On the defining matrices of Schubert varieties

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1 Introduction

Grassmannians and their Schubert subvarieties are fascinating objects of algebraic geometry and attracted many mathematicians. Their homogeneous coordinate rings are also attracting objects. They are commutative rings with combinatorial flavor. In fact, conbinatorial analysis of these rings is the germ of the theory of algebras with straightening law and Hodge algebras. They are generated by the maximal minors of certain matrices with university property and these matrices are also studied by many mathematicians.

In this article, we consider the group action on the rings generated by these matrices. And study the rings of (absolute) invariants, sagbi bases of these rings and properties of these rings together with the initial algebras of them. In the way, we define a new concept which we call “doset Hibi rings,” which are subrings of a Hibi ring of Veronese type, to analyze the initial algebra of the certain rings of invariants.

2 Preliminaries

In this article, all rings and algebras are assumed to be commutative with identity element. Let l, m, n be integers with $0 \leq l < m \leq n$ and k a field. For an $m \times n$ matrix M with entries in a k algebra S, we denote by $I_t(M)$ the ideal generated by the t-minors of M, by $M^{(\leq i)}$ the $i \times n$ matrix consisting of first i-rows of M, by $M_{\leq j}$ the $m \times j$ matrix consisting of first j-columns of M, by $\Gamma(M)$ the set of maximal minors of M and by $k[M]$ the k-algebra of S generated by the entries of M.

We define posets $\Gamma(m \times n)$ and $\Gamma'(m \times n)$ by $\Gamma(m \times n) := \{[c_1, \ldots, c_m] \mid 1 \leq c_1 < \cdots < c_m \leq n\}$ and $\Gamma'(m \times n) := \{[c_1, \ldots, c_r] \mid r \leq m, 1 \leq c_1 <$

$\cdots < c_r \leq n\}$. The order of $\Gamma'(m \times n)$ is defined by

$$\begin{array}{rl} & [c_1, \ldots, c_r] \leq [d_1, \ldots, d_s] \\ \overset{\text{def}}{\Longleftrightarrow} & r \geq s,\ c_i \leq d_i \text{ for } i = 1,\, 2,\, \ldots,\, s, \end{array}$$

and the order of $\Gamma(m \times n)$ is defined by that of $\Gamma'(m \times n)$. Note that $\Gamma'(m \times n)$ is a distributive lattice and $\Gamma(m \times n)$ is a sublattice of $\Gamma'(m \times n)$. For $[c_1, \ldots, c_r] \in \Gamma'(m \times n)$, we define its size to be r and denote it by $\text{size}[c_1, \ldots, c_r]$.

We also define the poset $\Delta(m \times n)$ by $\Delta(m \times n) := \{[\alpha|\beta] \mid \alpha, \beta \in \Gamma'(m \times n), \text{size}\alpha = \text{size}\beta\}$ and define the order of $\Delta(m \times n)$ by

$$\begin{array}{rl} & [\alpha|\beta] \leq [\alpha'|\beta'] \\ \overset{\text{def}}{\Longleftrightarrow} & \alpha \leq \alpha' \text{ in } \Gamma'(m \times m) \text{ and } \beta \leq \beta' \text{ in } \Gamma'(m \times n). \end{array}$$

Note also that $\Delta(m \times n)$ is a distributive lattice.

For $\gamma \in \Gamma(m \times n)$, we set $\Gamma(m \times n; \gamma) := \{\delta \in \Gamma(m \times n) \mid \delta \geq \gamma\}$. $\Gamma'(m \times n; \gamma)$ and $\Delta(m \times n; \delta)$ are defined similarly.

For an $m \times n$ matrix $M = (m_{ij})$ and $\delta = [\alpha|\beta] = [c_1, \ldots, c_r|d_1, \ldots, d_r] \in \Delta(m \times n)$, we denote the minor $\det(m_{c_i d_j})$ by δ_M or $[\alpha|\beta]_M$. We also denote the maximal minor $\det(m_{ic_j})$ of M by δ_M or $[c_1, \ldots, c_m]_M$, where $\delta = [c_1, \ldots, c_m] \in \Gamma(m \times n)$.

Now let V be an n-dimensional k-vector space. Then the set of m-dimensional subspaces of V has a structure of an algebraic variety, called the Grassmann variety. We denote this variety by $G_m(V)$. It is known that if $X = (X_{ij})$ is the $m \times n$ matrix of indeterminates, i.e., entries X_{ij} are independent indeterminates, then $k[\Gamma(X)]$ is the homogeneous coordinate ring of $G_m(V)$.

