BOUNDARY OF FREE PRODUCTS OF COARSELY CONVEX SPACES

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Abstract. In this paper, we compute (co)homologies of ideal boundaries of free products of geodesic coarsely convex spaces in terms of those of each of the components. The (co)homology theories we consider are K-theory, Alexander-Spanier cohomology, K-homology, and Steenrod homology. These computations lead to the computation of K-theory of the Roe algebra of free products of geodesic coarsely convex spaces via the coarse Baum–Connes conjecture.

1. Introduction

1.1. **Background.** This work arose from the study of index problem on the spaces of nonpositive curvature in the coarse geometric setting.

1.1.1. *Index theory and coarse geometry.* Let Y be a complete Spin manifold and let D be the Dirac operator acting on the spinor bundle on Y. If Y is closed, the dimension of the kernel and cokernel of D are finite, so the index of D is defined as the difference of them. Here we are interested in the case that Y is open, especially, Y is a universal cover of a closed aspherical manifold. In such cases, the above definition of index of D does not make sense. Higson and Roe [17] defined the *coarse index* of the Dirac type operator on Y as a class in the K-theory of the Roe algebra. To study this index, they constructed the *coarse assembly map*

$$A_p(Y)\colon KX_p(Y) \to K_p(C^*(Y)), \tag{1}$$

where $KX_p(Y)$ denotes the coarse K-homology of Y and $K_p(C^*(Y))$ denotes the K-theory of the Roe algebra $C^*(Y)$ of Y. The Dirac operator D determines a class in $KX_p(Y)$, and its index is the image of this class under the coarse assembly map.

Now we suppose that Y is coarsely convex as a metric space, then we have the ideal boundary ∂Y of Y, as described in Section 1.1.2. The K-theory $K_p(C^*(Y))$ is naturally isomorphic to the reduced K-homology $\tilde{K}_{p-1}(\partial Y)$ of the ideal boundary with shifting degree by one. This motivates us to compute $\tilde{K}_{p-1}(\partial Y)$ when Y is a free product of coarse convex spaces.

2020 *Mathematics Subject Classification.* Primary 51F30; Secondary 20F65, 58B34.

Key words and phrases. coarsely convex spaces, ideal boundary, cohomology,

The first author was supported by JSPS KAKENHI Grant number 24K06741. The second author was supported by JST, the establishment of university fellowships towards the creation of science technology innovations, Grant number JPMJFS2139.

1.1.2. *Coarse geometry of nonpositively curved spaces.* Gromov introduced a notion of negative curvature for general metric spaces, not necessarily Riemannian manifolds, thereby defining Gromov hyperbolic spaces. This led to the definition of hyperbolic groups, which are known to have many algebraic applications. Motivated by this success, it is natural to seek an appropriate notion of nonpositive curvature for general metric spaces.

Coarsely convex spaces are coarse geometric analogues of simply connected Riemannian manifolds of nonpositive sectional curvature introduced by Oguni and the first author. It was shown in [12] that the coarse assembly map (1) is an isomorphism for proper coarsely convex spaces, which verify the *coarse Baum–Connes conjecture* holds for them. In the proof, the ideal boundary played a central role.

The class of coarsely convex spaces includes geodesic Gromov hyperbolic spaces, CAT(0) spaces, Busemann spaces, and systolic complexes [20]. Descombes and Lang [7] showed that proper injective metric spaces admit geodesic convex bicombings. This implies that proper injective metric spaces are coarsely convex. Recently, Chalopin, Chepoi, Genevois, Hirai, and Osajda [6] showed that Helly graphs are coarsely dense in their injective hulls, and these injective hulls are proper. It follows that Helly graphs are coarsely convex.

In [11], we constructed free products of metric spaces, which generalizes the construction by Bridson-Haefliger [3, Theorem II.11.18]. These spaces can be regarded as a metric-space analogue of the group-theoretic free products. We showed that free products of geodesic coarsely convex spaces are also geodesic coarsely convex. In [10], the authors introduced trees of spaces, which are generalizations of free products of metric spaces. They showed that if each of the components of a tree of spaces is geodesic coarsely convex, then the tree of spaces is geodesic coarsely convex. This generalized the main result of [11] to trees of spaces.

1.2. **Main result and applications.** In this paper, we study the topology of ideal boundaries of trees of spaces whose components are coarsely convex. For this purpose, we introduce augmented spaces of trees of spaces. We determine the topological type of the ideal boundaries of augmented spaces. Then we compute K-theory, Alexander-Spanier cohomology, and K-homology of the ideal boundaries of the trees of spaces in terms of ideal boundaries of augmented spaces, and ideal boundaries of the components of the trees of spaces.

By applying Theorems 9.3 and 9.5 for free products, we have the following.

THEOREM 1.1. *Let X and Y be proper geodesic coarsely convex spaces with nets. Suppose that the net of X and that of Y are coarsely dense in X and Y, respectively, and, both X and Y are unbounded. Let $X * Y$ be the free product of X and Y. Let ∂X, ∂Y, and $\partial(X * Y)$, denote the ideal boundaries of X, Y, and $X * Y$, respectively. Let $\mathcal{C}$ denote the Cantor space.*

Let $M^* = (M^n)_{n\in\mathbb{N}}$ be the K-theory or the Alexander-Spanier cohomology. Then,

$$\tilde{M}^p(\partial(X * Y)) \cong \tilde{M}^p(\mathcal{C}) \oplus \bigoplus_{\mathbb{N}} \tilde{M}^p(\partial X) \oplus \bigoplus_{\mathbb{N}} \tilde{M}^p(\partial Y).$$

Let $M_* = (M_n)_{n\in\mathbb{N}}$ be the K-homology or the Steenrod homology. Then,

$$\tilde{M}_p(\partial(X * Y)) \cong \tilde{M}_p(\mathcal{C}) \times \prod_{\mathbb{N}} \tilde{M}_p(\partial X) \times \prod_{\mathbb{N}} \tilde{M}_p(\partial Y).$$

Here $\tilde{M}_p$ and $\tilde{M}^p$ denote the reduced (co)homology corresponding to M_p and M^p.

Here, we summarize the applications of Theorem 1.1. Let Y be a metric space. We denote by $C^*(Y)$ the Roe algebra of Y. As mentioned above, it is proved in [12] that the coarse Baum-Connes conjecture holds for proper coarsely convex spaces. It is also proved that the coarse K-homology of a coarsely convex space Y is isomorphic to the reduced K-homology of the ideal boundary ∂Y with shifting the degree by one [12, Theorem 6.7]. Therefore, we can compute the K-theory of the Roe algebra of a free product of coarsely convex spaces as follows.

For a topological space X, we denote by $\tilde{K}_p(X)$ its reduced K-homology, and for a C^*-algebra A, we denote by $K_p(A)$ the operator K-theory of A.

THEOREM 1.2. *Let X and Y be proper geodesic coarsely convex spaces with nets. Suppose that the net of X and that of Y are coarsely dense in X and Y, respectively, and, both X and Y are unbounded. We have*

$$K_p(C^*(X * Y)) \cong \tilde{K}_{p-1}(\mathcal{C}) \times \prod_{\mathbb{N}} \tilde{K}_{p-1}(\partial X) \times \prod_{\mathbb{N}} \tilde{K}_{p-1}(\partial Y).$$

COROLLARY 1.3. *Let X and Y be proper geodesic coarsely convex spaces with nets satisfying the condition of Theorem 1.2. We have*

$$K_p(C^*(X * Y)) \cong \tilde{K}_{p-1}(\mathcal{C}) \times \prod_{\mathbb{N}} K_p(C^*(X)) \times \prod_{\mathbb{N}} K_p(C^*(Y)).$$

It might be interesting to compare Corollary 1.3 with a formula for K-theories of the group C^* algebras of free products of groups [2, 10.11.11 (f) to (h)].

We also study the topological dimensions of the ideal boundaries of trees of spaces. Applying Theorem 12.1 for free products of metric spaces, we obtain a formula of the dimension.

THEOREM 1.4. *Let X and Y be geodesic coarsely convex spaces with nets.*

$$\dim(\partial(X * Y)) = \max\{\dim \partial X, \dim \partial Y\}.$$

Using Theorem 1.4, we obtain a formula for the cohomological dimension of the free products of groups acting on geodesic coarsely convex spaces properly and cocompactly (Corollary 12.5). This is a special case of the well-known formula on the cohomological dimensions of free products of groups (Theorem 12.6).

1.3. **Outline.** In Section 2, based on the work of Groves and Manning, we introduce combinatorial horoballs for metric spaces with lattices, and study the shape of geodesics in them.

In Sections 3 to 5, we review the coarsely convex bicombings and the construction of ideal boundaries. In this paper, we only deal with geodesic coarsely convex space, so in Section 5, we restrict results in previous sections to geodesic coarsely convex bicombings, and we improve some of them.

In Section 6, we prepare some technical lemmata to show the continuity of certain retractions on ideal boundaries, which play key roles in the computations of (co)homologies.

In Sections 7 and 8, we introduce trees of spaces and augmented spaces. We construct geodesic coarsely convex bicombings on them assuming that each component admits such bicombings.

In Section 9, we compute (co)homologies of ideal boundaries of trees of geodesic coarsely convex spaces by using Mayer-Vietoris exact sequences. Key tools are retractions on ideal boundaries. We give proofs of Theorems 1.1 and 1.2 and Corollary 9.7.

In Section 10, we construct the retractions used in Section 9 and show that they are continuous. Here we use the results in Section 6.

In Section 11, we study the topological type of the ideal boundary of the augmented spaces. We show (Theorem 11.4) that under certain conditions, the ideal boundaries of the augmented spaces are homeomorphic to the Cantor space.

In Section 12, we show a formula for the topological dimensions of the ideal boundaries of trees of geodesic coarsely convex spaces in terms of the dimensions of each component. In Section 12.2, we apply these results to the cohomological dimensions of the free products of groups.

2. Combinatorial and metric horoballs

All materials in this section are slight modifications of the work of Groves and Manning [16] on the combinatorial horoballs.

In the rest of this section, we assume that X is a geodesic metric space.

Definition 2.1. Let (X, d_X) be a metric space and $A \subset X$. Let $C \geq 0$.

(1) The subset A is C-discrete if for all $a, a' \in A$ with $a \neq a'$, we have $d_Z(a, a') \geq C$.
(2) The subset A is C-dense if for all $x \in X$, there exists $a \in A$ such that $d_Z(x, a) \leq C$.

Remark 2.2. A maximal C-discrete subset $A \subset X$ is $2C$-dense.

Definition 2.3. A metric space with a lattice is a pair $(X, X^{(0)})$ of metric space (X, d_X) and 1-discrete 2-dense subset $X^{(0)} \subset X$. We call $X^{(0)}$ *the lattice of* X.

Groves and Manning defined an augmented space and introduced an equivalent definition of relatively hyperbolic groups [16]. The augmented space is constructed by gluing a

"combinatorial horoballs" based on each $P \in \mathcal{P}$ onto the Cayley graph of G. They defined that G is hyperbolic relative to $\mathcal{P}$ if the augmented space is hyperbolic.

The construction presented here is closely modeled on Groves–Manning's combinatorial horoballs. They constructed horoballs based on an arbitrary connected graph, and showed that such a horoball is always δ–hyperbolic. A minor modification of the Groves–Manning construction produces connected horoballs based on any metric space.

DEFINITION 2.4. A *combinatorial horoball* $\mathcal{H}_{\mathsf{comb}}(X^{(0)})$ is a graph with a set of vertices $\mathcal{H}_{\mathsf{comb}}(X^{(0)})^{(0)}$ and a set of edges $\mathcal{H}_{\mathsf{comb}}(X^{(0)})^{(1)}$ defined as follows:

(1) $\mathcal{H}_{\mathsf{comb}}(X^{(0)})^{(0)} \coloneqq X^{(0)} \times (\mathbb{N} \cup \{0\})$.
(2) $\mathcal{H}_{\mathsf{comb}}(X^{(0)})^{(1)}$ contains the following two types of edges:
   (a) For each $x \in X^{(0)}$ and $l \in \mathbb{N} \cup \{0\}$, there exists a *vertical edge* connecting (x, l) and $(x, l+1)$.
   (b) For each $x, y \in X^{(0)}$ and $l \in \mathbb{N}$, if $0 < d_X(x, y) \leq 2^l$ holds for some $l \in \mathbb{N}$, there exists a *horizontal edge* connecting (x, l) and (y, l).

We endow $\mathcal{H}_{\mathsf{comb}}(X^{(0)})$ with a graph metric such that each edge has length one, and we consider $\mathcal{H}_{\mathsf{comb}}(X^{(0)})$ as a geodesic space.

Let $\mathcal{H}_{\mathsf{comb}}(X^{(0)})$ be a combinatorial horoball. A *horizontal segment* is a path consisting only of horizontal edges. A *vertical segment* is a path consisting only of vertical edges. Each $x \in \mathcal{H}_{\mathsf{comb}}(X^{(0)})^{(0)}$ can be identified with $x = (x_0, l)$, where $x_0 \in X^{(0)}$ and $l \in \mathbb{N} \cup \{0\}$. We say that the depth of x is l, denoted by $D(x) = l$. Moreover, let p be a horizontal segment. When the depth of a point on the horizontal segment p is l, we say that the depth of the horizontal segment p is l, denoted by $D(p) = l$.

