ON 3-2-1 VALUES OF FINITE MULTIPLE HARMONIC $q$-SERIES AT ROOTS OF UNITY

KH. HESSAMI PILEHROOD, T. HESSAMI PILEHROOD, AND R. TAURASO

Abstract. We mainly answer two open questions about finite multiple harmonic $q$-series on 3-2-1 indices at roots of unity, posed recently by H. Bachmann, Y. Takeyama, and K. Tasaka. Two conjectures regarding cyclic sums which generalize the given results are also provided.

1. Introduction.

For two $r$-tuples of non-negative integers $s = (s_1, \ldots, s_r)$ and $t = (t_1, \ldots, t_r)$ and a positive integer $n$, with a complex number $q$ satisfying $q^m \neq 1$ for $n > m > 0$, we define two classes of multiple $q$-harmonic sums

$$H_n(s; t; q) = H_n(s_1, \ldots, s_r; t_1, \ldots, t_r; q) = \sum_{1 \leq k_1 < \cdots < k_r \leq n} q^{k_1 t_1 + \cdots + k_r t_r} [k_1]_q \cdots [k_r]_q,$$

$$H_n^*(s; t; q) = H_n^*(s_1, \ldots, s_r; t_1, \ldots, t_r; q) = \sum_{1 \leq k_1 \leq \cdots \leq k_r \leq n} q^{k_1 t_1 + \cdots + k_r t_r} [k_1]_q \cdots [k_r]_q,$$

where

$$[k]_q = \frac{1 - q^k}{1 - q} = 1 + q + \cdots + q^{k-1}$$

is the $q$-analog of positive integer $k$. By convention, we put $H_n(\emptyset) = H_n^*(\emptyset) = 1$, and $H_n(s; t; q) = 0$ if $n < r$. The number $w(s) = \sum_{j=1}^r s_j$ is called the weight of the multiple harmonic sum.

For a primitive $n$-th root of unity $\zeta_n$, the following work [2], we adopt the notation

$$z_n(s; \zeta_n) = H_{n-1}(s; s - \{1\}^r; \zeta_n),$$

$$z_n^*(s; \zeta_n) = H_{n-1}^*(s; s - \{1\}^r; \zeta_n),$$

where $\{a\}^r$ denotes the $r$-tuple with $r$ consecutive copies of the letter $a$ (note that we reversed the order of summation for convenience in our settings).

2020 Mathematics Subject Classification. Primary 11M32; Secondary 05A30, 11B65, 11A07.

Key words and phrases. Multiple harmonic sums, $q$-analogs, roots of unity.
In [2], Bachmann, Takeyama, and Tasaka studied special values of $z_n\{\{k\}^r; \zeta_n\}$ and in particular for $k = 1, 2, 3$, showed that

\[
z_n(\{1\}^r; \zeta_n) = \frac{1}{n} \binom{n}{r+1} (1 - \zeta_n)^r, \tag{1}\n\]

\[
z_n(\{2\}^r; \zeta_n) = \frac{(-1)^r}{n(r+1)} \binom{n+r}{2r+1} (1 - \zeta_n)^{2r}, \tag{2}\n\]

\[
z_n(\{3\}^r; \zeta_n) = \frac{1}{n^2(r+1)} \left( \binom{n+2r+1}{3r+2} + (-1)^r \binom{n+r}{3r+2} \right) (1 - \zeta_n)^{3r}. \tag{3}\n\]

The authors of [2] also formulated two open questions for finite multiple harmonic $q$-series $z_n$ on 3-2-1 indices, namely,

\[
z_n(\{1\}^a, 2, \{1\}^b; \zeta_n) + z_n(\{1\}^b, 2, \{1\}^a; \zeta_n) = \frac{1}{n} \binom{n+1}{a+b+3} (1 - \zeta_n)^{a+b+2}, \tag{4}\n\]

\[
z_n(\{2\}^a, 3, \{2\}^b; \zeta_n) + z_n(\{2\}^b, 3, \{2\}^a; \zeta_n) = \frac{(-1)^{a+b}}{n(a+b+2)} \binom{n+a+b+1}{2(a+b)+3} (1 - \zeta_n)^{2(a+b)+3}. \tag{5}\n\]

In this paper, we prove the above relations and obtain related formulas for corresponding values of $\xi(s)$, which are defined as the limit values (see [1, Thm. 1.2])

\[
\xi(s) = \lim_{n \to \infty} z_n(s; e^{\frac{2\pi i}{n}}). \n\]

Note that when $n$ is a prime, formulas (4) as well as (2) and (3) follow from our results on $q$-congruences for multiple $q$-harmonic sums [3, Thm. 4.1, Thm. 5.1, Thm. 6.1, and Thm. 8.3], while formula (1) follows from [4, Cor. 2.2]. The methods of our paper [3] can be easily adjusted to prove (4) for arbitrary positive integer $n$.

