\nabla\boldsymbol{u}\nabla\varphi_j & \\
- \Delta\varphi_j\boldsymbol{u} - (\nabla\boldsymbol{u})^T\nabla\varphi_j - \nabla^2\varphi_j\boldsymbol{u} + \boldsymbol{u}\varphi_j & \quad \text{in } U_j \times (0,T) \\
\operatorname{div}(\boldsymbol{u}\varphi_j) = \nabla\varphi_j \cdot \boldsymbol{u} & \quad \text{in } U_j \times (0,T) \\
\varphi_j\boldsymbol{u}\cdot\boldsymbol{n} = 0, \quad B(\boldsymbol{u}\varphi_j) = \beta\boldsymbol{u}(\nabla\varphi_j\cdot\boldsymbol{n}) & \quad \text{on } \partial U_j \times (0,T) \\
\varphi_j\boldsymbol{u}(0) = \mathbf{0} & \quad \text{in } U_j.
\end{aligned}
\tag{3.3}
$$

We assume that $\nabla\varphi_j \cdot \boldsymbol{n}$ is extended from ∂U_j to the whole of the C^3-domain U_j, and for the extension denoted by $\Phi_j \in C^2(\overline{U_j})$ it holds $\|\Phi_j\|_{C^2} \le C(\tau_\Omega)$ uniformly in $j = 1, \dots, N$.

Now we apply the local maximal regularity estimate (2.22) to (3.3) and obtain (with all norms taken over $(0,T) \times U_j$)

$$
\begin{aligned}
&\|(\varphi_j\boldsymbol{u})_t\|_{L^q(L^q)} + \|\boldsymbol{u}\varphi_j\|_{L^q(W^{2,q})} + \|\nabla(\varphi_j(p - M_j))\|_{L^q(L^q)} \\
&\quad \le C\big(\|\boldsymbol{f}\varphi_j,\, (p - M_j)\nabla\varphi_j,\, \nabla\boldsymbol{u}\nabla\varphi_j,\, \Delta\varphi_j\boldsymbol{u},\, (\nabla\boldsymbol{u})^T\nabla\varphi_j,\, \nabla^2\varphi_j\boldsymbol{u},\, \boldsymbol{u}\varphi_j\|_{L^q(L^q)} \\
&\qquad + \|\nabla\varphi_j\cdot\boldsymbol{u},\, \boldsymbol{u}\varphi_j\|_{L^q(W^{1,q})} + \|\nabla\varphi_j\cdot\boldsymbol{u},\, \nabla\varphi_j\cdot\boldsymbol{u}_t\|_{L^q(\hat{W}^{-1,q})} + \|\boldsymbol{u}\Phi_j\|_{H^{1,1/2}_{q,q}}\big).
\end{aligned}
$$

The property (2.3) of φ_j yields the inequality

$$
\begin{aligned}
&\|\varphi_j\boldsymbol{u}_t,\, \varphi_j\boldsymbol{u},\, \varphi_j\nabla\boldsymbol{u},\, \varphi_j\nabla^2\boldsymbol{u},\, \varphi_j\nabla p\|_{L^q(L^q)} \\
&\quad \le C\big(\|\boldsymbol{f},\, p - M_j,\, \boldsymbol{u},\, \nabla\boldsymbol{u}\|_{L^q(L^q)} + \|\nabla\varphi_j\cdot\boldsymbol{u}_t\|_{L^q(\hat{W}^{-1,q})} + \|\boldsymbol{u}\|_{H^{1,1/2}_{q,q}}\big)
\end{aligned}
\tag{3.4}
$$

with $C = C(\tau_\Omega, T, q) > 0$. Note that $\|\boldsymbol{u}\Phi_j\|_{H^{1,1/2}_{q,q}} \le C(q,\tau_\Omega)\|\boldsymbol{u}\|_{H^{1,1/2}_{q,q}}$ since the operator $\langle D_t\rangle^{1/2}$ commutes with the multiplication by Φ_j.

Hence it remains to estimate in (3.4) the last two terms and the pressure term. For the last one we use Proposition 2.6 (i) and get with $\varepsilon \in (0,1)$

$$
\|\boldsymbol{u}\|_{H^{1,1/2}_{q,q}(U_j\times(0,T))} \le C\|\boldsymbol{u}\|_{L^q(0,T;W^{1,q}(U_j))} + \varepsilon\|\boldsymbol{u}_t\|_{L^q(0,T;L^q(U_j))}. \tag{3.5}
$$

For the pressure term, thanks to $\nabla p = \boldsymbol{f} + \Delta\boldsymbol{u} - \boldsymbol{u}_t$ and (2.8), we have

$$
\begin{aligned}
\|p - M_j\|_{L^q(U_j)} \le C(\tau_\Omega, q)\Bigg(& \|\boldsymbol{f}\|_{L^q(U_j)} + \|\nabla\boldsymbol{u}\|_{L^q(U_j)} \\
& + \sup\left\{\frac{|\langle\boldsymbol{u}_t, \psi\rangle_{U_j}|}{\|\nabla\psi\|_{L^{q'}(U_j)}} : \mathbf{0} \ne \psi \in W^{1,q'}_0(U_j)\right\}\Bigg).
\end{aligned}
$$

To estimate $|\langle \boldsymbol{u}_t, \psi\rangle_{U_j}|$ for $\psi \in W_0^{1,q'}(U_j)$ we use the interpolation inequality

$$\|v\|_{L^r(U_j)} \le \theta(1/\epsilon)^{1/\theta}\|v\|_{L^2(U_j)} + (1-\theta)\epsilon^{1/(1-\theta)}\|v\|_{L^q(U_j)},$$

with $r \in [2, q]$, $\theta \in [0,1]$, $\frac{1}{r} = \frac{\theta}{2} + \frac{1-\theta}{q}$. Let $r \in [2, q)$ be such that the embedding $W^{1,q'}(U_j) \hookrightarrow L^{r'}(U_j)$ holds. Then

$$\begin{aligned} |\langle \boldsymbol{u}_t, \psi\rangle_{U_j}| &\le \|\boldsymbol{u}_t\|_{L^r(U_j)}\|\psi\|_{L^{r'}(U_j)} \\ &\le \big(C(\tau_\Omega, q, \varepsilon)\|\boldsymbol{u}_t\|_{L^2(U_j)} + \varepsilon\|\boldsymbol{u}_t\|_{L^q(U_j)}\big)\|\nabla\psi\|_{L^{q'}(U_j)} \end{aligned} \tag{3.6}$$

which implies

$$\begin{aligned} \|p - M_j\|_{L^q(0,T;L^q(U_j))} \le C\big(&\|\boldsymbol{f}\|_{L^q(0,T;L^q(U_j))} + \|\nabla\boldsymbol{u}\|_{L^q(0,T;L^q(U_j))} \\ &+ \|\boldsymbol{u}_t\|_{L^q(0,T;L^2(U_j))}\big) + \varepsilon\|\boldsymbol{u}_t\|_{L^q(0,T;L^q(U_j))} \end{aligned} \tag{3.7}$$

