GENERALIZED COHEN-MACAULAY MONOMIAL IDEALS

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ABSTRACT. We give a generalization of Hochster's formula for local cohomologies of square-free monomial ideals to monomial ideals, which are not necessarily square-free. Using this formula, we give combinatorial characterizations of generalized Cohen-Macaulay monomial ideals. We also give other applications of the generalized Hochster's formula.

INTRODUCTION

Let K be a field and let $S = K[X_1, \ldots, X_n]$ be a polynomial ring with the standard grading. For a graded ideal $I \subset S$ we set $R = S/I$. We denote by x_i the image of X_i in R for $i = 1, \ldots, n$ and set $\mathfrak{m} = (x_1, \ldots, x_n)$, the unique graded maximal ideal. Also $H^i_{\mathfrak{m}}(R)$ denotes the local cohomology module of R with regard $\mathfrak{m}$. A residue class ring R is called a *generalized Cohen-Macaulay* ring (*generalized CM* ring), or *FLC* (Finite Length Cohomology) ring, if $H^i_{\mathfrak{m}}(R)$ has finite length for $i \neq \dim R$. In this case, we will call the ideal $I \subset S$ a *generalized CM ideal*.

As defining ideals of algebraic sets, we can find many examples of generalized CM ideals such as homogeneous coordinate rings of non-singular projective varieties. For monomial ideals, which are not directly related to algebraic sets, the notions of generalized CM rings and Buchsbaum rings [8] coincide in the square-free case and the combinatorial characterization of generalied CM square-free monomial ideals (Stanley-Reisner ideals) has been given in terms Buchsbaum simplicial complexes [6, 7, 8]. However, as far as the author is concerned, the case of non-square-free monomial ideals has not been studied very much, and the aim of this paper is to give combinatorial characterizations of generalized CM monomial ideals, which are not always square-free. Since we cannot use the geometric language of simplicial complexes as in the squarefree case, we will consider the characterization in terms of the exponents of variables in the monomial generators.

We first give a generalization of Hochster's formula on local cohomologies for square-free monomial ideals [5] to monomial ideals that are not necessarily square-free (Theorem 1). From this formula, we can easily deduce several already known and probably new facts on vanishing degrees of local cohomologies. In particular, the vanishing degrees of generalized CM monomial ideals (Proposition 1). This result allows us to deduce combinatorial characterizations of generalized CM monomial ideals in terms of the exponents of variables in the monomial generators (Theorem 2, Corollary 7 and Theorem 3). On the other hand, thanks to the generalized

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Hochster's formula we can compare local cohomologies for I and its radical $\sqrt{I}$ (Proposition 3 and 4), which, together with the combinatorial characterization of generalized CM property, suggests a method to construct generalized CM monomial ideals from Buchsbaum Stanley-Reisner ideals. Namely, by changing a square-free generator $X_{i_1} \cdots X_{i_\ell}$ of a Buchsbaum Stanley-Reisner ideal J to a monomial $X_{i_1}^{a_{i_1}} \cdots X_{i_\ell}^{a_\ell}$ ($a_j \in \mathbb{N}, j = 1, \dots, \ell$), we make a generator of a generalized CM monomial ideal I with $\sqrt{I} = J$, and the combinatorial characterization shows a right choice of the exponents a_j.

One way of the construction is changing all the occurrences of the variable X_i in the minimal set of generators to $X_i^{a_i}$ with a fixed exponent a_i, $i = 1, \dots, n$ (Example 1). In some specific case, we can show, using our combinatorial characterization, that this is the only way of construction (Example 2).

For a finite set S we denote by $\mid S \mid$ the cardinarity of S, and, for sets A and B, $A \subset B$ means that A is a subset of B, which may be equal to A. The author thanks Jürgen Herzog for valuable discussions and detailed comments on the early version of the paper.

The contents of this manuscript is essentially the same as the author's already published paper [9]. But there are a few improvements for this manuscript. First of all, the description of Theorem 2, Corollary 7 and Theorem 3 has been much improved by introducing the new combinatorial notions of *test pairs* and *purity condition*, which are extensions of the familiar notions of faces and purity of Stanley-Reisner simplicial complexes. Also Corollary 8 has been newly added. Although this is a well-known result, this shows that Theorem 3 is a natural extension of this familiar result in the squre-free case.

