NON-COMMUTATIVE DEFORMATIONS OF DERIVED MCKAY CORRESPONDENCE FOR A_n SINGULARITIES

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Contents

1. Introduction 1
2. NC schemes and (quasi-)coherent sheaves 4
3. NC deformation of CCR and NCCR for A_n 5
4. Qcoh($\mathcal{X}$) for A_n 7
4.1. Čech cohomology 7
4.2. Generator 9
4.3. Comparison of two cohomology groups 11
5. Comparison of End($\mathcal{T}$) and $\mathcal{S}$ 17
5.1. Decomposition of $\mathcal{S}$ by orthogonal idempotents 17
5.2. Decomposition of $\mathcal{A}$ by orthogonal idempotents 18
5.3. Algebra homomorphism $\mathcal{S} \to \mathcal{A}$ 20
5.4. Injectivity of ϕ 21
5.5. Surjectivity of ϕ 23
6. Derived equivalence 25
References 27

Abstract. Let $Y = \mathbf{C}^2/\mathbf{Z}_{n+1}$ be a surface singularity of type A_n, let $f : X \to Y$ be the minimal resolution, and let $S = k[u, v]\#\mathbf{Z}_{n+1}$ be a twisted group algebra. The derived McKay correspondence says that there is an equivalence of triangulated categories $D^b(\mathrm{coh}(X)) \cong D^b(\text{mod-}S)$ over Y. We will prove that this derived equivalence extends between the semi-universal non-commutative deformations of the commutative crepant resolution X and the non-commutative crepant resolution S.

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1. Introduction

The purpose of this paper is to show that the derived McKay correspondence extends under non-commutative (NC) deformations in an example. This kind of extension is expected naturally because the Fourier-Mukai equivalence extends under NC deformations in some examples ([15], [1]) and there is "Mukai implies McKay" principle ([3]).

The derived McKay correspondence conjecture is a relationship between commutative and NC crepant resolutions. Let Y be an algebraic variety with only Gorenstein singularities. A proper birational morphism $f : X \to Y$ from a smooth algebraic variety is said to be *crepant* if there is no discrepancy between the canonical divisors, i.e., $f^*K_Y = K_X$. Such f is called a *commutative crepant resolution* (CCR) in this paper. The non-commutative counterpart of CCR is a *non-commutative crepant resolution* (NCCR) which is defined as follows by [17] (also [18]): it is a coherent sheaf of NC associative $\mathcal{O}_Y$-algebras S which is Cohen-Macaulay, has finite global dimension, and has a form $S = \mathrm{End}_Y(M)$ for a reflexive $\mathcal{O}_Y$-module M. We note that CCR or NCCR may or may not exist, and they are not necessarily unique when they exist. There are also mixed cases of CCR and NCCR (cf. [8]).

The derived McKay correspondence conjecture claims that, if there exist both CCR X and NCCR S, then there is an $\mathcal{O}_Y$-linear equivalence of triangulated categories $D^b(\mathrm{coh}(X)) \cong D^b(\mathrm{coh}(S))$. We note that the underlying abelian categories $\mathrm{coh}(X)$ and $\mathrm{coh}(S)$ are completely different. The former is geometric while the latter is algebraic. For example, the set of simple objects over a point sheaf $\mathcal{O}_y$ for $y \in \mathrm{Sing}\ Y$ is infinite in $\mathrm{coh}(X)$ but finite in $\mathrm{coh}(S)$. There are positive answers to this conjecture in special cases, e.g., in lower dimensional cases ([7] in dimension 2 and [3] in dimension 3) and toric cases ([9]). There are generalizations of the derived McKay correspondence to non-Gorenstein singularities, e.g., KLT singularities, where the equivalences are replaced by fully faithful functors (cf. [8], [10]).

We consider in this paper the following question: when there is a derived McKay correspondence, is it extended under NC deformations of CCR and NCCR? A similar question was considered by [15] and [1] for Fourier-Mukai equivalences. According to [3], there is a principle that Mukai implies McKay, so it is natural to ask a similar question for McKay correspondences. We will confirm this expectation in an explicit example by calculation, namely we treat 2-dimensional toric singularities.

We consider the following case. Let $Y = k^2/G$ be a 2-dimensional toric singularity, where $k = \mathbf{C}$ and $G = \mathbf{Z}/(n+1)$ for a positive integer n, and the action of G is given by $g(u, v) = (\zeta u, \zeta^{-1} v)$ for coordinates (u, v) and $\zeta = e^{2\pi i/(n+1)}$. It is an isolated singularity of type A_n. CCR is the minimal resolution $f : X \to Y$, and NCCR is given by a twisted group algebra $S = k[u, v]\#G$, where the multiplication is defined by the rule $(ag)(bh) = ag(b)gh$ for $a, b \in k[u, v]$ and $g, h \in G$.

According to [4], we consider the semi-universal NC deformation of S given by $\mathcal{S} = k[s_0, \ldots, s_n]\langle u, v\rangle\#G/(uv - vu - \sum_{i=0}^{n} s_i g^i)$, where $s_0, \ldots, s_n$ are algebraically independent deformation parameters. It is a flat deformation of S over an affine space.

The NC deformation $\mathcal{X}$ of X we consider in this paper is a kind of an NC scheme. It is constructed by gluing NC affine rings in the same way as X is covered by commutative affine rings. Here we note that the localization is not possible in general for NC rings, so we do not consider a space with a

structure sheaf of rings. Instead an NC scheme $\mathcal{X}$ is defined as a collection of NC rings with gluing homomorphisms, which satisfy certain conditions (cocycle condition, flatness, and the birationality). $\mathcal{X}$ is flat over another polynomial ring $k[t_0, \dots, t_n]$. We note that the base ring is global, and it is not infinitesimal nor formal unlike the deformation quantization.

We then define the bounded derived categories of coherent modules on NC deformations of CCR and NCCR. The following is the main result:

Theorem 1.1. *Let Y be a 2-dimensional toric singularity of type A_n, let X and S be its CCR and NCCR, and let $\mathcal{X}$ and $\mathcal{S}$ be their semi-universal NC deformations over polynomial algebras $k[t_0, \dots, t_n]$ and $k[s_0, \dots, s_n]$ respectively. Then there is an equivalence of triangulated categories*

$$D^b(coh(\mathcal{X})) \cong D^b(coh(\mathcal{S}))$$

which is linear over a linear coordinate change $(t_0, \dots, t_n) \to (s_0, \dots, s_n)$.

The right hand side is the usual bounded derived category of finitely generated $\mathcal{S}$-modules. Since S is an NC algebra, we can write $D^b(\text{mod-}\mathcal{S})$ instead of $D^b(\text{coh}(\mathcal{S}))$. For the left hand side, we define an $\mathcal{X}$-module as a collection of modules over the NC rings with the compatibility isomorphisms. We will prove the theorem by showing that there is a tilting object in the category of coherent $\mathcal{X}$-modules, and its endomorphism ring is isomorphic to $\mathcal{S}$.

The outline of the paper is as follows. In §2, we define NC schemes and modules on them. We note that such NC schemes are already considered in [5] in full generality. Our theorem partly justifies our definition of NC schemes.

In §3, we construct NC deformations of CCR and NCCR for our toric surface singularities. The case for NCCR is the algebra $\mathcal{S}$ which is already constructed in [4], and the case for CCR is our new NC scheme $\mathcal{X}$.

We investigate the category of quasi-coherent modules on our $\mathcal{X}$ in §4. It is a Grothendieck abelian category, and we can consider cohomology groups based on injective resolutions. But they are difficult to calculate, so we will prove that they are isomorphic to Čech cohomology groups in the case which is needed. We will prove that there is a locally free coherent module $\mathcal{T}$ which is tilting, i.e., there are no higher cohomology groups. We also prove that $\mathcal{T}$ generates the derived category of coherent sheaves.

We will compare the endomorphism algebra $\text{End}(\mathcal{T})$ and the NC deformed algebra $\mathcal{S}$, and prove that they are isomorphic in §5. We will prove it by showing that they are both quiver algebras whose bases are of type $\tilde{A}_n$, and the sets of generators and relations are the same. Then we conclude the proof of Theorem 1.1 by the tilting theory.

The paper [12] treats a generalization of this paper and proves that a tilting bundle extends uniquely along any formal NC deformation of an NC

scheme. It follows that, if a crepant resolution of a scheme has a tilting bundle, then the corresponding NC crepant resolution deforms in a parallel way with any NC formal deformation of the original crepant resolution. We note that the NC deformations usually exist only formally while we treated global algebraic NC deformations in this paper. We do not know whether such global algebraic NC deformations exist even in the case of Kleinian singularities other than A_n.

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2. NC schemes and (quasi-)coherent sheaves

We will give temporary definitions of NC schemes and (quasi-)coherent sheaves on them which will be used in this paper.

Definition 2.1. Let I be a poset. An *NC scheme* X parametrized by I consists of associative NC algebras R_i for $i \in I$ and algebra homomorphisms $\phi^X_{ji} : R_i \to R_j$ for all $i < j$, called the *gluing homomorphisms*, which satisfy the following conditions:

(1) (cocycle condition) $\phi^X_{kj} \circ \phi^X_{ji} = \phi^X_{ki}$ for $i < j < k$.

(2) (flatness) R_j is flat as left and right R_i-modules through ϕ^X_{ji}.

(3) (birationality condition) The natural homorphism of R_j-modules $R_j \otimes_{R_i} R_j \to R_j$ is bijective for $i < j$.

We note that the existence of the maximum $\max\{i, j\} \in I$ for $i, j \in I$ is not required in this paper.

For example, let X be a (usual) algebraic variety and let $U_i = \operatorname{Spec} R_i$ for $i \in I$ be its open covering by affine open subsets. We define a partial order on I as follows: $i < j$ if $U_i \supset U_j$. Then the homomorphism $\phi^X_{ji} : R_i \to R_j$ is defined by the restrictions of functions. Since U_i and U_j are birationally equivalent, the condition (3) holds.

Definition 2.2. An *NC deformation* of an NC scheme $X = \{R_i\}_{i \in I}$ over a (commutative) local ring (B, M) is a pair $(\mathcal{X}, \alpha)$ consisting of an NC scheme $\mathcal{X} = \{\mathcal{R}_i\}_{i \in I}$ such that the algebras $\mathcal{R}_i$ are flat B-algebras for all i, and isomorphisms $\alpha_i : B/M \otimes_B \mathcal{R}_i \cong R_i$ for all i, which are compatible with the gluing homomorphisms $\phi^{\mathcal{X}}_{ji}$ and ϕ^X_{ji}.

Definition 2.3. A *quasi-coherent* module $M = (M_i, \psi^M_{ji})$ on an NC scheme $X = \{R_i\}$ consists of right R_i-modules M_i and gluing isomorphisms

$$\psi^M_{ji} : M_i \otimes_{R_i} R_j \to M_j$$

as R_j-modules for $i < j$ which satisfy the cocycle condition $\psi^M_{kj} \circ (\psi^M_{ji} \otimes_{R_j} \mathrm{Id}_{R_k}) = \psi^M_{ki}$ for $i < j < k$. If the M_i are finitely generated, then M is called *coherent*.

A homomorphism of quasi-coherent modules $h : M \to N$ is a collection of R_i-homomorphisms $h_i : M_i \to N_i$ which are compatible with gluing isomorphisms ψ_{ji}^M and ψ_{ji}^N.

Lemma 2.4. *(1) Let $h : M \to N$ be a homomorphism of quasi-coherent modules. Then $Ker(h) = (Ker(h_i))$ and $Coker(h) = (Coker(h_i))$ are also quasi-coherent modules.*

(2) The category of quasi-coherent modules $Qcoh(X)$ is an abelian category, and the category of coherent modules $coh(X)$ is an abelian subcategory.

Proof. By the flatness of $\phi_{ji} : R_i \to R_j$, we have isomorphisms $\mathrm{Ker}(h_i) \otimes_{R_i} R_j \cong \mathrm{Ker}(h_j)$ and $\mathrm{Coker}(h_i) \otimes_{R_i} R_j \cong \mathrm{Coker}(h_j)$. Therefore we have our assertions. □

We explain the relationship of this paper to [5]. The paper [5] considers more generally a presheaf of algebras $\mathcal{A} : \mathcal{U}^{\mathrm{op}} \to \mathrm{Alg}(k)$ from a small category $\mathcal{U}$ to the category of k-algebras which are not necessarily smooth. The category of quasi-coherent sheaves $\mathrm{Qch}(\mathcal{A})$ is defined as the descent category of a prestack of modules, a contravariant functor from $\mathcal{U}$ to the category of $\mathcal{A}(\mathcal{U})$-modules. Under the additional conditions of quasi-compactness and semi-separatedness defined in [5] as well as that $\mathcal{U}$ is a finite poset with binary meets and the exactness of the tensor product by $\mathcal{A}(\mathcal{U})$, $\mathrm{Qch}(A)$ becomes a Grothendieck category. Then the space of twisted deformations of $\mathcal{A}$ over $k[\epsilon]$ (the first order deformations) is described by the second Hochschild cohomology group $HH^2(\mathcal{A})$ using the Gerstenhaber-Schack complex, and is proved to be isomorphic to the space of the first order deformations of $\mathrm{Qch}(\mathcal{A})$ as abelian categories considered in [13]. The higher order deformations and the obstructions of extensions are not treated in [5].