DEFINITION 2.5. Let $(X, X^{(0)})$ be a metric space with a lattice. A *metric horoball* $\mathcal{H}(X, X^{(0)})$ is a quotient space of $X \sqcup \mathcal{H}_{\mathsf{comb}}(X^{(0)})$ by the equivalence relation generated by

$$\mathcal{H}_{\mathsf{comb}}(X^{(0)})^{(0)} \ni (x_0, 0) \sim x_0 \in X^{(0)} \subset X$$

for any $x_0 \in X^{(0)}$. We endow $\mathcal{H}(X, X^{(0)})$ with the quotient metric.

Geodesics in combinatorial horoballs are particularly easy to understand.

PROPOSITION 2.6 ([16], Lemma 3.10). *Let $x, y \in \mathcal{H}(X, X^{(0)})$. There exists a geodesic segment on $\mathcal{H}(X, X^{(0)})$ from x to y which consists of at most two vertical segments and a single horizontal segment of length at most 5. We say that this geodesic segment is a normal geodesic segment, denoted by $\mathtt{n}\gamma(x, y)$. Moreover, for each geodesic segment from x to y, denoted by $\gamma(x, y)$, we have*

$$d_H(\mathrm{Im}(\gamma(x, y)), \mathrm{Im}(\mathtt{n}\gamma(x, y))) \leq 4,$$

where d_H is the Hausdorff metric on X.

PROOF. Let $\gamma(x, y)$ be a geodesic segment from x to y. We denote the concatenation of the segments $p_1, p_2, \ldots, p_n$ by $\overline{p_1 p_2 \cdots p_n}$.

We can put

$$\gamma(x, y) = \overline{p_1 p_2 \cdots p_n}$$

such that for all i, we have that if p_i is a horizontal segment, then p_{i+1} is a vertical segment, and if p_i is a vertical segment, then p_{i+1} is a horizontal segment. We can assume that p_1 and p_n are vertical segments. Let $s(p_i)$ be an initial point of p_i and let $t(p_i)$ be a terminal point of p_i. Note that $t(p_i) = s(p_{i+1})$, $s(p_1) = x$, and $t(p_n) = y$.

We will show the proposition as a consequence of the following claims.

CLAIM 2.7. *Let D_M be the maximum depth of horizontal segments which compose $\gamma(x, y)$. If there exists a horizontal segment p which composes $\gamma(x, y)$ such that $D(p) < D_M$, then the length of p is* 1.

Let p_i be a horizontal segment with $D(p_i) < D_M$. Suppose the length of p_i is 2. Let $s(p_i) = (x_i, D(p_i))$ and $t(p_i) = (y_i, D(p_i))$. Note that p_{i+1} is a vertical segment and $t(p_i) = s(p_{i+1})$. We denote the depth of $t(p_{i+1})$ by D_1. Since $D(p_i) < D_1$, there exists a horizontal edge from (x_i, D_1) to (y_i, D_1), denoted by q_2. Let q_1 be the vertical segment from $s(p_i) = (x_i, D(p_i))$ to (x_i, D_1). The path $\overline{q_1 q_2}$ is a geodesic segment from $s(p_i)$ to $t(p_{i+1})$. The length of $\overline{q_1 q_2}$ is smaller than $\overline{p_i p_{i+1}}$. This is a contradiction.

CLAIM 2.8. *There does not exist a subgeodesic $\overline{h_1 v_1 h_2 v_2}$ such that h_i are horizontal segments and v_i are vertical segments.*

Suppose that there exists a subgeodesic $\overline{h_1 v_1 h_2 v_2}$ such that h_i are horizontal segments and v_i are vertical segments. Let $D(h_1) = D_1$, $D(h_2) = D_2$, and $D(t(v_2)) = D$. By Claim 2.7, the length of h_1 and h_2 are 1. We can put $s(h_1) = (x_1, D_1)$, $s(h_2) = (x_2, D_2)$, and $t(v_2) = (x_3, D)$

$$\begin{aligned} d_X(x_1, x_3) &\leq d_X(x_1, x_2) + d_X(x_2, x_3) \\ &\leq 2^{D_1} + 2^{D_2} \\ &\leq 2^{D_2+1} \leq 2^D. \end{aligned}$$

Therefore, there exists a horizontal edge q_2 from (x_1, D) to $t(v_2)$. Now, we define a path q_1 to be a vertical segment from $s(h_1) = (x_1, D_1)$ to (x_1, D). The length of the path $\overline{q_1 q_2}$ is smaller than $\overline{h_1 v_1 h_2 v_2}$. This is a contradiction.

Therefore, a geodesic segment $\gamma(x, y)$ has the following form:

$$\gamma(x, y) = \overline{v_1 h_1 v_2 h_2 v_3 h_3 v_4},$$

where v_i are vertical segments and h_i are horizontal segments. Note that h_2 is the deepest horizontal segment. By Claim 2.7 and Claim 2.8, we have that the length of h_i is 1 for

$i \in \{1, 3\}$. Put $s(h_1) = (x, D(h_1))$ and $t(h_3) = (y, D(h_3))$. We denote by v_1', a vertical segment from $(x, D(h_1))$ to (x, D_M), and denote by v_2', a vertical segment from (y, D_M) and $t(h_3) = (y, D(h_3))$. There exists a geodesic segment from (x, D_M) to (y, D_M), denoted by h_M. We replace $\overline{h_1 v_2 h_2 v_3 h_3}$ with $\overline{v_1' h_M v_2'}$. The replaced path is also a geodesic segment from x to y. Therefore, we obtain a normal geodesic segment from x to y.

For $x, y \in \mathcal{H}(X, X^{(0)})$, let $\mathtt{n}\gamma(x, y) = \overline{uhv}$, where u and v are vertical segments and h is a horizontal segment. Finally, we will show that the length of h is at most 5. Suppose that the length of h is 6. Let $s(h) = (p, D(h))$ and $t(h) = (q, D(h))$. We can connect $(p, D(h) + 1)$ and $(q, D(h) + 1)$ with three horizontal edges. Then, there exists a path of length 5 connecting $s(h)$ and $t(h)$. This is a contradiction.

Moreover, by the construction of $\mathtt{n}\gamma(x, y)$, for each geodesic segment from x to y, denoted by $\gamma(x, y)$, we have

$$d_H(\mathrm{Im}(\gamma(x, y)), \mathrm{Im}(\mathtt{n}\gamma(x, y))) \leq 4.$$

This completes the proof. □

PROPOSITION 2.9 ([16]). *$\mathcal{H}(X, X^{(0)})$ is a Gromov hyperbolic space.*

PROOF. Let $x, y, z \in X^{(0)}$. We denote by $[x, y]$, an image of a geodesic segment on $\mathcal{H}(X, X^{(0)})$ from x to y. Let $p_1 \in [x, y]$, $p_2 \in [y, z]$, and $p_3 \in [z, x]$. We define

$$\mathrm{diam}\{p_1, p_2, p_3\} \coloneqq \max\{d_X(p_i, p_j) \colon i \neq j\}.$$

One of the equivalent definitions of Gromov hyperbolicity is that the following quantity

$$\min\{\mathrm{diam}\{p_1, p_2, p_3\} \colon p_1 \in [x, y], p_2 \in [y, z], p_3 \in [z, x]\}$$

is bounded by a constant independent of choices of geodesic segments and geodesic triangles. See [14, 21.- Proposition] for details. We say that this quantity is the minimum diameter of geodesic triangle $[x, y] \cup [y, z] \cup [z, x]$.

By Proposition 2.6, there exists a normal geodesic segment connecting each two points, that is,

- $\mathtt{n}\gamma(x, y) = \overline{u_1 h_1 v_1}$,
- $\mathtt{n}\gamma(y, z) = \overline{u_2 h_2 v_2}$,
- $\mathtt{n}\gamma(z, x) = \overline{u_3 h_3 v_3}$,

where u_i and v_i are vertical segments and each h_i is a horizontal segment. Suppose $D(h_1) \leq D(h_2) \leq D(h_3)$.

It is easily shown that $\mathrm{Im}(u_1) \subset \mathrm{Im}(v_3)$ and $\mathrm{Im}(v_1) \subset \mathrm{Im}(u_2)$. Then, we have

$$\mathrm{Im}(h_1) \subset N(\mathrm{Im}(v_3), 3) \cup N(\mathrm{Im}(u_2), 3).$$

Therefore,

$$\min\{\mathrm{diam}\{p_1, p_2, p_3\} \colon p_1 \in \mathrm{Im}(\mathtt{n}\gamma(x, y)), p_2 \in \mathrm{Im}(\mathtt{n}\gamma(y, z)), p_3 \in \mathrm{Im}(\mathtt{n}\gamma(z, x))\}$$

is bounded by 5. By Proposition 2.6, for each geodesic segment from x to y, denoted by $\gamma(x,y)$, we have

$$d_H(\mathrm{Im}(\gamma(x,y)), \mathrm{Im}(\mathtt{n}\gamma(x,y))) \leq 4.$$

For any $x,y,z \in \mathcal{H}(X,X^{(0)})$ and any geodesic triangle $\Delta(x,y,z)$, the minimal diameter is bounded by 13. Therefore, $\mathcal{H}(X,X^{(0)})$ is a Gromov hyperbolic space. □

3. COARSELY CONVEX BICOMBING

Let (X,d_X) be a metric space. For $\lambda \geq 1$ and $k \geq 0$, a (λ,k)-*quasi-geodesic bicombing* on X is a map $\gamma\colon X \times X \times [0,1] \to X$ such that for $x,y \in X$, we have $\gamma(x,y,0) = x$, $\gamma(x,y,1) = y$, and

$$\lambda^{-1}\,|t-s|\,d_X(x,y) - k \leq d_X(\gamma(x,y,t),\gamma(x,y,s)) \leq \lambda\,|t-s|\,d_X(x,y) + k \quad (t,s \in [0,1]).$$

If the space X admits a (λ,k)-quasi-geodesic bicombing for some $\lambda \geq 1$ and $k > 0$, we say that X admits a quasi-geodesic bicombing. If a group G acts on X by isometries, a geodesic bicombing $\gamma : X \times X \times [0,1] \to X$ is *G-equivariant* if

$$g \cdot \gamma(x,y)(t) = \gamma(gx,gy)(t)$$

holds for any $g \in G$, $x,y \in X$, and $t \in [0,1]$.

Definition 3.1. Let $\lambda \geq 1$, $k \geq 0$, $E \geq 1$, and $C \geq 0$ be constants. Let $\theta\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a non-decreasing function.

A (λ,k,E,C,θ) *-coarsely convex bicombing* on a metric space (X,d_X) is a (λ,k)-quasi-geodesic bicombing $\gamma\colon X \times X \times [0,1] \to X$ with the following:

(i) Let $x_1,x_2,y_1,y_2 \in X$ and let $a,b \in [0,1]$. Set $y_1' := \gamma(x_1,y_1,a)$ and $y_2' := \gamma(x_2,y_2,b)$. Then, for $c \in [0,1]$, we have

$$d_X(\gamma(x_1,y_1,ca),\gamma(x_2,y_2,cb)) \leq (1-c)Ed_X(x_1,x_2) + cEd_X(y_1',y_2') + C.$$

(ii) Let $x_1,x_2,y_1,y_2 \in X$. Then for $t,s \in [0,1]$ we have

$$|td_X(x_1,y_1) - sd_X(x_2,y_2)| \leq \theta(d_X(x_1,x_2) + d_X(\gamma(x_1,y_1,t),\gamma(x_2,y_2,s))).$$

The following reparametrization is used to construct ideal boundaries in Section 4

Definition 3.2. Let X be a metric space and let $\gamma\colon X \times X \times [0,1] \to X$ be a (λ,k)-quasi-geodesic bicombing on X. A reparametrised bicombing of γ is a map

$$\mathtt{rp}\gamma\colon X \times X \times \mathbb{R}_{\geq 0} \to X$$

defined by

$$\mathtt{rp}\gamma(x,y,t) := \begin{cases} \gamma\left(x,y,t/d_X(x,y)\right) & \text{if } t \leq d_X(x,y) \\ y & \text{if } t > d_X(x,y). \end{cases}$$

4. Ideal boundary

In this section, we review the construction of the ideal boundary of a coarsely convex space.

Let (X, d_X) be a metric space equipped with a $(\lambda, k, E, C, \theta)$-coarsely convex bicombing $\gamma\colon X \times X \times [0,1] \to X$. We choose a base point $e \in X$.

4.1. Gromov product and sequential boundary.

Definition 4.1. Set $k_1 = \lambda + k$, $D = 2(1+E)k_1 + C$, and $D_1 = 2D + 2$. We define a product $(\cdot \mid \cdot)_e\colon X \times X \to \mathbb{R}_{\geq 0}$ by

$$(x \mid y)_e := \min\{d_X(e,x),\ d_X(e,y),\ \sup\{t \in \mathbb{R}_{\geq 0} : d_X(\mathtt{rp}\gamma(e,x,t), \mathtt{rp}\gamma(e,y,t)) \leq D_1\}\},$$

We abbreviate $(x \mid y)_e$ by $(x \mid y)$.

Lemma 4.2 ([12, Lemma 4.8]). *Set $D_2 := E(D_1 + 2k_1)$. For $x, y, z \in X$, we have*

$$(x \mid z) \geq D_2^{-1} \min\{(x \mid y), (y \mid z)\}.$$

We use a sequential model of the ideal boundary introduced in [13]. The idea is based on [15] and [14, Chapitre 7].