1. Local cohomologies of monomial ideals

1.1. Generalized Hochster's Formula. In this subsection, we give a natural extension of Hochster's formula on local cohomologies of Stanley-Reisiner ideals to monomial ideals. The proof goes along almost the same line as that for Stanley-Reisner ideals given, for example, in [3] chapter 5.3. But we will give a full detail for the readers' convenience.

Let $I \subset S$ be a monomial ideal, which is not necessarily square-free. Then we have

$$H^i_{\mathfrak{m}}(R) \cong H^i(C^\bullet)$$

where $C^\bullet$ is the Čech complex defined as follows:

$$C^\bullet : 0 \longrightarrow C^0 \longrightarrow C^1 \longrightarrow \cdots \longrightarrow C^n \longrightarrow 0, \qquad C^t = \bigoplus_{1 \le i_1 < \cdots < i_t \le n} R_{x_{i_1} \cdots x_{i_t}}.$$

and the differential $C^t \longrightarrow C^{t+1}$ of this complex is induced by

$$(-1)^s nat : R_{x_{i_1} \cdots x_{i_t}} \longrightarrow R_{x_{j_1} \cdots x_{j_{t+1}}} \qquad \text{with } \{i_1, \dots, i_t\} = \{j_1, \dots, \hat{h}_s, \dots, j_{t+1}\}$$

where *nat* is the natural homomorphism to localized rings and $R_{x_{i_1} \cdots x_{i_t}}$, for example, denotes localization by $x_{i_1}, \dots, x_{i_t}$.

We can consider a $\mathbb{Z}^n$-grading to $H^i_{\mathfrak{m}}(R)$, $C^\bullet$ and $R_{x_{i_1}\cdots x_{i_t}}$ induced by the multi grading of S. See for example [3] for more detailed information about this complex.

Now we will consider the degree a subcomplex $C^\bullet_a$ of $C^\bullet$ for any $a \in \mathbb{Z}^n$. Before that we will prepare the notation. For a monomial ideal $I \subset S$, we denote by $G(I)$ the minimal set of monomial generators. Let $u = X_1^{a_1} \cdots X_n^{a_n}$ be a monimial with $a_i \geq 0$ for all i, then we define $\nu_j(u) = a_j$ for $j = 1, \dots, n$, and $\mathrm{supp}(u) = \{i \mid a_i \neq 0\}$. We set $G_a = \{i \mid a_i < 0\}$ and $H_a = \{i \mid a_i > 0\}$ for $a \in \mathbb{Z}^n$.

Lemma 1. *Let $x = x_{i_1} \cdots x_{i_r}$ with $i_1 < \cdots < i_r$ and set $F = \mathrm{supp}(x)$. For all $a \in \mathbb{Z}^n$ we have $\dim_K(R_x)_a \leq 1$ and the following are equivalent*

(*i*) $(R_x)_a \cong K$

(*ii*) $F \supset G_a$ *and for all* $u \in G(I)$ *there exists* $j \notin F$ *such that* $\nu_j(u) > a_j \geq 0$.

Notice that the condition $a_i \geq 0$ in (ii) is redundant because this follows from the condition $F \supset G_a$. But it is written for the readers' convenience.

Proof. The proof of $\dim_K(R_x)_a \leq 1$ is verbatim the same as that of Lemma 5.3.6 (a) in [3]. Now we assume (i), i.e., $(R_x)_a \neq 0$. This is equivalent to the condition that there exists a monomial $\sigma \in R$ and $\ell \in \mathbb{N}$ such that

(*a*) $x^m \sigma \neq 0$ for all $m \in \mathbb{N}$, and

(*b*) $\deg \dfrac{\sigma}{x^\ell} = a$,

where deg denotes the multidegree. We know from (b) that we have $F \supset G_a$ because a negative degree $a_i(< 0)$ in a must come from the denominator of the fraction σ/x^ℓ and $F = \mathrm{supp}(x^\ell)$. Now we know that (a) is equivalent to the following condition: for all $u \in G(I)$ and for all $m \in \mathbb{N}$ we have $u \not| (X_{i_1}^m \cdots X_{i_r}^m)(X_1^{b_1} \cdots X_n^{b_n})$ where we set $\sigma = x_1^{b_1} \cdots x_n^{b_n}$ with some integers $b_j \geq 0$, $j = 1, \dots, n$. Namely, for all $u \in G(I)$ there exists $i \notin F$ such that $\nu_i(u) > b_i$. Furthermore, we know from the condition $F \supset G_a$ that we have $a_i = b_i$ for $i \notin F$ since by (b) non-negative degrees in a must come from σ. Consequently we obtain (ii).

