DEFINABILITY OF SINGULAR INTEGRAL OPERATORS ON MORREY-BANACH SPACES

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Abstract. We give a definition of singular integral operators on Morrey-Banach spaces which include Orlicz-Morrey spaces and Morrey spaces with variable exponents. The main result of this paper ensures that the singular integral operator is well defined on the Morrey-Banach spaces. Therefore, it provides a solid foundation for the study of singular integral operators on Morrey type spaces. As an application of our main result, we study the commutators of singular integral operators on Morrey-Banach spaces.

1. Introduction

In this paper, we aim to give an answer to the fundamental question for the study of singular integral operators on Morrey type spaces, that is, how to define the action of singular integral operator on functions belonging to a Morrey type space.

The classical Morrey spaces were introduced by Morrey [27] to study the solutions of some quasi-elliptic partial differential equations. Since the introduction of the classical Morrey spaces, several important results from Lebesgue spaces had been extended to the classical Morrey spaces such as the boundedness of the maximal operator [7], singular integral operators [28] and sublinear operators [23].

Recently, the studies of Morrey spaces is extending to Morrey space built on some non Lebesgue spaces such as Morrey-Lorentz spaces [3, 18, 31], Orlicz-Morrey spaces [11, 29, 28], Morrey spaces with variable exponents [1, 13, 15, 19, 22, 24, 25, 26, 33, 32]. On the other hand, for instance, in [15, 19], we are lack of a precise definition of the action of singular integral operators on the above mentioned Morrey type spaces. It is important to give a precise definition on the singular integral operators studied in the above mentioned results as it gives a solid foundation for us to study the boundedness of singular integral operators on Morrey spaces.

For the classical Morrey spaces, this fundamental question has been solved in [2] whereas the results in [2] rely on the pre-dual of the classical Morrey spaces.

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In this paper, we give an explicit expression for the definition of the singular integral operators on Morrey type spaces. This definition is independent of the pre-dual of the Morrey type space. That is, no matter whether we can identify the pre-dual of the above mentioned Morrey type spaces or not, our approach applies. Hence, our method directly applies to the Morrey-Lorentz spaces, the Orlicz-Morrey spaces and the Morrey spaces with variable exponent. Therefore, the main results of this paper give an unified approach of the study of the singular integral operators on Morrey spaces built on Banach function spaces.

We also see that the boundedness of singular integral operators come naturally from our definition of the action of singular integral operators on Morrey type spaces.

Additionally, our result provides a solid foundation for further studies of singular integral operators on Morrey spaces. As an application of our approach, we study the commutators of some singular integral operators on Morrey type spaces.

This paper is organized as follows. We give the definition of Morrey-Banach spaces in Section 2. This family includes Morrey-Lorentz spaces, the Orlicz-Morrey spaces and the Morrey spaces with variable exponents. The main result of this paper is given in Section 3. An application of our main result on the commutators is presented in Sections 4.

2. Morrey-Banach space

Let $B(z,r)=\{x\in\mathbb{R}^n:|x-z|<r\}$ denote the open ball with center $z\in\mathbb{R}^n$ and radius $r>0$. Let $\mathbb{B}=\{B(z,r):z\in\mathbb{R}^n,\ r>0\}$.

Let $\mathcal{M}(\mathbb{R}^n)$ and $L^1_{loc}(\mathbb{R}^n)$ denote the space of Lebesgue measurable functions and the space of locally integrable functions on $\mathbb{R}^n$, respectively. Let $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ be the space of Schwartz functions and the space of tempered distributions, respectively.

We recall the definition of Banach function space from [4, Chapter 1, Definitions 1.1 and 1.3].

Definition 2.1. A Banach space $X\subset\mathcal{M}(\mathbb{R}^n)$ is said to be a Banach function space (B.f.s.) on $\mathbb{R}^n$ if it satisfies

(1) $\|f\|_X=0 \Leftrightarrow f=0$ a.e.,
(2) $|g|\leq|f|$ a.e. $\Rightarrow \|g\|_X\leq\|f\|_X$,
(3) $0\leq f_n\uparrow f$ a.e. $\Rightarrow \|f_n\|_X\uparrow\|f\|_X$,
(4) $\chi_E\in\mathcal{M}(\mathbb{R}^n)$ and $|E|<\infty \Rightarrow \chi_E\in X$,
(5) $\chi_E\in\mathcal{M}(\mathbb{R}^n)$ and $|E|<\infty \Rightarrow \int_E|f(x)|dx<C_E\|f\|_X$, $\forall f\in X$ for some $C_E>0$.

The Lorentz spaces, the Orlicz spaces and the Lebesgue spaces of variable exponents are Banach function spaces, see [4, 12].

Furthermore, in view of Item (5) of Definition 2.1, for any B.f.s. X, we have $X\subset L^1_{loc}(\mathbb{R}^n)$.

We recall the definition of associate space from [4, Chapter 1, Definitions 2.1 and 2.3].

Definition 2.2. Let X be a B.f.s. The associate space of X, X', is the collection of all Lebesgue measurable function f such that

$$\|f\|_{X'} = \sup\left\{\left|\int f(t)g(t)dt\right| : g \in X,\ \|g\|_X \le 1\right\} < \infty.$$

According to [4, Chapter 1, Theorems 1.7 and 2.2], when X is a B.f.s., X' is also a B.f.s.

We have the Hölder inequality for X and X', see [4, Chapter 1, Theorem 2.4].

Theorem 2.1. *Let X be a B.f.s. Then, for any $f \in X$ and $g \in X'$, we have*

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \le \|f\|_X \|g\|_{X'}.$$

Definition 2.3. For any B.f.s. X, we write $X \in \mathbb{M}$ if the Hardy-Littlewood maximal operator M is bounded on X. We write $X \in \mathbb{M}'$ if M is bounded on X'.

As $X, X' \subset L^1_{loc}(\mathbb{R}^n)$, the Hardy-Littlewood maximal operator is well defined on X and X'.

The following result for $X \in \mathbb{M} \cup \mathbb{M}'$ is given in [14, Lemma 3.2]. For completeness, we also present the proof of the following lemma.