with $C = C(\tau_\Omega, T, q, \varepsilon)$ and any $\varepsilon \in (0,1)$. Moreover, the inequality (3.6), (2.3) and (2.7) for $v \in \hat{W}^{1,q'}(U_j)$ imply that

$$|\langle \nabla\varphi_j \cdot \boldsymbol{u}_t, v\rangle_{U_j}| \le \big(C(\tau_\Omega, q, \varepsilon)\|\boldsymbol{u}_t\|_{L^2(U_j)} + \varepsilon\|\boldsymbol{u}_t\|_{L^q(U_j)}\big)\|\nabla v\|_{L^{q'}(U_j)},$$

which yields for any $\varepsilon \in (0,1)$ and a constant $C = C(\tau_\Omega, T, q, \varepsilon)$

$$\|\nabla\varphi_j \cdot \boldsymbol{u}_t\|_{L^q(0,T;\hat{W}^{-1,q}(U_j))} \le C\|\boldsymbol{u}_t\|_{L^q(0,T;L^2(U_j))} + \varepsilon\|\boldsymbol{u}_t\|_{L^q(0,T;L^q(U_j))}. \tag{3.8}$$

From the estimates (3.4), (3.5), (3.7) and (3.8) we finally get that

$$\begin{aligned} &\|\varphi_j\boldsymbol{u}_t,\, \varphi_j\boldsymbol{u},\, \varphi_j\nabla\boldsymbol{u},\, \varphi_j\nabla^2\boldsymbol{u},\, \varphi_j\nabla p\|_{L^q(0,T;L^q(U_j))} \\ &\le C\big(\|\boldsymbol{f},\, \boldsymbol{u},\, \nabla\boldsymbol{u}\|_{L^q(0,T;L^q(U_j))} + \|\boldsymbol{u}_t\|_{L^q(0,T;L^2(U_j))}\big) + \varepsilon\|\boldsymbol{u}_t\|_{L^q(0,T;L^q(U_j))}, \end{aligned} \tag{3.9}$$

$j = 1, \dots, N$, with $C = C(\tau_\Omega, T, q, \varepsilon) > 0$.

Now, considering the q^{th} power of (3.9), we perform the summation over $j = 1, \dots, N$ and employ the property that at most $N_0 = N_0(\alpha, \beta, K)$ of the neighborhoods $U_1, \dots, U_N$ intersect, see Subsect. 2.1. In order to deal with the term $\|\boldsymbol{u}_t\|_{L^q(0,T;L^2(U_j))}$ we use the reverse Hölder inequality $\sum_j a_j^{q/2} \le \big(\sum_j a_j\big)^{q/2}$, $1 \le q/2$, with $a_j = \|\boldsymbol{u}_t(t,\cdot)\|^2_{L^2(U_j)}$. In the following we drop from time to time the designation of the time interval $(0,T)$ in the notation of function spaces and norms. We obtain

$$
\begin{aligned}
&\|\boldsymbol{u}_t,\, \boldsymbol{u},\, \nabla\boldsymbol{u},\, \nabla^2\boldsymbol{u},\, \nabla p\|^q_{L^q(0,T;L^q(\Omega))} \le \int_0^T\int_\Omega \Big(\Big(\sum_j \varphi_j|\boldsymbol{u}_t|\Big)^q + \\
&\quad + \Big(\sum_j \varphi_j|\boldsymbol{u}|\Big)^q + \Big(\sum_j \varphi_j|\nabla\boldsymbol{u}|\Big)^q + \Big(\sum_j \varphi_j|\nabla^2\boldsymbol{u}|\Big)^q + \Big(\sum_j \varphi_j|\nabla p|\Big)^q\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&\le N_0^{q/q'}\sum_j \int_0^T\int_\Omega \big(|\varphi_j\boldsymbol{u}_t|^q + |\varphi_j\boldsymbol{u}|^q + |\varphi_j\nabla\boldsymbol{u}|^q + |\varphi_j\nabla^2\boldsymbol{u}|^q + |\varphi_j\nabla p|^q\big)\,\mathrm{d}x\,\mathrm{d}t \\
&= N_0^{q/q'}\sum_j \big(\|\varphi_j\boldsymbol{u}_t,\, \varphi_j\boldsymbol{u},\, \varphi_j\nabla\boldsymbol{u},\, \varphi_j\nabla^2\boldsymbol{u},\, \varphi_j\nabla p\|^q_{L^q(L^q(U_j))}\big) \\
&\le N_0^{q/q'}\sum_j \big(C\big(\|\boldsymbol{f},\, \boldsymbol{u},\, \nabla\boldsymbol{u}\|^q_{L^q(L^q(U_j))} + \|\boldsymbol{u}_t\|^q_{L^q(L^2(U_j))}\big) + \varepsilon^q\|\boldsymbol{u}_t\|^q_{L^q(L^q(U_j))}\big) \\
&\le N_0^{q/q'}\big(C\big(N_0\|\boldsymbol{f},\, \boldsymbol{u},\, \nabla\boldsymbol{u}\|^q_{L^q(L^q(\Omega))} + N_0^{\frac{q}{2}}\|\boldsymbol{u}_t\|^q_{L^q(L^2(\Omega))}\big) + \varepsilon^q N_0\|\boldsymbol{u}_t\|^q_{L^q(L^q(\Omega))}\big)
\end{aligned}
$$

implying that

$$
\begin{aligned}
&\|\boldsymbol{u}_t, \boldsymbol{u},\, \nabla\boldsymbol{u},\, \nabla^2\boldsymbol{u},\, \nabla p\|_{L^q(0,T;L^q(\Omega))} \\
&\qquad \le C\big(\|\boldsymbol{f},\, \boldsymbol{u},\, \nabla\boldsymbol{u}\|_{L^q(L^q(\Omega))} + \|\boldsymbol{u}_t\|_{L^q(L^2(\Omega))}\big) + \varepsilon\|\boldsymbol{u}_t\|_{L^q(L^q(\Omega))}
\end{aligned}
\tag{3.10}
$$

with $C = C(\tau_\Omega, T, q, \varepsilon) > 0$, $\varepsilon \in (0,1)$.