Definition 4.3. Let

$$S_\infty(X) = \{(x_i) : (x_i \mid x_j) \to \infty \text{ as } i, j \to \infty\}$$

and define a relation $\sim$ on $S_\infty(X)$ as follows. For every $(x_i), (y_i) \in S_\infty(X)$, we have $(x_i) \sim (y_i)$ if

$$(x_i \mid y_i) \to \infty \text{ as } i \to \infty.$$

Lemma 4.4. *The relation $\sim$ is an equivalence relation on $S_\infty(X)$.*

Proof. It is clear that the relation $\sim$ is reflexive and symmetric. Lemma 4.2 implies that it is transitive. □

Definition 4.5. Let

$$\partial X = S_\infty(X)/\sim \qquad \text{and} \qquad \overline{X} = X \cup \partial X.$$

For $x \in X$ and a sequence (x_i) in X, we write $(x_i) \in x$ if $x_i = x$ for every $i \in \mathbb{N}$. We extend the Gromov product $(\cdot \mid \cdot)\colon X \times X \to \mathbb{R}_{\geq 0}$ to a symmetric function $(\cdot \mid \cdot)\colon \overline{X} \times \overline{X} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ by letting

$$(x \mid y) = \sup\{\liminf_{i \to \infty} (x_i \mid y_i) : (x_i) \in x, (y_i) \in y\} \tag{2}$$

for $x, y \in \overline{X}$.

For $n \in \mathbb{N}$, let

$$V_n = \{(x,y) \in \overline{X} \times \overline{X} : (x \mid y) > n\} \cup \{(x,y) \in X \times X : d_X(x,y) < 1/n\},$$
$$V_n[x] = \{y \in \overline{X} : (x,y) \in V_n\}, \quad (x \in \overline{X}).$$

By [13, Lemma 2.6], $\{V_n : n \in \mathbb{N}\}$ is a base of a metrizable uniformity on $\overline{X}$. See also [12, Section 4.3].

Definition 4.6. Let $\overline{X}$ be equipped with the topology $\mathcal{T}_{(\cdot|\cdot)}$ generated by the family $\{V_n[x] : x \in \overline{X}, n \in \mathbb{N}\}$. We call the subspace ∂X of $\overline{X}$ the *Gromov boundary* of X with respect to $(\cdot \mid \cdot)$. We also call ∂X the *ideal boundary* of X.

Lemma 4.7 ([13, Lemma 2.8, Theorem 2.9]). *We have the following:*

(i) The relative topology $\mathcal{T}_{(\cdot|\cdot)}|_X$ on X with respect to $\overline{X}$ coincides with the topology $\mathcal{T}_d$ induced by the metric d.
(ii) X is a dense open subset in $\overline{X}$.
(iii) If (X,d) is complete, then so is $(\overline{X},d)$.
(iv) If (X,d) is proper, then $(\overline{X},d)$ is compact.

4.2. **Combing at infinity.**

Lemma 4.8. *Suppose that X is proper. Then there exists a map*

$$\mathtt{rp}\bar{\gamma} \colon X \times \overline{X} \times \mathbb{R}_{\geq 0} \to X$$

satisfying the following:

(i) For $x, y \in X$, we have $\mathtt{rp}\bar{\gamma}(x,y,-) = \mathtt{rp}\gamma(x,y,-)$.
(ii) For $(e,x) \in X \times \partial X$, there exists a sequence (x_n) in X such that the sequence of maps $(\mathtt{rp}\gamma(e,x_n,-)|_{\mathbb{N}} \colon \mathbb{N} \to X)_n$ converges to $\mathtt{rp}\bar{\gamma}(e,x,-)|_{\mathbb{N}}$ pointwise.
(iii) For $(e,x) \in X \times \partial X$, we have

$$(\mathtt{rp}\bar{\gamma}(e,x,t) \mid x)_e \to \infty \quad as \quad t \to \infty.$$

(iv) For $(e,x) \in X \times \partial X$, the map $\mathtt{rp}\bar{\gamma}(e,x,-) \colon \mathbb{R}_{\geq 0} \to X$ is a (λ, k_1)-quasi geodesic, where $k_1 := \lambda + k$.

Proof. For $(x,y) \in X \times X$, we define $\mathtt{rp}\gamma(x,y,-)$ by the equality in (i). We use [12, Proposition 4.17] to construct maps $\mathtt{rp}\bar{\gamma}(e,x,-)$ for $(e,x) \in X \times \partial X$ satisfying (ii) and (iii). Finally, (iv) follows from [12, Lemma 4.1]. □

Definition 4.9. We call the map $\mathtt{rp}\bar{\gamma}$ given in Lemma 4.8 an *extended bicombing* on $X \times \overline{X}$ corresponding to γ. For $(e,x) \in X \times \partial X$, we abbreviate $\mathtt{rp}\bar{\gamma}(e,x,-)$ by $\gamma_e^x(-)$.

Lemma 4.10 ([12, Proposition 4.2]). *The extended bicombing $\mathtt{rp}\bar{\gamma}$ on $X \times \overline{X}$ satisfies the following:*

(1) *Let* $(e_1, x_1), (e_2, x_2) \in X \times \partial X$ *and let* $a, b \in \mathbb{R}_{\geq 0}$. *Set* $x_1' := \mathtt{rp}\bar{\gamma}(e_1, x_1, a)$ *and* $x_2' := \mathtt{rp}\bar{\gamma}(e_2, x_2, b)$. *Then, for* $c \in [0, 1]$, *we have*

$$d_X(\mathtt{rp}\bar{\gamma}(e_1, x_1, ca), \mathtt{rp}\bar{\gamma}(e_2, x_2, cb)) \leq (1-c)Ed_X(e_1, e_2) + cEd_X(x_1', x_2') + D$$

where $D := 2(1+E)k_1 + C$.

(2) *We define a non-decreasing function* $\tilde{\theta}\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *by* $\tilde{\theta}(t) := \theta(t+1) + 1$. *Let* $(e_1, x_1), (e_2, x_2) \in X \times \partial X$ *Then for* $t, s \in \mathbb{R}_{\geq 0}$, *we have*

$$|t - s| \leq \tilde{\theta}(d_X(e_1, e_2) + d_X(\mathtt{rp}\bar{\gamma}(e_1, x_1, t), \mathtt{rp}\bar{\gamma}(e_2, x_2, s))).$$

Definition 4.11. For $x, y \in \partial X$, we define

$$(\gamma_e^x \mid \gamma_e^y)_e := \sup\{t \in \mathbb{R}_{\geq 0} : d_X(\gamma_e^x(t), \gamma_e^y(t)) \leq D_1\}.$$

For $x \in \partial X$ and for $p \in X$, we define

$$(\gamma_e^x \mid p)_e = (p \mid \gamma_e^x)_e := \min\{d_X(e, p), \sup\{t \in \mathbb{R}_{\geq 0} : d_X(\mathtt{rp}\gamma(e, p, t), \gamma_e^x(t)) \leq D_1\}\}.$$

Lemma 4.12. *There exists a constant* $\Omega \geq 1$ *depending on* $\lambda, k, E, C, \theta(0)$ *such that the following holds:*

(1) *For* $x, y \in \partial X$, *we have*

$$(\gamma_e^x \mid \gamma_e^y) \leq (x \mid y) \leq \Omega(\gamma_e^x \mid \gamma_e^y).$$

(2) *For triplet* $x, y, z \in \partial X$, *we have*

$$(\gamma_e^x \mid \gamma_e^z) \geq \Omega^{-1} \min\{(\gamma_e^x \mid \gamma_e^y), (\gamma_e^y \mid \gamma_e^z)\}.$$

(3) *For triplet* $x, y, z \in \overline{X}$, *we have*

$$(x \mid z) \geq \Omega^{-1} \min\{(x \mid y), (y \mid z)\}.$$

(4) *Let* $x, y \in \partial X$. *For all* $t \in \mathbb{R}_{\geq 0}$ *with* $t \leq (\gamma_e^x \mid \gamma_e^y)$, *we have*

$$d_X(\gamma_e^x(t), \gamma_e^y(t)) \leq \Omega.$$

(5) *Let* $e \in X$ *and let* $x, y \in \overline{X}$. *If* $\mathtt{rp}\bar{\gamma}(e, x, a) = \mathtt{rp}\bar{\gamma}(e, y, b)$ *for some* $a, b \in \mathbb{R}_{\geq 0}$, *then for all* $t \in [0, \max\{a, b\}]$ *we have*

$$d_X(\mathtt{rp}\bar{\gamma}(e, x, t), \mathtt{rp}\bar{\gamma}(e, y, t)) \leq \Omega.$$

(6) *Let* $e \in X$ *and* $x \in \partial X$. *For* $v \in X$ *and* $t \in [0, 1]$, *we have*

$$(x \mid \gamma(e, v, t))_e \geq \Omega^{-1} \min\{(x \mid v)_e, td_X(e, v)\}.$$

Proof. (1) to (5) are [9, Lemma 2.9]. We give a proof of (6). We suppose that $(x \mid v) \leq td_X(e, v)$. Set $s = (x \mid v)$. By [12, Lemma 4.7], we have $d_X(\gamma_e^x(s), \mathtt{rp}\gamma(e, v, s)) \leq D_1 + k_1$. Set $s' \coloneqq s/(E(D_1 + k_1))$. Then,

$$d_X(\gamma_e^x(s'), \mathtt{rp}\gamma(e, v, s')) \leq \frac{E}{E(D_1 + k_1)} d_X(\gamma_e^x(s), \mathtt{rp}\gamma(e, v, s)) + D \leq D + 1 \leq D_1.$$

Thus $(x \mid \gamma(e,v,t)) \geq (E(D_1+k_1))^{-1}(x \mid v)$.

Next, we suppose that $(x \mid v) \geq td_X(e,v)$. Set $T = td_X(e,v)$. By [12, Lemma 4.7], we have $d_X(\gamma_e^x(T), \mathtt{rp}\gamma(e,v,T)) \leq E(D_1+k_1)+D$. Set $t' \coloneqq T/E(E(D_1+k_1)+D)$. Then,

$$d_X(\gamma_e^x(t'), \mathtt{rp}\gamma(e,v,t')) \leq \frac{E}{E(E(D_1+k_1)+D)} d_X(\gamma_e^x(T), \mathtt{rp}\gamma(e,v,T)) + D \leq D+1 \leq D_1$$

Thus $(x \mid \gamma(e,v,t)) \geq \{E(E(D_1+k_1)+D)\}^{-1} td_X(e,v)$. □

4.3. **Visual maps.** Let (X,d_X) and (Y,d_Y) be metric spaces. We say that a map $f\colon X \to Y$ is a *large scale Lipschitz* map, if there exists $L > 1$ such that

$$d_Y(f(x),f(y)) \leq Ld_X(x,y) + L.$$

Let γ_x and γ_y be coarsely convex bicombings on X and Y, respectively. Let $a \in X$ and $b \in Y$ be base points of X and Y, respectively.

Definition 4.13. We say that a large scale Lipschitz map $f\colon X \to Y$ is *visual* if for every pair of sequences $(x_n)_n, (y_n)_n$ in X with $(x_n \mid y_n)_a \to \infty$, we have $(f(x_n) \mid f(y_n))_b \to \infty$.

Let $f\colon X \to Y$ be a visual map. We define a map $\partial f\colon \partial X \to \partial Y$ as follows. For $x \in \partial X$, we choose a sequence $(x_n)_n$ representing x. Then we define $\partial f(x)$ to be the equivalence class of the sequence $(f(x_n))_n$. By [9, Proposition 3.5], $\partial f(x)$ does not depend on the choice of $(x_n)_n$. We say that ∂f is *induced* by f.

Proposition 4.14 ([9, Corollary 3.6]). *Let f be a visual map. Then the induced map $\partial f\colon \partial X \to \partial Y$ is continuous.*

5. Geodesic coarsely convex spaces

5.1. **Geodesic combing at infinity.** Let (X,d_X) be a metric space. A *geodesic bicombing* on X is a map $\gamma : X \times X \times [0,1] \to X$ such that for $x,y \in X$, we have $\gamma(x,y,0) = x$, $\gamma(x,y,1) = y$, and

$$d_X(\gamma(x,y,t),\gamma(x,y,s)) = |t-s|\, d_X(x,y) \quad (t,s \in [0,1]).$$

Definition 5.1. Let $E \geq 1$ and $C \geq 0$ be constants. An (E,C)-*geodesic coarsely convex bicombing* on a metric space (X,d_X) is a geodesic bicombing $\gamma\colon X \times X \times [0,1] \to X$ satisfying the following:

Let $x_1,x_2,y_1,y_2 \in X$ and let $a,b \in [0,1]$. Set $y_1' \coloneqq \gamma(x_1,y_1,a)$ and $y_2' \coloneqq \gamma(x_2,y_2,b)$. Then, for $c \in [0,1]$, we have

$$d_X(\gamma(x_1,y_1,ca),\gamma(x_2,y_2,cb)) \leq (1-c)Ed_X(x_1,x_2) + cEd_X(y_1',y_2') + C.$$

Lemma 5.2. *An (E,C)-geodesic coarsely convex bicombing γ on a metric space (X,d_X) is a $(1,0,E,C,\mathrm{id}_{\mathbb{R}_{\geq 0}})$-coarsely convex bicombing on (X,d_X).*

PROOF. It follows from the definition of the (E, C)-*geodesic coarsely convex bicombing*, that γ satisfies (i) of Definition 3.1.