Now we show the converse. Assume that we have (ii). Set $\tau = \prod_{i \in H_a} x_i^{a_i}$ and $\rho = \prod_{i \in G_a} x_i^{-a_i}$. Then since $F \supset G_a$ there exists $\ell \in \mathbb{N}$ and a monomial σ in R such that

$$x^\ell = \rho\sigma \tag{1}$$

Now we show that $\dfrac{\sigma\tau}{x^\ell} \neq 0$ in R_x. $\dfrac{\sigma\tau}{x^\ell} \neq 0$ is equivalent to the condition that $x^m(\sigma\tau) \neq 0$ for all $m \in \mathbb{N}$. As in the above discussion, this is equivalent to the condition

$$\text{for all } u \in G(I) \text{ there exists } i \notin F \text{ such that } \nu_i(u) > b_i \tag{2}$$

where we set $\sigma\tau = x_1^{b_1} \cdots x_n^{b_n}$ for some integers $b_j \geq 0$, $j = 1, \dots, n$. But by (1) we have $i \notin \mathrm{supp}(\sigma)$ for $i \notin F$, so that $b_i = \nu_i(\tau) = a_i(> 0)$ (i.e., $i \in H_a$) or $a_i = b_i = 0$ (i.e., $i \notin H_a \cup G_a$). Hence we can replace "$\nu_i(u) > b_i$" in (2) by "$\nu_i(u) > a_i \geq 0$"

and then (2) is assured by the assumption. Thus we have $\dfrac{\sigma\tau}{x^\ell} \neq 0$ in R_x. Therefore

$$\deg \frac{\sigma\tau}{x^\ell} = \deg \frac{\sigma\tau}{\rho\sigma} = \deg \prod_{i \in H_a \cup G_a} x_i^{a_i} = \deg x^a = a$$

as requied. $\square$

Let $a \in \mathbb{Z}^n$. By Lemma 1 we see that $(C^i)_a$ has a basis

$$\left\{ b_F : \begin{array}{l} F \supset G_a,\ |F| = i, \\ \text{and for all } u \in G(I) \text{ there exists } j \notin F \text{ such that } \nu_j(u) > a_j \geq 0 \end{array} \right\}.$$

Restricting the differentation of $C^\bullet$ to the ath graded piece, we obtain a complex $(C^\bullet)_a$ of finite dimensional K-vector spaces with differentation $\partial : (C^i)_a \longrightarrow (C^{i+1})_a$ given by $\partial(b_F) = \sum (-1)^{\sigma(F,F')} b_{F'}$ where the sum is taken over all F' such that $F' \supset F$ with $|F'| = i+1$ and for all $u \in G(I)$ there exists $j \notin F'$ such that $\nu_j(u) > a_j \geq 0$. Also we define $\sigma(F, F') = s$ if $F' = \{j_0, \ldots, j_i\}$ and $F = \{j_0, \ldots, \hat{j}_s, \ldots, j_i\}$. Then we describe the ath component of the local cohomology in terms of this subcomplex: $H^i_{\mathfrak{m}}(R)_a \cong H^i(C^\bullet) = H^i(C^\bullet_a)$.

Now we fix our notation on simplicial complex. A simplicial complex Δ on a finite set $[n] = \{1, \ldots, n\}$ is a collection of subsets of $[n]$ such that $F \in \Delta$ whenever $F \subset G$ for some $G \in \Delta$. Notice that, for the convenience in the later discussions, we do not assume the condition that $\{i\} \in \Delta$ for $i = 1, \ldots, n$. We define $\dim F = i$ if $| F |= i+1$ and $\dim \Delta = \max\{\dim F \mid F \in \Delta\}$, which will be called the dimension of Δ. If we assume a linear order on $[n]$, say $1 < 2 < \cdots < n$, then we will call Δ *oriented*, and in this case we always denote an element $F = \{i_1, \ldots, i_k\} \in \Delta$ with the orderd sequence $i_1 < \ldots < i_k$. For a given oriented simplicial complex of dimension $d-1$, we denote by $\mathcal{C}(\Delta)$ the augumented oriented chain complex of Δ:

$$\mathcal{C}(\Delta) : 0 \longrightarrow \mathcal{C}_{d-1} \xrightarrow{\partial} \mathcal{C}_{d-2} \longrightarrow \cdots \longrightarrow \mathcal{C}_0 \xrightarrow{\partial} \mathcal{C}_{-1} \longrightarrow 0$$

where

$$\mathcal{C}_i = \bigoplus_{F \in \Delta, \dim F = i} \mathbb{Z}F \qquad \text{and} \qquad \partial F = \sum_{j=0}^{i} (-1)^j F_j$$

for all $F \in \Delta$. Here we define $F_j = \{i_0, \ldots, \hat{i}_j, \ldots, i_k\}$ for $F = \{i_0, \ldots, i_k\}$. Now for an abelian group G, we define the ith reduced simplicial homology $\tilde{H}_i(\Delta; G)$ of Δ to be the ith homology of the complex $\mathcal{C}(\Delta) \otimes G$ for all i. Also we define the ith reduced simplicial cohomology $\tilde{H}^i(\Delta; G)$ of Δ to be the ith cohomology of the dual chain complex $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{C}(\Delta), G)$ for all i. Notice that we have

$$\tilde{H}_{-1}(\Delta; G) = \begin{cases} G & \text{if } \Delta = \{\emptyset\} \\ 0 & \text{otherwise} \end{cases},$$

and if $\Delta = \emptyset$ then $\dim \Delta = -1$ and $\tilde{H}_i(\Delta; G) = 0$ for all i.

Now we will establish an isomorphism between the complex $(C^\bullet)_a$, $a \in \mathbb{Z}^n$, and a dual chain complex. For any $a \in \mathbb{Z}^n$, we define a simplicial complex

$$\Delta_a = \left\{ F - G_a \mid \begin{array}{l} F \supset G_a, \text{ and} \\ \text{for all } u \in G(I) \text{ there exists } j \notin F \text{ such that } \nu_j(u) > a_j \geq 0 \end{array} \right\}.$$

Notice that we may have $\Delta_a = \emptyset$ for some $a \in \mathbb{Z}^n$.

Lemma 2. *For all $a \in \mathbb{Z}^n$ there exists an isomorphism of complexes*

$$\alpha^\bullet : (C^\bullet)_a \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathcal{C}(\Delta_a)[-j-1], K) \qquad j = |G_a|$$

where $\mathcal{C}(\Delta_a)[-j-1]$ means shifting the degree of $\mathcal{C}(\Delta_a)$ by $-j-1$.

Proof. The assignment $F \mapsto F - G_a$ induces an isomorphism $\alpha^\bullet : (C^\bullet)_a \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathcal{C}(\Delta_a)[-j-1], K)$ of K-vector spaces such that $b_F \mapsto \varphi_{F-G_a}$, where

$$\varphi_{F'}(F") = \begin{cases} 1 & \text{if } F' = F" \\ 0 & \text{otherwise.} \end{cases}$$

That this is a homomorphism of complexes can be checked in a straightforward way. □

Now we come to the main theorem in this section.

Theorem 1. *Let $I \subset S = K[X_1, \ldots, X_n]$ be a monomial ideal. Then the multigraded Hilbert series of the local cohomology modules of $R = S/I$ with respect to the $\mathbb{Z}^n$-grading is given by*

$$\mathrm{Hilb}(H^i_{\mathfrak{m}}(R), \mathbf{t}) = \sum_{F \in \Delta} \sum \dim_K \tilde{H}_{i-|F|-1}(\Delta_a; K) \mathbf{t}^a$$

where $\mathbf{t} = t_1 \cdots t_n$, the second sum runs over $a \in \mathbb{Z}^n$ such that $G_a = F$ and $a_j \leq \rho_j - 1$, $j = 1, \ldots, n$, with $\rho_j = \max\{\nu_j(u) \mid u \in G(I)\}$ for $j = 1, \ldots, n$, and Δ is the simplicial complex corresponding to the Stanley-Reisner ideal $\sqrt{I}$.