Choosing ε sufficiently small and using the estimates (2.9) and (2.10) we absorb the terms $\|\boldsymbol{u}_t\|_{L^q(L^q(\Omega))}$, $\|\nabla\boldsymbol{u}\|_{L^q(L^q(\Omega))}$ and finally $\|\boldsymbol{u}\|_{L^q(L^q(\Omega))}$ on the right-hand side of (3.10) by the left-hand side. Then we get the inequality

$$
\|\boldsymbol{u}_t,\, \boldsymbol{u},\, \nabla\boldsymbol{u},\, \nabla^2\boldsymbol{u},\, \nabla p\|_{L^q(0,T;L^q(\Omega))} \le C\big(\|\boldsymbol{f}\|_{L^q(L^q(\Omega))} + \|\boldsymbol{u}_t,\, \boldsymbol{u},\, \nabla^2\boldsymbol{u}\|_{L^q(L^2(\Omega))}\big).
$$

Considering Theorem 1.2 (iii) and (2.27) for $q = 2$ (recall Theorem 2.9(v)) the estimate

$$
\|\boldsymbol{u}_t,\, \boldsymbol{u},\, \nabla^2\boldsymbol{u}\|_{L^q(0,T;L^2(\Omega))} \le C\|\boldsymbol{f}\|_{L^q(L^2(\Omega))}
$$

with a constant $C = C(q, T)$ independent of Ω finally implies that

$$
\|\boldsymbol{u}_t,\, \boldsymbol{u},\, \nabla\boldsymbol{u},\, \nabla^2\boldsymbol{u},\, \nabla p\|_{L^q(0,T;L^q(\Omega))} \le C\big(\|\boldsymbol{f}\|_{L^q(L^q(\Omega))} + \|\boldsymbol{f}\|_{L^q(L^2(\Omega))}\big)
$$

with $C = C(\tau_\Omega, T, q) > 0$. Using the last estimate also for $q = 2$ we obtain the desired $\tilde{L}^q$-estimate (3.2) with a constant $C = C(\tau_\Omega, T, q) > 0$.

Moreover, from Theorem 1.2 (iii) we have

$$
\|\boldsymbol{u}_t,\, \boldsymbol{u},\, \tilde{A}_{q,B}\boldsymbol{u}\|_{L^q(0,T;\tilde{L}^q(\Omega))} \le C\|\boldsymbol{f}\|_{L^q(0,T;\tilde{L}^q(\Omega))} \tag{3.11}
$$

with $C = C(\tau_\Omega, T, q) > 0$.

Obviously, also the backward equation $-\boldsymbol{u}_t + \tilde{A}_{q,B}\boldsymbol{u} = \boldsymbol{f}$, $\boldsymbol{u}(T) = \boldsymbol{0}$, admits a unique solution satisfying (3.11) and (3.2) with $C = C(\tau_\Omega, T, q) > 0$.

3.2 The case $1 < s = q < 2$

Again we consider the Stokes system (3.1), but now for a right-hand side $\boldsymbol{f} \in L^q(0,T;\tilde{L}^q_\sigma(\Omega))$, $1 < q < 2$. The aim is to prove the *a priori* estimate (3.2). According to Theorem 2.9, there is a unique solution $\boldsymbol{u}(t) = \mathcal{J}_{q,q}\boldsymbol{f}(t) = \tilde{\mathcal{J}}_{q,q}\boldsymbol{f}(t)$. Recall that Ω is a bounded domain and hence $\tilde{L}^q(\Omega) = L^q(\Omega)$, $\tilde{P}_q = P_q$ and $\tilde{A}_{q,B} = A_{q,B}$.

For the proof we use a duality argument. First, note that the space

$$C_0^\infty(0,T;C_{0,\sigma}^\infty) = \{\boldsymbol{v} \in C_0^\infty(\Omega \times (0,T)) : \operatorname{div} \boldsymbol{v}(x,t) = 0 \text{ for all } t \in (0,T)\}$$

is dense (see [10]) in

$$\begin{aligned} L^{q'}(0,T;\tilde{L}^{q'}_\sigma(\Omega)) &= L^{q'}(0,T;L^{q'}_\sigma(\Omega)) \cap L^{q'}(0,T;L^2_\sigma(\Omega)) \\ &= (L^q(0,T;L^q_\sigma(\Omega)) + L^q(0,T;L^2_\sigma(\Omega)))' = (L^q(0,T;\tilde{L}^q_\sigma(\Omega)))'. \end{aligned}$$

Notice that also $\tilde{\mathcal{J}}_{q,q} = (\tilde{\mathcal{J}}'_{q',q'})'$ as well as $(\tilde{A}_{q,B})' = \tilde{A}_{q',B}$. Then for $\boldsymbol{u} = \tilde{\mathcal{J}}_{q,q}\boldsymbol{f}$ with $\boldsymbol{f} \in L^q(0,T;\tilde{L}^q_\sigma(\Omega))$ and $\boldsymbol{v} = \tilde{\mathcal{J}}_{q',q'}\boldsymbol{g}$ with $\boldsymbol{g} \in L^{q'}(0,T;\tilde{L}^{q'}_\sigma(\Omega))$ there holds the equality

$$\begin{aligned} \langle \boldsymbol{f},\, \tilde{A}_{q',B}\boldsymbol{v}\rangle_{T,\Omega} &= \langle \boldsymbol{u}_t + \tilde{A}_{q,B}\boldsymbol{u},\, \tilde{A}_{q',B}\boldsymbol{v}\rangle_{T,\Omega} \\ &= \langle \boldsymbol{u},\, (-\partial_t + \tilde{A}_{q',B})\tilde{A}_{q',B}\boldsymbol{v}\rangle_{T,\Omega} \\ &= \langle \tilde{A}_{q,B}\boldsymbol{u},\, -\boldsymbol{v}_t + \tilde{A}_{q',B}\boldsymbol{v}\rangle_{T,\Omega} = \langle \tilde{A}_{q,B}\boldsymbol{u},\, \boldsymbol{g}\rangle_{T,\Omega}; \end{aligned}$$

to justify the second step we mollify $\boldsymbol{u}$ and $\boldsymbol{v}$ in space and time separately and use an approximation argument. Thus, with $\boldsymbol{v}_g = \tilde{\mathcal{J}}_{q',q'}\boldsymbol{g}$ and the estimate (3.11) with $s = q$ replaced by $s' = q' > 2$ and $\boldsymbol{u}$ replaced by $\boldsymbol{v}_g$, we obtain that

$$\begin{aligned} \|\tilde{A}_{q,B}\boldsymbol{u}\|_{L^q(0,T;\tilde{L}^q(\Omega))} &\le c \sup\left\{ \frac{|\langle \tilde{A}_{q,B}\boldsymbol{u}, \boldsymbol{g}\rangle_{T,\Omega}|}{\|\boldsymbol{g}\|_{L^{q'}(\tilde{L}^{q'}(\Omega))}} : \boldsymbol{0} \neq \boldsymbol{g} \in L^{q'}(0,T;\tilde{L}^{q'}_\sigma(\Omega))\right\} \\ &\le c \sup\left\{ \frac{|\langle \tilde{A}_{q,B}\boldsymbol{u}, -\partial_t\boldsymbol{v}_g + \tilde{A}_{q',B}\boldsymbol{v}_g\rangle_{T,\Omega}|}{\|\boldsymbol{g}\|_{L^{q'}(\tilde{L}^{q'}(\Omega))}} : \boldsymbol{0} \neq \boldsymbol{g} \in L^{q'}(0,T;\tilde{L}^{q'}_\sigma(\Omega))\right\} \\ &\le C \sup\left\{ \frac{|\langle \boldsymbol{f}, \tilde{A}_{q',B}\boldsymbol{v}_g\rangle_{T,\Omega}|}{\|\tilde{A}_{q',B}\boldsymbol{v}_g\|_{L^{q'}(\tilde{L}^{q'}(\Omega))}} : \boldsymbol{0} \neq \boldsymbol{g} \in L^{q'}(0,T;\tilde{L}^{q'}_\sigma(\Omega))\right\} \\ &\le \|\boldsymbol{f}\|_{L^q(\tilde{L}^q(\Omega))} \end{aligned}$$

with a constant $C = C(\tau_\Omega, T, q)$. Moreover, from Theorem 1.2 (iii) we know that

$$\|\boldsymbol{u}\|_{W^{2,q}(\Omega)+W^{2,2}(\Omega)} \le C(\tau_\Omega, q)\big(\|\boldsymbol{u}\|_{L^q(\Omega)+L^2(\Omega)} + \|\tilde{A}_{q,B}\boldsymbol{u}\|_{L^q(\Omega)+L^2(\Omega)}\big).$$