Lemma 2.2. *Let X be a B.f.s. If $X \in \mathbb{M} \cup \mathbb{M}'$, then there exists a constant $C \ge 1$ such that*

$$|B| \le \|\chi_B\|_X \|\chi_B\|_{X'} \le C|B|, \quad \text{for all } B \in \mathbb{B} \tag{2.1}$$

where χ_B is the characteristic function of B.

Proof: According to the Lorentz-Luxemburg theorem [4, Chapter 1, Theorem 2.7], we have $X = X''$. Hence, it suffices to establish (2.1) with the assumption $X \in \mathbb{M}$.

Theorem 2.1 gives the first inequality in (2.1).

For any $B \in \mathbb{B}$, we consider the projection

$$(P_B g)(y) = \left(\frac{1}{|B|}\int_B |g(x)|dx\right)\chi_B(y).$$

Since $X \subset L^1_{loc}(\mathbb{R}^n)$, P_B is well defined on X.

There exists a constant $C > 0$ such that for any $D \in \mathbb{B}$, $P_D(f) \le C\,\mathrm{M}(f)$. Consequently, for any $g \in X$ with $\|g\|_X \le 1$,

$$\left(\frac{1}{|B|}\int_B |g(x)|dx\right)\|\chi_B\|_X \le \|P_B\|_{X\to X} \le \sup_{D\in\mathbb{B}} \|P_D\|_{X\to X} \le C\|\mathrm{M}\|_{X\to X}.$$

Definition 2.2 ensures that

$$\|\chi_B\|_{X'}\|\chi_B\|_X = \sup\left\{\Big|\int_B g(x)dx\Big|\|\chi_B\|_X : g\in X,\ \|g\|_X\le 1\right\}\le C|B|. \quad \blacksquare$$

Furthermore, for any $x\in\mathbb{R}^n$ and $r>0$, we have

$$\|\chi_{B(x,r)}\|_{X'}\le\|\chi_{B(x2,r)}\|_{X'}.$$

If $X\in\mathbb{M}\cup\mathbb{M}'$, then, (2.1) asserts that for any $x\in\mathbb{R}^n$ and $r>0$, we have

$$\|\chi_{B(x,2r)}\|_X\le C\frac{|B(x,2r)|}{\|\chi_{B(x,2r)}\|_{X'}}\le C\frac{|B(x,r)|}{\|\chi_{B(x,r)}\|_{X'}}\le C\|\chi_{B(x,r)}\|_X \tag{2.2}$$

for some $C>0$.

We now give the definition of Morrey-Banach spaces.

Definition 2.4. Let X be a B.f.s. and $u(y,r):\mathbb{R}^n\times(0,\infty)\to(0,\infty)$ be a Lebesgue measurable function. The Morrey-Banach space M_X^u consists of all $f\in\mathcal{M}(\mathbb{R}^n)$ satisfying

$$\|f\|_{M_X^u(\mathbb{R}^n)}=\sup_{y\in\mathbb{R}^n,r>0}\frac{1}{u(y,r)}\|\chi_{B(y,r)}f\|_X<\infty. \tag{2.3}$$

When X is the Orlicz space, M_X^u is the Orlicz-Morrey space. Similarly, when X is the Lebesgue space with variable exponent $L^{p(\cdot)}(\mathbb{R}^n)$, M_X^u is the Morrey space with variable exponent $\mathcal{M}_{p(\cdot),u}$. The reader is referred to [10, 12] and [16] for the definition of Lebesgue spaces with variable exponents $L^{p(\cdot)}(\mathbb{R}^n)$ and Morrey spaces with variable exponents $\mathcal{M}_{p(\cdot),u}$.

Next, we characterize the weight function $u(y,r)$ used in our main results.

Definition 2.5. Let X be a B.f.s. We say that a Lebesgue measurable function, $u(x,r):\mathbb{R}^n\times(0,\infty)\to(0,\infty)$, belongs to $u\in\mathbb{W}_X$ if there exists a constant $C>0$ such that for any $x\in\mathbb{R}^n$ and $r>0$, u fulfills

$$C\le u(x,r),\quad \forall x\in\mathbb{R}^n \quad\text{and}\quad r\ge 1, \tag{2.4}$$

$$\|\chi_{B(x,r)}\|_X\le Cu(x,r),\quad \forall x\in\mathbb{R}^n\quad\text{and}\quad r<1, \tag{2.5}$$

$$\sum_{j=0}^{\infty}\frac{\|\chi_{B(x,r)}\|_X}{\|\chi_{B(x,2^{j+1}r)}\|_X}u(x,2^{j+1}r)\le Cu(x,r). \tag{2.6}$$

Proposition 2.3. *Let X be a B.f.s. and $u(y,r):\mathbb{R}^n\times(0,\infty)\to(0,\infty)$ be a Lebesgue measurable function. If u satisfies (2.4) and (2.5), then for any $B\in\mathbb{B}$, $\chi_B\in M_X^u$.*

Proof: Let $B\in\mathbb{B}$, $x\in\mathbb{R}^n$ and $r>0$. When $r\ge 1$, Item (2) of Definition 2.1 and (2.4) give

$$\frac{1}{u(x,r)}\|\chi_B\chi_{B(x,r)}\|_X\le\frac{1}{u(x,r)}\|\chi_B\|_X\le C\|\chi_B\|_X \tag{2.7}$$

for some $C > 0$. When $r < 1$, Item (2) of Definition 2.1 and (2.5) yield

$$\frac{1}{u(x,r)}\|\chi_B\chi_{B(x,r)}\|_X \leq \frac{1}{u(x,r)}\|\chi_{B(x,r)}\|_X \leq C. \tag{2.8}$$

Therefore, (2.7) and (2.8) assure that

$$\|\chi_B\|_{M_X^u} = \sup_{B(x,r)\in\mathbb{B}} \frac{1}{u(x,r)}\|\chi_B\chi_{B(x,r)}\|_X < C + C\|\chi_B\|_X.$$

Item (4) of Definition 2.1 guarantees that $\chi_B \in M_X^u$. ∎

The above proposition shows that when $u \in \mathbb{W}_X$, M_X^u is nontrivial.