Proof. By Lemma 2 and universal coefficient theorem for simplicial (co)homology, we have

$$\begin{aligned} \mathrm{Hilb}(H^i_{\mathfrak{m}}(R), \mathbf{t}) &= \sum_{a \in \mathbb{Z}^n} \dim_K H^i_{\mathfrak{m}}(R)_a \mathbf{t}^a = \sum_{a \in \mathbb{Z}^n} \dim_K H^i(C^\bullet_a) \mathbf{t}^a \\ &= \sum_{a \in \mathbb{Z}^n} \dim_K \tilde{H}_{i-|G_a|-1}(\Delta_a; K) \mathbf{t}^a. \end{aligned}$$

It is clear from the definition that $\Delta_a = \emptyset$ if for all $j \notin G_a$ we have $a_j \geq \rho_j$. Moreover for all $a \in \mathbb{Z}^n$ with $a_j \geq \rho_j$ for at least one index $j \notin G_a$ we have $\dim_K \tilde{H}_{i-|G_a|-1}(\Delta_a; K) = 0$. To prove this fact we can assume without loss of generality that $a_1 \geq \rho_1$ and that $\Delta_a \neq \emptyset$. Then we have $1 \notin G_a$, and, for all $\sigma = (L - G_a) \in \Delta_a$ with $L \supset G_a$ and $1 \notin \sigma$, we have $\sigma \cup \{1\} \in \Delta_a$. In fact, since we have $\nu_1(u) \leq a_1$ for all $u \in G(I)$ the existence of $k \notin L$ with $\nu_k(u) > a_k$ implies

$k \notin L \cup \{1\}$. Consequently we know that Δ_a is a cone by the vertex $\{1\}$ so that, as is well known, we have $\tilde{H}_{i-|G_a|-1}(\Delta_a; K) = 0$ for all i as required. Thus we obtain

$$\operatorname{Hilb}(H^i_{\mathfrak{m}}(R), \mathbf{t}) = \sum_{\substack{a \in \mathbb{Z}^n \\ a_j \le \rho_j - 1 \\ j=1,\dots,n}} \dim_K \tilde{H}_{i-|G_a|-1}(\Delta_a; K)\mathbf{t}^a.$$

Now if $\Delta_a \neq \emptyset$, we must have $(G_a - G_a =)\ \emptyset \in \Delta_a$, i.e., for all $u \in G(I)$ there exists $j \notin G_a$ such that $\nu_j(u) > a_j \ge 0$, and this implies that $G_a \not\supset \operatorname{supp}(u)$ for all $u \in G(I)$, namely G_a is not a non-face of Δ, i.e., $G_a \in \Delta$. Thus we finally obtain the required formula. □

The original Hochster's formula is a special case of Theorem 1.

Corollary 1 (Hochster)**.** *Let Δ be a simplicial complex and let $K[\Delta]$ be the Stanley-Reisner ring corresponding to Δ. Then we have*

$$\operatorname{Hilb}(H^i_{\mathfrak{m}}(K[\Delta]), \mathbf{t}) = \sum_{F \in \Delta} \dim_K \tilde{H}_{i-|F|-1}(\operatorname{lk}_\Delta F; K) \prod_{j \in F} \frac{t_j^{-1}}{1 - t_j^{-1}},$$

where $\operatorname{lk}_\Delta F = \{G | F \cup G \in \Delta, F \cap G = \emptyset\}$.

Proof. By Theorem 1 we have

$$\operatorname{Hilb}(H^i_{\mathfrak{m}}(R), \mathbf{t}) = \sum_{F \in \Delta} \sum_{\substack{a \in \mathbb{Z}^n_- \\ G_a = F}} \dim_K \tilde{H}_{i-|F|-1}(\Delta_a; K)\mathbf{t}^a$$

where $\mathbb{Z}^n_- = \{a \in \mathbb{Z}^n | a_j \le 0 \text{ for } j = 1, \dots, n\}$ and

$$\begin{aligned} \Delta_a &= \left\{ F - G_a \mid \begin{array}{c} F \supset G_a, \text{ and for all } u \in G(I) \text{ there exists } j \notin F \\ \text{such that } j \in \operatorname{supp}(u) \text{ and } j \notin H_a \cup G_a \end{array} \right\}. \\ &= \{F - G_a \mid F \supset G_a, \text{ and for all } u \in G(I) \text{ we have } H_a \cup F \not\supset \operatorname{supp}(u)\}. \\ &= \{L \mid L \cap G_a = \emptyset, L \cup G_a \cup H_a \in \Delta\} = \operatorname{lk}_{\operatorname{st} H_a} G_a. \end{aligned}$$