Therefore, using the relation $\nabla p = \boldsymbol{f} - \boldsymbol{u}_t + \Delta\boldsymbol{u}$, from the preceding estimate and from (3.11) we get

$$\|\boldsymbol{u}_t,\, \boldsymbol{u},\, \nabla\boldsymbol{u},\, \nabla^2\boldsymbol{u},\, \nabla p\|_{L^q(0,T;\tilde{L}^q(\Omega))} \le C\|\boldsymbol{f}\|_{L^q(0,T;\tilde{L}^q(\Omega))}$$

with $C = C(\tau_\Omega, T, q) > 0$.

Hence Theorem 1.3 is proven for Ω bounded, $1 < s = q < \infty$ and $\boldsymbol{u}(0) = \boldsymbol{0}$.

4 The case of unbounded domains

Let now $\Omega \subset \mathbb{R}^n$ be an unbounded domain of uniform C^3-type τ_Ω, and let $\{\Omega_j\}_{j\in\mathbb{N}}$ be a sequence of bounded subdomains of uniform C^3-type τ_Ω as in Assumption 1.1. Given $1 < s = q < \infty$ and $\boldsymbol{f} \in L^q(0,T;\tilde{L}^q_\sigma(\Omega))$ we set $\boldsymbol{f}_j := \boldsymbol{f}|_{\Omega_j} \in L^q(0,T;\tilde{L}^q(\Omega_j))$. In view of Sect. 3 we consider for each $j \in \mathbb{N}$ the solution $(\boldsymbol{u}_j, \nabla p_j) \in \big(L^q(0,T;\tilde{\mathcal{D}}^q_B(\Omega_j)) \cap W^{1,q}(0,T;\tilde{L}^q_\sigma(\Omega_j))\big) \times L^q(0,T;\tilde{G}^q(\Omega_j))$ in $\Omega_j \times (0,T)$ of the instationary Stokes system with right-hand side $\boldsymbol{f}_j$:

$$\begin{aligned} \partial_t \boldsymbol{u}_j - \Delta \boldsymbol{u}_j + \nabla p_j = \boldsymbol{f}_j \qquad & \text{in } \Omega_j \times (0,T) \\ \operatorname{div} \boldsymbol{u}_j = 0 \qquad & \text{in } \Omega_j \times (0,T) \\ \boldsymbol{u}_j \cdot \boldsymbol{n}_j = 0,\ B(\boldsymbol{u}_j) = \boldsymbol{0} \qquad & \text{on } \partial\Omega_j \times (0,T) \\ \boldsymbol{u}_j(0) = \boldsymbol{0} \qquad & \text{in } \Omega_j, \end{aligned} \tag{4.1}$$

From Sect. 3 we know that $\boldsymbol{u}_j, p_j$ satisfy the estimate

$$\begin{aligned} \|\partial_t \boldsymbol{u}_j\|_{L^q(0,T;\tilde{L}^q(\Omega_j))} + \|\boldsymbol{u}_j\|_{L^q(0,T;\tilde{W}^{2,q}(\Omega_j))} + \|\nabla p_j\|_{L^q(0,T;\tilde{L}^q(\Omega_j))} \\ \le C\|\boldsymbol{f}_j\|_{L^q(0,T;\tilde{L}^q(\Omega_j))} \le C\|\boldsymbol{f}\|_{L^q(0,T;\tilde{L}^q(\Omega))} \end{aligned} \tag{4.2}$$

with $C = C(\tau_\Omega, T, q) > 0$ independent of $j \in \mathbb{N}$.

In the following, we use the notation $\widetilde{\boldsymbol{g}_j}$ for the extension of a function or a vector field $\boldsymbol{g}_j$ defined on $\Omega_j \times (0,T)$ by zero to the whole of $\Omega \times (0,T)$, *i.e.*,

$$\widetilde{\boldsymbol{g}_j}(x,t) = \begin{cases} \boldsymbol{g}_j(x,t) & \text{for } (x,t) \in \Omega_j \times (0,T), \\ \boldsymbol{0} & \text{for } (x,t) \in \Omega\backslash\Omega_j \times (0,T). \end{cases}$$

In particular, for $\boldsymbol{f}_j = \boldsymbol{f}|_{\Omega_j}$ we get, as $j \to \infty$,

$$\widetilde{\boldsymbol{f}_j} \to \boldsymbol{f} \quad \text{strongly in } L^q(0,T;\tilde{L}^q(\Omega)). \tag{4.3}$$

4.1 The case $2 \le s = q < \infty$

Let $2 \le q < \infty$ and let us consider the pressure first. Extending ∇p_j by zero to $\widetilde{\nabla p_j} \in L^q(0,T;\tilde{L}^q(\Omega))$ we get that

$$\|\widetilde{\nabla p_j}\|_{L^q(0,T;\tilde{L}^q(\Omega))} = \|\nabla p_j\|_{L^q(0,T;\tilde{L}^q(\Omega_j))} \le C\|\boldsymbol{f}\|_{L^q(0,T;\tilde{L}^q(\Omega))}$$

with the same constant as in (4.2). From the reflexivity of the Bochner spaces we obtain (at least for a not relabelled subsequence) that $\widetilde{\nabla p_j} \rightharpoonup Q$ weakly in $L^q(0,T;\tilde{L}^q(\Omega))$ and

$$\|Q\|_{L^q(0,T;\tilde{L}^q(\Omega))} \le \liminf_{j\to\infty} \|\widetilde{\nabla p_j}\|_{L^q(0,T;\tilde{L}^q(\Omega))} \le C\|\boldsymbol{f}\|_{L^q(0,T;\tilde{L}^q(\Omega))}, \tag{4.4}$$

again with a constant as in (4.2). To identify Q as a gradient field, let $\phi \in C_0^\infty(0,T;C_{0,\sigma}^\infty(\Omega))$. Without loss of generality we may assume that $\operatorname{supp}\phi(t) \subset \Omega_j$ for all $j \ge j_0$, $t \in (0,T)$. Then, since $\nabla p_j \in L^q(0,T;\tilde{G}^q(\Omega_j))$ and $\phi \in C_0^\infty(0,T;C_{0,\sigma}^\infty(\Omega_j))$, we get