Let $x_1, x_2, y_1, y_2 \in X$. Then for $t, s \in [0, 1]$, we have

$$|td_X(x_1, y_1) - sd_X(x_2, y_2)| \leq d_X(x_1, x_2) + d_X(\gamma(x_1, y_1, t), \gamma(x_2, y_2, s))$$

by the triangle inequality. Therefore, γ satisfies (ii) of Definition 3.1 with $\theta = \mathrm{id}_{\mathbb{R}_{\geq 0}}$. □

EXAMPLE 5.3. Let (X, d_X) be a geodesic δ-hyperbolic space. For $x, y \in X$, we choose a geodesic $\gamma_{x,y} \colon [0, d_X(x, y)] \to X$ such that $\gamma_{x,y}(0) = x$ and $\gamma_{x,y}(d_X(x, y)) = y$. We define $\gamma \colon X \times X \times [0, 1] \to X$ by

$$\gamma(x, y, t) := \gamma_{x,y}(t/d_X(x, y)).$$

Then γ is a $(1, 8\delta)$-geodesic coarsely convex bicombing. See [14, Chapitre 2. 25.- Proposition] for details.

Let X be a space equipped with an (E, C)-geodesic coarsely convex bicombing $\gamma \colon X \times X \times [0, 1] \to X$.

By Lemma 5.2 and Lemma 4.8, we have an extended bicombing on $X \times \overline{X}$. Since γ is a geodesic bicombing, we can upgrade it.

LEMMA 5.4. *Suppose that X is proper and γ is an (E, C)-geodesic coarsely convex bicombing on X. Then there exists a map*

$$\mathtt{rp}\bar{\gamma} \colon X \times \overline{X} \times \mathbb{R}_{\geq 0} \to X$$

satisfying the following:

1. *For $x, y \in X$, we have $\mathtt{rp}\bar{\gamma}(x, y, -) = \mathtt{rp}\gamma(x, y, -)$.*
2. *For each $(e, x) \in X \times \partial X$, there exists a sequence (x_n) in X such that the sequence of maps $(\mathtt{rp}\gamma(e, x_n, -) : \mathbb{R}_{\geq 0} \to X)_n$ converges to $\mathtt{rp}\bar{\gamma}(e, x, -)$ uniformly on compact sets.*
3. *For each $(e, x) \in X \times \partial X$, we have*

$$(\mathtt{rp}\bar{\gamma}(e, x, t) \mid x)_e \to \infty \quad \text{as} \quad t \to \infty.$$

4. *For each $(e, x) \in X \times \partial X$, the map $\mathtt{rp}\bar{\gamma}(e, x, -) \colon \mathbb{R}_{\geq 0} \to X$ is a geodesic.*

PROOF. We can use the same construction of geodesics as the one described in [14, 5-25 Théorème]. Here we only give a sketch of the proof.

Let $(e, x) \in X \times \partial X$. We choose $(x_n) \in S_\infty(X)$ such that $(x_n) \in x$. By the Arzelà–Ascoli Theorem and a standard diagonal argument, we can take a subsequence x_{n_k} such that the sequence of maps $(\mathtt{rp}\gamma(e, x_{n_k}, -) : \mathbb{R}_{\geq 0} \to X)_k$ converges uniformly on compact sets. This defines a map $\mathtt{rp}\bar{\gamma}(e, x, -) \colon \mathbb{R}_{\geq 0} \to X$. From the construction, it is easy to see that this map is a geodesic, and $(\mathtt{rp}\bar{\gamma}(e, x, t) \mid x)_e \to \infty$ as $t \to \infty$. □

DEFINITION 5.5. We call the map $\mathtt{rp}\bar{\gamma}$ given in Lemma 5.4 an *extended geodesic bicombing* on $X \times \overline{X}$ corresponding to γ. For $(e, x) \in X \times \partial X$, we abbreviate $\mathtt{rp}\bar{\gamma}(e, x, -)$ by γ_e^x.

LEMMA 5.6. *The extended geodesic bicombing $\mathtt{rp}\bar{\gamma}$ on $X \times \overline{X}$ satisfies the following: Let $(e_1, x_1), (e_2, x_2) \in X \times \partial X$ and let $a, b \in \mathbb{R}_{\geq 0}$. Set $x_1' := \mathtt{rp}\bar{\gamma}(e_1, x_1, a)$ and $x_2' := \mathtt{rp}\bar{\gamma}(e_2, x_2, b)$. Then, for $c \in [0, 1]$, we have*

$$d_X(\mathtt{rp}\bar{\gamma}(e_1, x_1, ca), \mathtt{rp}\bar{\gamma}(e_2, x_2, cb)) \leq (1 - c)Ed_X(e_1, e_2) + cEd_X(x_1', x_2') + C.$$

PROOF. The statement follows immediately from (ii) of Lemma 5.4. □

REMARK 5.7. All statements in Lemma 4.12 hold for γ_e^x.

LEMMA 5.8. *For $e, x, y \in X$, we have*

$$(x \mid y)_e \geq \frac{\min\{d_X(e, x), d_X(e, y)\}}{2Ed_X(x, y)}.$$

PROOF. Set $u := \min\{d_X(e, x), d_X(e, y)\}$. We have

$$d_X(\mathtt{rp}\gamma(e, x, u), \mathtt{rp}\gamma(e, y, u)) \leq d_X(x, y) + |d_X(e, x) - d_X(e, y)| \leq 2d_X(x, y).$$

Set $c := 1/(2Ed_X(x, y))$, we have

$$d_X(\mathtt{rp}\gamma(e, x, cu), \mathtt{rp}\gamma(e, y, cu)) \leq cEd_X(\mathtt{rp}\gamma(e, x, u), \mathtt{rp}\gamma(e, y, u)) + C \leq 1 + C.$$

Therefore,

$$(x \mid y)_e \geq cu = \frac{\min\{d_X(e, x), d_X(e, y)\}}{2Ed_X(x, y)}.$$

□

5.2. γ-convex subspaces.

DEFINITION 5.9. Let X be a metric space with (E, C)-geodesic coarsely convex bicombing $\gamma\colon X \times X \times [0, 1] \to X$. We say that a subspace $Y \subset X$ is γ-convex if for all $x, y \in Y$ and all $t \in [0, 1]$, we have $\gamma(x, y, t) \in Y$.

The following proposition can be shown by a simple argument.

PROPOSITION 5.10. *Let X be a metric space with (E, C)-geodesic coarsely convex bicombing and $Y \subset X$ be a γ-convex subspace. Then the restriction γ_Y of γ on $: Y \times Y \times [0, 1]$ is a (E, C)-geodesic coarsely convex bicombing on Y. Moreover, the inclusion $\iota\colon Y \hookrightarrow X$ extends to a topological embedding $\bar{\iota}\colon \overline{Y} \hookrightarrow \overline{X}$.*

COROLLARY 5.11. *Let X be a metric space with (E, C)-geodesic coarsely convex bicombing and $Y \subset X$ be a γ-convex subspace. Then the ideal boundary of Y with respect to γ_Y is homeomorphic to $\overline{\iota(Y)} \setminus \iota(Y)$, where $\overline{\iota(Y)}$ is the closure of $\iota(Y)$ in $\overline{X}$.*

6. Continuity at infinity

We study maps between geodesic coarsely convex spaces which are not necessarily continuous, although, induce continuous maps between ideal boundaries. The main result of this section will be used to prove Proposition 9.2, which is a key proposition for results in Section 9

Definition 6.1. Let X be a space with a geodesic coarsely convex bicombing. Let $A \subset X$ be a subspace and $\overline{A}$ be the closure of A in $\overline{X}$. Let U be a topological space. We say that a map $F\colon \overline{A} \to U$ is *continuous at infinity* if F is continuous at all $x \in \overline{A} \setminus A$.

Let (X, d_X) be a space equipped with an (E, C)-geodesic coarsely convex bicombing $\gamma\colon X \times X \times [0,1] \to X$.

Lemma 6.2. *Let $A \subset X$ be a subspace and $\overline{A}$ be the closure of A in $\overline{X}$. Let $f\colon \overline{A} \to \overline{X}$ be a map such that:*

(1) $\sup\{d_X(x, f(x)) : x \in A\} < \infty$, and
(2) for $x \in \partial X \cap \overline{A}$, $f(x) = x$.

Then f is continuous at infinity.

Proof. Let $x \in \partial X \cap \overline{A}$. We show that f is continuous at x. Set $K := \sup\{d_X(x, f(x)) : x \in A\}$. Let (x_n) be a sequence in $\overline{A}$ converging to x. We can assume without loss of generality that $x_n \in A$ for all n. Set $t_n := \min\{d_X(e, f(x_n)), d_X(e, x_n)\}$. Since $|d_X(e, f(x_n)) - d_X(e, x_n)| \leq K$, we have

$$d_X(\mathtt{rp}\gamma(e, f(x_n), t_n), \mathtt{rp}\gamma(e, x_n, t_n)) \leq K + d_X(x_n, f(x_n)) \leq 2K.$$

Set

$$p_n := \mathtt{rp}\gamma(e, f(x_n), (1/(2KE)t_n)),$$
$$q_n := \mathtt{rp}\gamma(e, x_n, (1/(2KE)t_n)).$$

We have

$$d_X(p_n, q_n) \leq \frac{1}{2KE} E d_X(\mathtt{rp}\gamma(e, f(x_n), t_n), \mathtt{rp}\gamma(e, x_n, t_n)) + C \leq C + 1$$

So we have $(x_n \mid f(x_n))_e \geq 1/(2KE)t_n$. Then,

$$(x \mid f(x_n))_e \geq \Omega^{-1} \min\{(x \mid x_n)_e, (x_n \mid f(x_n))_e\} \to \infty.$$

It follows that $(f(x_n))$ converges to x. □

6.1. **Image of the exponential map.** We fix a base point $e \in X$. We define a map $\exp\colon \partial X \times \mathbb{R}_{\geq 0} \to X$ by $\exp(x,t) := \mathtt{rp}\bar{\gamma}(e,x,t)$. Set

$$\exp(\mathcal{O}\partial X) := \{\mathtt{rp}\bar{\gamma}(e,x,t) : x \in \partial X, t \in \mathbb{R}_{\geq 0}\}$$
$$\overline{\exp(\mathcal{O}\partial X)} := \exp(\mathcal{O}\partial X) \cup \partial X.$$

We remark that $\exp(\mathcal{O}\partial X)$ is not necessarily coarsely dense in X. In [12, Section 5.5], a coarse equivalence map $\tilde{\varphi}\colon X \to \exp(\mathcal{O}\partial X)$ is constructed. We review the construction.

Let $X^{(0)}$ be a subset of X such that $X^{(0)}$ is 2-dense in X and 1-discrete, that is, for all $v, w \in X^{(0)}$, if $v \neq w$ then $d_X(v,w) \geq 1$, and, for all $v \in X$, there exists $v' \in X^{(0)}$ with $d_X(v,v') \leq 2$. Set $D_6 := 2D_1E + D$ and $Y := B_{D_6}(\exp(\mathcal{O}\partial X))$.

Let $\chi\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a proper map given in [12, Section 5.5]. By the construction of χ, for $v \in X^{(0)}$, we have $\mathtt{rp}\gamma(e,v,\chi(d_X(e,v))) \in Y$, so we define a map $\varphi\colon X^{(0)} \to Y$ by

$$\varphi(v) := \mathtt{rp}\gamma(e,v,\chi(d_X(e,v))).$$

We define a map $\overline{\varphi}\colon \overline{X^{(0)}} \to \overline{Y}$ by

$$\overline{\varphi}(v) := \begin{cases} \varphi(v) & \text{if } v \in X^{(0)} \\ v & \text{if } v \in \partial X. \end{cases}$$

Here we remark that $\overline{X^{(0)}} \setminus X^{(0)} = \partial X$ since $X^{(0)}$ is 2-dense in X.

LEMMA 6.3. *The map $\overline{\varphi}\colon \overline{X^{(0)}} \to \overline{Y}$ is continuous at infinity.*

PROOF. Let $x \in \partial X$. We show that $\overline{\varphi}$ is continuous at x. Let (x_n) be a sequence in $\overline{X^{(0)}}$ converging to x. We can assume without loss of generality that $x_n \in X^{(0)}$ for all n. We remark $(x \mid x_n)_e \to \infty$. By (6) of Lemma 4.12, we have

$$(x \mid \varphi(x_n))_e \geq \Omega^{-1} \min\{(x \mid x_n), \chi(d_X(e,x_n))\} \to \infty$$

since χ is proper. It follows that $(\varphi(x_n))$ converges to x. □

Let $\iota\colon X \to X^{(0)}$ be a restriction such that for $x \in X$, we have $d_X(x,\iota(x)) \leq 2$. Let $j\colon Y \to \exp(\mathcal{O}\partial X)$ be a retraction such that, for $x \in Y$, we have $d_X(x,j(x)) \leq D_6$. Here we do not require that ι and j are continuous. We define a map $\bar{\iota}\colon \overline{X} \to \overline{X^{(0)}}$ and $\bar{j}\colon \overline{Y} \to \overline{\exp(\mathcal{O}\partial X)}$ by

$$\bar{\iota}(v) := \begin{cases} \iota(v) & \text{if } v \in X^{(0)} \\ v & \text{if } v \in \partial X \end{cases} \quad \text{and} \quad \bar{j}(v) := \begin{cases} j(v) & \text{if } v \in Y \\ v & \text{if } v \in \partial X. \end{cases}$$

By Lemma 6.2, the maps $\bar{\iota}$ and $\bar{j}$ are continuous at infinity. We define a map $\Phi\colon \overline{X} \to \overline{\exp(\mathcal{O}\partial X)}$ by

$$\Phi(x) := \begin{cases} \bar{j} \circ \overline{\varphi} \circ \bar{\iota}(x) & \text{if } x \in X \\ x & \text{if } x \in \partial X. \end{cases}$$

LEMMA 6.4. *The map* $\Phi\colon \overline{X} \to \overline{\exp(\mathcal{O}\partial X)}$ *is continuous at infinity.*

6.2. **Angle map.** We define a map $\arg\colon \overline{\exp(\mathcal{O}\partial X)} \to \partial X$ as follows: For $x \in \exp(\mathcal{O}\partial X)$, we choose $(p_x, t_x) \in \partial X \times \mathbb{R}_{\geq 0}$ such that $x = \mathtt{rp}\bar{\gamma}(e, p_x, t_x)$. Then we define

$$\arg(x) = p_x.$$

LEMMA 6.5. *The map* $\arg\colon \exp(\mathcal{O}\partial X) \to \partial X$ *is continuous at infinity.*

We define a map $\Psi\colon \overline{X} \to \partial X$ by

$$\Psi(x) := \begin{cases} \arg \circ \Phi(x) & \text{if } x \in X \\ x & \text{if } x \in \partial X. \end{cases}$$

By Lemmata 6.4 and 6.5, we have the following.