We now give some characterizations of the class $\mathbb{W}_X$ when X is the Lebesgue space with variable exponent $L^{p(\cdot)}(\mathbb{R}^n)$. For simplicity, we refer the reader to [12, Chapters 3 and 4] for the definitions of $L^{p(\cdot)}(\mathbb{R}^n)$ and the class of log-Hölder continuous functions $\mathcal{P}^{\log}(\mathbb{R}^n)$.

Let $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $p_+ = \sup_{x\in\mathbb{R}^n} p(x) < \infty$ and $p_- = \inf_{x\in\mathbb{R}^n} p(x) > 1$. According to [15, Proposition 2.5 and Lemma 6.3], we find that for any $p > p_+$, there is a constant $C > 0$ such that for any $y \in \mathbb{R}^n$, $r > 0$ and $j \in \mathbb{N}$, we have

$$\frac{\|\chi_{B(y,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(y,2^jr)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \leq C\left(\frac{|B(y,r)|}{|B(y,2^jr)|}\right)^{1/p} = C\frac{\|\chi_{B(y,r)}\|_{L^p}}{\|\chi_{B(y,2^jr)}\|_{L^p}}. \tag{2.9}$$

Let $0 \leq \theta < 1$ and $u_\theta(x,r) = \|\chi_{B(x,r)}\|^\theta_{L^{p(\cdot)}(\mathbb{R}^n)}$. For any $p > p_+$, (2.9) yields a constant $C > 0$ such that

$$\begin{aligned}\sum_{j=0}^{\infty} \frac{\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(x,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\frac{u(x,2^{j+1}r)}{u(x,r)} &= \sum_{j=0}^{\infty}\left(\frac{\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(x,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\right)^{1-\theta} \\ &\leq C\sum_{j=0}^{\infty} 2^{-jn(1-\theta)/p} \leq C. \end{aligned} \tag{2.10}$$

Therefore, u_θ satisfies (2.6). Furthermore, in view of [12, Corollary 4.5.9], we have

$$\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \approx \begin{cases} |B(x,r)|^{\frac{1}{p(x)}}, & |B(x,r)| \leq 2^n \\ |B(x,r)|^{\frac{1}{p_\infty}}, & |B(x,r)| \geq 1, \end{cases} \tag{2.11}$$

where $p_\infty = \lim_{x\to\infty} p(x)$ and the existence of this limit is assured by the definition of log-Hölder continuous function.

Hence, (2.11) guarantees that when $r \geq 1$, we have

$$u_\theta(x,r) = \|\chi_{B(x,r)}\|^\theta_{L^{p(\cdot)}(\mathbb{R}^n)} \geq C|B(x,r)|^{\frac{\theta}{p_+}} > C$$

for some $C > 0$.

Moreover, (2.11) also yields a constant $K > 0$ such that $\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq K$ for all $r < 1$. Consequently, there exists a $C > 0$ such that

$$u_\theta(x,r) = \|\chi_{B(x,r)}\|^\theta_{L^{p(\cdot)}(\mathbb{R}^n)} \geq C\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}, \quad 0 < r < 1.$$

Therefore, $u_\theta \in \mathbb{W}_{L^{p(\cdot)}(\mathbb{R}^n)}$.

Let $p > p_+$ and $u \in \mathbb{W}_{L^p}$. (2.9) shows that u also satisfies (2.6) for $X = L^{p(\cdot)}(\mathbb{R}^n)$. In addition, (2.11) ensures that for any $r \in (0,1)$

$$u(x,r) > C|B(x,r)|^{\frac{1}{p}} > C|B(x,r)|^{\frac{1}{p(x)}} \geq C\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

because $|B(x,r)| < K$ for some constant $K > 0$ and $\frac{1}{p} < \frac{1}{p_+} \leq \frac{1}{p(x)}$, $\forall x \in \mathbb{R}^n$. Therefore, $\mathbb{W}_{L^p} \subset \mathbb{W}_{L^{p(\cdot)}(\mathbb{R}^n)}$ provided that $p > p_+$.

Moreover, we find that the above inclusion is proper. Precisely, for any $p > p_+$, there is a $\theta \in (0,1)$ such that $u_\theta \notin \mathbb{W}_{L^p}$. Let $\theta \in (0,1)$ be selected so that $\theta p > p_+$.

For any $N \in \mathbb{N}$, we have

$$\sum_{j=0}^{N} \frac{\|\chi_{B(0,r)}\|_{L^p}}{\|\chi_{B(0,2^{j+1}r)}\|_{L^p}} \frac{u(0,2^{j+1}r)}{u(0,r)} = C\sum_{j=0}^{N} 2^{-\frac{jn}{p}} \frac{\|\chi_{B(0,2^{j+1}r)}\|^{\theta}_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(0,r)}\|^{\theta}_{L^{p(\cdot)}(\mathbb{R}^n)}}.$$

Let $r \in (0,1)$ and $N = \frac{\ln(\frac{2}{rc_n})}{\ln 2} - 1$ where c_n is the Lebesgue measure of the unit ball in $\mathbb{R}^n$. For any $j \leq N$, we have $|B(0,2^{j+1}r)| \leq 2^n$. Consequently, (2.11) asserts that

$$\begin{aligned}\sum_{j=0}^{N} 2^{-\frac{jn}{p}} \frac{\|\chi_{B(0,2^{j+1}r)}\|^{\theta}_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(0,r)}\|^{\theta}_{L^{p(\cdot)}(\mathbb{R}^n)}} &\geq C\sum_{j=0}^{N} 2^{-\frac{jn}{p}} \frac{|B(0,2^{j+1}r)|^{\frac{\theta}{p(0)}}}{|B(0,r)|^{\frac{\theta}{p(0)}}} \\ &= C\sum_{j=0}^{N} 2^{jn(\frac{\theta}{p(0)}-\frac{1}{p})}\end{aligned}$$

for some $C > 0$.