**Theorem 1.1.** For all non-negative integers $a, b$ and any $n$-th primitive root of unity $\zeta_n$,

\[
z_n(\{2\}^a, 3, \{2\}^b; \zeta_n) + z_n(\{2\}^b, 3, \{2\}^a; \zeta_n) = \frac{(-1)^{a+b}}{n(a+b+2)} \binom{n+a+b+1}{2(a+b)+3} (1 - \zeta_n)^{2(a+b)+3}. \tag{6}\n\]

**Theorem 1.2.** For all non-negative integers $a, b$ and any $n$-th primitive root of unity $\zeta_n$,

\[
z_n(\{1\}^a, 2, \{1\}^b; \zeta_n) + z_n(\{1\}^b, 2, \{1\}^a; \zeta_n) = -\frac{1}{n} \binom{n+1}{a+b+3} (1 - \zeta_n)^{a+b+2}. \tag{7}\n\]

The complex numbers $\xi(s)$ are of interest in view of their connections to the finite and symmetric multiple zeta values as was shown in [1]. After letting $\zeta_n = e^{\frac{2\pi i}{n}}$ in Theorem 1.1 and Theorem 1.2, and by noting that for $j, k \in \mathbb{N}$,

\[
\lim_{n \to \infty} \binom{n+j}{k} \frac{k!}{n^k} = 1, \quad \text{and} \quad \lim_{n \to \infty} n(1 - e^{-\frac{2\pi i}{n}}) = -2\pi i, \n\]

we obtain the following corollary.

**Corollary 1.1.** For all non-negative integers $a, b$,

\[
\xi(\{1\}^a, 2, \{1\}^b) + \xi(\{1\}^b, 2, \{1\}^a) = -\frac{(-2\pi i)^{a+b+2}}{(a+b+3)!}. \tag{8}\n\]

and

\[
\xi(\{2\}^a, 3, \{2\}^b) + \xi(\{2\}^b, 3, \{2\}^a) = 0. \tag{9}\n\]
Note that the last relation can also be readily obtained from the definition of the symmetric multiple zeta values (see, for example, [1, Def. 2.5]).

Finally, we put forward the following conjectures regarding cyclic sums of multiple \( q \)-harmonic sums \( z_n \) at roots of unity, which generalize both of the theorems above.

**Conjecture 1.1** (Cyclic-sum). Let \( d_0, d_1, \ldots, d_t \) be non-negative integers. Then

(i) For every integer \( n > r \), where \( r = \sum_{j=0}^{t} d_j + 2t \), and any primitive root of unity \( \zeta_n \),

\[
\sum_{j=0}^{t} z_n \left( \{1\}^{d_j}, 2, \{1\}^{d_{j+1}}, 2, \ldots, 2, \{1\}^{d_{j+t}} \right) = \frac{(-1)^t}{n} \sum_{r=1}^{n} \left( n + t \right) \left( 1 - \zeta_n \right)^r.
\]

(ii) For every integer \( n > r \), where \( r = \sum_{j=0}^{t} 2d_j + 3t \), and any primitive root of unity \( \zeta_n \),

\[
\sum_{j=0}^{t} z_n \left( \{2\}^{d_j}, 3, \{2\}^{d_{j+1}}, 3, \ldots, 3, \{2\}^{d_{j+t}} \right) \in (1 - \zeta_n)^r \mathbb{Q}.
\]

In both sums above it is understood that \( d_j = d_k \) if \( j \equiv k \) modulo \( t + 1 \).

Note that the case \( t = 1 \) follows from Theorem 1.1 and Theorem 1.2. The case of arbitrary \( t \) when all \( d_j \) are zeros follows from (2) and (3).

2. Proof of Theorem 1.1.

Let \( \overline{s} = (s_r, s_{r-1}, \ldots, s_1) \) denote the reverse of \( s = (s_1, \ldots, s_{r-1}, s_r) \). Then we have the following relations.