PROPOSITION 6.6. *The map* $\Psi\colon \overline{X} \to \partial X$ *is continuous at infinity.*

7. Trees of spaces

7.1. **Trees of spaces.**

DEFINITION 7.1. Let (Z, d_Z) be a geodesic metric space and T be a bipartite tree. Set $V(T) := K \sqcup L$. Let $\xi\colon Z \to T$ be a continuous map. We suppose the following:

(1) For each $l \in L$, the inverse image $\xi^{-1}(l)$ consists of a single point.
(2) For each $k \in K$, the inverse image $X_k := \xi^{-1}(\mathsf{star}(k))$ is a geodesic space.

Then we say that Z is a *tree of spaces associated with the map* $\xi\colon Z \to T$ *and the bipartite structure* $V(T) = K \sqcup L$.

DEFINITION 7.2. Let (Z, d_Z) be a tree of spaces associated with the map $\xi\colon Z \to T$ and the bipartite structure $V(T) = K \sqcup L$.

Let $E' \geq 1$ and $C' \geq 0$ be constants. We suppose that for each $k \in K$, the inverse image $X_k := \xi^{-1}(\mathsf{star}(k))$ admits an (E', C')-geodesic coarsely convex bicombing

$$\gamma_k\colon X_k \times X_k \times [0,1] \to X_k.$$

Then we say that Z is a $(\xi, K, L, \{\gamma_k\}_{k\in K})$*-tree of geodesic coarsely convex spaces.*

DEFINITION 7.3. We say that a metric space Z is *a tree of geodesic coarsely convex spaces* if Z is $(\xi, K, L, \{\gamma_k\}_{k\in K})$-tree of geodesic coarsely convex space for some $(\xi, K, L, \{\gamma_k\}_{k\in K})$.

THEOREM 7.4 ([10]). *Let* (Z, d_Z) *be a* $(\xi, K, L, \{\gamma_k\}_{k\in K})$*-tree of geodesic coarsely convex space. For* $k \in K$*, set* $X_k := \xi^{-1}(\mathsf{star}(k))$.

Then the space Z admits an (E, C)-geodesic coarsely convex bicombing

$$\gamma\colon Z \times Z \times [0,1] \to Z$$

where $E = E' + 2$ and $C = 6C'$. The restriction of γ to $X_k \times X_k \times [0,1]$ is equal to γ_k.

DEFINITION 7.5. Let (Z, d_Z) be a $(\xi, K, L, \{\gamma_k\}_{k\in K})$-tree of geodesic coarsely convex space. We call the bicombing γ on Z constructed in Theorem 7.4 *the geodesic coarsely convex bicombing associated with the family* $\{\gamma_k\}$.

REMARK 7.6. Let (Z, d_Z) be a $(\xi, K, L, \{\gamma_k\}_{k\in K})$-tree of geodesic coarsely convex space, and let γ be the geodesic coarsely convex bicombing on Z associated with the family $\{\gamma_k\}$. For $k \in K$, set $X_k := \xi^{-1}(\mathsf{star}(k))$.

By the construction, it is clear that for each $k \in K$, the subspace X_k is γ-convex in Z. So by Corollary 5.11, the boundary of X_k in $\overline{Z}$ is homeomorphic to ∂X_k.

7.2. **Free products of metric spaces.** In [11], we introduce free products of metric spaces, which can be regarded as trees of metric spaces.

DEFINITION 7.7. Let (X, d_X) be a metric space and let X_0 be a set with the discrete topology. Let $i_X : X_0 \to X$ be a proper map. We choose a *base point* $e_X \in i_X(X_0)$. We call (X_0, i_X, e_X) a *net* of X. For $x_0 \in X_0$, we denote by $\overline{x_0}$ the image $i_X(x_0)$. We say that (X, d_X, X_0, i_X, e_X) is *a metric space with a net*.

REMARK 7.8. We often abbreviate specifying a net (X_0, i_X, e_X) and say that (X, d_X) is a metric space with a net.

EXAMPLE 7.9. Let (X, d_X, e_X) be a metric space with a base point e_X. Let G be a group acting on X properly by isometries. We define a map $\mathsf{orb}\colon G \to X$ by $\mathsf{orb}(g) := g \cdot e_X$. Then $(G, o(e_X), e_X)$ is a net of X, called *G-net*.

Let (X, d_X) and (Y, d_Y) be metric spaces with nets. In [11, Section 1], we constructed the free product $X * Y$ of X and Y. In [10], we constructed a bipartite tree T and continuous map $\xi\colon X * Y \to T$ such that $X * Y$ is a tree of spaces associated with ξ.

EXAMPLE 7.10. Let G and H be groups acting on metric spaces X and Y. We consider a G-net of X, and an H-net of Y, respectively. Then the free product $G * H$ acts on $X * Y$ properly. If the action of G and H are cocompact, then so is the action of $G * H$.

8. AUGMENTED SPACE

8.1. **n-th augmented spaces.** Let (Z, d_Z) be a tree of spaces associated with the map $\xi\colon Z \to T$ and the bipartite structure $V(T) = K \sqcup L$. For each $k \in K$, we denote by X_k the preimage $\xi^{-1}(\mathsf{star}(k))$.

We assume that for each $k \in K$, the set $X_k \cap \xi^{-1}(L)$ is 1-discrete. For $k \in K$, we choose a 1-discrete, 2-dense subset $X_k^{(0)} \subset X_k$ such that

$$X_k \cap \xi^{-1}(L) \subset X_k^{(0)}. \tag{3}$$

So the pair $(X_k, X_k^{(0)})$ is a space with a lattice. We choose a bijection $k\colon \mathbb{N} \to K$. Under these assumptions, we define *n-th augmented spaces*.

Definition 8.1. Let $n \in \mathbb{N} \sqcup \{0\}$. The *$n$-th augmented space $A_n(Z)$ of Z* is a tree of spaces associated with the map $A_n(\xi)\colon A_n(Z) \to T$ satisfying the following:

(1) For $i \in \mathbb{N}$ with $i \leq n$, the space $A_n(\xi)^{-1}(\mathsf{star}(k(i)))$ is isometric to $X_{k(i)}$.
(2) For $i \in \mathbb{N}$ with $i > n$, the space $A_n(\xi)^{-1}(\mathsf{star}(k(i)))$ is isometric to the metric horoball $\mathcal{H}(X_{k(i)}, X_{k(i)}^{(0)})$ of the space with a lattice $(X_{k(i)}, X_{k(i)}^{(0)})$.

We call the 0-th augmented space $A_0(Z)$ *the augmented space*, and we denote it by $A(Z)$.

Remark 8.2. In the above definition, the coarse equivalence class of the n-th augmented space does not depend on the choice of lattices $X_k^{(0)}$. Therefore, when we consider n-th augmented spaces, we implicitly choose lattices, and we say that $A_n(Z)$ is the n-th augmented space of Z, without mentioning lattices. We abbreviate $\mathcal{H}(X_k, X_k^{(0)})$ by $\mathcal{H}(X_k)$.

The augmented space $A(Z)$ does not depend on the choice of the bijection $k\colon \mathbb{N} \to K$.

We can explicitly construct an n-th augmented space $A_n(Z)$ from a tree of spaces Z by choosing lattices that satisfy Assumption (3) and gluing combinatorial horoballs along lattices by the way described in Definition 2.5.

8.2. **Bicombings on n-th augmented spaces.** Let (Z, d_Z) be a $(\xi, K, L, \{\gamma_k\}_{k\in K})$-tree of geodesic coarsely convex spaces. Let γ be the geodesic coarsely convex bicombing associated with the family $\{\gamma_k\}$. We choose a pair of constants (E', C') such that γ is (E', C') geodesic coarsely convex.

For $k \in K$, set $X_k \colon = \xi^{-1}(\mathsf{star}(k))$. For $k \in K$, we define an $(E', C' + 8\delta)$-geodesic coarsely convex bicombing $A_n(\gamma_k)$ on $A_n(\xi)^{-1}(\mathsf{star}(k)) = A_n(X_k)$ by the following:

(1) if $i \leq n$ then $A_n(\gamma_{k(i)}) = \gamma_{k(i)}$,
(2) Suppose $i > n$. Note that $A_n(X_{k(i)})$ is equal to $\mathcal{H}(X_{k(i)})$. For $x, y \in A_n(X_{k(i)})$, we define a map

$$A_n(\gamma_{k(i)})(x, y, -)\colon [0, 1] \to A_n(X_{k(i)})$$

to be the normal geodesic segment from x to y given in Proposition 2.6.

We choose a bijection $k\colon \mathbb{N} \to K$. Let $A_n(Z)$ be the n-th augmented space of Z. Then $A_n(Z)$ is a $(A_n(\xi), K, L, \{A_n(\gamma_k)\}_{k\in K})$-tree of geodesic coarsely convex space. Let $A_n(\gamma)$ be a geodesic coarsely convex bicombing on $A_n(Z)$ associated with $\{A_n(\gamma_k)\}$.

DEFINITION 8.3. We call the bicombing $A_n(Z)$ constructed in the above an *augmented bicombing associated with the family* $\{\gamma_k\}_{k\in K}$.

REMARK 8.4. For $i \leq n$, $X_{k(i)}$ is an $A_n(\gamma)$-convex subspace, so by Corollary 5.11, $\partial X_{k(i)}$ is a closed subspace of $\partial A_n(Z)$.

For $i > n$, $\mathcal{H}(X_{k(i)})$ is an $A_n(\gamma)$-convex subspace, the boundary $\partial\mathcal{H}(X_{k(i)}) = \overline{\mathcal{H}(X_{k(i)})} \setminus \mathcal{H}(X_{k(i)})$ consists of a single point. We call this the *center of the horoball* $\mathcal{H}(X_{k(i)})$, and denote by $c_{k(i)}$.

9. (co)Homology of the Ideal boundary of a tree of spaces

Let (Z, d_Z) be a $(\xi, K, L, \{\gamma_k\}_{k\in K})$-tree of geodesic coarsely convex spaces. We fix a bijection $k\colon \mathbb{N} \to K$. Let $A(Z)$ be the augmented space of Z, and let $A_n(\gamma)$ be the augmented bicombing on $A_n(Z)$ associated with the family $\{\gamma_k\}_{k\in K}$.

REMARK 9.1. It is clear that the inclusion $\iota_n\colon A_n(Z) \hookrightarrow A_{n-1}(Z)$ is a visual map in the sense of Definition 4.13. By Proposition 4.14, it induces a continuous map

$$\partial\iota_n\colon \partial A_n(Z) \to \partial A_{n-1}(Z).$$

The family $\{\partial A_n(Z), \partial\iota_n\}$ forms a projective system, and the projective limit is ∂Z, that is,

$$\partial Z = \varprojlim_n \partial A_n(Z).$$

The following is a key proposition for computing (co)homologies of ∂Z.

PROPOSITION 9.2. *For $n \in \mathbb{N}$, there exists a continuous retraction*

$$s_{k(n)}\colon \partial A_n(Z) \to \partial X_{k(n)}.$$

We will prove Proposition 9.2 in Section 10

9.1. **Cohomology.** Let $M^* = (M^n)_{n\in\mathbb{N}}$ be the K-theory or the Alexander-Spanier cohomology, and let $\tilde{M}^*$ be the reduced one. For a compact Hausdorff space Y, and $e \in Y$, we have $\tilde{M}^*(Y) \cong M^*(Y, \{e\})$.