Now fix a complete flag $0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n = V$ of subspaces of V.

Definition 2.1 For $[a_1, \ldots, a_m] \in \Gamma(m \times n)$, we define $\Omega(a_1, \ldots, a_m) := \{W \in G_m(V) \mid \dim(W \cap V_{a_i}) \geq i \text{ for } i = 1,\, 2,\, \ldots,\, m\}$.

It is known that $\Omega(a_1, \ldots, a_m)$ is subvariety of $G_m(V)$ and called the Schubert subvariety of $G_m(V)$.

Set $b_i := n - a_{m-i+1} + 1$ for $i = 1, \ldots, m$ and $\gamma := [b_1, b_2, \ldots, b_m] \in \Gamma(m \times n)$. It is known that there is the universal $m \times n$ matrix Z_γ with $I_i((Z_\gamma)_{\leq b_i - 1}) = (0)$ for $i = 1, \ldots, m$. That is,

(1) $I_i((Z_\gamma)_{\leq b_i - 1}) = (0)$ for $i = 1, \ldots, m$ and

(2) if $I_i(M_{\leq b_i - 1}) = (0)$ for $i = 1, \ldots, m$ for a matrix with entries in some k-algebra S, then there is a unique k-algebra homomorphism $k[Z_\gamma] \longrightarrow S$ which maps Z_γ to M.

And it is also known that $k[Z_\gamma]$ is the homogeneous coordinate ring of $\Omega(a_1,\ldots,a_m)$.

Set

$$W := \begin{pmatrix} W_{11} & W_{12} & \cdots & W_{1m} \\ W_{21} & W_{22} & \cdots & W_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ W_{m1} & W_{m2} & \cdots & W_{mm} \end{pmatrix}$$

and

$$U_\gamma := \begin{pmatrix} 0 & \cdots & 0 & U_{1b_1} & \cdots & U_{1b_2-1} & U_{1b_2} & \cdots & \cdots & U_{1b_m} & \cdots & U_{1n} \\ 0 & \cdots & 0 & 0 & \cdots & 0 & U_{2b_2} & \cdots & \cdots & U_{2b_m} & \cdots & U_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & U_{mb_m} & \cdots & U_{mn} \end{pmatrix}.$$

where W_{ij} and U_{ij} are independent indeterminates. Then

Theorem 2.2 ([Miy]) *WU_γ has the universal property above.*

So we set $Z_\gamma := WU_\gamma$ in the following.

3 Sagbi bases of the rings of invariants

For any $g \in \mathrm{GL}(m,k)$, $I_i((gZ_\gamma)_{\leq b_i-1}) = (0)$ for $i = 1, \ldots, m$. So there is a k-automorphism of $k[Z_\gamma]$ sending Z_γ to gZ_γ, i.e., any subgroup of $\mathrm{GL}(m,k)$ acts on $k[Z_\gamma]$.

Definition 3.1 $f \in k[Z_\gamma]$ is called an absolute $\mathrm{SL}(m)$-invariant if f is an $\mathrm{SL}(m,B)$-invariant in $B[Z_\gamma]$ for any k-algebra B. The set of absolute $\mathrm{SL}(m)$ invariants is denoted by $k[Z_\gamma]^{\mathrm{SL}(m,-)}$. Absolute $\mathrm{O}(m)$ and $\mathrm{SO}(m)$ invariants are difined similarly.

In this section, we study the rings of invariants or absolute invariants of several subgroups of $\mathrm{SL}(m)$.

Set

$$G_1 := \left\{ \begin{pmatrix} E_l & O \\ * & P \end{pmatrix} \mid P \in \mathrm{SL}(m-l,k), \right\} \text{ and}$$
$$G_2 := \left\{ \begin{pmatrix} Q & O \\ * & P \end{pmatrix} \mid Q \in \mathrm{SL}(l,k), P \in \mathrm{SL}(m-l,k), \right\}.$$

First we recall the following results.

Theorem 3.2 *Let m' be an integer with $m' \geq m$ and X an $m \times m'$ matirx of indeterminates. Then*

(1) *(DeConcini-Procesi [DP])* $k[X]^{\mathrm{SL}(m,-)} = k[\Gamma(X)]$.

(2) *(Goto-Hayasaka-Kurano-Nakamura [GHKN]) If k is an infinite field, then* $k[X]^{G_1} = k[X^{(\leq l)}, \Gamma(X)]$.