3. NC deformation of CCR and NCCR for A_n

Let Y be a 2-dimensional singularity of type A_n. Y has a commutative crepant resolution (CCR) X and a non-commutative crepant resolution (NCCR) S. We will describe NC deformations $\mathcal{X}$ and $\mathcal{S}$ of X and S, respectively.

We consider a quotient variety $Y = (\mathrm{Spec}\ k[u, v])/G$, where a cyclic group $G = \mathbf{Z}/(n+1)$ acts on an affine plane by $(u, v) \mapsto (\zeta u, \zeta^{-1} v)$ with $\zeta = e^{2\pi i/(n+1)}$. Y is a toric variety with an isolated singularity of type A_n, and the minimal resolution $f : X \to Y$ is a toric morphism. X is a union of affine planes $U_i = \mathrm{Spec}\ R_i \cong k^2$ with $R_i = k[x_i, y_i]$ $(0 \le i \le n)$, which are glued together by the following transformation rules:

$$x_{i+1} = x_i^2 y_i, \ \ y_{i+1} = x_i^{-1}.$$

We deform these affine planes to non-commutative planes, and then glue them to obtain an NC variety $\mathcal{X}$. The sheaf of bivectors $\bigwedge^2 T_X \cong \mathcal{O}_X$ has a nowhere vanishing section, and $\mathcal{X}$ contains the corresponding purely

non-commutative deformation. Since $H^2(X, \mathcal{O}_X) \cong 0$, there is no twisted deformation (cf. [11]).

We define a poset

$$I = \{0, 1, \dots, n\} \cup \{(0,1), (1,2), \dots, (n-1, n)\}$$

with the order $i, i+1 < (i, i+1)$. The corresponding affine open subsets are the U_i $(0 \leq i \leq n)$ and the $U_{i,i+1} = \mathrm{Spec}\ R_{i,i+1} \cong k^* \times k$ with $R_{i,i+1} = k[x_i, x_i^{-1}, y_i]$ $(0 \leq i < n)$. We consider only intersections $U_{i,i+1} = U_i \cap U_{i+1}$ and ignore other intersections.

We define deformed affine rings $\mathcal{R}_i$ by

$$\mathcal{R}_i = k[t_0, \dots, t_n]\langle x_i, y_i\rangle / (x_i y_i - y_i x_i - t_0)$$

where $t_0, \dots, t_n$ are deformation parameters, and define an NC variety $\mathcal{X}$ by gluing these NC rings with the deformed transformation rules:

$$x_{i+1} = x_i^2 y_i + t_{i+1} x_i, \ \ y_{i+1} = x_i^{-1}$$

for $0 \leq i < n$. The parameters $t_1, \dots, t_n$ correspond to the commutative deformations of X constructed by changing the gluing, and t_0 corresponds to the purely non-commutative deformation.

More precisely, we define

$$\begin{aligned}\mathcal{R}_{i,i+1} &= k[t_0, \dots, t_n]\langle x_i, x_i^{-1}, y_i\rangle / (x_i y_i - y_i x_i - t_0) \\ &:= k[t_0, \dots, t_n]\langle x_i, y_i, y_{i+1}\rangle / (x_i y_i - y_i x_i - t_0, x_i y_{i+1} - 1, y_{i+1} x_i - 1).\end{aligned}$$

The gluing is given by the natural homomorphisms

$$\phi_i^+ : \mathcal{R}_i \to \mathcal{R}_{i,i+1}, \ \ \phi_i^- : \mathcal{R}_{i+1} \to \mathcal{R}_{i,i+1}.$$

We note that the commutativity relations of the x_i, y_i are compatible with gluing. Indeed we have

$$\begin{aligned}x_{i+1} y_{i+1} - y_{i+1} x_{i+1} &= x_i^2 y_i x_i^{-1} + t_{i+1} - (x_i y_i - t_{i+1}) \\ &= x_i (y_i x_i + t_0) x_i^{-1} - x_i y_i = t_0.\end{aligned}$$

Lemma 3.1. *(1) $\mathcal{R}_i$ is Noetherian.*
(2) ϕ_i^+, ϕ_i^- are injective.
(3) $y_i y_{i+1} - y_{i+1} y_i = t_0 y_{i+1}^2$.
(4) $\mathcal{R}_{i,i+1} = \bigcup_{l \geq 0} \mathcal{R}_i y_{i+1}^l$.
(5) Let $\mathcal{M}_i$ be a right $\mathcal{R}_i$-module. Then

$$Ker(\mathcal{M}_i \to \mathcal{M}_i \otimes_{\mathcal{R}_i} \mathcal{R}_{i,i+1}) = \mathcal{M}_{i,x_i\text{-}tor} := \{m \in \mathcal{M}_i \mid \exists l \geq 0, m x_i^l = 0\},$$

$$\mathcal{M}_i \otimes_{\mathcal{R}_i} \mathcal{R}_{i,i+1} = \bigcup_{l \geq 0} (\mathcal{M}_i / \mathcal{M}_{i,x_i\text{-}tor}) y_{i+1}^l.$$

(6) ϕ_i^+, ϕ_i^- are flat.
(7) The birationality conditions hold:

$$\mathcal{R}_{i,i+1} \otimes_{\mathcal{R}_i} \mathcal{R}_{i,i+1} \cong \mathcal{R}_{i,i+1}, \ \ \mathcal{R}_{i,i+1} \otimes_{\mathcal{R}_{i+1}} \mathcal{R}_{i,i+1} \cong \mathcal{R}_{i,i+1}.$$

Proof. (1) We define degrees on $\mathcal{R}_i$ by $\deg(x_i, y_i, t_j) = (1, 1, 0)$, and define a filtration by $F_d(\mathcal{R}_i) = \{r \in \mathcal{R}_i \mid \deg(r) \leq d\}$. Then we have $\mathrm{Gr}^F(\mathcal{R}_i) \cong k[t_0, \ldots, t_n][x_i, y_i]$, hence $\mathcal{R}_i$ is Noetherian.

(2) Since $\mathrm{Gr}(\phi_i^+) : \mathrm{Gr}(\mathcal{R}_i) \to \mathrm{Gr}(\mathcal{R}_{i,i+1}) \cong k[t_0, \ldots, t_n][x_i, y_i, x_i^{-1}]$ is injective, so is ϕ_i^+.

(3) This follows from $x_i(y_iy_{i+1} - y_{i+1}y_i)x_i = x_iy_i - y_ix_i = t_0$.

(4) and (5) are corollaries of (3).

(6) Let $\mathcal{M}_i \to \mathcal{N}_i$ be an injective homomorphism of $\mathcal{R}_i$-modules. Then $\mathcal{M}_i/\mathcal{M}_{i,x_i\text{-tor}} \to \mathcal{N}_i/\mathcal{N}_{i,x_i\text{-tor}}$ is injective, hence so is $\mathcal{M}_i \otimes_{\mathcal{R}_i} \mathcal{R}_{i,i+1} \to \mathcal{N}_i \otimes_{\mathcal{R}_i} \mathcal{R}_{i,i+1}$. Thus $\mathcal{R}_{i,i+1}$ is flat over $\mathcal{R}_i$.

(7) is a corollary of (5). □

Next we define an NCCR and its NC deformation for A_n following [4]. We have $Y = \mathrm{Spec}\ k[u, v]^G$, where $G = \mathbf{Z}/(n+1)$ acts by $(u, v) \mapsto (\zeta u, \zeta^{-1}v)$ with $\zeta = e^{2\pi i/(n+1)}$. A non-commutative crepant resolution (NCCR) of Y is given by a *twisted group ring* $S = k[u, v]\#G$, where the multiplication is defined as $ag \cdot bh = ag(b)gh$ for $a, b \in k[u, v]$ and $g, h \in G$.

According to [4], we define the "versal" NC deformation $\mathcal{S}$ of S by

$$\mathcal{S} = k[s_0, \ldots, s_n]\langle u, v\rangle \#G/(uv - vu - \sum_{i=0}^{n} s_i g^i)$$

where $s_0, \ldots, s_n$ are deformation parameters and g is a generator of G. $s_1, \ldots, s_n$ parametrize commutative deformations and s_0 corresponds to a purely non-commutative deformation. Indeed, the ring of invariants $e\mathcal{S}e$ of $\mathcal{S}$ for $e = \sum_{i=0}^{n} g^i/(n+1)$ becomes commutative when we put $s_0 = 0$ (loc. cit.).

4. Qcoh($\mathcal{X}$) for A_n

We will investigate the structure of the category Qcoh($\mathcal{X}$) of quasi-coherent modules over the NC deformation $\mathcal{X}$ for the singular surface Y of type A_n.

4.1. **Čech cohomology.** We will define Čech cohomology groups for NC schemes in this subsection. They will be compared with the cohomology groups defined by injective resolutions later. We consider here only 1-dimensional special case. The general case will be treated in a subsequent paper.

We recall that a quasi-coherent module $\mathcal{M}$ on $\mathcal{X}$ is a collection of right $\mathcal{R}_i$-modules $\mathcal{M}_i$ and right $\mathcal{R}_{i,i+1}$-modules $\mathcal{M}_{i,i+1}$ together with isomorphisms

$$\psi_{i,+}^{\mathcal{M}} : \mathcal{M}_i \otimes_{\mathcal{R}_i} \mathcal{R}_{i,i+1} \to \mathcal{M}_{i,i+1},$$
$$\psi_{i,-}^{\mathcal{M}} : \mathcal{M}_i \otimes_{\mathcal{R}_i} \mathcal{R}_{i-1,i} \to \mathcal{M}_{i-1,i}.$$

A homomorphism $h : \mathcal{M} \to \mathcal{N}$ consists of $\mathcal{R}_i$-homomorphisms $h_i : \mathcal{M}_i \to \mathcal{N}_i$ and $\mathcal{R}_{i,i+1}$-homomorphisms $h_{i,i+1} : \mathcal{M}_{i,i+1} \to \mathcal{N}_{i,i+1}$ such that the following

diagrams are commutative:

$$\begin{array}{rcl}
\mathcal{M}_i \otimes_{\mathcal{R}_i} \mathcal{R}_{i,i+1} & \xrightarrow{\psi^{\mathcal{M}}_{i,+}} & \mathcal{M}_{i,i+1} \\
{\scriptstyle h_i\otimes_{\mathcal{R}_i}\mathcal{R}_{i,i+1}}\downarrow & & \downarrow{\scriptstyle h_{i,i+1}} \\
\mathcal{N}_i \otimes_{\mathcal{R}_i} \mathcal{R}_{i,i+1} & \xrightarrow{\psi^{\mathcal{N}}_{i,+}} & \mathcal{N}_{i,i+1}, \\
\mathcal{M}_i \otimes_{\mathcal{R}_i} \mathcal{R}_{i-1,i} & \xrightarrow{\psi^{\mathcal{M}}_{i,-}} & \mathcal{M}_{i-1,i} \\
{\scriptstyle h_i\otimes_{\mathcal{R}_i}\mathcal{R}_{i-1,i}}\downarrow & & \downarrow{\scriptstyle h_{i-1,i}} \\
\mathcal{N}_i \otimes_{\mathcal{R}_i} \mathcal{R}_{i-1,i} & \xrightarrow{\psi^{\mathcal{N}}_{i,-}} & \mathcal{N}_{i-1,i}.
\end{array}$$

The following is an immediate consequence of the definition:

Lemma 4.1. *Let $\mathcal{M},\mathcal{N} \in Qcoh(\mathcal{X})$ be quasi-coherent modules. Define a k-homomorphism*

$$\begin{aligned}
\Delta : \prod_{i=0}^{n} Hom_{\mathcal{R}_i}(\mathcal{M}_i,\mathcal{N}_i) &\to \prod_{i=1}^{n} Hom_{\mathcal{R}_{i-1,i}}(\mathcal{M}_{i-1,i},\mathcal{N}_{i-1,i}) \\
(h_0,\dots,h_n) &\mapsto (g_{0,1},\dots,g_{n-1,n})
\end{aligned}$$

by

$$\begin{aligned}
g_{i-1,i} =& \psi^{\mathcal{N}}_{i-1,+} \circ (h_{i-1} \otimes_{\mathcal{R}_{i-1}} \mathcal{R}_{i-1,i}) \circ (\psi^{\mathcal{M}}_{i-1,+})^{-1} \\
&- \psi^{\mathcal{N}}_{i,-} \circ (h_i \otimes_{\mathcal{R}_i} \mathcal{R}_{i-1,i}) \circ (\psi^{\mathcal{M}}_{i,-})^{-1}.
\end{aligned}$$

Then

$$Hom_{\mathcal{X}}(\mathcal{M},\mathcal{N}) \cong Ker(\Delta).$$

Definition 4.2. We define the first Čech cohomology group by

$$\check{\mathrm{C}}\mathrm{Hom}^1_{\mathcal{X}}(\mathcal{M},\mathcal{N}) \cong \mathrm{Coker}(\Delta).$$

Definition 4.3. A locally free module $\mathcal{M}$ of rank 1 on $\mathcal{X}$ is a coherent module such that $\mathcal{M}_i \cong \mathcal{R}_i$ as right $\mathcal{R}_i$-modules for all i. A *divisorial sheaf* $\mathcal{D} = \mathcal{R}(\sum_{i=1}^{n} d_i D_i)$ is a locally free module of rank 1 defined by gluing homomorphisms

$$\psi^{\mathcal{D}}_{i,-}(r) = x_{i-1}^{d_i} r, \;\; \psi^{\mathcal{D}}_{i,+}(r) = r$$

for $1 \le i \le n$ and $0 \le i < n$, respectively.