We consider the long exact sequence for the triple $(\partial A_n(Z), \partial X_{k(n)}, \{e_n\})$.

$$\begin{aligned}\dots \to M^p(\partial A_n(Z), \partial X_{k(n)}) \to M^p(\partial A_n(Z), \{e_n\}) \to M^p(\partial X_{k(n)}, \{e_n\})\\ \to M^{p+1}(\partial A_n(Z), \partial X_{k(n)}) \to \cdots\end{aligned}$$

By Proposition 9.2, the above exact sequence splits, so we have

$$0 \to M^p(\partial A_n(Z), \partial X_{k(n)}) \to \tilde{M}^p(\partial A_n(Z)) \to \tilde{M}^p(\partial X_{k(n)}) \to 0. \tag{4}$$

By the strong excision axiom, we have

$$\begin{aligned} M^p(\partial A_n(Z), \partial X_{k(n)}) &\cong M^p((\partial A_n(Z) \setminus \partial X_{k(n)})^+, \{c_{k(n)}\}) \\ &\cong M^p(\partial A_{n-1}(Z), \{c_{k(n)}\}) \\ &= \tilde{M}^p(\partial A_{n-1}(Z)). \end{aligned}$$

Here $(\partial A_n(Z) \setminus \partial X_{k(n)})^+$ denotes the one-point compactification

$$(\partial A_n(Z) \setminus \partial X_{k(n)}) \sqcup \{c_{k(n)}\}.$$

It follows that $\tilde{M}^p(\partial A_n(Z)) \cong \tilde{M}^p(\partial A_{n-1}(Z)) \oplus \tilde{M}^p(\partial X_{k(n)})$. By the continuity of M^*, we have $\tilde{M}^p(\partial Z) \cong \varinjlim \tilde{M}^p(\partial A_n(Z))$. Here we summarize the computation.

THEOREM 9.3. *Let Z be a $(\xi, K, L, \{\gamma_k\}_{k\in K})$-tree of geodesic coarsely convex spaces. Suppose that Z is proper. Set $X_k := \xi^{-1}(\mathsf{star}(k))$. Let $A(Z)$ be the augmented space of Z. Then,*

$$\tilde{M}^p(\partial Z) \cong \tilde{M}^p(\partial A(Z)) \oplus \bigoplus_{k\in K} \tilde{M}^p(\partial X_k).$$

Here we remark that the topological type of $\partial A(Z)$ is described in Theorem 11.4.

9.2. **Homology.** Let $M_* = (M_n)_{n\in\mathbb{N}}$ be the K-homology or the Steenrod homology, and let $\tilde{M}_*$ be the reduced one. For a compact Hausdorff space X, and $e \in X$, we have $\tilde{M}_*(X) \cong M_*(X, \{e\})$.

THEOREM 9.4 ([19, Theorem 4]). *Let $Y_1 \leftarrow Y_2 \leftarrow Y_3 \leftarrow \cdots$ be a projective system of compact metric spaces. Then there exists an exact sequence*

$$0 \to \varprojlim{}^1 M_{p+1}(Y_i) \to M_p(\varprojlim Y_i) \to \varprojlim M_p(Y_i) \to 0.$$

By the same argument as the one in Section 9.1, we obtain

$$\tilde{M}_p(\partial A_n(Z)) \cong \tilde{M}_p(\partial A_{n-1}(Z)) \oplus \tilde{M}_p(\partial X_{k(n)}).$$

By Theorem 9.4, we have

$$0 \to \varprojlim{}^1 M_{p+1}(\partial A_n(Z)) \to M_p(\partial Z) \to \varprojlim M_p(\partial A_n(Z)) \to 0.$$

By Proposition 9.2, the maps $M_{p+1}(\partial A_{n+1}(Z)) \to M_{p+1}(\partial A_n(Z))$ are surjective, so the Mittag-Leffler condition holds. Thus, $\varprojlim{}^1 M_{p+1}(\partial A_n(Z))$ vanishes.

THEOREM 9.5. *Let Z be a $(\xi, K, L, \{\gamma_k\}_{k\in K})$-tree of geodesic coarsely convex spaces. Set $X_k := \xi^{-1}(\mathsf{star}(k))$. Suppose that Z is proper. Let $A(Z)$ be the augmented space of Z. Then,*

$$\tilde{M}_p(\partial Z) \cong \tilde{M}_p(\partial A(Z)) \times \prod_{k\in K} \tilde{M}_p(\partial X_k).$$

9.3. **K-theory of the Roe algebra.** Let V be a proper metric space. We denote by $C^*(V)$ the Roe algebra of V. We also denote by $KX_*(V)$ the coarse K-homology of V. For detail, see [17], [21] and [18].

Suppose that V is a proper coarsely convex space. By [12, Theorem 1.3], the following coarse assembly map is an isomorphism.

$$\mu_*\colon KX_*(V) \to K_*(C^*(V)).$$

By [12, Theorem 6.7], the following transgression map is an isomorphism.

$$T_{\partial V}\colon KX_*(V) \to \tilde{K}_{*-1}(\partial V).$$

Thus, we have an isomorphism

$$T_{\partial V} \circ \mu_*^{-1}\colon K_*(C^*(V)) \to \tilde{K}_{*-1}(\partial V). \tag{5}$$

THEOREM 9.6. *Let Z be a $(\xi, K, L, \{\gamma_k\}_{k\in K})$-tree of geodesic coarsely convex spaces. Set $X_k := \xi^{-1}(\mathsf{star}(k))$. Suppose that Z is proper. Let $A(Z)$ be the augmented space of Z. Then,*

$$K_p(C^*(Z)) \cong \tilde{K}_{p-1}(\partial A(Z)) \times \prod_{k\in K} \tilde{K}_{p-1}(\partial X_k).$$

PROOF. Combining Theorem 9.5 for K-homology with an isomorphism

$$K_*(C^*(Z)) \cong \tilde{K}_{*-1}(\partial Z)$$

obtained by applying (5) for geodesic coarsely convex space Z, we obtain the desired result. □

COROLLARY 9.7. *Let Z be the tree of geodesic coarsely convex spaces described in Theorem 9.6. We have*

$$K_p(C^*(Z)) \cong \tilde{K}_{p-1}(\partial A(Z)) \times \prod_{k\in K} K_p(C^*(X_k)).$$

PROOF. Combining Theorem 9.6 with isomorphisms

$$K_*(C^*(X)) \cong \tilde{K}_{*-1}(\partial X) \quad \text{and} \quad K_*(C^*(Y)) \cong \tilde{K}_{*-1}(\partial Y),$$

we obtain the desired result. □

9.4. **Case of free products.**

Proof of Theorems 1.1 and 1.2 and Corollary 1.3. Let X and Y be proper geodesic coarsely convex spaces with nets. Suppose that the net of X and that of Y are coarsely dense in X and Y, respectively. Let $X * Y$ be the free product of X and Y. Then $X * Y$ is a tree of space. Since the net of X and Y are coarsely dense in X and Y, respectively, by Theorem 11.4, the ideal boundary $\partial A(X * Y)$ of the augmented space has no isolated point. So, $\partial A(X * Y)$ is homeomorphic to the Cantor space $\mathcal{C}$. Then applying Theorems 9.3, 9.5 and 9.6 and Corollary 9.7 for $X * Y$, we obtain the desired result. □

10. Construction of retractions

Let (Z, d_Z) be a $(\xi, K, L, \{\gamma_k\}_{k\in K})$-tree of geodesic coarsely convex space. We suppose that Z is proper. Let γ be the geodesic coarsely convex bicombing on Z associated with the family $\{\gamma_k\}_{k\in K}$. By Remark 7.6, we can regard ∂X_k as a closed subspace of ∂Z. The purpose of this section is to construct a retraction $s_k\colon \partial Z \to \partial X_k$.

Let $k \in K$. For $x \in \partial Z$, set

$$t(k;x) := \sup\{t \in \mathbb{R} : \mathtt{rp}\bar{\gamma}(e, x, t) \in X_k\}.$$

We define a map $\pi_k\colon \partial Z \to \overline{X_k}$ by

$$\pi_k(x) := \mathtt{rp}\bar{\gamma}(e, x, t(k;x)).$$

Remark 10.1. For $x \in \partial Z\backslash\partial X_k$, we have $t(k;x) = d(e, \mathtt{rp}\bar{\gamma}(e, x, t(k;x)))$. For $x \in \partial X_k$, we have $t(k;x) = \infty$ and $\pi_k(x) = \mathtt{rp}\bar{\gamma}(e, x, \infty) = x$.

Lemma 10.2. *We fix $k \in K$. Let $e \in X_k$ be a base point of X_k, and let $v, w \in X_k$ such that $v \neq w$. Then for $x \in \pi_k^{-1}(v)$ and $y \in \pi_k^{-1}(w)$, we have*

$$(x \mid y)_e < \Omega(\max\{d_Z(e, v), d_Z(e, w)\} + \Omega).$$

Here Ω is a constant given in Lemma 4.12.

Proof. We define $t \in \mathbb{R}$ by

$$t = \max\{t(k;x), t(k;y)\} + \Omega = \max\{d_Z(e, v), d_Z(e, w)\} + \Omega$$

The geodesic from $\gamma_e^x(t)$ to $\gamma_e^y(t)$ is obtained by concatenating $\gamma(\gamma_e^x(t), v, -)$, $\gamma(v, w, -)$ and $\gamma(w, \gamma_e^y(t), -)$. So we have

$$\begin{aligned} d_Z(\gamma_e^x(t), \gamma_e^y(t)) &= d_Z(\gamma_e^x(t), v) + d_Z(v, w) + d_Z(w, \gamma_e^y(t)) \\ &\geq t - t(k;x) + d_Z(v, w) + t - t(k;y) \\ &\geq \Omega. \end{aligned}$$

Thus by (4) of Lemma 4.12, we have $(\gamma_e^x \mid \gamma_e^y)_e < t$. Then by (2) of Lemma 4.12, we have $(x \mid y)_e \leq \Omega(\gamma_e^x \mid \gamma_e^y)_e \leq \Omega t$. □

Lemma 10.3. *We fix $k \in K$. Let $e \in X_k$ be a base point of X_k, and let $v \in X_k$. Then for $x \in \partial X_k$ and $y \in \pi_k^{-1}(v)$, we have*

$$(x \mid y)_e < \Omega(d_Z(e, v) + \Omega).$$

Here Ω is a constant given in Lemma 4.12.

Proof. We can prove the lemma by a similar argument as the proof of Lemma 10.2. □

Lemma 10.4. *The map $\pi_k\colon \partial Z \to \overline{X_k}$ is continuous.*

PROOF. Let $x \in \partial Z$. We will show that π_k is continuous at x. First, we suppose that $x \in \partial Z \setminus \partial X_k$. Set $v = \pi_k(x)$. Then by Lemma 10.2, $\pi_k^{-1}(v)$ is an open set. This implies that π_k is continuous at x.

Now we suppose that $x \in \partial X_k$. We remark that $\pi_k(x) = x$. For $r \in \mathbb{R}_{\geq 0}$, set $V_r[x] := \{y \in \overline{X} : (x \mid y) > r\}$. The family $\{V_n[x] : n \in \mathbb{N}\}$ forms a system of the fundamental neighborhood of x. We will show that for each $n \in \mathbb{N}$, there exists $T \in \mathbb{R}_{\geq 0}$ such that

$$\pi_k(V_T[x]) \subset V_{n-1}[\pi_k(x)] = V_{n-1}[x]. \tag{6}$$

We fix $n \in \mathbb{N}$. Set $T = E\Omega^2(n+1) + \Omega^2$. Let $y \in V_T[x]$. Thus, $(x \mid y)_e > T$. Set $v = \pi_k(y)$. By Lemma 10.3, we have

$$t(k; y) = d_Z(e, v) \geq \Omega^{-1}(x \mid y)_e - \Omega > E\Omega(n+1) > n+1.$$

Set $T' \coloneqq E\Omega(n+1)$. By item (4) of Lemma 4.12, we have $d_Z(\gamma_e^x(T'), \gamma_e^y(T')) \leq \Omega$. Then we have

$$d_Z(\gamma_e^y(n+1), \gamma_e^x(n+1)) \leq \frac{1}{E\Omega} E d_Z(\gamma_e^y(T'), \gamma_e^x(T')) + C \leq 1 + C.$$

Moreover, we have

$$\begin{aligned} d_Z(\gamma_e^y(n+1), \mathtt{rp}\gamma(e, v, n+1)) &\leq \frac{n+1}{t(k;y)} E d_Z(\gamma_e^y(t(k;y)), \mathtt{rp}\gamma(e, v, t(k;y))) + C \\ &= C. \end{aligned}$$

Here we used that $\gamma_e^y(t(k;y)) = v = \mathtt{rp}\gamma(e, v, t(k;y)))$ and that $n+1 < t(k;y)$. Combining these, we obtain

$$d_Z(\gamma_e^x(n+1), \mathtt{rp}\gamma(e, v, n+1)) \leq 2C + 1.$$

It follows that $(x \mid v) > n$. Thus, $v = \pi_k(y) \in V_{n-1}[x]$. Therefore, (6) holds. This implies that π_k is continuous at x. □

Let $\Psi_k \colon \overline{X_k} \to \partial X_k$ be the map constructed in Section 6.2.

DEFINITION 10.5. We define a retraction $s_k \colon \partial Z \to \partial X_k$ by

$$s_k = \Psi_k \circ \pi_k.$$

PROPOSITION 10.6. *The map $s_k \colon \partial Z \to \partial X_k$ is continuous.*

PROOF. Let $x \in \partial Z$. We will show that π_k is continuous at x.