Since $\frac{\theta}{p(0)} - \frac{1}{p} > 0$, $\sum_{j=0}^{N} 2^{jn(\frac{\theta}{p(0)}-\frac{1}{p})}$ diverges as $N \to \infty$. Furthermore, since $N \to \infty$ as $r \to 0^+$. Therefore, there does not exist a constant $C > 0$ such that for any $0 < r < 1$,

$$\sum_{j=0}^{\infty} \frac{\|\chi_{B(0,r)}\|_{L^p}}{\|\chi_{B(0,2^{j+1}r)}\|_{L^p}} \frac{u(0,2^{j+1}r)}{u(0,r)} < C.$$

That is, $u_\theta \notin \mathbb{W}_{L^p}$. Hence, for any $p > p_+$, we have $\mathbb{W}_{L^p} \subsetneqq \mathbb{W}_{L^{p(\cdot)}(\mathbb{R}^n)}$.

3. Main result

In this section, we present the main result of this paper, the definition of singular integral operators on Morrey-Banach spaces.

Let $T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ be a linear operator. We say that T is a singular integral operator if its kernel $K(x,y)$ satisfies

$$|K(x,y)| \leq \frac{1}{|x-y|^n}, \quad x \neq y. \tag{3.1}$$

The action of T on $f \in L^1_{loc}(\mathbb{R}^n)$ is given by

$$Tf(x) = \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} K(x,y) f(y) dy \tag{3.2}$$

whenever the limit exists. It is easy to see that if $K(x,\cdot)f(\cdot)$ is integrable, then by using dominated convergence theorem, the limit exists and $Tf(x)$ is well defined.

The definition (3.2) is named as the truncated approach [35, Chapter I, Section 7] or [34, Chapter 2, Section 4]. It is shown in [5] that whenever K satisfies some conditions such as cancellation conditions, the limit (3.2) exists for $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$. This result had been generalized to Lebesgue spaces with variable exponents, see [12, Corollary 6.3.13].

Notice that in view of the definition of Morrey-Banach spaces, to study the action of singular integral operator T on $f \in M^u_X$, we need to estimate Tf on a neighborhood of $x \in \mathbb{R}^n$. On the other hand, the definition of T in (3.2) is given in term of pointwise limit. Therefore, we cannot directly use it to study the boundedness of T on M^u_X.

The following definition overcomes this difficulty by giving the definition of Tf for $f \in M^u_X$ in a neighborhood of $x \in \mathbb{R}^n$.

We are now ready to extend the definition of $T(f)$ when $f \in M^u_X$ and T is the singular integral operator defined by (3.2).

Definition 3.1. Let X be a B.f.s. with $X \in \mathbb{M} \cup \mathbb{M}'$ and $u \in \mathbb{W}_X$. Let T be a singular integral operator defined by (3.2). Suppose that T is bounded on X. For any $f \in M^u_X$ and $x \in B(z,r) \in \mathbb{B}$, we define

$$(\mathcal{T}f)(x) = (T(\chi_{B(z,2r)}f))(x) + \int_{\mathbb{R}^n \setminus B(z,2r)} K(x,y) f(y) dy. \tag{3.3}$$

We need to show that $\mathcal{T}f$ is well defined. That is, the above definition is independent of $B(z,r)$.

The following is the main result of this paper. It shows that $\mathcal{T}$ is well defined on M^u_X.

Theorem 3.1. *Let T be a singular integral operator defined by (3.2). Let X be a B.f.s. with $X \in \mathbb{M} \cup \mathbb{M}'$ and $u \in \mathbb{W}_X$. If T is a bounded linear operator on X, then $\mathcal{T}$ is a well defined linear operator on M^u_X.*

Moreover, for any $f \in X \cap M^u_X$, we have $T(f) = \mathcal{T}(f)$. Hence, $\mathcal{T}$ is an extension of T.

Proof: Let $f \in M^u_X$ and $B(z,r) \in \mathbb{B}$. As T is bounded on X and $\chi_{B(z,2r)}f \in X$, $T(\chi_{B(z,2r)}f)$ is well defined.

Next, we show that there is a constant $C > 0$ such that for any $x \in B(z,r)$, we have

$$\int_{\mathbb{R}^n \setminus B(z,2r)} |K(x,y)||f(y)| dy \le C \|f\|_{M^u_X} \frac{u(z,r)}{\|\chi_{B(z,r)}\|_X}. \tag{3.4}$$

Write $f_j = \chi_{B(z,2^{j+1}r)\setminus B(z,2^j r)} f$, $j \in \mathbb{N}\setminus\{0\}$. We have

$$\int_{\mathbb{R}^n\setminus B(z,2r)} |K(x,y)||f(y)|dy \leq \sum_{j=1}^{\infty} \int_{\mathbb{R}^n\setminus B(z,2r)} |K(x,y)||f_j(y)|dy$$
$$= \sum_{j=1}^{\infty} E_j(x).$$

In view of (3.1), the Hölder inequality yields

$$E_j(x) \leq C \int_{B(z,2^{j+1}r)\setminus B(z,2^j r)} \frac{1}{|x-y|^n} |f_j(y)|dy$$
$$\leq C 2^{-(j+1)n} r^{-n} \|f_j\|_X \|\chi_{B(z,2^{j+1}r)}\|_{X'}$$

for some $C > 0$. Lemma 2.2 gives

$$E_j(x) \leq C \frac{u(z,2^{j+1}r)}{\|\chi_{B(z,2^{j+1}r)}\|_X} \frac{1}{u(z,2^{j+1}r)} \|\chi_{B(z,2^{j+1}r)} f\|_X$$
$$\leq C \frac{u(z,2^{j+1}r)}{\|\chi_{B(z,2^{j+1}r)}\|_X} \|f\|_{M_X^u}.$$

Since $u \in \mathbb{W}_X$, (2.6) assures that

$$\sum_{j=1}^{\infty} E_j(x) \leq C\|f\|_{M_X^u} \sum_{j=1}^{\infty} \frac{u(z,2^{j+1}r)}{\|\chi_{B(z,2^{j+1}r)}\|_X} \leq C\|f\|_{M_X^u} \frac{u(z,r)}{\|\chi_{B(z,r)}\|_X}.$$

Therefore, $\chi_{\mathbb{R}^n\setminus B(z,2r)}|K(x,y)||f(y)|$ is integrable. That is, the second term on the right hand side of (3.3) is well defined.