$$\langle Q,\phi\rangle_{T,\Omega} = \lim_{j\to\infty}\langle\widetilde{\nabla p_j},\phi\rangle_{T,\Omega} = \lim_{j\ge j_0}\langle\nabla p_j,\phi\rangle_{T,\Omega_j} = 0,$$

and the de Rham argument yields the existence of a gradient ∇p such that

$$Q = \nabla p \in L^q(0,T;\tilde{G}^q(\Omega)). \tag{4.5}$$

Let us now consider the velocity fields

$$\boldsymbol{u}_j \in L^q(0,T;\tilde{\mathcal{D}}_B^q(\Omega_j)) \cap W^{1,q}(0,T;\tilde{L}_\sigma^q(\Omega_j))$$

with $\tilde{\mathcal{D}}_B^q(\Omega_j) = \tilde{L}_\sigma^q(\Omega_j) \cap \tilde{W}_{\mathrm{B}}^{2,q}(\Omega_j)$. We extend each component of $\boldsymbol{u}_j, \nabla\boldsymbol{u}_j, \nabla^2\boldsymbol{u}_j \in L^q(0,T;\tilde{L}^q(\Omega_j))$ by zero to the whole of Ω yielding extensions

$$\widetilde{\boldsymbol{u}_j} \in W^{1,q}(0,T;\tilde{L}_\sigma^q(\Omega)), \quad \widetilde{\nabla\boldsymbol{u}_j},\ \widetilde{\nabla^2\boldsymbol{u}_j} \in L^q(0,T;\tilde{L}^q(\Omega));$$

we note that the extensions of $\nabla\boldsymbol{u}_j, \nabla^2\boldsymbol{u}_j$ are $\tilde{L}^q(\Omega)$-functions in space and need not be derivatives with respect to the spatial variable. Furthermore, since $\|\partial_t\widetilde{\boldsymbol{u}_j}\|_{L^q(0,T;\tilde{L}^q(\Omega))} = \|\partial_t\boldsymbol{u}_j\|_{L^q(0,T;\tilde{L}^q(\Omega_j))}$ and

$$\|\widetilde{\boldsymbol{u}_j}\|_{L^q(0,T;\tilde{L}^q(\Omega))} + \|\widetilde{\nabla\boldsymbol{u}_j}\|_{L^q(0,T;\tilde{L}^q(\Omega))} + \|\widetilde{\nabla^2\boldsymbol{u}_j}\|_{L^q(0,T;\tilde{L}^q(\Omega))} = \|\boldsymbol{u}_j\|_{L^q(0,T;\tilde{W}^{2,q}(\Omega_j))}$$

(4.2) implies that

$$\|\partial_t\widetilde{\boldsymbol{u}_j},\, \widetilde{\boldsymbol{u}_j},\, \widetilde{\nabla\boldsymbol{u}_j},\, \widetilde{\nabla^2\boldsymbol{u}_j}\|_{L^q(0,T;\tilde{L}^q(\Omega))} \le C(\tau_\Omega,T,q)\|\boldsymbol{f}\|_{L^q(0,T;\tilde{L}^q(\Omega))}. \tag{4.6}$$

From the estimate (4.6) which is uniform in $j \in \mathbb{N}$ we get the weak convergences (at least for not relabelled subsequences)

$$\partial_t\widetilde{\boldsymbol{u}_j} \rightharpoonup \widehat{\boldsymbol{u}},\ \widetilde{\boldsymbol{u}_j} \rightharpoonup \boldsymbol{u},\ \widetilde{\nabla\boldsymbol{u}_j} \rightharpoonup \nabla\boldsymbol{u},\ \widetilde{\nabla^2\boldsymbol{u}_j} \rightharpoonup \nabla^2\boldsymbol{u} \text{ in } L^q(0,T;\tilde{L}^q(\Omega)). \tag{4.7}$$

Note that all convergences are meant componentwise and that the weak limits of the extended gradients are easily seen to be spatial derivatives of $\boldsymbol{u}$. Moreover,

$\widehat{\boldsymbol{u}} = \boldsymbol{u}_t$ and $\operatorname{div} \boldsymbol{u} = 0$. From (4.7), the lower semicontinuity of norms as well as from (4.6) it follows that

$$\|\boldsymbol{u}_t\|_{L^q(0,T;\tilde{L}^q(\Omega))} + \|\boldsymbol{u}\|_{L^q(0,T;\tilde{W}^{2,q}(\Omega))} \leq C(\tau_\Omega, T, q)\|\boldsymbol{f}\|_{L^q(0,T;\tilde{L}^q(\Omega))}. \tag{4.8}$$

In particular, $\boldsymbol{u} \in L^q(0,T;\tilde{W}^{2,q}(\Omega))$ and $\boldsymbol{u}_t \in L^q(0,T;\tilde{L}^q_\sigma(\Omega))$.

Next we show that $\boldsymbol{u}$ satisfies the Robin boundary condition $B(\boldsymbol{u}) = \boldsymbol{0}$ on $\partial\Omega$. This boundary condition is understood as follows: Let

$$\phi \in C_0^\infty(0,T;C^\infty_{0,\boldsymbol{n}}(\overline{\Omega})) := \{\varphi \in C_0^\infty((0,T)\times\overline{\Omega}) : \varphi\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega\times(0,T)\}.$$

Then, since ϕ is tangential on $\partial\Omega$, there holds the identity

$$\begin{aligned} 0 = \langle B(\boldsymbol{u}), \phi\rangle_{T,\partial\Omega} =& \langle \alpha\boldsymbol{u}, \phi\rangle_{T,\partial\Omega} + \beta\langle \mathbf{S}(\boldsymbol{u})\boldsymbol{n}, \phi\rangle_{T,\partial\Omega} \\ =& \langle \alpha\boldsymbol{u}, \phi\rangle_{T,\partial\Omega} + \beta\langle \mathbf{S}(\boldsymbol{u}), \nabla\phi\rangle_{T,\Omega} + \beta\langle \operatorname{div}\mathbf{S}(\boldsymbol{u}), \phi\rangle_{T,\Omega}. \end{aligned} \tag{4.9}$$

Assume that $\operatorname{supp}\phi(t)\cap\overline{\Omega} \subset \overline{\Omega_j}$ and $\operatorname{dist}(\operatorname{supp}\phi(t), \partial\Omega_j\backslash\Gamma_j) > 0$ for all $j \geq j_0$. Thus we also have $\phi \in C_0^\infty(0,T;C^\infty_{0,\boldsymbol{n}}(\overline{\Omega_j}))$ with $\phi\cdot\boldsymbol{n}_j|_{\Gamma_j} = \phi\cdot\boldsymbol{n}|_{\Gamma_j} = 0$ and $\phi|_{\partial\Omega_j\backslash\Gamma_j} = \boldsymbol{0}$. Since all extensions "$\sim$" are meant componentwise we obtain that

$$\begin{aligned} \langle \mathbf{S}(\boldsymbol{u})\boldsymbol{n}, \phi\rangle_{T,\partial\Omega} &= \langle \mathbf{S}(\boldsymbol{u}), \nabla\phi\rangle_{T,\Omega} + \langle \Delta\boldsymbol{u}, \phi\rangle_{T,\Omega} \\ &= \lim_{j\to\infty}\left(\langle \widetilde{\mathbf{S}(\boldsymbol{u}_j)}, \nabla\phi\rangle_{T,\Omega} + \langle \widetilde{\Delta\boldsymbol{u}_j}, \phi\rangle_{T,\Omega}\right) \\ &= \lim_{j\geq j_0}\left(\langle \mathbf{S}(\boldsymbol{u}_j), \nabla\phi\rangle_{T,\Omega_j} + \langle \Delta\boldsymbol{u}_j, \phi\rangle_{T,\Omega_j}\right) \\ &= \lim_{j\geq j_0}\langle \mathbf{S}(\boldsymbol{u}_j)\boldsymbol{n}_j, \phi\rangle_{T,\partial\Omega_j}. \end{aligned}$$