Let x_n be a sequence in ∂Z converging to x. Since π_k is continuous, the sequence $\pi_k(x_n)$ converges to $\pi_k(x) \in \partial X_k$. Then by Proposition 6.6, the sequence $(\Psi_k(\pi_k(x_n)))$ converges to $\Psi_k(\pi_k(x))$. □

Proof of Proposition 9.2. Applying Proposition 10.6 to the tree of spaces $A_n(Z)$ associated with the map $A_n(\xi) \colon A_n(Z) \to T$, we obtain the desired result. □

11. Ideal boundaries of the augmented spaces

Let (Z, d_Z) be a $(\xi, K, L, \{\gamma_k\}_{k\in K})$-tree of geodesic coarsely convex space. We suppose that Z is proper.

Let $A(Z)$ be the augmented space of Z. For simplicity, we denote by γ an augmented bicombing on $A(Z)$ associated with the family $\{\gamma_k\}_{k\in K}$.

For $x \in \overline{A(Z)}$, set $W_n[x] = \{y \in \partial A(Z) : (\gamma_e^x \mid \gamma_e^y)_e \geq n\}$. Recall that $\mathcal{H}_{\mathsf{comb}}(X_k^{(0)})^{(0)}$ denotes the vertex set of the combinatorial horoball $\mathcal{H}_{\mathsf{comb}}(X_k^{(0)})$. We consider the family

$$\mathcal{T} := \bigsqcup_{k\in K} \{W_n[x] : x \in \mathcal{H}_{\mathsf{comb}}(X_k^{(0)})^{(0)}, n \in \mathbb{N}\}.$$

Since Z is proper, for any bounded subset $B \subset Z$, the cardinality of the set $B \cap X_k^{(0)}$ is finite. It follows that $\mathcal{T}$ is countable.

Lemma 11.1. *The family $\mathcal{T}$ is a countable family of open basis of $\partial A(Z)$.*

Proof. We have already seen that $\mathcal{T}$ is countable. Let $x \in \partial A(Z)$, and $n \in \mathbb{N}$. We will show that there exists $U \in \mathcal{T}$ such that

$$x \in U \subset V_n[x] \cap \partial A(Z).$$

We choose $t \in \mathbb{R}_{\geq 0}$ such that:

$$t \geq 4E\Omega^3 n + 2,$$

$$\mathtt{rp}^{\bar\gamma}(e, x, t) = \gamma_e^x(t) \in \mathcal{H}(X_k, X_k^{(0)}) \text{ for some } k \in K.$$

There exists $x' \in \mathcal{H}_{\mathsf{comb}}(X_k^{(0)})^{(0)}$ such that $d_{A(Z)}(\gamma_e^x(t), x') \leq 2$ since $\mathcal{H}_{\mathsf{comb}}(X_k^{(0)})^{(0)}$ is 2-dense in $\mathcal{H}(X_k, X_k^{(0)})$. By Lemma 5.8, we have

$$(\gamma_e^x(t) \mid x') \geq \frac{\min\{d_{A(Z)}(e, \gamma_e^x(t)),\, d_{A(Z)}(e, x')\}}{2E d_{A(Z)}(\gamma_e^x(t), x')} \geq \frac{t-2}{4E}.$$

So

$$(\gamma_e^x \mid x') \geq \Omega^{-1} \min\{(\gamma_e^x \mid \gamma_e^x(t)), (\gamma_e^x(t) \mid x')\} \geq \frac{t-2}{4E\Omega} \geq \Omega^2 n.$$

Set $t' = \Omega^2 n$. Then for all $y \in W_{t'}[x']$,

$$(x \mid y) \geq \Omega^{-2} \min\{(x \mid \gamma_e^x), (\gamma_e^x \mid x'), (x' \mid y)\} \geq n.$$

Here we used that $(x' \mid y) \geq (\gamma_e^{x'} \mid \gamma_e^y) \geq \Omega^2 n$. Therefore, $x \in W_{t'}[x'] \subset V_n[x] \cap \partial A(Z)$. This implies that $\mathcal{T}$ is an open basis of $\partial A(Z)$. □

Lemma 11.2. *Each $W \in \mathcal{T}$ is a clopen set.*

Proof. Let $k \in K$. Let $x \in \mathcal{H}_{\mathsf{comb}}(X_k^{(0)})^{(0)}$ and $n \in \mathbb{N}$. We will show that $W_n[x]$ is closed. Let $y \in \partial A(Z) \setminus W_n[x]$. We will show that there exists $T > 0$ such that

$$W_{2T}[y] \subset \partial A(Z) \setminus W_n[x].$$

For $p \in \partial A(Z)$, set

$$s(k;p) := \inf\{t \in \mathbb{R} : \mathtt{rp}\bar{\gamma}(e,p,t) \in X_k^{(0)}\}, \quad t(k;p) := \sup\{t \in \mathbb{R} : \mathtt{rp}\bar{\gamma}(e,p,t) \in X_k\}.$$

Here we use the convention that $\inf \emptyset = \infty$ and $\sup \emptyset = -\infty$. First, we suppose $y = c_k \in \overline{\mathcal{H}_{\mathsf{comb}}(X_k^{(0)})^{(0)}}$. Then $s(k;y) < \infty$ and $t(k;y) = \infty$. We remark that the geodesic $\mathtt{rp}\bar{\gamma}(\gamma_e^y(s(k;y)), y, -)$ consists of vertical line in $\mathcal{H}(X_k, X_k^{(0)})$. Let h be the depth of x in $\mathcal{H}_{\mathsf{comb}}(X_k^{(0)})^{(0)}$. Set $T := h + s(k;y) + n + D_1$. Let $y' \in W_{2T}[y]$. We have

$$\mathtt{rp}\bar{\gamma}(e,y,t) = \mathtt{rp}\bar{\gamma}(e,y',t) \qquad (\forall t \in [0,T]).$$

See fig. 1. If $y' \in W_n[x]$, then

$$d_Z(\gamma_e^x(n), \mathtt{rp}\bar{\gamma}(e,y,n)) = d_Z(\gamma_e^x(n), \mathtt{rp}\bar{\gamma}(e,y',n)) \le D_1.$$

Thus $(\gamma_x^x \mid \gamma_e^y) \ge n$. This contradicts $y \notin W_n[x]$. Therefore, $y' \notin W_n[x]$, so we have $W_{2T}[y] \cap W_n[x] = \emptyset$.

Next, we suppose $y \notin \overline{\mathcal{H}_{\mathsf{comb}}(X_k^{(0)})^{(0)}}$. If $s(k;y) = \infty$, then set $T = 2n + D_1$. It is easy to see that $W_{2T}[y] \cap W_n[x] = \emptyset$. If $s(k;y) < \infty$, then set $T = t(k;y) + 2n + D_1$. For $y' \in W_{2T}[y]$, we have

$$\mathtt{rp}\bar{\gamma}(e,y,t) = \mathtt{rp}\bar{\gamma}(e,y',t) \qquad (\forall t \in [0,T]).$$

See fig. 2. Then by a similar argument as before, $y' \notin W_n[x]$. Therefore, $W_{2T}[y] \cap W_n[x] = \emptyset$. □

LEMMA 11.3. *Let $k \in K$. The center c_k of the horoball $\mathcal{H}(X_k)$ is an isolated point if and only if $X_k \cap \xi^{-1}(L^\circ)$ is bounded, where L° denotes a set of vertices $l \in L$ which are adjacent to at least two different vertices in K.*

PROOF. For $p \in \partial A(Z)$, set

$$t(k;p) := \sup\{t \in \mathbb{R} : \mathtt{rp}\bar{\gamma}(e,p,t) \in X_k\}.$$

First, we suppose that c_k is an accumulation point. Thus, there exists a sequence (x_n) in $\partial A(Z) \setminus \{c_k\}$ converging to c_k. Then we see that $t(k;x_n) \to \infty$. We remark that $\mathtt{rp}\bar{\gamma}(e, x_n, t(k;x_n)) \in X_k \cap \xi^{-1}(L^\circ)$. Therefore, $X_k \cap \xi^{-1}(L^\circ)$ is unbounded.

Next, we suppose that $X_k \cap \xi^{-1}(L^\circ)$ is unbounded. Then there exists an unbounded sequence (v_n) in $X_k \cap \xi^{-1}(L^\circ)$. By the definition of L°, for each v_n, there exists $k_n \in K \setminus \{k\}$ which are adjacent to $\xi(v_n)$. Then we see that c_{k_n} converges to c_k. See fig. 3 . □

THEOREM 11.4. *Let Z be a tree of geodesic coarsely convex spaces, and let $A(Z)$ be the augmented space of Z.*

(1) Let $\partial A(Z)_{\mathsf{iso}}$ denote the set of isolated points in $\partial A(Z)$. Then,

$$\partial A(Z)_{\mathsf{iso}} = \{c_k : k \in K,\ X_k \cap \xi^{-1}(L^\circ) \text{ is bounded }\}.$$

(2) If $\partial A(Z)$ has no isolated point, then $\partial A(Z)$ is the Cantor space.

PROOF. By Lemma 11.3,

$$\partial A(Z)_{\mathsf{iso}} \cap \{c_k : k \in K\} = \{c_k : k \in K,\ X_k \cap \xi^{-1}(L) \text{ is bounded }\}.$$

Let $x \in \partial A(Z)_{\mathsf{iso}} \setminus \{c_k : k \in K\}$. Then there exists a sequence $\{t_n\} \subset \mathbb{R}_{\geq 0}$ and $\{k_n\} \subset K$ such that $\mathtt{rp}\bar{\gamma}(e, x, t_n) \in X_{k_n}$ for all $n \in \mathbb{N}$, and $k_n \neq k_m$ for all $n \neq m$. Then the sequence (c_{k_n}) converges to x. Thus, (1) holds.

By Lemmata 11.1 and 11.2, $\partial A(Z)$ has a countable basis consisting of clopen sets. Thus, if $\partial A(Z)$ has no isolated points, $\partial A(Z)$ is homeomorphic to the Cantor space by [4]. □

REMARK 11.5. It is known by [1, Lemma 4.6(2)] that the augmented space $A(Z)$ is a Gromov hyperbolic space. The ideal boundary of $A(Z)$ constructed by the geodesic bicombing coincides with the Gromov boundary of $A(Z)$ as a Gromov hyperbolic space.

12. Topological dimension of ideal boundaries

In this section, we study the topological dimensions of ideal boundaries of trees of spaces. Then we apply the results to the cohomological dimension of free products of groups.

12.1. **Topological dimension.**

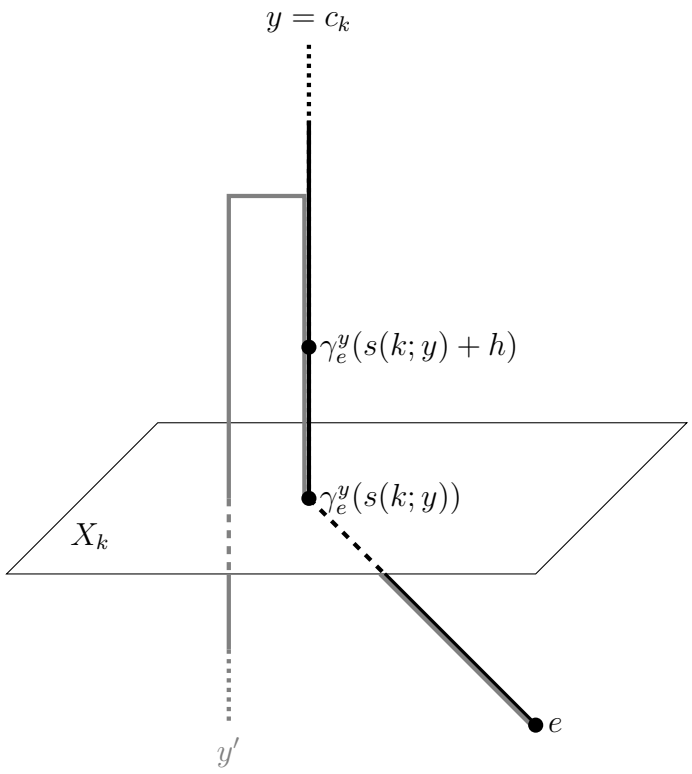


FIGURE 1. geodesics γ_e^y and $\gamma_e^{y'}$ where $y = c_k$

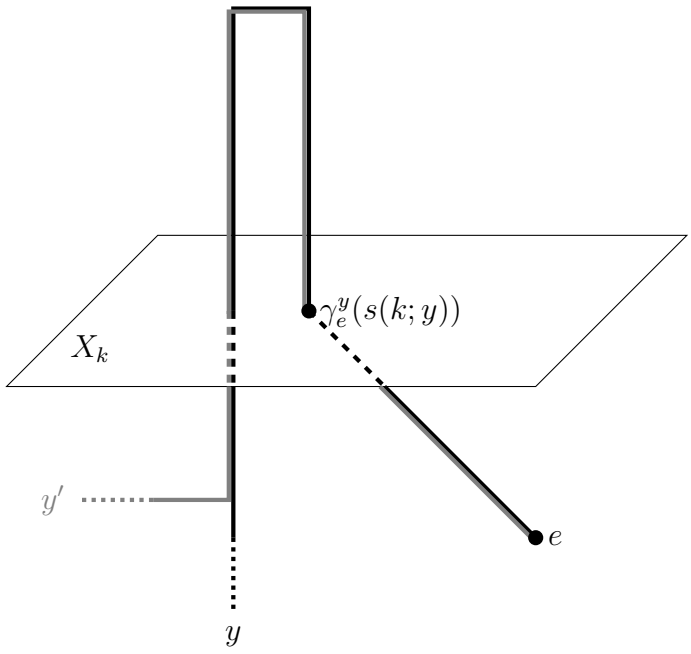


FIGURE 2. geodesics γ_e^y and $\gamma_e^{y'}$ where $y \neq c_k$

THEOREM 12.1. *Let (Z, d_Z) be a $(\xi, K, L, \{\gamma_k\}_{k\in K})$-tree of geodesic coarsely convex space. Suppose that Z is proper. For $k \in K$, set $X_k := \xi^{-1}(\mathsf{star}(k))$. Then,*

$$\dim(\partial Z) = \sup\{\dim(\partial X_k) : k \in K\}.$$

Theorem 1.4 is obtained by applying Theorem 12.1 to free products of metric spaces.