**Lemma 2.1.** Let \( s = (s_1, \ldots, s_r) \) and \( t = (t_1, \ldots, t_r) \) be two \( r \)-tuples of non-negative integers, and \( \zeta_n \) be an \( n \)-th primitive root of unity. Then

\[
H_{n-1}(s; t; \zeta_n) = (-1)^{w(s)} H_{n-1}(\overline{s}; \overline{t}; \zeta_n),
\]

\[
H^*_n(s; t; \zeta_n) = (-1)^{w(s)} H^*_n(\overline{s}; \overline{t}; \zeta_n),
\]

and in particular,

\[
z_n(s; \zeta_n) = (-1)^{w(s)} H_{n-1}(\overline{s}; \{1\}^r; \zeta_n),
\]

\[
z^*_n(s; \zeta_n) = (-1)^{w(s)} H^*_n(\overline{s}; \{1\}^r; \zeta_n).
\]

**Proof.** Replacing each \( k_i \) by \( n - k_i \) and reversing the order of summation, we get

\[
H_{n-1}(s; t; \zeta_n) = \sum_{0 < k_1 < \cdots < k_r < n} z_n \left( k_1 \zeta_n, \ldots, k_r \zeta_n \right) \sum_{0 < n - k_1 < \cdots < n - k_r < n} \zeta_n^{t_1(n-k_1)+\cdots+t_r(n-k_r)}
\]

\[
= \sum_{0 < k_r < \cdots < k_1 < n} \zeta_n^{-t_1 k_1 + \cdots - t_r k_r} \cdot (-1)^{w(s)} z_n \left( k_1 \zeta_n, \ldots, k_r \zeta_n \right) \zeta_n^{t_1(n-k_1)+\cdots+t_r(n-k_r)}
\]

\[
= (-1)^{w(s)} H_{n-1}(\overline{s}; \overline{t}; \zeta_n),
\]

where we used the identity

\[
[n - k_i] \zeta_n = \frac{1 - \zeta_n^{-k_i}}{1 - \zeta_n} = \frac{1 - \zeta_n^{-k_i}}{1 - \zeta_n} = -\zeta_n^{-k_i} [k_i] \zeta_n.
\]

Setting \( t = s - \{1\}^r \), we get (5). The proofs for the multiple harmonic star sums are similar. \( \square \)
Proof of Theorem 1.1. We have
\[ z_n(\{2^a, 3, 2^b; \zeta_n\}) + (1 - \zeta_n)z_n(\{2^{a+b+1}\}) \]
\[ = \sum_{0<k_1<\ldots<k_a} \frac{\zeta_{k_1+\ldots+k_a}}{[k_1]_\zeta \ldots [k_a]_\zeta} \sum_{k_a<k_{a+1}<k_{a+2}} \left( \frac{\zeta_{2k_{a+1}}}{[k_{a+1}]_\zeta^2} + \frac{(1 - \zeta_n)\zeta_{k_{a+1}}}{[k_{a+1}]_\zeta} \right) \]
\[ \times \sum_{k_{a+1}<k_{a+2}<\ldots<k_{a+b+1}<n} \frac{\zeta_{k_{a+2}+\ldots+k_{a+b+1}}}{[k_{a+2}]_\zeta^2 \ldots [k_{a+b+1}]_\zeta^2} \]
\[ = H_{n-1}(\{2^a, 3, 2^b; \zeta_n\}) = -z_n(\{2^b, 3, 2^a; \zeta_n\}), \]
where in the last equality we used (5). Hence
\[ z_n(\{2^a, 3, 2^b; \zeta_n\}) + z_n(\{2^b, 3, 2^a; \zeta_n\}) = -(1 - \zeta_n)z_n(\{2^{a+b+1}\}), \]
which, by (2), implies the theorem. \qed

3. Proof of Theorem 1.2.

The \(q\)-binomial coefficient, or Gaussian coefficient, when \(q\) is specified to a primitive root of unity has the following properties.

Lemma 3.1. Let \(n > 1\) be a positive integer. Then for any primitive \(n\)-th root of unity \(\zeta_n\) and \(1 \leq k < n\),
\[ \left[ \frac{n-1}{k} \right]_{\zeta_n} = (-1)^k \zeta_n^{-\left(\frac{k+1}{2}\right)}. \]

Proof. We have
\[ \left[ \frac{n-1}{k} \right]_{\zeta_n} = \prod_{j=1}^k \left[ \frac{n-j}{j} \right]_{\zeta_n} = \prod_{j=1}^k \frac{1 - \zeta_n^{n-j}}{1 - \zeta_n^j} = \prod_{j=1}^k \frac{1 - \zeta_n^{-j}}{1 - \zeta_n^{-j}} = \prod_{j=1}^k (-\zeta_n^{-j}) = (-1)^k \zeta_n^{-\left(\frac{k+1}{2}\right)}. \]

The proof of Theorem 1.2 is based on the following multiple \(q\)-binomial identity.