(3) *(DeConcini-Procesi [DP])* $k[X]^{\mathrm{O}(m,-)} = k[{}^tXX]$.

(4) *(DeConcini-Procesi [DP])* $k[X]^{\mathrm{SO}(m,-)} = k[{}^tXX, \Gamma(X)]$.

Corollary 3.3 (1) $k[W]^{\mathrm{SL}(m,-)} = k[\det W]$.

(2) *If k is an infinite field, then* $k[W]^{G_1} = k[W^{(\leq l)}, \det W]$.

(3) $k[W]^{\mathrm{O}(m,-)} = k[{}^tWW]$.

(4) $k[W]^{\mathrm{SO}(m,-)} = k[{}^tWW, \det W]$.

Lemma 3.4 *If k is an infinite field, then* $k[W]^{G_2} = k[\Gamma(W^{\leq l}), \det W]$.

proof Since $G_1 \subset G_2$, we see that $k[W]^{G_2} \subset k[W]^{G_1} = k[W^{(\leq l)}, \det W]$. So if $f \in k[W]^{G_2}$, we can write

$$f = f_0 + f_1(\det W) + \cdots + f_u(\det W)^u,$$

where $f_i \in k[W^{(\leq l)}]$. Since $\det W$ is transcendental over $k[W^{(\leq l)}]$ and since $f \in k[W]^{G_2}$, we see that $f_i \in k[W]^{\mathrm{SL}(l,k)}$. for $i = 0, \ldots, u$. Therefore $f \in k[\Gamma(W^{(\leq l)})]$ for $i = 0, \ldots, u$ (see e.g. [BV]) and we see that $f \in k[\Gamma(W^{(\leq l)}), \det W]$. The opposite inclusion relation is clear. ∎

Now we introduce degree lexicographic order on the polynomial ring $k[W, U_\gamma]$ by $W_{11} > W_{21} > \cdots > W_{m1} > W_{12} > \cdots > W_{mm} > U_{1b_1} > U_{1b_1+1} > \cdots > U_{1n} > U_{2b_2} > \cdots > U_{mn}$. Then

Theorem 3.5 (1) $\{\delta_{Z_\gamma} \mid \delta \in \Gamma(m \times n; \gamma)\}$ *is a sagbi basis of* $k[\det W, U_\gamma] \cap k[Z_\gamma]$. *In particular,* $k[Z_\gamma]^{\mathrm{SL}(m,-)} = k[\det W, U_\gamma] \cap k[Z_\gamma] = k[\Gamma(Z_\gamma)]$.

(2) $\{\delta_{Z_\gamma} \mid \delta \in \Gamma(m \times n; \gamma)\} \cup \{[\alpha|\beta]_{Z_\gamma^{(\leq l)}} \mid [\alpha|\beta] \in \Delta(l \times n; [1, \ldots, l|b_1, \ldots, b_l])\}$ *is a sagbi basis of* $k[W^{(\leq l)}, \det W, U_\gamma] \cap k[Z_\gamma]$. *In particular,* $k[Z_\gamma]^{G_1} = k[W^{(\leq l)}, \det W, U_\gamma] \cap k[Z_\gamma] = k[Z_\gamma^{(\leq l)}, \Gamma(Z_\gamma)]$.

(3) $\{\delta_{Z_\gamma} \mid \delta \in \Gamma(m \times n; \gamma)\} \cup \{\delta'_{Z_\gamma^{(\leq l)}} \mid \delta' \in \Gamma(l \times n; [b_1, \ldots, b_l])\}$ *is a sagbi basis of* $k[\Gamma(W^{(\leq l)}), \det W, U_\gamma] \cap k[Z_\gamma]$. *In particular,* $k[Z_\gamma]^{G_2} = k[\Gamma(W^{(\leq l)}), \det W, U_\gamma] \cap k[Z_\gamma] = k[\Gamma(Z_\gamma^{(\leq l)}), \Gamma(Z_\gamma)]$.

(4) $\{[\alpha|\beta]_{{}^tZ_\gamma Z_\gamma} \mid \alpha, \beta \in \Gamma'(m \times n; \gamma),\ \alpha \leq \beta,\ \mathrm{size}\alpha = \mathrm{size}\beta\}$ *is a sagbi basis of* $k[{}^tWW, U_\gamma] \cap k[Z_\gamma]$. *In particular,* $k[Z_\gamma]^{\mathrm{O}(m,-)} = k[{}^tWW, U_\gamma] \cap k[Z_\gamma] = k[{}^tZ_\gamma Z_\gamma]$.