Finally, it remains to show that the definition is independent of $B(z,r) \in \mathbb{B}$. That is, for any $x \in B(z,r)\cap B(w,R)$ with $B(z,r), B(w,R) \in \mathbb{B}$ and $B(z,r)\cap B(w,R) \neq \emptyset$, we have

$$(T(\chi_{B(z,2r)}f))(x) + \int_{\mathbb{R}^n\setminus B(z,2r)} K(x,y)f(y)dy$$
$$= (T(\chi_{B(w,2R)}f))(x) + \int_{\mathbb{R}^n\setminus B(w,2R)} K(x,y)f(y)dy. \tag{3.5}$$

Let $B(s,M) \in \mathbb{B}$ be selected so that $B(z,2r)\cup B(w,2R) \subset B(s,M)$. According to (3.4), for any $x \in B(z,r)\cap B(w,R)$, both

$$\chi_{B(s,M)\setminus B(z,2r)}|K(x,y)||f(y)| \quad \text{and} \quad \chi_{B(s,M)\setminus B(w,2R)}|K(x,y)||f(y)|$$

are integrable. Therefore, for any $x \in B(z,r) \cap B(w,R)$,

$$\begin{aligned}(T\chi_{B(s,M)\setminus B(z,2r)}f)(x) &= \lim_{\epsilon\to 0}\int_{\substack{B(s,M)\setminus B(z,2r)\\ |x-y|>\epsilon}} K(x,y)f(y)dy\\ &= \int_{B(s,M)\setminus B(z,2r)} K(x,y)f(y)dy,\\ (T\chi_{B(s,M)\setminus B(w,2R)}f)(x) &= \lim_{\epsilon\to 0}\int_{\substack{B(s,M)\setminus B(w,2R)\\ |x-y|>\epsilon}} K(x,y)f(y)dy\\ &= \int_{B(s,M)\setminus B(w,2R)} K(x,y)f(y)dy.\end{aligned}$$

Since $\chi_{B(z,2r)}f, \chi_{B(s,M)\setminus\chi_{B(z,2r)}}f \in X$, the linearity of T on X yields

(3.6)

$$\begin{aligned}&(T(\chi_{B(z,2r)}f))(x) + \int_{\mathbb{R}^n\setminus B(z,2r)} K(x,y)f(y)dy\\ &= (T(\chi_{B(z,2r)}f))(x) + \int_{B(s,M)\setminus B(z,2r)} K(x,y)f(y)dy + \int_{\mathbb{R}^n\setminus B(s,M)} K(x,y)f(y)dy\\ &= (T(\chi_{B(z,2r)}f))(x) + (T(\chi_{B(s,M)\setminus\chi_{B(z,2r)}}f))(x) + \int_{\mathbb{R}^n\setminus B(s,M)} K(x,y)f(y)dy\\ &= (T(\chi_{B(s,M)}f))(x) + \int_{\mathbb{R}^n\setminus B(s,M)} K(x,y)f(y)dy.\end{aligned}$$

Similarly, we also have

$$\begin{aligned}(T(\chi_{B(w,2R)}f))(x) &+ \int_{\mathbb{R}^n\setminus B(w,2R)} K(x,y)f(y)dy\\ &= (T(\chi_{B(s,M)}f))(x) + \int_{\mathbb{R}^n\setminus B(s,M)} K(x,y)f(y)dy.\end{aligned} \tag{3.7}$$

Therefore, (3.6) and (3.7) yield (3.5). That is, $\mathcal{T}f$ is well defined when $f \in M^u_X$.

Obviously, according to (3.3), $\mathcal{T}$ is a linear operator on M^u_X.

When $f \in X \cap M^u_X$, (3.4) guarantees that $K(x,\cdot)\chi_{\mathbb{R}^n\setminus B(z,2r)}(\cdot)f(\cdot)$ is integrable, therefore,

$$(T(\chi_{\mathbb{R}^n\setminus B(z,2r)}f))(x) = \int_{\mathbb{R}^n\setminus B(z,2r)} K(x,y)f(y)dy.$$

Consequently, $\chi_{\mathbb{R}^n\setminus B(z,2r)}f \in X$ and the linearity of T on X assure that

$$(\mathcal{T}f)(x) = (T(\chi_{B(z,2r)}f))(x) + (T(\chi_{\mathbb{R}^n\setminus B(z,2r)}f))(x) = (Tf)(x).$$

That is, on $X \cap M^u_X$, $\mathcal{T}$ reduces to T. Therefore, $\mathcal{T}$ is an extension of T.
∎

With the precise definition of singular integral operators acting on M^u_X, it is now reasonable to study the boundedness of the singular integral operators on M^u_X.

Theorem 3.2. *Let T be a singular integral operator. Let X be a B.f.s. and $u \in \mathbb{W}_X$. If $X \in \mathbb{M} \cup \mathbb{M}'$ and T is a bounded linear operator on X, then $\mathcal{T}$ is bounded on M^u_X.*

Proof: In view of Theorem 3.1 and (3.4), for any $x \in B(z,r) \in \mathbb{B}$, we have

$$\begin{aligned}
&|\chi_{B(z,r)}(x)(\mathcal{T}f)(x)| \\
&\le |\chi_{B(z,r)}(x)(T(\chi_{B(z,2r)}f))(x)| + \chi_{B(z,r)}(x)\int_{\mathbb{R}^n\setminus B(z,2r)} |K(x,y)f(y)|dy \\
&\le |\chi_{B(z,r)}(x)(T(\chi_{B(z,2r)}f))(x)| + C\chi_{B(z,r)}(x)\|f\|_{M^u_X}\frac{u(z,r)}{\|\chi_{B(z,r)}\|_X}
\end{aligned}$$