By analogy, trace theorems and compact embeddings for Bochner spaces imply that $\langle \boldsymbol{u}, \phi\rangle_{T,\partial\Omega} = \lim_{j\geq j_0}\langle \boldsymbol{u}_j, \phi\rangle_{T,\partial\Omega_j}$. Since $B(\boldsymbol{u}_j) = 0$ on $\partial\Omega_j$ we conclude that

$$\langle B(\boldsymbol{u}), \phi\rangle_{T,\partial\Omega} = 0 \quad \text{for all } \phi \in C_0^\infty(0,T;C^\infty_{0,\boldsymbol{n}}(\overline{\Omega})),$$

i.e., $B(\boldsymbol{u}) = 0$ on $\partial\Omega\times(0,T)$. Thus $\boldsymbol{u} \in L^q(0,T;(W^{2,q}_B(\Omega)\cap L^q_\sigma(\Omega))\cap(W^{2,2}_B(\Omega)\cap L^2_\sigma(\Omega))) = L^q(0,T;\tilde{\mathcal{D}}^q_B(\Omega))$ and $\boldsymbol{u}_t \in L^q(0,T;\tilde{L}^q_\sigma(\Omega))$.

From the weak convergence properties of $\partial_t\widetilde{\boldsymbol{u}_j}, \widetilde{\Delta\boldsymbol{u}_j}, \widetilde{\nabla p_j}$ and $\widetilde{\boldsymbol{f}}_j$, see *e.g.* (4.7) and (4.3), it follows immediately that $\boldsymbol{u}_t - \Delta\boldsymbol{u} + \nabla p = \boldsymbol{f}$ in $\Omega\times(0,T)$ in the sense of distributions. Moreover, by (4.7), for all $\varphi \in C_0^1([0,T);\tilde{L}^{q'}(\Omega))$

$$\begin{aligned} -\langle \widetilde{\boldsymbol{u}_j}(0), \varphi(0)\rangle_\Omega &= \int_0^T \langle \partial_t\widetilde{\boldsymbol{u}_j}, \varphi\rangle_\Omega\,\mathrm{d}t + \int_0^T \langle \widetilde{\boldsymbol{u}_j}, \varphi_t\rangle_\Omega\,\mathrm{d}t \\ &\to \int_0^T \langle \boldsymbol{u}_t, \varphi\rangle_\Omega\,\mathrm{d}t + \int_0^T \langle \boldsymbol{u}, \varphi_t\rangle_\Omega\,\mathrm{d}t = -\langle \boldsymbol{u}(0), \varphi(0)\rangle_\Omega. \end{aligned}$$

Thus we obtain that $\mathbf{0} = \widetilde{\boldsymbol{u}_j}(0) \rightharpoonup \boldsymbol{u}(0) = \mathbf{0}$ weakly in $\tilde{L}^q(\Omega)$. Furthermore, combining (4.8), (4.4) and (4.5) we get the inequality

$$\|\boldsymbol{u}_t\|_{L^q(0,T;\tilde{L}^q(\Omega))} + \|\boldsymbol{u}\|_{L^q(0,T;\tilde{W}^{2,q}(\Omega))} + \|\nabla p\|_{L^q(0,T;\tilde{L}^q(\Omega))} \leq C\|\boldsymbol{f}\|_{L^q(0,T;\tilde{L}^q(\Omega))} \quad (4.10)$$

with $C = C(\tau_\Omega, T, q) > 0$.

4.2 The case $1 < s = q < 2$

Now let $1 < q < 2$. We again start with the pressure gradient. In this case we have $\nabla p_j \in L^q(0,T; G^q(\Omega_j) + G^2(\Omega_j))$ and hence we can choose $\nabla p_j^1 \in L^q(0,T; G^q(\Omega_j))$ and $\nabla p_j^2 \in L^q(0,T; G^2(\Omega_j))$ such that $\nabla p_j = \nabla p_j^1 + \nabla p_j^2$ and - by a reflexivity argument -

$$\|\nabla p_j\|_{L^q(0,T;\tilde{L}^q(\Omega_j))} = \|\nabla p_j^1\|_{L^q(0,T;L^q(\Omega_j))} + \|\nabla p_j^2\|_{L^q(0,T;L^2(\Omega_j))}.$$

Extending ∇p_j^1 and ∇p_j^2 by zero to functions $\widetilde{\nabla p_j^1}$ and $\widetilde{\nabla p_j^2}$ to $\Omega \times (0,T)$, from (4.2) we have with the same constant $C = C(\tau_\Omega, T, q)$

$$\|\widetilde{\nabla p_j^1}\|_{L^q(0,T;L^q(\Omega))} = \|\nabla p_j^1\|_{L^q(0,T;L^q(\Omega_j))} \leq C\|\boldsymbol{f}\|_{L^q(0,T;\tilde{L}^q(\Omega))},$$
$$\|\widetilde{\nabla p_j^2}\|_{L^q(0,T;L^2(\Omega))} = \|\nabla p_j^2\|_{L^q(0,T;L^2(\Omega_j))} \leq C\|\boldsymbol{f}\|_{L^q(0,T;\tilde{L}^q(\Omega))}.$$

This implies that in the weak sense (at least for subsequences)

$$\widetilde{\nabla p_j^1} \rightharpoonup Q^1 \text{ in } L^q(0,T;L^q(\Omega)), \quad \widetilde{\nabla p_j^2} \rightharpoonup Q^2 \text{ in } L^q(0,T;L^2(\Omega)) \quad (4.11)$$

and

$$\|Q^1\|_{L^q(0,T;L^q(\Omega))} + \|Q^2\|_{L^q(0,T;L^2(\Omega))} \leq C\|\boldsymbol{f}\|_{L^q(0,T;\tilde{L}^q(\Omega))}$$

again with a constant as in (4.2).