The notation "D_i" does not have an independent meaning, but it comes from a prime divisor on the minimal resolution X which intersects transversally to the exceptional curve C_i.

Lemma 4.4. *Let $\mathcal{D}$ be a divisorial sheaf and let $0 \to \mathcal{N} \to \mathcal{N}' \to \mathcal{N}'' \to 0$ be an exact sequence of quasi-coherent $\mathcal{R}$-modules. Then there is a long exact sequence*

$$\begin{aligned}
0 \to Hom_{\mathcal{X}}(\mathcal{D},\mathcal{N}) \to Hom_{\mathcal{X}}(\mathcal{D},\mathcal{N}') \to Hom_{\mathcal{X}}(\mathcal{D},\mathcal{N}'') \\
\to \check{C}Hom^1_{\mathcal{X}}(\mathcal{D},\mathcal{N}) \to \check{C}Hom^1_{\mathcal{X}}(\mathcal{D},\mathcal{N}') \to \check{C}Hom^1_{\mathcal{X}}(\mathcal{D},\mathcal{N}'') \to 0.
\end{aligned}$$

Proof. Since $\mathcal{D}_i \cong \mathcal{R}_i$, we have an isomorphism as k-modules $\mathrm{Hom}_{\mathcal{R}_i}(\mathcal{D}_i, \mathcal{N}_i) \to \mathcal{N}_i$ given by $h_i \mapsto h_i(1)$. Hence we have the following commutative diagram of exact sequences:

$$\begin{array}{ccccccc} 0 \to \prod_{i=0}^n (\mathcal{D}_i, \mathcal{N}_i) & \longrightarrow & \prod_{i=0}^n (\mathcal{D}_i, \mathcal{N}'_i) & \longrightarrow & \prod_{i=0}^n (\mathcal{D}_i, \mathcal{N}''_i) \to 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0 \to \prod_{i=1}^n (\mathcal{D}_{i-1,i}, \mathcal{N}_{i-1,i}) & \longrightarrow & \prod_{i=1}^n (\mathcal{D}_{i-1,i}, \mathcal{N}'_{i-1,i}) & \longrightarrow & \prod_{i=1}^n (\mathcal{D}_{i-1,i}, \mathcal{N}''_{i-1,i}) \to 0 \end{array}$$

where we use abbreviations $(\mathcal{D}_i, \mathcal{N}_i) = \mathrm{Hom}_{\mathcal{R}_i}(\mathcal{D}_i, \mathcal{N}_i)$ and $(\mathcal{D}_{i-1,i}, \mathcal{N}_{i-1,i}) = \mathrm{Hom}_{\mathcal{R}_{i-1,i}}(\mathcal{D}_{i-1,i}, \mathcal{N}_{i-1,i}))$, etc. Then our assertion is a consequence of the snake lemma. □

4.2. **Generator.** We will prove in this subsection that $\mathrm{Qcoh}(\mathcal{X})$ is generated by an object $\bigoplus_{m_1,\dots,m_n \in \mathbf{Z}} \mathcal{R}(\sum_{i=1}^n m_i D_i)$.

Lemma 4.5. *Let $\mathcal{M}$ be a non-zero quasi-coherent module. Then there exist integers d_i^0 such that $Hom_{\mathcal{X}}(\mathcal{R}(-\sum_i d_i D_i), \mathcal{M}) \neq 0$ if $d_i \geq d_i^0$.*

Proof. Since $\mathcal{M} \ncong 0$, there is i such that $\mathcal{M}_i \ncong 0$. Then there is a non-zero homomorphism $h_i : \mathcal{R}_i \to \mathcal{M}_i$. We will extend h_i to neighboring $\mathcal{M}_j$'s inductively. We set $h_i(1) = m_i$.

We take sufficiently large positive integers d_i^0 and d_{i+1}^0 such that, for $d_i \geq d_i^0$ and $d_{i+1} \geq d_{i+1}^0$,

$$\psi_{i,-}^{\mathcal{M}}(m_i) x_{i-1}^{d_i} = m_{i-1} \otimes 1 \in \mathrm{Im}(\mathcal{M}_{i-1} \to \mathcal{M}_{i-1} \otimes_{\mathcal{R}_{i-1}} \mathcal{R}_{i-1,i}),$$
$$\psi_{i,+}^{\mathcal{M}}(m_i) y_{i+1}^{d_{i+1}} = m_{i+1} \otimes 1 \in \mathrm{Im}(\mathcal{M}_{i+1} \to \mathcal{M}_{i+1} \otimes_{\mathcal{R}_{i+1}} \mathcal{R}_{i,i+1})$$

where we consider only $\psi_{i,\pm}^{\mathcal{M}}$ when $i+1 \leq n$ or $i-1 \geq 0$, respectively. Then we define homomorphisms

$$h_* : \mathcal{R}_*(-d_i D_i - d_{i+1} D_{i+1}) \to \mathcal{M}_*$$

for $* = i-1, (i-1,i), i, (i,i+1), i+1$ by

$$h_{i-1}(r) = m_{i-1} r, \;\; h_i(r) = m_i r, \;\; h_{i+1}(r) = m_{i+1} r,$$
$$h_{i-1,i}(r) = \psi_{i,-}^{\mathcal{M}}(m_i x_{i-1}^{d_i} r), \;\; h_{i,i+1}(r) = \psi_{i,+}^{\mathcal{M}}(m_i r).$$

We check the compatibilities:

$$\begin{aligned} &\psi_{i-1.+}^{\mathcal{M}} h_{i-1}(r) = \psi_{i-1,+}^{\mathcal{M}}(m_{i-1} r) = \psi_{i,-}^{\mathcal{M}}(m_i x_{i-1}^{d_i} r) \\ &= h_{i-1,i}(r) = h_{i-1,i} \psi_{i-1,+}^{\mathcal{D}}(r), \\ &\psi_{i,-}^{\mathcal{M}} h_i(r) = \psi_{i,-}^{\mathcal{M}}(m_i r) = \psi_{i,-}^{\mathcal{M}}(m_i x_{i-1}^{d_i} x_{i-1}^{-d_i} r) \\ &= h_{i-1,i}(x_{i-1}^{-d_i} r) = h_{i-1,i} \psi_{i,-}^{\mathcal{D}}(r), \\ &\psi_{i,+}^{\mathcal{M}} h_i(r) = \psi_{i,+}^{\mathcal{M}}(m_i r) = h_{i,i+1}(r) = h_{i,i+1} \psi_{i,+}^{\mathcal{D}}(r), \\ &\psi_{i+1,-}^{\mathcal{M}} h_{i+1}(r) = \psi_{i+1,-}^{\mathcal{M}}(m_{i+1} r) = \psi_{i,+}^{\mathcal{M}}(m_i y_{i+1}^{d_{i+1}} r) \\ &= h_{i,i+1}(x_i^{-d_{i+1}} r) = h_{i,i+1} \psi_{i+1,-}^{\mathcal{D}}(r), \end{aligned}$$

where we denote $\mathcal{D} = \mathcal{R}(-d_iD_i - d_{i+1}D_{i+1})$. Thus h_i is extended to h_{i-1}, h_{i+1}, etc. in a compatible way. Here we note that we use the same symbol h_i to denote homomorphisms $\mathcal{R}_i \to \mathcal{M}_i$ and $\mathcal{R}_i(-d_iD_i - d_{i+1}D_{i+1}) \to \mathcal{M}_i$ by identifying $\mathcal{R}_i = \mathcal{R}_i(-d_iD_i - d_{i+1}D_{i+1})$. We repeat this process to obtain the h_j for all j. □

Lemma 4.6. *Let $\mathcal{M}$ be a coherent $\mathcal{R}$-module.*

(1) Then there exist integers d_i^0 such that $\check{C}Hom^1_{\mathcal{X}}(\mathcal{R}(-\sum_i d_iD_i), \mathcal{M}) = 0$ if $d_i \geq d_i^0$.

(2) There is a surjective homomorphism from a finite direct sum of divisorial sheaves to $\mathcal{M}$.

Proof. First we prove (1) in the case when $\mathcal{M}$ is itself a divisorial sheaf; we assume that $\mathcal{M} = \mathcal{R}(-\sum_i d_i'D_i)$ for some integers d_i'. Then we have

$$\begin{aligned}
&\check{\mathrm{C}}\mathrm{Hom}^1_{\mathcal{X}}(\mathcal{R}(-\sum_i d_iD_i), \mathcal{R}(-\sum_i d_i'D_i)) \\
&= \mathrm{Coker}(\Delta : \prod_{i=0}^{n} \mathrm{Hom}_{\mathcal{R}_i}(\mathcal{R}_i, \mathcal{R}_i) \to \prod_{i=1}^{n} \mathrm{Hom}_{\mathcal{R}_{i-1,i}}(\mathcal{R}_{i-1,i}, \mathcal{R}_{i-1,i})) \\
&= \mathrm{Coker}(\Delta : \prod_{i=0}^{n} \mathcal{R}_i \to \prod_{i=1}^{n} \mathcal{R}_{i-1,i})
\end{aligned}$$

since $\mathrm{Hom}_{\mathcal{R}_i}(\mathcal{R}_i, \mathcal{R}_i) \cong \mathcal{R}_i$, etc., as k-modules, by sending $h_i : \mathcal{R}_i \to \mathcal{R}_i$ to $h_i(1)$. The map $\Delta : (h_0, \ldots, h_n) \mapsto (g_{0,1}, \ldots, g_{n-1,n})$ is given by

$$g_{i-1,i}(1) = h_{i-1}(1) - x_{i-1}^{-d_i'} h_i(1) x_{i-1}^{d_i}.$$

Suppose that we are given the $g_{i-1,i}(1) \in \mathcal{R}_{i-1,i}$ for $1 \leq i \leq n$, and we look for the $h_i(1) \in \mathcal{R}_i$ for $0 \leq i \leq n$ which induce the $g_{i-1.i}$. We proceed by induction on i. We set $h_0(1) = 0$. We take d_1 large enough so that $h_1(1) = -y_1^{-d_1'} g_{0,1}(1) y_1^{d_1} \in \mathcal{R}_1$. Then we take d_2 large enough so that $h_2(1) = -y_2^{-d_2'}(g_{1,2}(1) - h_1(1)) y_2^{d_2} \in \mathcal{R}_2$. In this way, we determine the d_i and the h_i to obtain our first claim.

Now we take any coherent module $\mathcal{M}$ and prove (1) and (2) simultaneously. We will construct a strictly increasing sequence of submodules $\{\mathcal{M}^t\}$ of $\mathcal{M}$ and a weakly increasing sequence of integers $\{d_i^t\}$ and $\{e_i^t\}$ such that the following hold:

(1t) $\check{\mathrm{C}}\mathrm{Hom}^1_{\mathcal{X}}(\mathcal{R}(-\sum_i d_iD_i), \mathcal{M}^t) = 0$ if $d_i \geq d_i^t$,

(2t) There are surjective homomorphisms $\bigoplus_{j=1}^{t} \mathcal{R}(-\sum_i e_i^j D_i) \to \mathcal{M}^t$.

Since the $\mathcal{R}_i$ are Noetherian, there is a t such that $\mathcal{M}^t = \mathcal{M}$, which is our claim.

We let $M^0 = 0$. Assuming that we have already $\mathcal{M}^t$ and the d_i^t, we will construct $\mathcal{M}^{t+1}$ and the d_i^{t+1}. We take large integers $e_i^t \geq d_i^t$ such that there is a non-zero homomorphism $h_t' : \mathcal{R}(-\sum_i e_i^t D_i) \to \mathcal{M}/\mathcal{M}^t$. By

the induction assumption, we have $\check{\mathrm{C}}\mathrm{Hom}^1_{\mathcal{X}}(\mathcal{R}(-\sum_i e_i^t D_i), \mathcal{M}^t) = 0$. Hence there is a homomorphism $h_t : \mathcal{R}(-\sum_i e_i^t D_i) \to \mathcal{M}$ which induces h'_t. We define $\mathcal{M}^{t+1}$ by $\mathrm{Im}(h_t) = \mathcal{M}^{t+1}/\mathcal{M}^t$. Then we have (2).