Applying the norm $\|\cdot\|_X$ on both sides, the boundedness of T on X guarantees that

$$\begin{aligned}
\|\chi_{B(z,r)}\mathcal{T}f\|_X &\le \|T(\chi_{B(z,2r)}f)\|_X + C\|\chi_{B(z,r)}\|_X\|f\|_{M^u_X}\frac{u(z,r)}{\|\chi_{B(z,r)}\|_X} \\
&\le C\|\chi_{B(z,2r)}f\|_X + C\|f\|_{M^u_X}u(z,r).
\end{aligned}$$

Next, we show that there is a constant $C > 0$ such that for any $z \in \mathbb{R}^n$ and $r > 0$, we have

$$u(z,2r) \le Cu(z,r). \tag{3.8}$$

In view of (2.6), we obtain

$$\frac{\|\chi_{B(z,r)}\|_X}{\|\chi_{B(z,2r)}\|_X}u(z,2r) \le \sum_{j=0}^{\infty}\frac{\|\chi_{B(z,r)}\|_X}{\|\chi_{B(z,2^{j+1}r)}\|_X}u(z,2^{j+1}r) \le Cu(z,r).$$

By using (2.2), we have

$$u(z,2r) \le C\frac{\|\chi_{B(z,r)}\|_X}{\|\chi_{B(z,2r)}\|_X}u(z,2r) \le Cu(z,r)$$

for some $C > 0$. Therefore, we establish (3.8).

Consequently, (3.8) yields

$$\frac{1}{u(z,r)}\|\chi_{B(z,r)}\mathcal{T}f\|_X \le C\frac{1}{u(z,2r)}\|\chi_{B(z,2r)}f\|_X + C\|f\|_{M^u_X} \le C\|f\|_{M^u_X}.$$

By taking supremum over $B(z,r) \in \mathbb{B}$, we obtain the boundedness of $\mathcal{T}$ on M^u_X. ∎

Our method is also used in [21] to study singular integral operators with rough kernels.

Our approach does not only apply to singular integral operators, with some simple modifications, it can be used to study the commutator of singular integral operators in the subsequent section.

4. Commutators

In this section, we use our definition for singular integral operator T on M^u_X to study the commutator $[T, b]$. The reader is referred to [6, 8] for some details about commutators of singular integral operators on Lebesgue spaces and its applications.

The study of commutators involves the function space of bounded mean oscillations BMO. We say that a locally integrable function f belongs to BMO if

$$\|f\|_{BMO} = \sup_{B\in\mathbb{B}} \frac{\|\chi_B(f - f_B)\|_{L^1}}{|B|} < \infty$$

where $f_B = \frac{1}{|B|}\int_B f(y)dy$.

We recall some characterizations of BMO as these characterization are related to the study of commutator.

The BMO can also be characterized by B.f.s., see [14, Theorem 2.3].

Theorem 4.1. *Let $X \in \mathbb{M}'$. Then, the norm*

$$\|f\|_{BMO_X} = \sup_{B\in\mathbb{B}} \frac{\|\chi_B(f - f_B)\|_X}{\|\chi_B\|_X}$$

and $\|\cdot\|_{BMO}$ are mutually equivalent.

Let $b \in BMO$ and T be defined by (3.2). Suppose that T is bounded on X, the commutator $[T, b]$ is defined as

$$[T, b]f = T(bf) - b(Tf), \quad f \in X.$$

We now ready to study the boundedness of the commutator $[T, b]$ on M^u_X via the formula (3.3).

Definition 4.1. Let X be a B.f.s. with $X \in \mathbb{M} \cup \mathbb{M}'$, $u \in \mathbb{W}_X$ and $b \in BMO$. Let T be a singular integral operator defined by (3.2). Suppose that T and $[T, b]$ are bounded on X. For any $f \in M^u_X$ and $x \in B(z, r) \in \mathbb{B}$, we define

(4.1)

$$[\mathcal{T}, b]f(x) = [T, b](\chi_{B(z,2r)}f)(x) + \int_{\mathbb{R}^n\setminus B(z,2r)} (b(y) - b(x))K(x, y)f(y)dy.$$

Theorem 4.2. *Let T be a singular integral operator defined by (3.2) and $b \in BMO$. Let X be a B.f.s. with $X \in \mathbb{M}$ and $u \in \mathbb{W}_X$. Suppose that there exists a constant $C > 0$ such that for any $x \in \mathbb{R}^n$ and $r > 0$, u fulfills*

$$\sum_{j=0}^{\infty}(j+1)\frac{\|\chi_{B(x,r)}\|_X}{\|\chi_{B(x,2^{j+1}r)}\|_X}u(x, 2^{j+1}r) \le Cu(x, r). \tag{4.2}$$

If T and $[T, b]$ are bounded on X, then $[\mathcal{T}, b]$ is well defined on M^u_X.

In addition, for any $f \in X \cap M^u_X$, we have $[T, b]f = [\mathcal{T}, b]f$.

Proof: Similar to the proof of Theorem 3, we find that our result follows when, for any $x \in B(z,r)$,

$$\chi_{\mathbb{R}^n \setminus B(z,2r)}(\cdot)(b(\cdot)-b(x))K(x,\cdot)f(\cdot)$$

is integrable.