(5) $\{[\alpha|\beta]_{{}^tZ_\gamma Z_\gamma} \mid \alpha, \beta \in \Gamma'(m \times n; \gamma),\ \alpha \leq \beta,\ \mathrm{size}\alpha = \mathrm{size}\beta\} \cup \{\delta_{Z_\gamma} \mid \delta \in \Gamma(m \times n; \gamma)\}$ *is a sagbi basis of* $k[{}^tWW, \det W, U_\gamma] \cap k[Z_\gamma]$. *In particular,* $k[Z_\gamma]^{\mathrm{SO}(m,-)} = k[{}^tWW, \det W, U_\gamma] \cap k[Z_\gamma] = k[{}^tZ_\gamma Z_\gamma, \Gamma(Z_\gamma)]$.

4 Doset Hibi rings and their generalizations

In this section, we introduce a new concept "doset Hibi ring" and its generalization, which are certain subrings of a Hibi ring.

First we recall the concept of Hibi ring. Let H be a finite distributive lattice, P the set of join-irreducible elements of H (i.e., elements of H such that $x = \alpha \vee \beta \Rightarrow x = \alpha$ or $x = \beta$), $\{X_\alpha\}_{\alpha \in H}$ a family of indeterminates indexed by H and $\{T_a\}_{a \in P}$ a family of indeterminates indexed by P.

Definition 4.1 (Hibi [Hib]) $\mathcal{R}_k(H) := k[\prod_{x \leq \alpha} T_x \mid \alpha \in H]$.

Nowadays, $\mathcal{R}_k(H)$ is called the Hibi ring.

Theorem 4.2 (Hibi [Hib]) *$\mathcal{R}_k(H)$ is a homogeneous ASL (algebra with straightening law, ordinal Hodge algebra in the terminology of [DEP]) over k generated by H with straightening relation $\alpha\beta = (\alpha \wedge \beta)(\alpha \vee \beta)$ for any α, $\beta \in H$ with $\alpha \not\sim \beta$. That is $\mathcal{R}_k(H) \simeq k[X_\alpha \mid \alpha \in H]/(X_\alpha X_\beta - X_{\alpha\wedge\beta}X_{\alpha\vee\beta} \mid \alpha, \beta \in H)$ by $X_\alpha \mapsto \prod_{x \leq \alpha} T_x$.*

Set $\overline{\mathcal{T}}(P) := \{\nu\colon P \to \boldsymbol{N} \mid a \leq b \Rightarrow \nu(a) \geq \nu(b)\}$. Then

Theorem 4.3 (Hibi) *$\mathcal{R}_k(H)$ is a free k-module with basis $\{T^\nu \mid \nu \in \overline{\mathcal{T}}(P)\}$, where $T^\nu := \prod_{x \in P} T_x^{\nu(x)}$.*

Let us recall the definition of dosets ([DEP]). Set $\mathrm{Diag}_H := \{(\alpha, \alpha) \mid \alpha \in H\}$ and $\mathcal{O}_H := \{(\alpha, \beta) \mid \alpha, \beta \in H,\ \alpha \leq \beta\}$.

Definition 4.4 (DeConcini-Eisenbud-Procesi [DEP]) A set $D \subset H \times H$ is called a doset if

(1) $\mathrm{Diag}_H \subset D \subset \mathcal{O}_H$ and

(2) if $\alpha_1 \leq \alpha_2 \leq \alpha_3$, then

$$(\alpha_1, \alpha_3) \in D \Leftrightarrow (\alpha_1, \alpha_2) \in D \text{ and } (\alpha_2, \alpha_3) \in D.$$

Now let L be a distributive lattice, $\varphi\colon H \to L$ a surjective lattice homomorphism. We define $D := \{(\alpha, \beta) \mid \alpha \le \beta,\ \varphi(\alpha) = \varphi(\beta)\}$. Then it is easily verified that D is a doset.

Definition 4.5 Doset Hibi ring over k defined by H, φ is the subalgebra of $\mathcal{R}_k(H)$ generated by $\{\alpha\beta \mid (\alpha, \beta) \in D\}$.