We take integers $d_i^{t+1} \geq e_i^t$ such that $\check{\mathrm{C}}\mathrm{Hom}^1_{\mathcal{X}}(\mathcal{R}(-\sum_i d_i D_i), \mathcal{R}(-\sum_i e_i^t D_i)) = 0$ if $d_i \geq d_i^{t+1}$. Then by Lemma 4.4, we have $\check{\mathrm{C}}\mathrm{Hom}^1_{\mathcal{X}}(\mathcal{R}(-\sum_i d_i D_i), \mathrm{Im}(h_t)) = 0$, hence $\check{\mathrm{C}}\mathrm{Hom}^1_{\mathcal{X}}(\mathcal{R}(-\sum_i d_i D_i), \mathcal{M}^{t+1}) = 0$ for $d_i \geq d_i^{t+1}$. This is (1). □

Corollary 4.7. *Any object in coh($\mathcal{R}$) has a locally free resolution whose terms are direct sums of divisorial sheaves.*

The following is a special case of [5]:

Proposition 4.8. *Qcoh($\mathcal{R}$) is a Grothendieck category ([6]).*

Proof. We check the conditions. The coproduct $\bigoplus_{\lambda\in\Lambda} \mathcal{M}_\lambda$ and the direct limit $\varinjlim_\lambda \mathcal{M}_\lambda$ always exist. For exact sequences $0 \to \mathcal{L}_\lambda \to \mathcal{M}_\lambda \to \mathcal{N}_\lambda \to 0$, we have an exact sequence

$$0 \to \varinjlim_\lambda \mathcal{L}_\lambda \to \varinjlim_\lambda \mathcal{M}_\lambda \to \varinjlim_\lambda \mathcal{N}_\lambda \to 0.$$

Finally, let $\mathcal{G} = \bigoplus_{m_1,\dots,m_n\in\mathbf{Z}} \mathcal{R}(\sum_{i=1}^n m_i D_i)$. Then, for any object $\mathcal{M} \in \mathrm{Qcoh}(\mathcal{R})$, there exists an index set Λ with a surjective homomorphism $\mathcal{G}^{(\Lambda)} \to \mathcal{M}$. □

Corollary 4.9. *Qcoh($\mathcal{R}$) has enough injectives.*

4.3. **Comparison of two cohomology groups.** Now that $\mathrm{Qcoh}(\mathcal{R})$ is a Grothendieck category, we can define the right derived functor $R\mathrm{Hom}_{\mathcal{X}}(\mathcal{M}, \mathcal{N})$ by using an injective resolution of $\mathcal{N}$. For a divisorial sheaf $\mathcal{D}$, we will describe the cohomology groups $\mathrm{Hom}^p_{\mathcal{X}}(\mathcal{D}, \mathcal{N}) = H^p(R\mathrm{Hom}_{\mathcal{X}}(\mathcal{D}, \mathcal{N}))$ by using the Cěch cohomology.

Lemma 4.10. *Let $\mathcal{D}$ be a divisorial sheaf and let $\mathcal{N}$ be a quasi-coherent module. Then there is an injective homomorphism*

$$Hom^1_{\mathcal{X}}(\mathcal{D}, \mathcal{N}) \to \check{C}Hom^1_{\mathcal{X}}(\mathcal{D}, \mathcal{N}).$$

Moreover, for a homomorphism of quasi-coherent modules $\mathcal{N} \to \mathcal{N}'$, there is a commutative diagram

$$\begin{array}{ccc} Hom^1_{\mathcal{X}}(\mathcal{D}, \mathcal{N}) & \longrightarrow & \check{C}Hom^1_{\mathcal{X}}(\mathcal{D}, \mathcal{N}) \\ \downarrow & & \downarrow \\ Hom^1_{\mathcal{X}}(\mathcal{D}, \mathcal{N}') & \longrightarrow & \check{C}Hom^1_{\mathcal{X}}(\mathcal{D}, \mathcal{N}'). \end{array}$$

Proof. We embed $\mathcal{N}$ into an exact sequence $0 \to \mathcal{N} \to \mathcal{I} \to \mathcal{L} \to 0$ where $\mathcal{I}$ is an injective module. Then we have a commutative diagram of exact

sequences

$$\begin{array}{ccccccc}
(\mathcal{D},\mathcal{I}) & \longrightarrow & (\mathcal{D},\mathcal{L}) & \longrightarrow & (\mathcal{D},\mathcal{N})^1 & \longrightarrow & 0 \\
= \downarrow & & = \downarrow & & & & \\
(\mathcal{D},\mathcal{I}) & \longrightarrow & (\mathcal{D},\mathcal{L}) & \longrightarrow & \check{C}(\mathcal{D},\mathcal{N})^1 & \longrightarrow & \check{C}(\mathcal{D},\mathcal{I})^1
\end{array}$$

because $(\mathcal{D},\mathcal{I})^1 = 0$, where we abbreviate $(\mathcal{D},\mathcal{N})^p = \mathrm{Hom}^p_{\mathcal{X}}(\mathcal{D},\mathcal{N})$, $\check{C}(\mathcal{D},\mathcal{N})^p = \check{\mathrm{C}}\mathrm{Hom}^1_{\mathcal{X}}(\mathcal{D},\mathcal{N})$, etc. Hence we obtain our injection.

From a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{N} & \longrightarrow & \mathcal{I} & \longrightarrow & \mathcal{L} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{N}' & \longrightarrow & \mathcal{I}' & \longrightarrow & \mathcal{L}' & \longrightarrow & 0
\end{array}$$

we obtain the second result. □

Let $\mathcal{N}$ be a quasi-coherent $\mathcal{R}$-module. We define $\mathcal{N}^{(i-1,i)} \in \mathrm{Qcoh}(\mathcal{R})$ by

$$\begin{aligned}
\mathcal{N}^{(i-1,i)}_{i-1} &= \mathcal{N}^{(i-1,i)}_{i-1,i} = \mathcal{N}^{(i-1,i)}_{i} = \mathcal{N}_{i-1,i}, \\
\mathcal{N}^{(i-1,i)}_{j} &= \mathcal{N}^{(i-1,i)}_{k-1,k} = \mathcal{N}^{(i-1,i)}_{\eta} = \mathcal{N}_{\eta},
\end{aligned}$$

for $j \neq i-1, i$ and $k \neq i$. Indeed, by the birationality condition, we confirm that $\mathcal{N}^{(i-1,i)} \in \mathrm{Qcoh}(\mathcal{R})$. Let $\mathcal{N}^e = \bigoplus_{i=1}^n \mathcal{N}^{(i-1,i)}$.

Let $\mathcal{N}^t$ be the kernel of the natural homomorphism $\mathcal{N} \to \mathcal{N}^e$. Then we have $\mathcal{N}^t_{i-1,i} = 0$ for all i. Hence $\check{\mathrm{C}}\mathrm{Hom}^1(\mathcal{M},\mathcal{N}^t) = 0$ for any $\mathcal{M}$.

Lemma 4.11. *$\check{C}Hom^1_{\mathcal{X}}(\mathcal{D},\mathcal{N}^e) = 0$.*

Proof. This is because $\mathrm{Hom}_{\mathcal{R}_i}(\mathcal{D}_i,\mathcal{N}^e_i) \to \mathrm{Hom}_{\mathcal{R}_{i-1,i}}(\mathcal{D}_{i-1,i},\mathcal{N}^e_{i-1,i}))$ is surjective for all i. □

Lemma 4.12. *Let $\mathcal{D}$ be a divisorial sheaf and let $\mathcal{N}$ be a quasi-coherent module. Then there is an injective homomorphism*

$$\check{C}Hom^1_{\mathcal{X}}(\mathcal{D},\mathcal{N}) \to Hom^1_{\mathcal{X}}(\mathcal{D},\mathcal{N}).$$

Moreover, for a homomorphism of quasi-coherent modules $\mathcal{N} \to \mathcal{N}'$, there is a commutative diagram

$$\begin{array}{ccc}
\check{C}Hom^1_{\mathcal{X}}(\mathcal{D},\mathcal{N}) & \longrightarrow & Hom^1_{\mathcal{X}}(\mathcal{D},\mathcal{N}) \\
\downarrow & & \downarrow \\
\check{C}Hom^1_{\mathcal{X}}(\mathcal{D},\mathcal{N}') & \longrightarrow & Hom^1_{\mathcal{X}}(\mathcal{D},\mathcal{N}').
\end{array}$$

Proof. Let $\mathcal{N}^f = \mathcal{N}/\mathcal{N}^t$. We have an exact sequence $0 \to \mathcal{N}^f \to \mathcal{N}^e \to \mathcal{L} \to 0$ for some $\mathcal{L}$. Then we have a commutative diagram of exact sequences

$$\begin{array}{ccccccc} (\mathcal{D},\mathcal{N}^e) & \longrightarrow & (\mathcal{D},\mathcal{L}) & \longrightarrow & \check{C}(\mathcal{D},\mathcal{N}^f)^1 & \longrightarrow & \check{C}(\mathcal{D},\mathcal{N}^e)^1 \\ =\downarrow & & =\downarrow & & & & \\ (\mathcal{D},\mathcal{N}^e) & \longrightarrow & (\mathcal{D},\mathcal{L}) & \longrightarrow & (\mathcal{D},\mathcal{N}^f)^1 & \longrightarrow & (\mathcal{D},\mathcal{N}^e)^1. \end{array}$$

Since $\check{C}(\mathcal{D},\mathcal{N}^e)^1 = 0$, we obtain our injection $\check{\mathrm{C}}\mathrm{Hom}^1(\mathcal{D},\mathcal{N}^f) \to \mathrm{Hom}^1(\mathcal{D},\mathcal{N}^f)$.

From an exact sequence

$$0 \to \mathcal{N}^t \to \mathcal{N} \to \mathcal{N}^f \to 0$$

and $\check{C}(\mathcal{D},\mathcal{N}^t)^1 = 0$, we deduce that $\check{C}(\mathcal{D},\mathcal{N})^1 \cong \check{C}(\mathcal{D},\mathcal{N}^f)^1$. Hence we have an injection $\check{C}(\mathcal{D},\mathcal{N})^1 \to (\mathcal{D},\mathcal{N})^1$. The functoriality is obvious. □

Corollary 4.13. *Let $\mathcal{D}$ be a divisorial sheaf and let $\mathcal{N}$ be a quasi-coherent module. Then there is a functorial isomorphism*

$$Hom^1_{\mathcal{X}}(\mathcal{D},\mathcal{N}) \to \check{C}Hom^1_{\mathcal{X}}(\mathcal{D},\mathcal{N}).$$

Proof. By Lemma 4.12, we have $\check{\mathrm{C}}\mathrm{Hom}^1(\mathcal{D},\mathcal{I}) = 0$ for an injective module $\mathcal{I}$. Then by the commutative diagram in the proof of Lemma 4.10, we obtain our isomorphism. □

Corollary 4.14. *Let $\mathcal{D}$ be a divisorial sheaf and let $\mathcal{N}$ be quasi-coherent module. Then $Hom^1_{\mathcal{X}}(\mathcal{D},\mathcal{N}) \cong 0$ for $p \neq 0, 1$.*

Proof. We consider an exact sequence

$$0 \to \mathcal{N} \to \mathcal{I} \to \mathcal{N}' \to 0$$

where $\mathcal{I}$ is an injective object. Since

$$\check{\mathrm{C}}\mathrm{Hom}^1(\mathcal{D},\mathcal{I}) \cong \mathrm{Hom}^1(\mathcal{D},\mathcal{I}) \cong 0$$

we deduce that

$$\mathrm{Hom}^1(\mathcal{D},\mathcal{N}') \cong \check{\mathrm{C}}\mathrm{Hom}^1(\mathcal{D},\mathcal{N}') \cong 0.$$

Then $\mathrm{Hom}^2(\mathcal{D},\mathcal{N}) \cong 0$. Since $\mathcal{N}$ is general, we also have $\mathrm{Hom}^2(\mathcal{D},\mathcal{N}') \cong 0$, hence $\mathrm{Hom}^3(\mathcal{D},\mathcal{N}) \cong 0$. If we repeat this process, we obtain our result. □

Lemma 4.15. *Let $\mathcal{D} = \mathcal{R}(\sum d_i D_i)$ and $\mathcal{D}' = \mathcal{R}(\sum d'_i D_i)$ be divisorial sheaves. Assume that $d'_i \geq d_i$ for all i except possibly $d'_{i_0} = d_{i_0} - 1$ for one i_0. Then $\check{C}Hom^1_{\mathcal{X}}(\mathcal{D},\mathcal{D}') = 0$.*