Let $\phi \in C_0^\infty(0,T; C_{0,\sigma}^\infty(\Omega))$ and assume that $\operatorname{supp}\phi(t) \subset \Omega_j$ for all $j \geq j_0$, $t \in (0,T)$. Then we get that

$$\langle Q^1, \phi\rangle_{T,\Omega} = \lim_{j\geq j_0} \langle \nabla p_j^1, \phi\rangle_{T,\Omega_j} = 0, \quad \langle Q^2, \phi\rangle_{T,\Omega} = \lim_{j\geq j_0} \langle \nabla p_j^2, \phi\rangle_{T,\Omega_j} = 0,$$

and the de Rham argument yields the existence of gradients ∇p^1, ∇p^2 such that $Q^1 = \nabla p^1 \in L^q(0,T; G^q(\Omega))$ and $Q^2 = \nabla p^2 \in L^q(0,T; G^2(\Omega))$. Defining $\nabla p = \nabla p^1 + \nabla p^2$ we obtain $\nabla p \in L^q(0,T; \tilde{G}^q(\Omega))$ satisfying

$$\|\nabla p\|_{L^q(0,T;\tilde{L}^q(\Omega))} \leq C\|\boldsymbol{f}\|_{L^q(0,T;\tilde{L}^q(\Omega))} \quad (4.12)$$

with a constant $C = C(\tau_\Omega, T, q) > 0$.

Let us now concentrate on the velocity fields

$$\boldsymbol{u}_j \in L^q(0,T;\tilde{\mathcal{D}}^q_B(\Omega_j)),\ \partial_t \boldsymbol{u}_j \in L^q(0,T;\tilde{L}^q_\sigma(\Omega_j)).$$

Recall that $\tilde{\mathcal{D}}^q_B(\Omega_j) = \tilde{L}^q_\sigma(\Omega_j) \cap \tilde{W}^{2,q}_B(\Omega_j)$. We choose a decomposition $\boldsymbol{u}_j = \boldsymbol{u}_j^1 + \boldsymbol{u}_j^2$ with $\boldsymbol{u}_j^1 \in L^q(0,T;\mathcal{D}^q_B(\Omega_j))$, $\boldsymbol{u}_j^2 \in L^q(0,T;\mathcal{D}^2_B(\Omega_j))$ satisfying

$$\|\boldsymbol{u}_j\|_{L^q(0,T;\tilde{W}^{2,q}(\Omega_j))} = \|\boldsymbol{u}_j^1\|_{L^q(0,T;W^{2,q}(\Omega_j))} + \|\boldsymbol{u}_j^2\|_{L^q(0,T;W^{2,2}(\Omega_j))}. \tag{4.13}$$

In the similar way as before we extend $\boldsymbol{u}_j^1,\ \nabla\boldsymbol{u}_j^1,\ \nabla^2\boldsymbol{u}_j^1 \in L^q(0,T;L^q(\Omega_j))$ and $\boldsymbol{u}_j^2,\ \nabla\boldsymbol{u}_j^2,\ \nabla^2\boldsymbol{u}_j^2 \in L^q(0,T;L^2(\Omega_j))$ by zero to Ω and obtain extensions

$$\widetilde{\boldsymbol{u}_j^1},\ \widetilde{\nabla\boldsymbol{u}_j^1},\ \widetilde{\nabla^2\boldsymbol{u}_j^1} \in L^q(0,T;L^q(\Omega)),\quad \widetilde{\boldsymbol{u}_j^2},\ \widetilde{\nabla\boldsymbol{u}_j^2},\ \widetilde{\nabla^2\boldsymbol{u}_j^2} \in L^q(0,T;L^2(\Omega)).$$

By (4.2) we have (omitting the designation of the interval $0,T$)

$$\begin{aligned}
\|\partial_t(\widetilde{\boldsymbol{u}_j^1}+\widetilde{\boldsymbol{u}_j^2})\|_{L^q(\tilde{L}^q(\Omega))} &= \sup\left\{\frac{|\langle\partial_t(\widetilde{\boldsymbol{u}_j^1}+\widetilde{\boldsymbol{u}_j^2}),\varphi\rangle_{T,\Omega}|}{\|\varphi\|_{L^{q'}(\tilde{L}^{q'}(\Omega))}} : 0\neq\varphi\in L^{q'}(\tilde{L}^{q'}(\Omega))\right\}\\
&\le \sup\left\{\frac{|\langle\partial_t(\boldsymbol{u}_j^1+\boldsymbol{u}_j^2),\varphi\rangle_{T,\Omega_j}|}{\|\varphi\|_{L^{q'}(\tilde{L}^{q'}(\Omega_j))}} : 0\neq\varphi\in L^{q'}(\tilde{L}^{q'}(\Omega_j))\right\}\\
&= \|\partial_t\boldsymbol{u}_j\|_{L^q(\tilde{L}^q(\Omega_j))}\\
&\le C\|\boldsymbol{f}\|_{L^q(\tilde{L}^q(\Omega))}
\end{aligned}$$

as well as due to (4.13)

$$\begin{aligned}
&\|\widetilde{\boldsymbol{u}_j^1},\ \widetilde{\nabla\boldsymbol{u}_j^1},\widetilde{\nabla^2\boldsymbol{u}_j^1}\|_{L^q(L^q(\Omega))} + \|\widetilde{\boldsymbol{u}_j^2},\ \widetilde{\nabla\boldsymbol{u}_j^2},\ \widetilde{\nabla^2\boldsymbol{u}_j^2}\|_{L^q(L^2(\Omega))}\\
&\qquad = \|\boldsymbol{u}_j\|_{L^q(\tilde{W}^{2,q}(\Omega_j))} \le C(\tau_\Omega,T,q)\|\boldsymbol{f}\|_{L^q(\tilde{L}^q(\Omega))}.
\end{aligned}$$

From these uniform estimate we obtain the weak convergences

$$\begin{aligned}
&\widetilde{\boldsymbol{u}_j^1} \rightharpoonup \boldsymbol{u}^1,\ \widetilde{\nabla\boldsymbol{u}_j^1} \rightharpoonup \nabla\boldsymbol{u}^1,\ \widetilde{\nabla^2\boldsymbol{u}_j^1} \rightharpoonup \nabla^2\boldsymbol{u}^1 \quad \text{in } L^q(0,T;L^q(\Omega)),\\
&\widetilde{\boldsymbol{u}_j^2} \rightharpoonup \boldsymbol{u}^2,\ \widetilde{\nabla\boldsymbol{u}_j^2} \rightharpoonup \nabla\boldsymbol{u}^2,\ \widetilde{\nabla^2\boldsymbol{u}_j^2} \rightharpoonup \nabla^2\boldsymbol{u}^2 \quad \text{in } L^q(0,T;L^2(\Omega)).
\end{aligned} \tag{4.14}$$

Defining $\boldsymbol{u} := \boldsymbol{u}_1 + \boldsymbol{u}_2$ we see that

$$\widetilde{\boldsymbol{u}_j} \rightharpoonup \boldsymbol{u},\ \ \partial_t\widetilde{\boldsymbol{u}_j} \rightharpoonup \widehat{\boldsymbol{u}} = \boldsymbol{u}_t \quad \text{weakly in } L^q(0,T;\tilde{L}^q_\sigma(\Omega)). \tag{4.15}$$

Hence we already have $\boldsymbol{u} \in L^q(0,T;L^q_\sigma(\Omega)\cap W^{2,q}(\Omega)) + L^q(0,T;L^2_\sigma(\Omega)\cap W^{2,2}(\Omega))$, $\boldsymbol{u}_t \in L^q(0,T;\tilde{L}^q_\sigma(\Omega))$. Moreover, from the previous estimates, the lower weak semicontinuity of norms and from (4.12) we get the *a priori* estimate (4.10).