Note that if (α, β), $(\alpha', \beta') \in D$, then $\alpha\beta\alpha'\beta' = (\alpha \wedge \alpha')(\alpha \vee \alpha')(\beta \wedge \beta')(\beta \vee \beta') = (\alpha \wedge \alpha')((\alpha \vee \alpha') \wedge (\beta \wedge \beta'))((\alpha \vee \alpha') \vee (\beta \wedge \beta'))(\beta \vee \beta')$ and $\varphi((\alpha \vee \alpha') \wedge (\beta \wedge \beta')) = (\varphi(\alpha) \vee \varphi(\alpha')) \wedge (\varphi(\beta) \wedge \varphi(\beta')) = (\varphi(\alpha) \vee \varphi(\alpha')) \wedge (\varphi(\alpha) \wedge \varphi(\alpha')) = \varphi(\alpha) \wedge \varphi(\alpha') = \varphi(\alpha \wedge \alpha')$ since $\varphi(\alpha) = \varphi(\beta)$ and $\varphi(\alpha') = \varphi(\beta')$. So $(\alpha \wedge \alpha', (\alpha \vee \alpha') \wedge (\beta \wedge \beta')) \in D$. And we see that $((\alpha \vee \alpha') \vee (\beta \wedge \beta'), \beta \vee \beta') \in D$ by the same way. Therefore by repeated application of this "straightening relation", we see that the doset Hibi ring is a free k-module with basis $\{\alpha_1\alpha_2 \cdots \alpha_{2r} \mid \alpha_1 \le \alpha_2 \le \cdots \le \alpha_{2r},\ (\alpha_{2i-1}, \alpha_{2i}) \in D \text{ for } i = 1, \ldots, r\}$.

Now let Q be the set of join-irreducible elements of L. For $x \in Q$, we set $\alpha_x := \bigwedge_{\beta \in \varphi^{-1}(x)} \beta$. It is directly verified that α_x is a join-irreducible element of H, i.e., $\alpha_x \in P$ for any $x \in Q$. So if we set $\overline{\mathcal{T}}' := \{\nu \in \overline{\mathcal{T}}(P) \mid \nu(\alpha_x) \equiv 0 \pmod 2 \text{ for } \forall x \in Q\}$, then

Theorem 4.6 *Doset Hibi ring is a free k-module with basis $\{T^\nu \mid \nu \in \overline{\mathcal{T}}'\}$.*

The key point of the proof is the fact that $\alpha_x \le \beta \Leftrightarrow x \le \varphi(\beta)$ for $\beta \in H$ and $x \in Q$.

Next we generalize the concept of doset Hibi rings. Suppose that L is decomposed into two sublattices as $L = L_1 \oplus L_2$, i.e., $L = L_1 \cup L_2$ (disjoint union) with L_i is join and meet closed for $i = 1, 2$ and $\xi \in L_1$, $\eta \in L_2 \Rightarrow \xi < \eta$.

Definition 4.7 The generalized doset Hibi ring over k defined by H, L_1, L_2 is the subalgebra of $\mathcal{R}_k(H)$ generated by $\{\alpha\beta \mid (\alpha, \beta) \in D\} \cup \{\alpha \mid \varphi(\alpha) \in L_1\}$.

It is easily verified that the generalized doset Hibi ring defined by H, L_1, L_2, is a free k-module with basis $\{\alpha_1\alpha_2 \cdots \alpha_r\beta_1\beta_2 \cdots \beta_{2s} \mid \alpha_1 \le \cdots \le \alpha_r \le \beta_1 \le \cdots \le \beta_{2s},\ \alpha_i \in L_1,\ \beta_j \in L_2 \text{ and } \varphi(\beta_{2j-1}) = \varphi(\beta_{2j})\}$.

Since L_i is a distributive lattice for $i = 1, 2$, we denote the set of join-irreducible elements of L_i by Q_i. Then if we put $\overline{\mathcal{T}}'' := \{\nu \in \overline{\mathcal{T}}(P) \mid \nu(\alpha_x) \equiv 0 \pmod 2 \text{ for } \forall x \in Q_2\}$, we see the following

Theorem 4.8 *Generalized doset Hibi ring is a free k-module with basis $\{T^\nu \mid \nu \in \overline{\mathcal{T}}''\}$.*

Hibi rings, doset Hibi rings and generalized doset Hibi rings are affine semigroup rings of $k[T_a \mid a \in P]$. Let us recall the following results of affine semigroup rings by Hochster.

Theorem 4.9 (Hochster [Hoc]) *Let S be a finitely generated submonoid of $\boldsymbol{N}^r$. Then*

(1) $k[S]$ is normal if and only if $S = \boldsymbol{R}_{\geq 0} S \cap \boldsymbol{Z} S$.