Proof. We have

$$\mathcal{D}_i \cong \mathcal{D}'_i \cong \mathcal{R}_i = k[t_0,\dots,t_n]\langle x_i, y_i\rangle/(x_i y_i - y_i x_i - t_0),$$

$$\mathcal{D}_{i,i+1} \cong \mathcal{D}'_{i,i+1} \cong \mathcal{R}_{i,i+1} = k[t_0,\dots,t_n]\langle x_i, y_i, x_i^{-1}\rangle/(x_i y_i - y_i x_i - t_0)$$

with gluing

$$\psi_{i,-}\mathcal{D}(r) = x_{i-1}^{d_i} r,\ \ \psi^{\mathcal{D}}_{i,+}(r) = r,\ \ \psi^{\mathcal{D}'}_{i,-}(r) = x_{i-1}^{d'_i} r,\ \ \psi^{\mathcal{D}'}_{i,+}(r) = r$$

and a transformation rule

$$x_{i+1} = x_i^2 y_i + t_{i+1} x_i, \;\; y_{i+1} = x_i^{-1}.$$

Taking homomorphisms $h_{i-1,i} : \mathcal{D}_{i-1,i} \to \mathcal{D}'_{i-1,i}$ for $1 \leq i \leq n$, we look for homomorphisms $h_i : \mathcal{D}_i \to \mathcal{D}'_i$ for $0 \leq i \leq n$ which induce the $h_{i-1,i}$ in the sense that

$$h_{i-1,i} = h_{i-1} \otimes_{\mathcal{R}_{i-1}} \mathcal{R}_{i-1,i} - h_i \otimes_{\mathcal{R}_i} \mathcal{R}_{i-1,i}$$

for all i, that is

$$h_{i-1,i}(r) = h_{i-1}(r) - x_{i-1}^{d'_i} h_i(y_i^{d_i} r).$$

We write

$$h_i(r) = \sum_{l,m \geq 0} a^i_{l,m} x_i^l y_i^m r, \;\; h_{i-1,i}(r) = \sum_{l \geq 0, m \in \mathbf{Z}} b^i_{l,m} x_i^l y_i^m r$$

for some $a^i_{l,m}, b^i_{l,m} \in k[t_0, \ldots, t_n]$. Then our condition becomes

$$\sum_{l \geq 0, m \in \mathbf{Z}} b^i_{l,m} x_i^l y_i^m = \sum_{l', m' \geq 0} a^{i-1}_{l',m'} x_{i-1}^{l'} y_{i-1}^{m'} - \sum_{l'', m'' \geq 0} a^i_{l'',m''} x_{i-1}^{d'_i} x_i^{l''} y_i^{m''+d_i}.$$

We can write

$$x_{i-1}^{l'} y_{i-1}^{m'} = x_i^{m'} y_i^{2m'-l'} + \text{ lower terms},$$
$$x_{i-1}^{d'_i} x_i^{l''} y_i^{m''+d_i} = x_i^{l''} y_i^{m''+d_i-d'_i} + \text{ lower terms},$$

with respect to the bidegree defined by $\deg(x_i, y_i) = (1, 1)$ so that x_i and y_i become commutative modulo lower degree terms. We note that the lower terms have smaller bidegrees by $(-e, -e)$ for some positive integers e.

We will prove that all the $b^i_{l,m}$ are realized by some choices of the $a^{i-1}_{l',m'}$ and the $a^i_{l'',m''}$. We proceed by the descending induction on the total degree $l + m$.

We start with the case where $d'_i \geq d_i$ for all i. We proceed be the descending induction on i. First we consider the case $i = n$. The terms $b^n_{l,m}$ with $l, m \geq 0$ are taken care of modulo lower terms by the $a^n_{l'',m''}$, while the remaining terms $b^n_{l,m}$ with $l \geq 0$ and $m < 0$ are by $a^{n-1}_{l',m'}$ with $l' > 2m' \geq 0$.

We note that the lower degree terms are added to the given $b^n_{l,m}$ with lower degrees and treated by the $a^{n-1}_{l',m'}$ and the $a^n_{l'',m''}$ with lower degrees. There are only finitely many non-zero $b^n_{l,m}$ initially, hence such l, m are bounded. Then l', m', l'', m'' are also bounded, even after taking care of lower degree terms. Therefore we can obtain our claim by induction on the total degree.

Next we assume that the case $i = i_1$ is proved for some $i_1 \leq n$ in the sense that the terms $b^{i_1}_{l,m}$ are taken care of modulo lower terms by the $a^{i_1}_{l'',m''}$ with $0 \leq (n - i_1)l'' \leq (n - i_1 + 1)m''$ and the $a^{i_1-1}_{l',m'}$ with $(n - i_1 + 1)l' > (n - i_1 + 2)m' \geq 0$. Thus we have still terms $a^{i_1-1}_{l',m'}$ with $0 \leq (n - i_1 + 1)l' \leq (n - i_1 + 2)m'$ to be used for the $b^{i_1-1}_{l,m}$.

We will prove that all the $b^{i_1-1}_{l,m}$ are realized by some choices of the $a^{i_1-1}_{l'',m''}$ with $0 \leq (n-i_1+1)l'' \leq (n-i_1+2)m''$ and the $a^{i_1-2}_{l',m'}$ with $(n-i_1+2)l' > (n-i_1+3)m' \geq 0$. We proceed by the descending induction on the total degree $l+m$ again. Indeed the $b^{i_1-1}_{l,m}$ with $0 \leq (n-i_1+1)l \leq (n-i_1+2)m$ are realized by the $a^{i_1-1}_{l'',m''}$ with $0 \leq (n-i_1+1)l'' \leq (n-i_1+2)m''$. And the $b^{i_1-1}_{l,m}$ with $(n-i_1+1)l > (n-i_1+2)m$ and $l \geq 0$ are realized by the $a^{i_1-2}_{l',m'}$ with $(n-i_1+2)l' > (n-i_1+3)m' \geq 0$, because $(n-i_1+1)m' > (n-i_1+2)(2m'-l')$ is equivalent to $(n-i_1+2)l' > (n-i_1+3)m'$. Therefore our assertion for the case where $d'_i \geq d_i$ for all i is proved.

Next we consider the case where $d'_i \geq d_i$ except $i = i_0$ and $d'_{i_0} = d_{i_0} - 1$. The argument is similar, and we only check the difference from the previous argument. We proceed by the descending induction on i. We prove by the previous argument that the terms $b^i_{l,m}$ for $i > i_0$ are taken care of modulo lower terms by the $a^i_{l'',m''}$ with $0 \leq (n-i)l'' \leq (n-i+1)m''$ and the $a^{i-1}_{l',m'}$ with $(n-i+1)l' > (n-i+2)m' \geq 0$.

Then we consider the case $i = i_0$. We can use the remaining terms $a^{i_0}_{l'',m''}$ with $0 \leq (n-i_0)l'' \leq (n-i_0+1)m''$ for the $b^{i_0}_{l,m}$. Then the terms $b^{i_0}_{l,m}$ with $0 \leq (n-i_0)l \leq (n-i_0+1)(m-1)$ are realized modulo lower terms by such $a^{i_0}_{l'',m''}$. The remaining terms $b^{i_0}_{l,m}$ with $(n-i_0)l > (n-i_0+1)(m-1)$ and $l \geq 0$ are realized modulo lower terms by the $a^{i_0-1}_{l',m'}$ with $(n-i_0+1)(l'+1) > (n-i_0+2)m' \geq 0$, because the last inequality is equivalent to $(n-i_0)m' > (n-i_0+1)(2m'-l'-1)$ and $m' \geq 0$. Here we note that the forbidden case $l' = -1$ does not happen. We note that, if we had $d'_{i_0} = d_{i_0} - 2$, then the case $l' = -1$ would appear.

We have the remaining terms $a^{i_0-1}_{l'',m''}$ with $(n-i_0+1)l'' + n - i_0 + 1 \leq (n-i_0+2)m''$ and $l'', m'' \geq 0$. Then the $b^{i_0-1}_{l,m}$ with $(n-i_0+1)l+n-i_0+1 \leq (n-i_0+2)m$ and $l \geq 0$ are taken care of modulo lower terms by such $a^i_{l'',m''}$. We see that the remaining $b^{i_0-1}_{l,m}$ are those with $(n-i_0+1)l + n - i_0 + 1 > (n-i_0+2)m$ and $l \geq 0$. Then they are realized modulo lower terms by the $a^{i_0-1}_{l',m'}$ with $(n-i_0+2)l' + (n-i_0+1) > (n-i_0+3)m' \geq 0$, $l' \geq 0$, because the last inequality is equivalent to $(n-i_0+1)m' + n - i_0 + 1 > (n-i_0+2)(2m'-l')$ and $l', m' \geq 0$. We note that $l' < 0$ does not happen, because $(n-i_0+2)l' + (n-i_0+1) < 0$ otherwise.

For $i < i_0$, we claim that the terms $b^i_{l,m}$ are taken care of modulo lower terms by the $a^i_{l'',m''}$ with $(n-i)l'' + (n-i_0+1) \leq (n-i+1)m''$, $l'' \geq 0$, and the $a^{i-1}_{l',m'}$ with $(n-i+1)l' + (n-i_0+1) > (n-i+2)m' \geq 0$, $l' \geq 0$. We proceed by the descending induction on i. The case $i = i_0 - 1$ is already proved.

Suppose that the case for $i = i_1$ is proved. Then we can still use the $a^{i_1-1}_{l'',m''}$ with $(n-i_1+1)l''+(n-i_0+1) \leq (n-i_1+2)m''$, $l'' \geq 0$, $m'' \geq 0$ for the $b^{i_1-1}_{l,m}$

with $(n-i_1+1)l+(n-i_0+1) \leq (n-i_1+2)m$, $l \geq 0$. We note that $m \geq 0$ is automatic. The remaining $b_{l,m}^{i_1-1}$ with $(n-i_1+1)l+(n-i_0+1) > (n-i_1+2)m$, $l \geq 0$ are taken care by the $a_{l',m'}^{i_1-2}$ with $(n-i_1+2)l'+(n-i_0+1) > (n-i_1+3)m' \geq 0$, $l' \geq 0$, because the last inequality is equivalent to $(n-i_1+1)m'+n-i_0+1 > (n-i_1+2)(2m'-l')$ and $m' \geq 0$. We note that $l' \geq 0$ is again automatic. Therefore we complete the proof by the descending induction on i. □

The following says that $\mathcal{T}$ is a tilting locally free module:

Corollary 4.16. *Let $\mathcal{T} = \mathcal{R} \oplus \bigoplus_{i=1}^{n} \mathcal{R}(-D_i)$. Then $Hom^p(T,T) = 0$ for $p \neq 0$.*

We need the following lemma later:

Lemma 4.17. *There exists an exact sequence of $\mathcal{R}$-modules*

$$0 \to \mathcal{R}(\sum d_i D_i) \to \mathcal{R}(\sum d_i D_i + D_j) \oplus \mathcal{R}(\sum d_i D_i + D_k)$$
$$\to \mathcal{R}(\sum d_i D_i + D_j + D_k) \to 0$$

for any integers $j \leq k$, where the first and the second arrows (h, h') and (g, g') are given by

$$h = (\dots, h_{j-1,j}, h_j, h_{j,j+1}, \dots) = (\dots, x_{j-1}, 1, 1, \dots),$$
$$h' = (\dots, h'_{k-1,k}, h'_k, h'_{k,k+1}, \dots) = (\dots, 1, y_k, y_k, \dots),$$
$$g = (\dots, g_{k-1,j}, g_k, g_{k,k+1}, \dots) = (\dots, 1, y_k, y_k, \dots),$$
$$g' = (\dots, g'_{j-1,j}, g'_j, g'_{j,j+1}, \dots) = (\dots, -x_{j-1}, -1, -1, \dots).$$

Proof. First we check that these maps are compatible with the gluing maps: h, h' are given by

$$\begin{array}{ccccc}
\mathcal{R}_{j-1,j} & \xleftarrow{x_{j-1}^{d_j}} & \mathcal{R}_j & \xrightarrow{1} & \mathcal{R}_{j,j+1} \\
{\scriptstyle x_{j-1}}\downarrow & & {\scriptstyle 1}\downarrow & & {\scriptstyle 1}\downarrow \\
\mathcal{R}_{j-1,j} & \xleftarrow{x_{j-1}^{d_j+1}} & \mathcal{R}_j & \xrightarrow{1} & \mathcal{R}_{j,j+1}, \\
\\
\mathcal{R}_{k-1,k} & \xleftarrow{x_{k-1}^{d_k}} & \mathcal{R}_k & \xrightarrow{1} & \mathcal{R}_{k,k+1} \\
{\scriptstyle 1}\downarrow & & {\scriptstyle y_k}\downarrow & & {\scriptstyle y_k}\downarrow \\
\mathcal{R}_{k-1,k} & \xleftarrow{x_{k-1}^{d_k+1}} & \mathcal{R}_k & \xrightarrow{1} & \mathcal{R}_{k,k+1}.
\end{array}$$

and g, g' are similar. We note that the dots have more complicated expressions.

It is easy to see that the first arrow is injective, the second surjective, and the composite vanishes. We will check the exactness at the middle term.

For example, assume that $j = k$ and $g_j(u) + g'_j(v) = 0$, i.e., $y_j u - v = 0$. Then $v = h'_j(u)$. Other cases are similar. □

5. Comparison of End($\mathcal{T}$) and $\mathcal{S}$

We will compare associative algebras $\mathcal{S} = k[s_0, \dots, s_n]\langle u, v\rangle \# G/(uv - vu - s)$ and $\mathcal{A} = \mathrm{End}(\mathcal{T})$, where $\mathcal{T} = \mathcal{R} \oplus \bigoplus_{i=1}^n \mathcal{R}(-D_i)$. We will prove that there is an isomorphism of algebras $\mathcal{S} \to \mathrm{End}(\mathcal{T})$ over a linear coordinate change from the parameters s_i to the t_j.