Let (Z, d_Z) be a $(\xi, K, L, \{\gamma_k\}_{k\in K})$-tree of geodesic coarsely convex space. We fix a bijection $k\colon \mathbb{N} \to K$. Let $A(Z)$ be the augmented space of Z.

For simplicity, we denote by γ an augmented bicombing on $A_n(Z)$ associated with the family $\{\gamma_k\}_{k\in K}$.

We introduce a new neighbourhood basis. For $x \in \partial A_n(Z)$ and $r \geq 0$, set

$$W_r[x] := \{y \in \partial A_n(Z)\colon (\gamma_e^x \mid \gamma_e^y) \geq r\}.$$

By (1) of Lemma 4.12, the family $\{W_r[x] : r > 0\}$ forms a neighbourhood basis of x.

LEMMA 12.2. *Let $A_n(Z)$ be the n-th augmented space as above.*

$$\dim \partial A_n(Z) \leq \max\{\dim \partial X_{k(i)} : 1 \leq i \leq n\}.$$

PROOF. We suppose that $N := \max\{\dim \partial X_{k(i)} : 1 \leq i \leq n\} < \infty$. Let $\mathcal{U} = \{U_\alpha\}_{\alpha\in\Lambda}$ be a finite open cover of $\partial A_n(Z)$. We can suppose without loss of generality that $\mathcal{U}$ consists of neighbourhood basis, that is, for $\alpha \in \Lambda$, there exists $(x_\alpha, n_\alpha) \in A_n(Z) \times \mathbb{N}$ such that

$$U_\alpha = W_{n_\alpha}[x_\alpha] = \{y \in \partial A_n(Z) : (\gamma_e^{x_\alpha} \mid \gamma_e^y) > n_\alpha\}.$$

Furthermore, since $A_n(Z)$ is normal, and each $\partial X_{k(i)}$ is closed, we can assume that for each $\alpha \in \Lambda$, there exists at most one $i \in \{1, \dots, n\}$ such that $U_\alpha \cap \partial X_{k(i)} \neq \emptyset$. Set

$$\Lambda_i := \{\alpha : U_\alpha \cap \partial X_{k(i)} \neq \emptyset\}.$$

We can assume that for $i \neq i'$ and for $\alpha \in \Lambda_i$ and $\alpha' \in \Lambda_{i'}$, we have

$$\overline{U_\alpha} \cap \overline{U_{\alpha'}} = \emptyset. \tag{7}$$

By taking a common refinement of the cover

$$\{U_\alpha : \alpha \in \Lambda_i\} \quad \text{and} \quad \{W_n[x] : (x, n) \in X_{k(i)} \times \mathbb{N}\},$$

we can assume that for $\alpha \in \Lambda_i$, we have $(x_\alpha, n_\alpha) \in X_{k(i)} \times \mathbb{N}$.

Since $\dim \partial X_{k(i)} \leq N$, the cover $\{U_\alpha : \alpha \in \Lambda_i\}$ has a refinement $\mathcal{V}_i$ of order $N + 1$.

Now set

$$F_i := \bigcup_{\alpha \in \Lambda_i} U_\alpha = \bigcup_{V \in \mathcal{V}_i} V, \qquad Z' := \partial A_n(Z) \setminus \left(\bigcup_{1 \leq i \leq n} F_i \right).$$

Since F_i's are open, Z' is closed, so Z' is a compact metrizable space.

Claim 12.3. *For each $i = 1, \dots, n$, F_i is closed. Thus, Z' is open.*

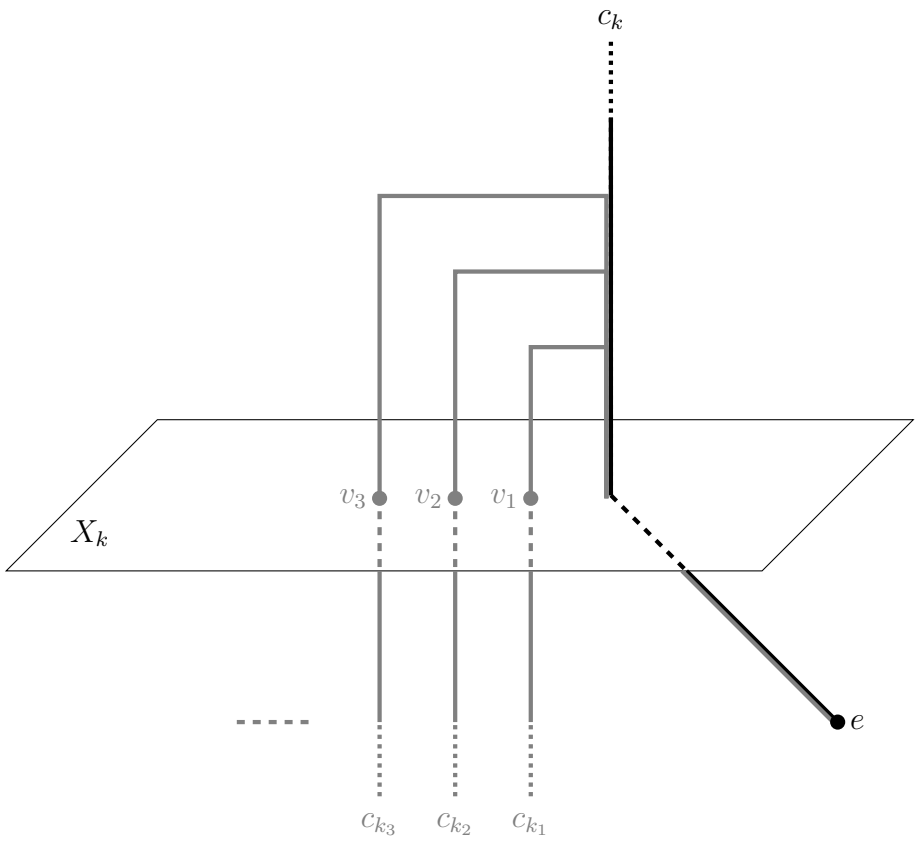


Figure 3. c_{k_n} converges to c_k

We will show the claim by a similar argument as in the proof of Lemma 11.2.

Let $x \in \partial A_n(Z) \setminus F_i$. We will show that there exists $T_x > 0$ such that $W_{T_x}[x] \cap F_i = \emptyset$. Since Λ_i is a finite set, it is enough to show that for each $\alpha \in \Lambda_i$, there exists $T_{x,\alpha} > 0$ such that $W_{2T_{x,\alpha}}[x] \cap U_\alpha = \emptyset$. We remark that $U_\alpha = W_{n_\alpha}[x_\alpha] \cap \partial A_n(Z)$ with $(x_\alpha, n_\alpha) \in X_k \times \mathbb{N}$.

For $k = k(i) \in K$ with $i \geq n$, let $c_k = c_{k(i)}$ be the center of the horoball $\mathcal{H}(X_k)$. First, we suppose that $x = c_k$ for some $k \in K \setminus \{k(i) : 1 \leq i \leq n\}$. Set

$$s(k; x) := \inf\{t \in \mathbb{R} : \mathtt{rp}\bar{\gamma}(e, x, t) \in X_k^{(0)}\}.$$

The geodesic $\mathtt{rp}\bar{\gamma}(\gamma_e^x(s(k;x)), x, -)$ consists of the vertical line. Set $T_{x,\alpha} := s(k;x) + 4D_1 + n_\alpha$. Then for $y \in W_{2T_{x,\alpha}}[x]$, we have

$$\mathtt{rp}\bar{\gamma}(e, x, t) = \mathtt{rp}\bar{\gamma}(e, y, t) \quad (\forall t \in [0, T_{x,\alpha}])$$

If $y \in U_\alpha = W_{n_\alpha}[x_\alpha]$, then

$$d_Z(\mathtt{rp}\bar{\gamma}(e, x_\alpha, n_\alpha), \mathtt{rp}\bar{\gamma}(e, x, n_\alpha)) = d_Z(\mathtt{rp}\bar{\gamma}(e, x_\alpha, n_\alpha), \mathtt{rp}\bar{\gamma}(e, y, n_\alpha)) \leq D_1.$$

Here the last inequality holds because $(\gamma_e^{x_\alpha} \mid \gamma_e^y) \geq n_\alpha$ and the map

$$t \mapsto d_Z(\mathtt{rp}\gamma(e, x_\alpha, t), \mathtt{rp}\bar{\gamma}(e, y, t))$$

is strictly increasing for $t \geq s(k;x)$. Thus, $(\gamma_e^x \mid \gamma_e^{x_\alpha}) \geq n_\alpha$. This contradicts $x \notin U_\alpha$. Therefore, $y \notin U_\alpha$, so we have

$$W_{2T_{x,\alpha}}[x] \cap U_\alpha = \emptyset.$$

Next, we suppose $x \neq c_k$ for all $k \in K$. Then there exists $k \in K \setminus \{k(i) : 1 \leq i \leq n\}$ such that

$$s(k; x) := \inf\{t \in \mathbb{R} : \mathtt{rp}\bar{\gamma}(e, x, t) \in X_k\}$$

is finite, and $d_Z(\mathtt{rp}\bar{\gamma}(e, x, s(k;x)), x_\alpha) > 2n_\alpha$. Set $T_{x,\alpha} = 2(s(k;x) + n_\alpha + D_1)$. Then for all $y \in W_{2T_{x,\alpha}}[x]$, we have

$$\mathtt{rp}\bar{\gamma}(e, x, t) = \mathtt{rp}\bar{\gamma}(e, y, t) \quad (\forall t \in [0, T_{x,\alpha}]).$$

By the same argument as before, $y \notin U_\alpha$, so we have $W_{2T_{x,\alpha}}[x] \cap U_\alpha = \emptyset$. This completes a proof of the claim.

Now by a similar argument as in Section 11, we see that Z' has a countable basis consisting of clopen sets. Thus, Z' is zero dimensional. So the open cover

$$\{U_\alpha \cap Z' : \alpha \notin \Lambda_i,\ 1 \leq i \leq n\}$$

has a refinement $\mathcal{V}_\infty$ of order 1. It follows that $\mathcal{U}$ has a refinement $\mathcal{V}_\infty \cup \bigcup_i \mathcal{V}_i$ of order $N+1$. Thus, $\dim \partial A_n(Z) \leq N$. □

Proof of Theorem 12.1. Since each ∂X_k is a closed subset, we have

$$\dim(\partial Z) \geq \sup\{\dim(\partial X_k) : k \in K\}.$$

By a formula for the topological dimensions of projective limits in Engelking's book [8, Theorem 1.13.4], we have

$$\dim(\partial Z) \leq \sup_n \dim \partial A_n(Z).$$

By Lemma 12.2, we have

$$\sup_n \dim \partial A_n(Z) \leq \sup_n \dim \partial X_{k(n)}.$$

This completes the proof. □

12.2. **Cohomological dimensions of free products of groups.** For a group G, we denote by $\operatorname{cd} G$ the cohomological dimension of G.

The following theorem relates the cohomological dimensions of group G acting on coarsely convex space X and the topological dimension of the ideal boundary ∂X.

THEOREM 12.4 ([12, Corollary 8.10]). *Let G be a group acting geometrically on a proper coarsely convex space X. If G admits a finite model for the classifying space BG, then*

$$\operatorname{cd} G = \dim \partial X + 1.$$

COROLLARY 12.5. *Let G and H be groups acting properly and cocompactly on geodesic coarsely convex spaces X and Y respectively. Suppose G and H admit finite models for the classifying spaces BG and BH respectively. Then we have*

$$\operatorname{cd}(G * H) = \max\{\operatorname{cd} G, \operatorname{cd} H\}.$$

PROOF. We consider the free product of the spaces $X * Y$ described in Example 7.10. Then the free product of the groups $G * H$ acts on $X * Y$ properly and cocompactly.

Since the free product $G * H$ is hyperbolic relative to $\{G, H\}$, by [12, Theorem A.1], $G * H$ admits a finite model for the classifying spaces $B(G * H)$. By Theorem 12.4, we have

$$\operatorname{cd}(G) = \dim \partial X + 1, \quad \operatorname{cd}(H) = \dim \partial Y + 1, \quad \operatorname{cd}(G * H) = \dim \partial(X * Y) + 1.$$

Therefore $\operatorname{cd}(G * H) = \max\{\operatorname{cd} G, \operatorname{cd} H\}$. □

We remark that Corollary 12.5 is a special case of the following more general formula for the cohomological dimensions of free products.

THEOREM 12.6 ([5, VIII (2.4) Proposition, (a) and (c)]). *Let G and H be groups. Let $G * H$ be the free product of G and H. Then we have*

$$\operatorname{cd}(G * H) = \max\{\operatorname{cd} G, \operatorname{cd} H\}.$$

Acknowledgment

We would like to thank the referee for carefully reading our manuscript and for providing valuable comments and suggestions, which helped us improve the paper.

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