(2) If $k[S]$ is normal, the $k[S]$ is Cohen-Macaulay.

Corollary 4.10 *Hibi rings, doset Hibi rings and generalized doset Hibi rings are normal Cohen-Macaulay.*

Now we apply these results to the initial algebras of the rings of (absolute) invariants.

Set $L := \{1, 2, \ldots, m\}$ with reverse order. Then $\varphi :=$ size is a surjective lattice homomorphism from $\Gamma'(m \times n)$ to L. Set $L_1 = \{m\}$ and $L_2 = \{1, \ldots, m-1\}$. Then

Theorem 4.11 *(1) $\operatorname{in} k[\Gamma(Z_\gamma)]$ is the Hibi ring on $\Gamma(m \times n; \gamma)$.*

(2) $\operatorname{in} k[Z_\gamma^{(\leq l)}, \Gamma(Z_\gamma)]$ is the Hibi ring on $\Gamma(m \times n; \gamma) \cup \Delta(l \times n; [1, \ldots, l | b_1, \ldots, b_l])$.

(3) $\operatorname{in} k[\Gamma(Z_\gamma^{(\leq l)}), \Gamma(Z_\gamma)]$ is the Hibi ring on $\Gamma(m \times n; \gamma) \cup \Gamma(l \times n; [b_1, \ldots, b_l])$.

(4) $\operatorname{in} k[{}^t Z_\gamma Z_\gamma]$ is the doset Hibi ring on $\Gamma'(m \times n; \gamma)$ and L.

(5) $\operatorname{in} k[{}^t Z_\gamma Z_\gamma, \Gamma(Z_\gamma)]$ is the generalized doset Hibi ring on $\Gamma'(m \times n; \gamma)$ and L_1, L_2.

Corollary 4.12 *All the rings above and their original rings are normal and Cohen-Macaulay.*

5 Gorenstein property

In this section, we state criteria of Gorenstein property of the rings of (absolute) invariants.

First we recall the following result of Stanley.

Theorem 5.1 (Stanley [Sta]) *Let A be a graded Cohen-Macaulay domain over k. Then A is Gorenstein if and only if $H(A, \lambda^{-1}) = (-1)^d \lambda^\rho H(A, \lambda)$ for some $\rho \in \boldsymbol{Z}$, where $H(A, -)$ is the Hilbert series of A.*

Note that if A is a graded subring of a polynomial ring with monomial order, then A and $\operatorname{in} A$ have the same Hilbert function. So if $\operatorname{in} A$ is Cohen-Macaulay, then A is Gorenstein if and only if $\operatorname{in} A$ is Gorenstein.

Next we recall the Stanley's description of the canonical module of a normal affine semigroup ring.

Theorem 5.2 (Stanley [Sta]) *If S is a finitely generated submonoid of $\boldsymbol{N}^r$ such that $A = k[S]$ is normal, then $K_A = \bigoplus_{\nu \in \operatorname{relint} \boldsymbol{R}_{\geq 0} S \cap S} kT^\nu$, where K_A is the canonical module of A.*

Let H be a finite distributive lattice, P the set of join-irreducible elements of H. Set

$$\mathcal{T}(P) := \{\nu \in \overline{\mathcal{T}}(P) \mid a < b \Rightarrow \nu(a) > \nu(b) \text{ and } \nu(a) > 0 \text{ for } \forall a \in P\},$$

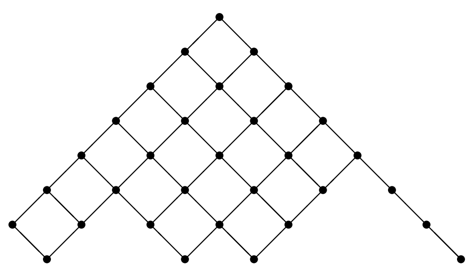
$\mathcal{T}' := \overline{\mathcal{T}}' \cap \mathcal{T}(P)$ and $\mathcal{T}'' := \overline{\mathcal{T}}'' \cap \mathcal{T}(P)$. Then with the notation defined in the previous section, the canonical module of the Hibi ring, the doset Hibi ring and the generalized doset Hibi ring are the free k-modules with basis $\{T^\nu \mid \nu \in \mathcal{T}(P)\}$, $\{T^\nu \mid \nu \in \mathcal{T}'\}$ and $\{T^\nu \mid \nu \in \mathcal{T}''\}$ respectively. In particular,

Corollary 5.3 (Hibi [Hib]) *$\mathcal{R}_k(H)$ is Gorenstein if and only if P is pure.*

Now let P_1 be the set of join-irreducible elements of $\Gamma(m \times n; \gamma)$, P_2 the set of join-irreducible elements of $\Gamma(m \times n; \gamma) \cup \Delta(l \times n; [1, \ldots, l | b_1, \ldots, b_l])$, P_3 the set of join-irreducible elements of $\Gamma(m \times n; \gamma) \cup \Gamma(l \times n; [b_1, \ldots, b_l])$, P_4 the set of join-irreducible elements of $\Gamma'(m \times n; \gamma)$.