5.1. **Decomposition of $\mathcal{S}$ by orthogonal idempotents.** $\mathcal{S}$ has natural orthogonal idempotents corresponding to irreducible representations of G. We define

$$e_i^s = \sum_{j=0}^n \zeta^{ij} g^j/(n+1) \in \mathcal{S}$$

for $0 \le i \le n$, where $\zeta = e^{2\pi i/(n+1)}$ and g is a generator of $\mathbf{Z}/(n+1)$. We regard the index i of e_i^s is an element of $\mathbf{Z}/(n+1)$, i.e., we consider it only modulo $n+1$, we use the convention $e_{i+n+1}^s = e_i^s$.

The following is clear from the definition:

Lemma 5.1. *(0)* $e_0^s = e$.
(1) $1 = \sum_{i=0}^n e_i^s$.
(2) $(e_i^s)^2 = e_i^s$, *and* $e_i^s e_j^s = 0$ *for* $i \ne j$.
(3) $e_i^s u = u e_{i+1}^s$ *and* $e_i^s v = v e_{i-1}^s$.

Thus we have a decomposition as $k[s_0, \dots, s_n]$-module:

$$\mathcal{S} = \bigoplus_{i,j=0}^n \mathcal{S}_{i,j}, \;\; \mathcal{S}_{i,j} = e_i^s \mathcal{S} e_j^s.$$

We define elements $\alpha_{i,i+1}^s \in \mathcal{S}_{i,i+1}$ and $\beta_{i+1,i}^s \in \mathcal{S}_{i+1,i}$ by

$$\alpha_{i,i+1}^s = e_i^s u e_{i+1}^s, \;\; \beta_{i+1,i}^s = e_{i+1}^s v e_i^s$$

for $0 \le i \le n$.

Lemma 5.2. *(1)* $u = \sum_{i=0}^n \alpha_{i,i+1}^s$ *and* $v = \sum_{i=0}^n \beta_{i+1,i}^s$.
(2) $\mathcal{S}$ *is generated as a* $k[s_0, \dots, s_n]$*-algebra by the* e_i^s, $\alpha_{i,i+1}^s$ *and* $\beta_{i+1,i}^s$ *for* $0 \le i \le n$.
(3) There are following relations:

$$\alpha_{i,i+1}^s \beta_{i+1,i}^s = u v e_i^s,$$
$$\beta_{i+1,i}^s \alpha_{i,i+1}^s = v u e_{i+1}^s,$$
$$\alpha_{i,i+1}^s \alpha_{i+1,i+2}^s \dots \alpha_{i+n-1,i+n}^s \alpha_{i+n,i+n+1}^s = u^{n+1} e_i^s,$$
$$\beta_{i,i-1}^s \beta_{i-1,i-2}^s \dots \beta_{i-n+1,i-n}^s \beta_{i-n,i-n-1}^s = v^{n+1} e_i^s.$$

Proof. (1) If $j \neq i+1$, then

$$e_i^s u e_j^s = e_i^s e_i^s u e_j^s = e_i^s u e_{i+1}^s e_j^s = 0.$$

Hence

$$u = \sum_{i,j=0}^{n} e_i^s u e_j^s = \sum_{i=0}^{n} e_i^s u e_{i+1}^s.$$

The assertion for v is similarly proved.

(2) The claim is clear.

(3) The assertion follows from $uve_i^s = e_i^s uv$, $u^{n+1}e_i^s = e_i^s u^{n+1}$ and $v^{n+1}e_i^s = e_i^s v^{n+1}$. □

5.2. **Decomposition of $\mathcal{A}$ by orthogonal idempotents.** We will determine the structure of the endomorphism algebra $\mathcal{A} = \mathrm{End}_{\mathcal{R}}(\mathcal{T})$ of the tilting object $\mathcal{T} = \mathcal{R} \oplus \bigoplus_{i=1}^{n} \mathcal{R}(-D_i)$.

$\mathcal{A}$ has natural orthogonal idempotents $e_i^t \in \mathcal{A}$ for $0 \leq i \leq n$ defined by the projections

$$e_i^t : \mathcal{T} \to \mathcal{R}(-D_i) \to \mathcal{T}$$

where we set $D_0 = 0$ and $\mathcal{R}(-D_0) = \mathcal{R}$. We have a decomposition of the identity to orthogonal idempotents $\sum_{i=0}^{n} e_i^t = 1 \in \mathcal{A}$. Thus we have a decomposition as $k[t_0, \ldots, t_n]$-modules:

$$\mathcal{A} = \bigoplus_{i,j=0}^{n} \mathcal{A}_{i,j}, \;\; \mathcal{A}_{i,j} = e_i^t \mathcal{A} e_j^t.$$

We note that we use again the convention $e_i^t = e_{i+n+1}^t$.

We will express any element $h \in \mathcal{A}_{i,j}$ by a sequence of functions $(h_0, h_1, \ldots, h_n)$, where $h_k \in \mathcal{R}_k$, such that $h_k(r) = h_k r$, because other components $h_{k-1,k}$ are determined by the h_k.

Lemma 5.3. *There are elements $\alpha_{i,i+1}^t \in \mathcal{A}_{i,i+1}$, $\beta_{i+1,i}^t \in \mathcal{A}_{i+1,i}$ for $0 \leq i \leq n$ such that*

$$(\alpha_{i,i+1}^t)_k = \begin{cases} x_k y_k + (i-k-1)t_0 + t_{k+1} + \cdots + t_i, & (0 \leq k < i), \\ x_i, & (k = i), \\ 1, & (i < k \leq n). \end{cases}$$

$$(\beta_{i+1,i}^t)_k = \begin{cases} 1, & (0 \leq k < i), \\ y_i, & (k = i), \\ x_k y_k - (k-i)t_0 - t_{i+1} - \cdots - t_k, & (i < k \leq n). \end{cases}$$

Proof. Since $x_{i+1} = x_i^2 y_i + t_{i+1} x_i$ and $y_{i+1} = x_i^{-1}$, we have

$$x_{i+1} y_{i+1} = x_i y_i + t_0 + t_{i+1}.$$

The following is a part of the diagram for the homomorphism $\alpha^t_{i,i+1} : \mathcal{R}(-D_{i+1}) \to \mathcal{R}(-D_i)$:

$$\begin{array}{ccccccccc}
\mathcal{R}_{i-1} & \xrightarrow{1} & \mathcal{R}_{i-1,i} & \xleftarrow{1} & \mathcal{R}_i & \xrightarrow{1} & \mathcal{R}_{i,i+1} & \xleftarrow{x_i^{-1}} & R_{i+1} \\
{\scriptstyle x_{i-1}y_{i-1}+t_i}\downarrow & & {\scriptstyle x_{i-1}y_{i-1}+t_i}\downarrow & & {\scriptstyle x_i}\downarrow & & {\scriptstyle x_i}\downarrow & & {\scriptstyle 1}\downarrow \\
\mathcal{R}_{i-1} & \xrightarrow{1} & \mathcal{R}_{i-1,i} & \xleftarrow{x_{i-1}^{-1}} & \mathcal{R}_i & \xrightarrow{1} & \mathcal{R}_{i,i+1} & \xleftarrow{1} & R_{i+1}.
\end{array}$$

We confirm the commutativity of the middle left square by $x_{i-1}^{-1}x_i = y_ix_i = x_{i-1}y_{i-1} + t_i$. We note that the case $i = 0, n$ are also confirmed by the following diagrams for $\alpha^t_{0,1}$ and $\alpha^t_{n,0}$:

$$\begin{array}{ccccc}
\mathcal{R}_0 & \xrightarrow{1} & \mathcal{R}_{0,1} & \xleftarrow{x_0^{-1}} & \mathcal{R}_1 \\
{\scriptstyle x_0}\downarrow & & {\scriptstyle x_0}\downarrow & & {\scriptstyle 1}\downarrow \\
\mathcal{R}_0 & \xrightarrow{1} & \mathcal{R}_{0,1} & \xleftarrow{1} & \mathcal{R}_1,
\end{array}$$

$$\begin{array}{ccccc}
\mathcal{R}_{n-1} & \xrightarrow{1} & \mathcal{R}_{n-1,n} & \xleftarrow{1} & \mathcal{R}_n \\
{\scriptstyle x_{n-1}y_{n-1}+t_n}\downarrow & & {\scriptstyle x_{n-1}y_{n-1}+t_n}\downarrow & & {\scriptstyle x_n}\downarrow \\
\mathcal{R}_{n-1} & \xrightarrow{1} & \mathcal{R}_{n-1,n} & \xleftarrow{x_{n-1}^{-1}} & \mathcal{R}_n.
\end{array}$$

Then the assertion for $k < i - 1$ is confirmed by the descending induction on k by

$$x_{k+1}y_{k+1} + (i-k-2)t_0 + t_{k+2} + \cdots + t_i = x_ky_k + (i-k-1)t_0 + t_{k+1} + \cdots + t_i.$$

The assertion for $\beta^t_{i+1,i}$ with $\beta^t_{1,0}$ and $\beta^t_{0,n}$ is similarly confirmed by the following diagrams:

$$\begin{array}{ccccccccc}
\mathcal{R}_{i-1} & \xrightarrow{1} & \mathcal{R}_{i-1,i} & \xleftarrow{x_{i-1}^{-1}} & \mathcal{R}_i & \xrightarrow{1} & \mathcal{R}_{i,i+1} & \xleftarrow{1} & R_{i+1} \\
{\scriptstyle 1}\downarrow & & {\scriptstyle 1}\downarrow & & {\scriptstyle y_i}\downarrow & & {\scriptstyle y_i}\downarrow & & \downarrow{\scriptstyle x_{i+1}y_{i+1}-t_0-t_{i+1}} \\
\mathcal{R}_{i-1} & \xrightarrow{1} & \mathcal{R}_{i-1,i} & \xleftarrow{1} & \mathcal{R}_i & \xrightarrow{1} & \mathcal{R}_{i,i+1} & \xleftarrow{x_i^{-1}} & R_{i+1}.
\end{array}$$

$$\begin{array}{ccccc}
\mathcal{R}_0 & \xrightarrow{1} & \mathcal{R}_{0,1} & \xleftarrow{1} & \mathcal{R}_1 \\
{\scriptstyle y_0}\downarrow & & {\scriptstyle y_0}\downarrow & & \downarrow{\scriptstyle x_0y_0-t_0-t_1} \\
\mathcal{R}_0 & \xrightarrow{1} & \mathcal{R}_{0,1} & \xleftarrow{x_0^{-1}} & \mathcal{R}_1.
\end{array}$$

$$\begin{array}{ccccc}
\mathcal{R}_{n-1} & \xrightarrow{1} & \mathcal{R}_{n-1,n} & \xleftarrow{x_{n-1}^{-1}} & \mathcal{R}_n \\
{\scriptstyle 1}\downarrow & & {\scriptstyle 1}\downarrow & & {\scriptstyle y_n}\downarrow \\
\mathcal{R}_{n-1} & \xrightarrow{1} & \mathcal{R}_{n-1,n} & \xleftarrow{1} & \mathcal{R}_n
\end{array}$$

and

$$x_{k+1}y_{k+1} - (k+1-i)t_0 - t_{i+1} - \cdots - t_{k+1} = x_k y_k - (k-i)t_0 - t_{i+1} - \cdots - t_k.$$

□

Definition 5.4. We define $u^t, v^t \in \mathcal{A}$ by

$$u^t = \sum_{i=0}^{n} \alpha^t_{i,i+1}, \quad v^t = \sum_{i=0}^{n} \beta^t_{i+1,i}.$$

Then we have $\alpha^t_{i,i+1} = e^t_i u e^t_{i+1}$, $\beta^t_{i+1,i} = e^t_{i+1} v e^t_i$.