We write $f_j = \chi_{B(z,2^{j+1}r)\setminus B(z,2^j r)} f$, $j \in \mathbb{N}\setminus\{0\}$ and

$$b_{B(z,2^{j+1}r)} = \frac{1}{|B(z,2^{j+1}r)|}\int_{B(z,2^{j+1},r)} b(w)dw$$

As $X \in \mathbb{M}$, the Lorentz-Luxemburg theorem [4, Chapter 1, Theorem 2.7] guarantees that $(X')' = X \in \mathbb{M}$. That is, $X' \in \mathbb{M}'$. According to Theorem 4.1, we find that

$$\|\chi_{B(z,r)}(b(\cdot)-b_{B(z,r)})\|_{X'} \le C\|b\|_{BMO}\|\chi_{B(z,r)}\|_{X'}, \tag{4.3}$$

and

$$\|\chi_{B(z,2^{j+1}r)}(b(\cdot)-b_{B(z,2^{j+1}r)})\|_{X'} \le C\|b\|_{BMO}\|\chi_{B(z,2^{j+1},r)}\|_{X'}$$

for some $C > 0$. Furthermore, we have

$$|b_{B(z,2^{j+1}r)} - b_{B(z,r)}| \le C(j+1)\|f\|_{BMO},$$

see [35, p.141]. Thus,

$$\|\chi_{B(z,2^{j+1}r)}(b(\cdot)-b_{B(z,r)})\|_{X'} \le C(j+1)\|b\|_{BMO}\|\chi_{B(z,2^{j+1},r)}\|_{X'} \tag{4.4}$$

Write

$$E_j(x) = \int_{B(z,2^{j+1}r)\setminus B(z,2^j r)} |b(y)-b(x)||K(x,y)||f(y)|dy.$$

We have

$$\begin{aligned} E_j(x) &\le \int_{B(z,2^{j+1}r)\setminus B(z,2^j r)} |b(y)-b_{B(z,r)}||K(x,y)||f(y)|dy \\ &\quad+ \int_{B(z,2^{j+1}r)\setminus B(z,2^j r)} |b_{B(z,r)}-b(x)||K(x,y)||f(y)|dy \\ &= I_j + II_j. \end{aligned}$$

Lemma 2.2 and (4.4) yield

$$\begin{aligned} I_j &\le \int_{B(z,2^{j+1}r)\setminus B(z,2^j r)} |b(y)-b_{B(z,r)}||K(x,y)||f(y)|dy \\ &\le C2^{-(j+1)}r^{-n}\|\chi_{B(z,2^{j+1}r)}f\|_X\|\chi_{B(z,2^{j+1}r)}(b-b_{B(z,r)})\|_{X'} \\ &\le C(j+1)\|b\|_{BMO}\frac{u(z,2^{j+1}r)}{\|\chi_{B(z,2^{j+1}r)}\|_X}\frac{1}{u(z,2^{j+1}r)}\|\chi_{B(z,2^{j+1}r)}f\|_X \\ &\le C(j+1)\|b\|_{BMO}\frac{u(z,2^{j+1}r)}{\|\chi_{B(z,2^{j+1}r)}\|_X}\|f\|_{M^u_X}. \end{aligned}$$

Next, Lemma 2.2 gives

$$\begin{aligned}
II_j &\le \int_{B(z,2^{j+1}r)\backslash B(z,2^j r)} |b_{B(z,r)} - b(x)||K(x,y)||f(y)|dy \\
&\le C|b(x) - b_{B(z,r)}|2^{-(j+1)}r^{-n}\|\chi_{B(z,2^{j+1}r)}f\|_X\|\chi_{B(z,2^{j+1}r)}\|_{X'} \\
&\le C|b(x) - b_{B(z,r)}|\frac{u(z,2^{j+1}r)}{\|\chi_{B(z,2^{j+1}r)}\|_X}\|f\|_{M^u_X}.
\end{aligned}$$

Therefore, for any $x \in B(z,r)$, (4.2) assures that

$$\begin{aligned}
(4.5)\quad &\int_{\mathbb{R}^n\backslash B(z,2r)} |b(y) - b(x)||K(x,y)||f(y)|dy \\
&\le \sum_{j=1}^{\infty} E_j \le \sum_{j=1}^{\infty}(I_j + II_j) \\
&\le C\sum_{j=1}^{\infty}(j+1)\|b\|_{BMO}\frac{u(z,2^{j+1}r)}{\|\chi_{B(z,2^{j+1}r)}\|_X}\|f\|_{M^u_X} \\
&+ C\sum_{j=1}^{\infty}|b(x) - b_{B(z,r)}|\frac{u(z,2^{j+1}r)}{\|\chi_{B(z,2^{j+1}r)}\|_X}\|f\|_{M^u_X} \\
&\le C\|b\|_{BMO}\frac{u(z,r)}{\|\chi_{B(z,r)}\|_X}\|f\|_{M^u_X} + |b(x) - b_{B(z,r)}|\frac{u(z,r)}{\|\chi_{B(z,r)}\|_X}\|f\|_{M^u_X} \\
&< \infty
\end{aligned}$$

because $b \in BMO$ implies that $b(\cdot) - b_{B(z,r)}$ is integrable on $B(z,r)$ and, hence, $b(\cdot) - b_{B(z,r)}$ is finite almost everywhere on $B(z,r)$.

Similar to the proof of Theorem 3, for any $x \in B(z,r) \cap B(w,R)$ with $B(z,r), B(w,R) \in \mathbb{B}$ and $B(z,r) \cap B(w,R) \neq \emptyset$, select a $B(s,M) \in \mathbb{B}$ so that $B(z,2r) \cup B(w,2R) \subset B(s,M)$.

We have

$$\begin{aligned}
[T,b](\chi_{B(s,M)\backslash B(z,2r)}f) &= \int_{B(s,M)\backslash B(z,2r)} (b(y) - b(x))K(x,y)f(y)dy, \\
[T,b](\chi_{B(s,M)\backslash B(w,2R)}f) &= \int_{B(s,M)\backslash B(w,2R)} (b(y) - b(x))K(x,y)f(y)dy
\end{aligned}$$

because $\chi_{B(s,M)\backslash B(z,2r)}(\cdot)(b(\cdot) - b(x))K(x,\cdot)f(\cdot)$ and $\chi_{B(s,M)\backslash B(w,2R)}(\cdot)(b(\cdot) - b(x))K(x,\cdot)f(\cdot)$ are integrable.