An element $[c_1, \ldots, c_m] \in \Gamma(m \times n; \gamma)$ is a non-minimal join-irreducible element of $\Gamma(m \times n; \gamma)$ if and only if there is unique i such that $c_i > b_i$ and $c_i > c_{i-1} + 1$, where $c_0 := 0$. For such $[c_1, \ldots, c_m]$, set $p := n - c_i - (m - i) = |\{j \mid j > c_i, j \notin \{c_1, \ldots, c_m\}\}|$ and $q := i - 1 = |\{j \mid j < c_i, j \in \{c_1, \ldots, c_m\}\}|$. This makes an order reversing map form $P_1 \setminus \{\text{unique minimal element}\}$ to $\boldsymbol{N} \times \boldsymbol{N}$.

Example 5.4 Let $m = 7$, $n = 15$ and $\gamma = [1, 5, 6, 7, 9, 12, 13]$. Then the Hasse diagram of $P_1 \setminus \{\gamma\}$ is the following.



$[1,5,6,11,12,13,14]$ is a join-irreducible element with $i=4$, $p=1$, $q=3$.

Minimal elements of $P_1 \setminus \{\gamma\}$ are $[1,5,6,7,9,12,14]$, $[1,5,6,7,10,12,13]$, $[1,5,6,8,9,12,13]$ and $[2,5,6,7,9,12,13]$.

Set $\{u \mid b_u + 1 < b_{u+1}\} = \{u_1, \ldots, u_t\}$ with $u_1 < \cdots < u_t$, where $b_{m+1} := n+1$ and $\chi_0 := \{1, 2, \ldots, b_1 - 1\}$, $B_1 := \{b_1, b_2, \ldots, b_{u_1}\}$, $\chi_1 := \{b_{u_1}+1, b_{u_1}+2, \ldots, b_{u_1+1}-1\}$, $B_2 := \{b_{u_1+1}, b_{u_1+2}, \ldots, b_{u_2}\}$, $\chi_2 := \{b_{u_2}+1, b_{u_2}+2, \ldots, b_{u_2+1}-1\}$, $B_3 := \{b_{u_2+1}, b_{u_2+2}, \ldots, b_{u_3}\}$, $\ldots$, $B_t := \{b_{u_{t-1}+1}, b_{u_{t-1}+2}, \ldots, b_{u_t}\}$, $\chi_t := \{b_{u_t}+1, b_{u_t}+2, \ldots, b_{u_t+1}-1\}$ and $B_{t+1} := \{b_{u_t+1}, b_{u_t+2}, \ldots, b_m\}$.

Example 5.5 For the above example $t=4$, $u_1=1$, $u_2=4$, $u_3=5$, $u_4=7$ and $\chi_0=\emptyset$, $B_1=\{1\}$, $\chi_1=\{2,3,4\}$, $B_2=\{5,6,7\}$, $\chi_2=\{8\}$, $B_3=\{9\}$, $\chi_3=\{10,11\}$, $B_4=\{12,13\}$, $\chi_4=\{14,15\}$ and $B_5=\emptyset$.

So

Theorem 5.6 (Svanes [Sva]) *$k[\Gamma(Z_\gamma)]$ is Gorenstein if and only if $|B_i| = |\chi_{i-1}|$ for $i = 2, 3, \ldots, t$, i.e., $n - b_{u_i} - m + 2u_i - 2$ is constant.*

Consider P_3 and P_2 next. Non-minimal elements of P_3 are

(1) non-minimal join-irreducible element of P_1,

(2) join-irreducible element $[c_1, \ldots, c_l]$ of $\Gamma(l \times n; [b_1, \ldots, b_l])$ with $c_l > n - m + l$ or

(3) $[b_1, \ldots, b_l]$.

These are also elements of P_2. And P_2 has elements of fourth type.