We can express u^t and v^t as matrix forms when they are restricted to an affine chart $\mathcal{R}_0$:

Lemma 5.5.

$$u^t|_{\mathcal{R}_0} = \begin{pmatrix} 0 & x_0 & 0 & \dots & 0 \\ 0 & 0 & x_0y_0 + t_1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & x_0y_0 + (n-2)t_0 + t_1 + \cdots + t_{n-1} \\ x_0y_0 + (n-1)t_0 + t_1 + \cdots + t_n & 0 & 0 & \dots & 0 \end{pmatrix}$$

$$v^t|_{\mathcal{R}_0} = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ y_0 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

We have the following identities:

Corollary 5.6.

$$\begin{aligned} u^t v^t|_{\mathcal{R}_0} &= diag(x_0y_0, x_0y_0 + t_1, x_0y_0 + t_0 + t_1 + t_2, \\ &\quad \dots, x_0y_0 + (n-1)t_0 + t_1 + \cdots + t_n), \\ v^t u^t|_{\mathcal{R}_0} &= diag(x_0y_0 + (n-1)t_0 + t_1 + \cdots + t_n, x_0y_0 - t_0, x_0y_0 + t_1, \\ &\quad \dots, x_0y_0 + (n-2)t_0 + t_1 + \cdots + t_{n-1}), \\ (u^t v^t - v^t u^t)|_{\mathcal{R}_0} &= diag(-(n-1)t_0 - t_1 - \cdots - t_n, t_0 + t_1, t_0 + t_2, \dots, t_0 + t_n). \end{aligned}$$

5.3. **Algebra homomorphism $\mathcal{S} \to \mathcal{A}$.** We defined $e^s_i = \frac{1}{n+1}\sum_{j=0}^{n} \zeta^{ij} g^j$. Thus we have $g^j = \sum_{i=0}^{n} \zeta^{-ij} e^s_i$. Now we define

$$g^t = \sum_{i=0}^{n} \zeta^{-i} e^t_i.$$

Lemma 5.7. *(1) $(g^t)^{n+1} = 1$ and $(g^t)^j \neq 1$ for $0 < j < n+1$.*

(2) $e^t_i = \sum_{j=0}^{n} \zeta^{ij} (g^t)^j/(n+1)$.

(3) $g^t u^t = \zeta u^t g^t$ and $g^t v^t = \zeta^{-1} v^t g^t$.

Proof. (1) This is because $(g^t)^j = \sum_{i=0}^n \zeta^{-ij} e_i^t$.

(2) We have $\sum_{j=0}^n \zeta^{ij}(g^t)^j = \sum_{j,k=0}^n \zeta^{ij}\zeta^{-jk} e_k^t = (n+1)e_i^t$.

(3) We will prove that $g^t u^t e_{i+1}^t = \zeta u^t g^t e_{i+1}^t$ and $g^t v^t e_i^t = \zeta^{-1} v^t g^t e_i^t$ for all i. Indeed

$$g^t u^t e_{i+1}^t = \zeta^{-i} e_i^t \alpha_{i,i+1}^t e_{i+1}^t = \zeta \alpha_{i,i+1} \zeta^{-i-1} e_{i+1}^t = \zeta u^t g^t e_{i+1}^t,$$
$$g^t v^t e_i^t = \zeta^{-i-1} e_{i+1}^t \beta_{i+1,i}^t e_i^t = \zeta^{-1} \beta_{i+1,i}^t \zeta^{-i} e_i^t = \zeta^{-1} v^t g^t e_i^t.$$

□

We define $w_0, \dots, w_n$ by

$$s_i = \frac{1}{n+1} \sum_{j=0}^n \zeta^{ij} w_j.$$

We note that this linear transformation is invertible. Then we compare two formulas:

- $\mathcal{S}$: $uv - vu = \sum_{i=0}^n s_i g^i = \sum_{j=0}^n w_j e_j^s$.
- $\mathcal{X}$: $u^t v^t - v^t u^t = (-(n-1)t_0 - t_1 - \cdots - t_n)e_0^t + (t_0 + t_1)e_1^t + (t_0 + t_2)e_2^t + \cdots + (t_0 + t_n)e_n^t$.

From the comparison, we define another invertible linear transformation:

$$w_0 = -(n-1)t_0 - t_1 - \cdots - t_n, \;\; w_1 = t_0 + t_1, \dots, w_n = t_0 + t_n.$$

We define an algebra homomorphism

$$\phi : \mathcal{S} \to \mathcal{A}$$

over the linear coordinate change $k[s_0, \dots, s_n] \to k[t_0, \dots, t_n]$ defined above by

$$\phi(e_i^s) = e_i^t, \;\; \phi(u) = u^t, \;\; \phi(v) = v^t, \;\; \phi(g) = g^t.$$

By Corollary 5.6, ϕ is a well defined algebra homomorphism.

Theorem 5.8. *$\phi : \mathcal{S} \to \mathcal{A}$ is an algebra isomorphism.*

We will prove that ϕ is injective and surjective in the following subsections.

5.4. **Injectivity of ϕ.** We call the subalgebra $e\mathcal{S}e$ of $\mathcal{S}$ the *ring of G-invariants*, where $e = e_0^s$ ([4]). We denote the subalgebra $e_0^t \mathcal{A} e_0^t = \mathrm{Hom}(\mathcal{R}, \mathcal{R})$ by $\Gamma(\mathcal{R})$, and call it the *ring of global functions*.

Lemma 5.9. *The injectivity of ϕ is reduced to the injectivity of its restriction $\phi_e : e\mathcal{S}e \to \Gamma(\mathcal{R})$.*

Proof. Assume that $\mathrm{Ker}(\phi) \neq 0$. Since ϕ preserves idempotents, we have $\phi(e_i^s \mathcal{S} e_j^s) \subset e_i^t \mathcal{A} e_j^t$. Thus there are i, j such that $\mathrm{Ker}(\phi) \cap e_i^s \mathcal{S} e_j^s \neq 0$.

If we set $\deg(s_i, g, u, v) = (0, 0, 1, 1)$ and introduce a filtration $F_p(\mathcal{S}) = \{x \in \mathcal{S} \mid \deg(x) \leq p\}$, then $\mathrm{Gr}(\mathcal{S}) \cong k[s_0, \dots, s_n][u, v] \# G$. Hence $\mathcal{S}$ has no non-zero divisor. By multiplying u^i and v^j from the left and the right respectively, we obtain $\mathrm{Ker}(\phi) \cap e\mathcal{S}e \neq 0$, because we have $u^i e_i^s \mathcal{S} e_j^s v^j \subset e\mathcal{S}e$. □

We define $x^t_{i,i}, y^t_{i,i}, z^t_{i,i} \in \mathcal{A}_{i,i}$ for $0 \leq i \leq n$ by

$$x^t_{i,i} = e^t_i (u^t)^{n+1} e^t_i, \;\; y^t_{i,i} = e^t_i (v^t)^{n+1} e^t_i, \;\; z^t_{i,i} = e^t_i u^t v^t e^t_i.$$

In particular we denote $x^t = x^t_{0,0}, y^t = y^t_{0,0}, z^t = z^t_{0,0}$. We have $x^t, y^t, z^t \in \Gamma(\mathcal{R})$.

Lemma 5.10. *(1)*

$$(\alpha^t_{i,i+1}\alpha^t_{i+1,i+2}\cdots\alpha^t_{i+n-1,i+n}\alpha^t_{i+n,i+n+1})_n = x_n,$$
$$(\beta^t_{i,i-1}\beta^t_{i-1,i-2}\cdots\beta^t_{i-n+1,i-n}\beta^t_{i-n,i-n-1})_0 = y_0.$$

(2)

$$(\alpha^t_{i,i+1}\beta^t_{i+1,i})_0 = \begin{cases} x_0 y_0, & (i=0), \\ x_0 y_0 + (i-1)t_0 + t_1 + \cdots + t_i, & (0 < i \leq n), \end{cases}$$
$$(\alpha^t_{i,i+1}\beta^t_{i+1,i})_n = \begin{cases} x_n y_n - (n-i)t_0 - t_{i+1} - \cdots - t_n, & (0 \leq i < n), \\ x_n y_n, & (i=n). \end{cases}$$

Proof. The assertion follows from

$$(\alpha^t_{i,i+1})_0 = \begin{cases} x_0, & (i=0), \\ x_0 y_0 + (i-1)t_0 + t_1 + \cdots + t_i, & (i \neq 0), \end{cases},$$
$$(\alpha^t_{i,i+1})_n = \begin{cases} 1, & (i \neq n), \\ x_n, & (i=n), \end{cases}$$
$$(\beta^t_{i+1,i})_0 = \begin{cases} y_0, & (i=0), \\ 1, & (i \neq 0), \end{cases},$$
$$(\beta^t_{i+1,i})_n = \begin{cases} x_n y_n - (n-i)t_0 - t_{i+1} - \cdots - t_n, & (i \neq n), \\ y_n, & (i=n). \end{cases}$$

□

Corollary 5.11.

$$(x^t)_0 = x_0 \prod_{i=1}^{n} (x_0 y_0 + (i-1)t_0 + t_1 \cdots + t_i), \;\; (y^t)_0 = y_0, \;\; (z^t)_0 = x_0 y_0,$$
$$(x^t)_n = x_n, \;\; (y^t)_n = y_n \prod_{i=1}^{n} (x_n y_n - (n+1-i)t_0 - t_i \cdots - t_n),$$
$$(z^t)_n = x_n y_n - n t_0 - t_1 - \cdots - t_n.$$

We note that the products $\prod_{i=1}^n$ in the above formula do not depend on the order of the factors.

Proof. Indeed we have

$$e_0^t(u^t)^{n+1}e_0^t = \alpha_{0,1}^t\alpha_{1,2}^t\dots\alpha_{n-1,n}^t\alpha_{n,0}^t,$$
$$e_0^t(v^t)^{n+1}e_0^t = \beta_{0,n}^t\beta_{n,n-1}^t\dots\beta_{2,1}^t\beta_{1,0}^t,$$
$$e_0^tu^tv^te_0^t = \alpha_{0,1}^t\beta_{1,0}^t.$$

□

Proof of the injectivity. We define

$$x^s = eu^{n+1}e,\ \ y^s = ev^{n+1}e,\ \ z^s = euve.$$

Then we have

$$\phi(x^s) = x^t,\ \ \phi(y^s) = y^t,\ \ \phi(z^s) = z^t.$$

We recall that, if we define a filtration on $\mathcal{S}$ by setting degrees $\deg(s_i, g, u, v) = (0, 0, 1, 1)$, then $\mathrm{Gr}(\mathcal{S}) \cong k[s_0, \dots, s_n][u, v]\#G$. We define a filtration on $\mathcal{R}_0$ by setting degrees

$$\deg(t_i, x_0, y_0) = (0, -n+1, n+1).$$

Then we have $\mathrm{Gr}(\mathcal{R}_0) \cong k[t_0, \dots, t_n][x_0, y_0]$. Moreover $\phi_e : e\mathcal{S}e \to \Gamma(\mathcal{R})$ preserves the degrees, so that ϕ_e induces a homomorphism $\mathrm{Gr}(\phi_e) : \mathrm{Gr}(e\mathcal{S}e) \to \mathrm{Gr}(\mathcal{R}_0)$.

We note that $e\mathcal{S}e$ is generated by x^s, y^s and z^s over $k[s_0, \dots, s_n]$. The only relation among generators x^s, y^s, z^s in $\mathrm{Gr}(\mathcal{S}) \cong k[s_0, \dots, s_n][u, v]^G$ is $x^sy^s \equiv (z^s)^{n+1}$. We also note that the only relation among $(x^t)_0$, $(y^t)_0$, $(z^t)_0$ in $\mathrm{Gr}(\mathcal{R}_0)$ is $(x^t)_0(y^t)_0 \equiv (z^t)_0^{n+1}$. Therefore $\mathrm{Gr}(\phi_e)$ is injective, hence so is ϕ_e. □

5.5. **Surjectivity of ϕ.** We will prove the surjectivity. First we consider a $k[t_0, \dots, t_n]$-algebra $\mathcal{A}_{i,i} = \mathrm{End}(\mathcal{R}(-D_i))$ for $0 \le i \le n$. We define $x_{i,i}^t, y_{i,i}^t, z_{i,i}^t \in \mathcal{A}_{i,i}$ by

$$x_{i,i}^t = e_i^t(u^t)^{n+1}e_i^t,\ \ y_{i,i}^t = e_i^t(v^t)^{n+1}e_i^t,\ \ z_{i,i}^t = e_i^tu^tv^te_i^t.$$

These elements are contained in the image of ϕ. We claim that $\mathcal{A}_{i,i}$ is generated by $x_{i,i}^t, y_{i,i}^t, z_{i,i}^t$ as an algebra over $k[t_0, \dots, t_n]$.

Lemma 5.12.

$$(y_{i,i}^t)_0 = y_0,\ \ (z_{i,i}^t)_0 = \begin{cases} x_0y_0, & (i = 0), \\ x_0y_0 + (i-1)t_0 + t_1 + \dots + t_i, & (0 < i \le n), \end{cases}$$

$$(x_{i,i}^t)_n = x_n,\ \ (z_{i,i}^t)_n = \begin{cases} x_ny_n - (n-i)t_0 - t_{i+1} - \dots - t_n, & (0 \le i < n), \\ x_ny_n, & (i = n). \end{cases}$$

Proof. Indeed we have

$$e_i^t(u^t)^{n+1}e_i^t = \alpha_{i,i+1}^t\alpha_{i+1,i+2}^t\dots\alpha_{i+n-1,i+n}^t\alpha_{i+n,i}^t,$$
$$e_i^t(v^t)^{n+1}e_i^t = \beta_{i,i-1}^t\beta_{i-1,i-2}^t\dots\beta_{i+2,i+1}^t\beta_{i+1,i}^t,$$
$$e_i^tu^tv^te_i^t = \alpha_{i,i+1}^t\beta_{i+1,i}^t.$$

□

Proof of the surjectivity. We first prove the surjectivity of ϕ for $\mathcal{A}_{i,i}$. We take an arbitrary $h \in \mathcal{A}_{i,i} = \mathrm{Hom}(\mathcal{R}(-D_i), \mathcal{R}(-D_i))$, and let $r_k = h_k(1)$. We define degree by $\deg(t_j, x_k, y_k) = (0, 1, -1)$. Then all relations in $\mathcal{R}_k$ as well as gluings for $\mathcal{R}(-D_i)$ are homogeneous. Hence we may assume that the r_k are homogeneous with a fixed degree $\deg(r)$.