In the next step we verify the Robin boundary condition for $\boldsymbol{u}^1$ and $\boldsymbol{u}^2$ and hence for $\boldsymbol{u}$. Let $\phi \in C_0^\infty(0,T;C^\infty_{0,\boldsymbol{n}}(\overline{\Omega}))$ and assume that $\operatorname{supp}\phi(t) \subset \overline{\Omega_j}$,

$\text{dist}(\text{supp}\,\phi(t), \partial\Omega_j \setminus \Gamma_j) > 0$ for all $j \geq j_0$. Thus $\phi \in C_0^\infty(0,T; C_{0,\boldsymbol{n}}^\infty(\overline{\Omega_j}))$ with $\phi \cdot \boldsymbol{n}_j|_{\Gamma_j} = \phi \cdot \boldsymbol{n}|_{\Gamma_j} = 0$ and $\phi|_{\partial\Omega_j\setminus\Gamma_j} = \mathbf{0}$. From (4.14), noting that $\boldsymbol{u}_j^1 \in L^q(0,T; W_B^{2,q}(\Omega_j)$ and $\boldsymbol{u}_j^2 \in L^q(0,T; W_B^{2,2}(\Omega_j))$, we get as in Subsect. 4.2 that

$$\langle \mathbf{S}(\boldsymbol{u}^i)\boldsymbol{n}, \phi\rangle_{T,\partial\Omega} = \lim_{j\geq j_0} \langle \mathbf{S}(\boldsymbol{u}_j^i)\boldsymbol{n}_j, \phi\rangle_{T,\partial\Omega_j} \quad \text{for } i = 1, 2,$$

and hence that $\langle \mathbf{S}(\boldsymbol{u})\boldsymbol{n}, \phi\rangle_{T,\partial\Omega} = \langle \mathbf{S}(\boldsymbol{u}^1)\boldsymbol{n}, \phi\rangle_{T,\partial\Omega} + \langle \mathbf{S}(\boldsymbol{u}^2)\boldsymbol{n}, \phi\rangle_{T,\partial\Omega}$. Since due to compact embeddings and trace theorems $\langle \boldsymbol{u}^i, \phi\rangle_{T,\partial\Omega} = \lim_{j\geq j_0}\langle \boldsymbol{u}_j^i, \phi\rangle_{T,\partial\Omega_j}$, $i = 1, 2$, and $B(\boldsymbol{u}_j) = \mathbf{0}$ on $\partial\Omega_j$ we conclude that $B(\boldsymbol{u}) = \mathbf{0}$ on $\partial\Omega$.

Altogether, for $1 < q < 2$ we proved that $\boldsymbol{u} = \boldsymbol{u}^1 + \boldsymbol{u}^2 \in L^q(0,T; \tilde{\mathcal{D}}_B^q(\Omega))$.

Again it is immediate that $\boldsymbol{u}$, ∇p satisfy the Stokes equation $\boldsymbol{u}_t - \Delta\boldsymbol{u} + \nabla p = \boldsymbol{f}$, $\text{div}\,\boldsymbol{u} = 0$ in the sense of distributions as well as the desired *a priori* estimate (4.10) with a constant $C = C(\tau_\Omega, T, q) > 0$. As in the case $q \geq 2$ we also have $\mathbf{0} = \widetilde{\boldsymbol{u}_j}(0) \rightharpoonup \boldsymbol{u}(0) = \mathbf{0}$ weakly in $\tilde{L}^q(\Omega)$.

We note that similar results for the backward Stokes system $-\boldsymbol{u}_t + \tilde{A}_q\boldsymbol{u} = \boldsymbol{f}$, $\boldsymbol{u}(T) = \mathbf{0}$, can be deduced in an analogous way.

4.3 End of the proof of Theorem 1.3

The case $s \neq q$ follows from an abstract extrapolation argument, see [3, page 191] and [5], where we have to consider the shifted operator $\delta + \tilde{A}_{q,B}$, $\delta > 0$. This argument shows that if (4.10) holds for some $s \in (1,\infty)$, *i.e.* here with $s = q$, then it holds for all $s \in (1,\infty)$.

To show uniqueness let us assume that $\boldsymbol{u} \in L^s(0,T; \mathcal{D}(\tilde{A}_{q,B}))$ satisfies $\boldsymbol{u}_t + \tilde{A}_{q,B}\boldsymbol{u} = \mathbf{0}$, $\boldsymbol{u}(0) = \mathbf{0}$. Then for given $\boldsymbol{g} \in L^{s'}(0,T; \tilde{L}^{q'}(\Omega))$ there is a solution $\boldsymbol{v} \in L^{s'}(0,T; \mathcal{D}(\tilde{A}_{q',B}))$ of $-\boldsymbol{v}_t + \tilde{A}_{q',B}\boldsymbol{v} = \tilde{P}_{q'}\boldsymbol{g}$, $\boldsymbol{v}(T) = \mathbf{0}$. So we have

$$\langle \boldsymbol{u}, \boldsymbol{g}\rangle_{T,\Omega} = \langle \boldsymbol{u}, \tilde{P}_{q'}\boldsymbol{g}\rangle_{T,\Omega} = \langle \boldsymbol{u}, -\boldsymbol{v}_t + \tilde{A}_{q',B}\boldsymbol{v}\rangle_{T,\Omega} = \langle \boldsymbol{u}_t + \tilde{A}_{q,B}\boldsymbol{u}, \boldsymbol{v}\rangle_{T,\Omega} = 0$$

for all $\boldsymbol{g} \in L^{s'}(0,T; \tilde{L}^{q'}(\Omega))$ and thus $\boldsymbol{u} = \mathbf{0}$.

Finally, considering the inhomogeneous equation $\boldsymbol{u}_t + \tilde{A}_q\boldsymbol{u} = \boldsymbol{f}$, $\boldsymbol{u}(0) = \boldsymbol{u}_0 \in \mathcal{D}(\tilde{A}_{q,B})$ we solve the equation $\boldsymbol{v}_t + \tilde{A}_{q,B}\boldsymbol{v} = \boldsymbol{F} := \boldsymbol{f} - \tilde{A}_{q,B}\boldsymbol{u}_0$, $\boldsymbol{v}(0) = \mathbf{0}$. Then $\boldsymbol{u}(t) = \boldsymbol{v}(t) + \boldsymbol{u}_0$ is the desired solution satisfying the maximal regularity estimate with right-hand side $C\big(\|\boldsymbol{f}\|_{L^s(0,T;\tilde{L}^q(\Omega))} + \|\boldsymbol{u}_0\|_{\mathcal{D}(\tilde{A}_{q,B})}\big)$, where as before $C = C(\tau_\Omega, T, q) > 0$.

Now the proof of Theorem 1.3 is complete. □

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