Theorem A ([3], Thm. 8.1) Let \(n, s_1, \ldots, s_r\) be positive integers. Then
\[ \sum_{k=1}^n \left[ \frac{n}{k} \right]_q (-1)^k q^{\left(\frac{k+1}{2}\right)} \sum_{1 \leq k_1 < k_2 < \ldots < k_r = k} \prod_{i=1}^r q^{(s_i-1)k_i} = \prod_{j_i < j_{i+1}, i \in I} w_{j_i}^{j_i}, \]
where \(w = w(s) = \sum_{i=1}^r s_i, I = \{s_1, s_1 + s_2, \ldots, s_1 + s_2 + \cdots + s_{r-1}\}\), and the sum on the right is taken over all integers \(j_1, \ldots, j_w\) satisfying the conditions \(1 \leq j_i \leq n, j_i < j_{i+1}\) for \(i \in I\), and \(j_i \leq j_{i+1}\) otherwise.

From Theorem A we get a kind of duality for finite multiple \(q\)-harmonic sums \(z_n\) at roots of unity.

Theorem 3.1. Let \(n, s_1, \ldots, s_r\) be positive integers. Then
\[ z_n(s; \zeta_n) = (-1)^r \prod_{1 \leq j_1 < j_2 < \ldots < j_w < n} \frac{\zeta_{j_i}^i}{[j_{i+1}]_{\zeta_n}}, \]
where $w = w(s) = \sum_{j=1}^{r} s_j$, $I = \{s_1, s_1 + s_2, \ldots, s_1 + s_2 + \cdots + s_{r-1}\}$, and the sum on the right is taken over all integers $j_1, \ldots, j_w$ satisfying the conditions $1 \leq j_i \leq n$, $j_i < j_{i+1}$ for $i \in I$, and $j_i \leq j_{i+1}$ otherwise.

Proof. To get (7), we replace $n$ by $n - 1$, and $q$ by a primitive root of unity $\zeta_n$ in Theorem A, and apply Lemma 3.1. \hfill \Box

Proof of Theorem 1.2. Let $s = (\{1\}^a, 2, \{1\}^b)$ and $w = w(s) = a + b + 2$. Applying Theorem 3.1 and noticing that $I = \{1, 2, \ldots, a, a + 2, a + 3, \ldots, a + b + 1\}$, we get

$$-z_n(\{1\}^a, 2, \{1\}^b; \zeta_n) = (-1)^{a+b} \sum_{1 \leq j_1 < j_2 < \ldots < j_{a+2} < j_{a+3} < \ldots < j_{w} < n} \prod_{i=1}^{w} \frac{\zeta_n^{j_i}}{[j_i]_{\zeta_n}}$$

$$= (-1)^{a+b} \sum_{0 < n - j_1 < \ldots < n - j_{a+1} \leq n - j_{a+2} < n - j_{a+3} < \ldots < n - j_{w} < n} \prod_{i=1}^{w} \frac{\zeta_n^{n-j_i}}{[n - j_i]_{\zeta_n}}.$$

Applying identity (6), we get

$$-z_n(\{1\}^a, 2, \{1\}^b; \zeta_n) = \sum_{n > j_1 > \ldots > j_{a+1} > j_{a+2} > j_{a+3} > \ldots > j_{w} \geq 1} \frac{1}{[j_1]_{\zeta_n} \cdots [j_a]_{\zeta_n} [j_{a+1}]_{\zeta_n}^2 [j_{a+3}]_{\zeta_n} \cdots [j_{w}]_{\zeta_n}} + z_n(\{1\}^w; \zeta_n).$$

Noticing that

$$\frac{1}{[j_{a+1}]_{\zeta_n}^2} = \frac{(1 - \zeta_n)[j_{a+1}]_{\zeta_n} + \zeta_n^{j_{a+1}}}{[j_{a+1}]_{\zeta_n}^2} = \frac{1 - \zeta_n}{[j_{a+1}]_{\zeta_n}^2} + \frac{\zeta_n^{j_{a+1}}}{[j_{a+1}]_{\zeta_n}^2},$$

we obtain

$$-z_n(\{1\}^a, 2, \{1\}^b; \zeta_n) = (1 - \zeta_n)z_n(\{1\}^{w-1}; \zeta_n) + z_n(\{1\}^b, 2, \{1\}^a; \zeta_n) + z_n(\{1\}^w; \zeta_n).$$

Therefore, by (1),

$$z_n(\{1\}^a, 2, \{1\}^b; \zeta_n) + z_n(\{1\}^b, 2, \{1\}^a; \zeta_n) = -(1 - \zeta_n)z_n(\{1\}^{w-1}; \zeta_n) - z_n(\{1\}^w; \zeta_n)$$

$$= -\frac{1}{n} \left(1 - \zeta_n\right)\binom{n}{w} (1 - \zeta_n)^{w-1} - \frac{1}{n} \left(\frac{n}{w + 1}\right) (1 - \zeta_n)^w$$

$$= -\frac{1}{n} \left(\frac{n}{w + 1}\right) (1 - \zeta_n)^w. \quad \Box$$

References