We claim that, if $\deg(r) \leq 0$, then we can express h by using $y_t^{(i)}, z_t^{(i)} \in \mathcal{A}_{i,i}$ with coefficients in $k[t_0, \ldots, t_n]$. For this purpose, we look at the affine chart $\mathcal{R}_0$. We consider the secondary grading in $\mathcal{R}_0$ defined by deg'$(t_j, x_0, y_0) = (0, 1, 1)$. Then we can write

$$r_0 = a_0(x_0y_0)^b y_0^c + \text{ lower deg' terms}$$

for $a_0 \in k[t_0, \ldots, t_n]$ and $b, c \geq 0$ using the commutation relation $x_0y_0 = y_0x_0 + t_0$, because $\deg(r) \leq 0$. Let $h' = h - a_0(z_{i,i}^t)^b(y_{i,i}^t)^c \in \mathcal{A}_{i,i}$. Then h' is again homogeneous of degree $-c$, and the secondary degree is less than $2b + c$. Therefore we infer that h is expressed by $y_{i,i}^t, z_{i,i}^t$ using induction on the secondary degree.

We assume next that $\deg(r) > 0$. We consider the secondary grading in $\mathcal{R}_n$ defined by deg"$(t_j, x_n, y_n) = (0, 1, 1)$. Then we can write

$$r_n = a_n x_n^b (x_n y_n)^c + \text{ lower deg" terms}$$

for $a_n \in k[t_0, \ldots, t_n]$ and $b, c \geq 0$ using the commutation relation $x_n y_n = y_n x_n + t_0$, because $\deg(r) > 0$. Let $h'' = h - a_n(x_{i,i}^t)^b(z_{i,i}^t)^c \in \mathcal{A}_{i,i}$. Then h'' is again homogeneous of degree b, and secondary degree less than $b + 2c$. Therefore we conclude that h is expressed by $x_{i,i}^t, z_{i,i}^t$ using induction on the secondary degree.

Now we consider $\mathcal{A}_{i,j}$ for $i \neq j$. We take $h \in \mathcal{A}_{i,j}$ and let $r_k = h_k(1)$ as before. We may assume that the r_k are homogeneous, but the gluing does not preserve the degree. Indeed we have

$$\deg(r_n) = \deg(r_0) + \begin{cases} 0, & (i, j \neq 0), \\ -1, & (i = 0), \\ 1, & (j = 0). \end{cases}$$

Assume first that (i) or (ii) holds:

(i) $\deg(r_0) \leq 0$. $0 \neq j < i$ or $i = 0$.
(ii) $\deg(r_0) < 0$. $0 \neq i < j$ or $j = 0$.

We have

$$(\beta_{i,i-1}^t \ldots \beta_{j+1,j}^t)_0 = \begin{cases} 1, & (0 < j < i \text{ or } i = 0), \\ y_0, & (0 < i < j \text{ or } j = 0). \end{cases}$$

We can write

$$r_0 = a_0(x_0y_0)^b y_0^c + \text{ lower deg' terms}$$

with $a_0 \in k[t_0, \dots, t_n]$, $b \geq 0$, and (i) $c \geq 0$ or (ii) $c > 0$. We set

$$h' = h - a_0 (z^t_{i,i})^b (y^t_{i,i})^{c'} \beta^t_{i,i-1} \dots \beta^t_{j+1,j} \in \mathcal{A}_{i,j}$$

where (i) $c' = c$, or (ii) $c' = c - 1$. Then h'_0 is again homogeneous of degree $-c$, and the secondary degree is less than $2b + c$. Therefore we infer that h is expressed by $y^t_{i,i}, z^t_{i,i}$ and the $\beta_{k+1,k}$ using induction on the secondary degree.

Assume next that (i') or (ii') holds:

(i') $\deg(r_n) \geq 0$. $i < j$ or $j = 0$.
(ii') $\deg(r_n) > 0$. $0 \neq j < i$.

We note that the cases (i), (ii), (i'), (ii') exhaust all cases. Indeed, in the case $i, j \neq 0$, if $j < i$, then (i) or (ii') holds, and if $i < j$, then (i') or (ii) holds. In the case $i = 0$, if (i) does not hold, then then (i') holds. In the case $j = 0$, if (ii) does not hold, then (i') holds.

We have

$$(\alpha^t_{i,i+1} \dots \alpha^t_{j-1,j})_n = \begin{cases} 1, & (i < j), \\ x_n, & (j < i). \end{cases}$$

We can write

$$r_n = a_n (x_n y_n)^b x_n^c + \text{ lower deg" terms}$$

with $a_n \in k[t_0, \dots, t_n]$, $b \geq 0$, and (i') $c \geq 0$ or (ii') $c > 0$. We set

$$h' = h - a_n (z^t_{i,i})^b (x^t_{i,i})^{c'} \alpha^t_{i,i+1} \dots \alpha^t_{j-1,j} \in \mathcal{A}_{i,j}$$

where (i) $c' = c$, or (ii) $c' = c - 1$. Then h'_n is again homogeneous of degree c, and the secondary degree is less than $2b + c$. Therefore we infer that h is expressed by $x^t_{i,i}, z^t_{i,i}$ and the $\alpha_{k,k+1}$ using induction on the secondary degree. □

Remark 5.13. We have $s_0 = \frac{1}{n+1} \sum_{j=0}^n w_j = t_0$. The commutative deformations of S and X are defined by $s_0 = 0$ and $t_0 = 0$ respectively. Thus the parameters s_0 and t_0 correspond to the direction of purely non-commutative deformations.

6. Derived equivalence

Let $\mathcal{T}$ be a coherent locally free $\mathcal{R}$-module. It is called a *tilting generator* of $D^b(\mathrm{coh}(\mathcal{R}))$ if the following conditions are satisfied:

(1) (tilting) $\mathrm{Hom}(\mathcal{T}, \mathcal{T}[m]) = 0$ for $m \neq 0$.

(2) (generator) For any object $x \in D^-(\mathrm{coh}(\mathcal{R}))$, if $\mathrm{Hom}(\mathcal{T}, x[m]) = 0$ for all integers m, then $x \cong 0$.

Theorem 6.1. *$\mathcal{T} = \mathcal{R} \oplus \bigoplus_{i=1}^n \mathcal{R}(-D_i)$ is a tilting generator of $D^-(coh(\mathcal{R}))$.*

Proof. We know already that $\mathcal{T}$ is a tilting object. We will prove that, for any non-zero object $x \in D^-(\mathrm{coh}(\mathcal{R}))$, there exists an integer m such that $\mathrm{Hom}(\mathcal{T}, x[m]) \neq 0$. x is quasi-isomorphic to the following type of complex

$$\dots \longrightarrow \mathcal{M}^{p-1} \xrightarrow{h} \mathcal{M}^p \longrightarrow 0$$

such that h is not surjective. By Lemmas 4.5 and 4.6, there exists a divisorial sheaf $\mathcal{D} = \mathcal{R}(\sum d_i D_i)$ such that $\mathrm{Hom}(\mathcal{D}, \mathrm{Coker}(h)) \neq 0$ and $\mathrm{Hom}(\mathcal{D}, \mathrm{Im}(h)[1]) = 0$. Then it follows that there is a homomorphism $\mathcal{D} \to \mathcal{M}^p$ which does not factor through h. Therefore we have $\mathrm{Hom}(\mathcal{D}, x[p]) \neq 0$.

On the other hand, assume that $\mathrm{Hom}(\mathcal{T}, x[m]) = 0$ for all m. Then by Lemma 4.17, we have $\mathrm{Hom}(\mathcal{D}, x[m]) = 0$ for all divisorial sheaves $\mathcal{D}$, a contradiction. Therefore we have our claim. □

Let $\mathcal{A} = \mathrm{End}(\mathcal{T})$ be the endomorphism algebra. Since it is isomorphic to $\mathcal{S}$, it is a Noetherian algebra.

Lemma 6.2. *Let M be a coherent module on $\mathcal{X}$. Then $Hom(\mathcal{T}, \mathcal{M}[p])$ is a finitely generated right $\mathcal{A}$-modules for $p = 0, 1$, and $Hom(\mathcal{T}, \mathcal{M}[p]) = 0$ for $p \neq 0, 1$.*

Proof. We know already that $\mathrm{Hom}(\mathcal{T}, \mathcal{M}[p]) = 0$ for $p \neq 0, 1$.

$\mathrm{Hom}(\mathcal{T}, \mathcal{R})$ and $\mathrm{Hom}(\mathcal{T}, \mathcal{R}(-D_i))$ are right ideals of the Noetherian ring $\mathcal{A}$, hence finitely generated. We know also already that $\mathrm{Hom}(\mathcal{T}, \mathcal{R}[1]) = \mathrm{Hom}(\mathcal{T}, \mathcal{R}(-D_i)[1]) = 0$.

By the exact sequences in Lemma 4.17, we can deduce inductively that $\mathrm{Hom}(\mathcal{T}, \mathcal{M}[p])$ are finitely generated for all divisorial sheaves $\mathcal{M}$.

Finally let $\mathcal{M}$ be an arbitrary coherent module. Then there is an exact sequence $0 \to \mathcal{N} \to \mathcal{P} \to \mathcal{M} \to 0$ where $\mathcal{P}$ is a finite direct sum of divisorial sheaves. First it follows that $\mathrm{Hom}(\mathcal{T}, \mathcal{M}[1])$ is finitely generated. Then $\mathrm{Hom}(\mathcal{T}, \mathcal{N}[1])$ is also finitely generated because $\mathcal{M}$ is arbitrary. Then it follows that $\mathrm{Hom}(\mathcal{T}, \mathcal{M})$ is also finitely generated. □

Now we can define the following functors:

$$\Phi : D^-(\mathrm{coh}(\mathcal{X})) \to D^-(\text{mod-}\mathcal{A}), \ \ \Phi(\bullet) = R\mathrm{Hom}_{\mathcal{X}}(\mathcal{T}, \bullet),$$
$$\Psi : D^-(\text{mod-}\mathcal{A}) \to D^-(\mathrm{coh}(\mathcal{X})), \ \ \Psi(\bullet) = \bullet \otimes^{\mathbf{L}}_{\mathcal{A}} \mathcal{T}.$$

By the same argument as in [16] Lemma 3.3, we prove the following:

Theorem 6.3 ([2], [14], [16])**.** *(1) Φ is an equivalence and Ψ is its quasi-inverse.*

(2) Φ induces an equivalence $D^b(coh(\mathcal{X})) \to D^b(mod\text{-}\mathcal{A})$.

Proof. This is the same as the proof of [16] Lemma 3.3. We recall it briefly for the convenience of the reader.

(0) Ψ is the left adjoint of Φ:

$$R\mathrm{Hom}_{\mathcal{X}}(y \otimes^{\mathbf{L}}_{\mathcal{A}} \mathcal{T}, x) \cong R\mathrm{Hom}_{\mathcal{A}}(y, R\mathrm{Hom}_{\mathcal{X}}(\mathcal{T}, x)).$$

(1) First we see that $\Phi\Psi \cong \mathrm{Id}_{D^-(\text{mod-}\mathcal{A})}$, i.e.,

$$R\mathrm{Hom}(\mathcal{T}, \bullet \otimes^{\mathbf{L}}_{\mathcal{A}} \mathcal{T}) \cong \bullet.$$

This is true for free modules, hence for all $D^-(\text{mod-}\mathcal{A})$.

In order to prove that the adjunction morphism $\Psi\Phi \to \mathrm{Id}_{D^-(\mathrm{coh}(\mathcal{X}))}$ is also an isomorphism, we use the generator condition that $\Phi(c) \cong 0$ implies $c \cong 0$ for $c \in D^-(\mathrm{coh}(\mathcal{X}))$. From a distinguished triangle

$$\Psi\Phi(x) \to x \to c \to \Psi\Phi(x)[1]$$

and $\Phi\Psi\Phi(x) \cong \Phi(x)$, we obtain $\Phi(c) \cong 0$.

(2) First we have that $\Phi(D^b(\mathrm{coh}(\mathcal{X}))) \subset D^b(\text{mod-}\mathcal{A})$ by Lemma 6.2. For an A-module y, we consider a natural morphism $\alpha : \tau_{<-m}\Psi(y) \to \Psi(y)$. Since Φ is bounded, $\Phi(\alpha) : \Phi(\tau_{<-m}\Psi(y)) \to \Phi\Psi(y) \cong y$ vanishes for large m. Hence $\alpha = 0$, and $\Psi(y)$ is bounded. □

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