Consequently,

$$\begin{aligned}
&[T,b](\chi_{B(z,2r)}f)(x)+\int_{\mathbb{R}^n\setminus B(z,2r)}(b(y)-b(x))K(x,y)f(y)dy\\
&=[T,b](\chi_{B(s,M)}f)(x)+\int_{\mathbb{R}^n\setminus B(s,M)}(b(y)-b(x))K(x,y)f(y)dy\\
&=[T,b](\chi_{B(w,2R)}f)(x)+\int_{\mathbb{R}^n\setminus B(w,2R)}(b(y)-b(x))K(x,y)f(y)dy.
\end{aligned}$$

Therefore, $[\mathcal{T},b]$ is well defined.

Moreover, for any $f\in X\cap M^u_X$ and $x\in B(z,r)$, $\chi_{\mathbb{R}^n\setminus B(z,2r)}(\cdot)(b(\cdot)-b(x))K(x,\cdot)f(\cdot)$ is integrable, the linearity of $[T,b]$ guarantees that

$$\begin{aligned}
[\mathcal{T},b]f(x)&=[T,b](\chi_{B(z,2r)}f)(x)+\int_{\mathbb{R}^n\setminus B(z,2r)}(b(y)-b(x))K(x,y)f(y)dy\\
&=[T,b](\chi_{B(z,2r)}f)(x)+[T,b](\chi_{\mathbb{R}^n\setminus B(z,2r)}f)(x)=[T,b]f(x). \qquad \blacksquare
\end{aligned}$$

With some modifications on (4.2), the above method can also be used to show that the higher order commutator

$$[\mathcal{T},b]^kf(x)=[T,b]^k(\chi_{B(z,2r)}f)(x)+\int_{\mathbb{R}^n\setminus B(z,2r)}(b(y)-b(x))^kK(x,y)f(y)dy,$$

$k\in\mathbb{N}$, is well defined on M^u_X, for brevity, we leave the details to the reader.

Since $[\mathcal{T},b]$ is well defined on M^u_X, we are allowed to study the boundedness of $[\mathcal{T},b]$ on M^u_X.

Theorem 4.3. *Let T be a singular integral operator defined by (3.2) and $b\in BMO$. Let X be a B.f.s. with $X\in\mathbb{M}\cap\mathbb{M}'$ and $u\in\mathbb{W}_X$. If T are bounded on X, $u\in\mathbb{W}_X$ satisfies (4.2) and*

$$\|[T,b]f\|_X\le C\|b\|_{BMO}\|f\|_X$$

for some $C>0$, then $[\mathcal{T},b]$ is bounded on M^u_X.

Proof: Let $f\in M^u_X$ and $B(z,r)\in\mathbb{B}$. Since $[T,b]$ is bounded on X, we have

$$\|\chi_{B(z,r)}[T,b](\chi_{B(z,r)}f)\|_X\le\|[T,b](\chi_{B(z,r)}f)\|_X\le C\|\chi_{B(z,r)}f\|_X,$$

for some $C>0$.

To deal with the second term on (4.1, (4.5) guarantees that

$$\begin{aligned}
&\chi_{B(z,r)}(x)\int_{\mathbb{R}^n\setminus B(z,2r)}|b(y)-b(x)||K(x,y)||f(y)|dy\\
&\le C\chi_{B(z,r)}(x)\|f\|_{M^u_X}\frac{u(z,r)}{\|\chi_{B(z,r)}\|_X}\\
&+C|b(x)-b_{B(z,r)}|\chi_{B(z,r)}(x)\|f\|_{M^u_X}\frac{u(z,r)}{\|\chi_{B(z,r)}\|_X}.
\end{aligned}$$

Since $X \in \mathbb{M}'$, Theorem 4.1 yields

$$\begin{aligned}
&\frac{1}{u(z,r)}\|\chi_{B(z,r)}[\mathcal{T},b]f\|_X \\
&\leq C\frac{1}{u(z,r)}\|\chi_{B(z,r)}f\|_X + C\|f\|_{M^u_X} + C\frac{\|\chi_{B(z,r)}|b-b_{B(z,r)}|\|_X}{\|\chi_{B(z,r)}\|_X}\|f\|_{M^u_X} \\
&\leq C\frac{1}{u(z,r)}\|b\|_{BMO}\|\chi_{B(z,r)}f\|_X + C\|b\|_{BMO}\|f\|_{M^u_X} + C\|b\|_{BMO}\|f\|_{M^u_X} \\
&\leq C\|b\|_{BMO}\|f\|_{M^u_X}
\end{aligned}$$

for some $C > 0$. By taking supremum over $B \in \mathbb{B}$, we establish the boundedness of $[\mathcal{T}, b]$ on M^u_X. ▪

In view of [9, Corollary 2.10], we have the boundedness of $[T, b]$ on the Lebesgue spaces with variable exponents $L^{p(\cdot)}(\mathbb{R}^n)$. The above result give an extension of the this boundedness result to Morrey spaces with variable exponents $\mathcal{M}_{p(\cdot),u}$.

Corollary 4.4. *Let T be a singular integral operator and $b \in BMO$. Suppose that $p(\cdot) : \mathbb{R}^n \to (1,\infty)$ satisfies $L^{p(\cdot)}(\mathbb{R}^n) \in \mathbb{M} \cap \mathbb{M}'$, T is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ and $u \in \mathbb{W}_{L^{p(\cdot)}(\mathbb{R}^n)}$ satisfies (4.2). Then, $[\mathcal{T}, b]$ is bounded on $\mathcal{M}_{p(\cdot),u}$.*

Let $p(\cdot)$ be a log-Hölder continuous function with $1 < p_- \leq p_+ < \infty$, $0 \leq \theta < 1$ and $u_\theta(x,r) = \|\chi_{B(x,r)}\|^\theta_{L^{p(\cdot)}(\mathbb{R}^n)}$. As an application of the above result, we find that if T and $L^{p(\cdot)}(\mathbb{R}^n)$ satisfy the conditions in Corollary 4.4, then there is a constant $C > 0$ such that for any $f \in \mathcal{M}_{p(\cdot),u_\theta}$

$$\|[\mathcal{T}, b]f\|_{\mathcal{M}_{p(\cdot),u_\theta}} \leq C\|f\|_{\mathcal{M}_{p(\cdot),u_\theta}}.$$

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