(4) Join-irreducible element of $\Delta(l \times n; [1, \ldots, l | b_1, \ldots, b_l])$ with size $\leq l-1$.

So by considering the purity of P_2 and P_3, we see the following

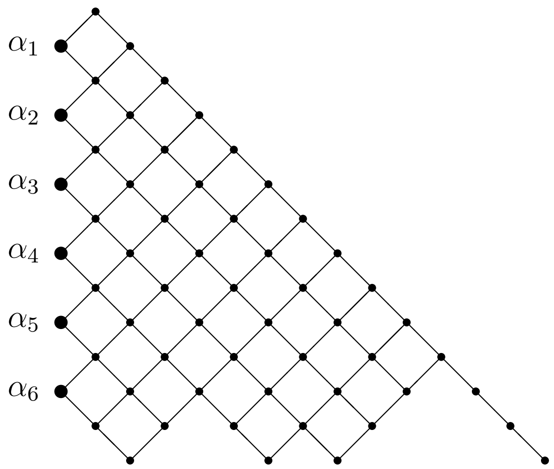
Theorem 5.7 *(1) If $u_t > l$, then $k[Z_\gamma^{(\leq l)}, \Gamma(Z_\gamma)]$ is Gorenstein if and only if $|B_i| = |\chi_{i-1}|$ for $i = 2, 3, \ldots, t$.*

(2) If $u_t \leq l$, then $k[Z_\gamma^{(\leq l)}, \Gamma(Z_\gamma)]$ is Gorenstein if and only if $m - l = n - b_{u_i} - m + 2u_i - 1$ for any $i = 1, 2, \ldots, t$.

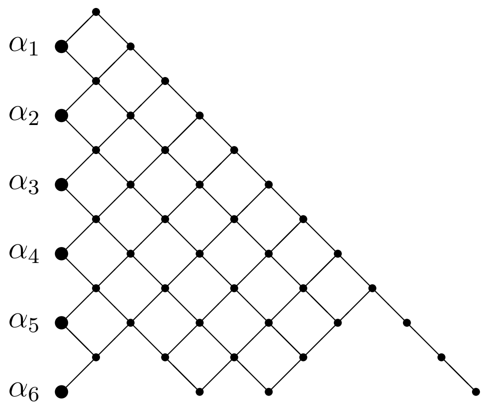
Theorem 5.8 *$k[\Gamma(Z_\gamma^{(\leq l)}), \Gamma(Z_\gamma)]$ is Gorenstein if and only if $k[Z_\gamma^{(\leq l)}, \Gamma(Z_\gamma)]$ is Gorenstein.*

Now consider the Gorenstein property of the rings of absolute O(m) and SO(m)-invariants. The Hasse diagram of P_4 is the one adding some planer lattice upper left side of the Hasse diagram of P_1.

Example 5.9 Let $m = 7$, $n = 15$ and $\gamma = [1, 5, 6, 7, 9, 12, 13]$. Then the Hasse diagram of $P_4 \setminus \{\gamma\}$ is the following.



Example 5.10 Change n to 13 and keep m and γ to be the same. That is, $m = 7$, $n = 13$ and $\gamma = [1, 5, 6, 7, 9, 12, 13]$. Then the Hasse diagram of $P_4 \setminus \{\gamma\}$ is the following.



By examining the positions of α_i ($i = 1, \ldots, m-1$) and γ, we see the following results.

Theorem 5.11 (Conca [Con]) *(1) If $b_m < n$, then $k[{}^tZ_\gamma Z_\gamma]$ is Gorenstein if and only if $|B_i| = |\chi_{i-1}|$ for $i = 2, 3, \ldots, t$ and $|\chi_t| \equiv 1 \pmod 2$.*

(2) If $b_m = n$, then $k[{}^t Z_\gamma Z_\gamma]$ is Gorenstein if and only if $|B_i| = |\chi_{i-1}|$ for $i = 2, 3, \dots, t$ and $|B_{t+1}| = |\chi_t| - 1$.

Theorem 5.12 *(1) If $b_m < n$, then $k[{}^t Z_\gamma Z_\gamma, \Gamma(Z_\gamma)]$ is Gorenstein if and only if $|B_i| = |\chi_{i-1}|$ for $i = 2, 3, \dots, t$.*

(2) If $b_m = n$, then $k[{}^t Z_\gamma Z_\gamma, \Gamma(Z_\gamma)]$ is Gorenstein if and only if $|B_i| = |\chi_{i-1}|$ for $i = 2, 3, \dots, t+1$.

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