Then the rest of the proof is exactly as in Theorem 5.3.8 [3]. □

1.2. **Vanishing degrees of local cohomolgies.** In this subsection, we give some easy consequences of Theorem 1. We define $a_i(R) = \max\{j | H^i_{\mathfrak{m}}(R)_j \neq 0\}$ if $H^i_{\mathfrak{m}}(R) \neq 0$ and $a_i(R) = -\infty$ if $H^i_{\mathfrak{m}}(R) = 0$. Similarly, we define and $b_i(R) = \inf\{j | H^i_{\mathfrak{m}}(R)_j \neq 0\}$ if $H^i_{\mathfrak{m}}(R) \neq 0$ and $b_i(R) = +\infty$ if $H^i_{\mathfrak{m}}(R) = 0$.

Recall that $\rho_j = \max\{\nu_j(u) \mid u \in G(I)\}$ for $j = 1, \dots, n$.

Corollary 2. *Let $I \subset S = K[X_1, \dots, X_n]$ be a monomial ideal. Then $a_i(R) \le \sum_{j=1}^n \rho_j - n$ for all i.*

Proof. By Theorem 1, the terms in $\operatorname{Hilb}(H^i_{\mathfrak{m}}(R), \mathbf{t})$ with the highest total degree are at most $\dim_K \tilde{H}_{i-|F|-1}(\Delta_a; K)\mathbf{t}^a$ with $a_j = \rho_j - 1$ for $j = 1, \dots, n$. Thus the total degree is at most $\sum_j \rho_j - n$. □

From Corollary 2, we can recover the following well known result.

Corollary 3. *Let $I \subset S$ be a generalized CM Stanley-Reisner ideal. Then $a_i(R) \leq 0$ for all i.*

Proof. If I is square-free, then $\rho_j \leq 1$ for $j = 1, \dots, n$. □

For a Stanley-Reisner generalized CM ideal $I \subset S$ with $\dim R = d$, it is well known that it is Buchsbaum and $b_i(R) \geq 0$ for all $i (\neq d)$. The following theorem extends this result to monomial ideals in general.

Proposition 1. *Let $I \subset S = K[X_1, \dots, X_n]$ be a monomial ideal. Then following are equivalent:*

(*i*) $\ell(H^i_{\mathfrak{m}}(S/I)) < \infty$

(*ii*) $H^i_{\mathfrak{m}}(S/I)_a = 0$ *for all* $a \in \mathbb{Z}^n$ *with* $G_a \neq \emptyset$, *in particular* $b_i(S/I) \geq 0$

(*iii*) $\tilde{H}_{i-|G_a|-1}(\Delta_a; K) = 0$ *for all* $a \in \mathbb{Z}^n$ *with* $a_j \leq \rho_j - 1$ $(j = 1, \dots, n)$ *and* $\emptyset \neq G_a \in \Delta$.

Proof. The equivalence of (*ii*) and (*iii*) are immediate from Theorem 1. We will prove the equivalence of (*i*) and (*iii*). Assume that $\ell(H^i_{\mathfrak{m}}(S/I)) < \infty$. Assume also that there exists $a \in \mathbb{Z}^n$ such that $a_j \leq \rho_j - 1$ $(j = 1, \dots, n)$, $\emptyset \neq G_a \in \Delta$ and $\tilde{H}_{i-|G_a|-1}(\Delta_a; K) \neq 0$. Now observe that by the definition of Δ_a, the condition is independent of the values a_j for $j \in G_a$. This means that the total degree $j = \sum_{k=1}^n a_k$ can be any negative integer so that $H^i_{\mathfrak{m}}(R)$ is not of finite length, which contradicts the assumption. Thus we must have $\tilde{H}_{i-|G_a|-1}(\Delta_a; K) = 0$ for all such $a \in \mathbb{Z}^n$. The converse implication is straightforward. □

Corollary 4. *Let $I \subset S = K[X_1, \dots, X_n]$ be a generalized CM monomial ideal with* $\dim R = d(> 0)$. *Then $b_i(R) \geq 0$ for all $i(\neq d)$.*

For a generalized CM ring R, there exists an integer $k \in \mathbb{Z}$, $k \geq 1$, such that $\mathfrak{m}^k H^i_{\mathfrak{m}}(R) = 0$ for $i \neq \dim R$. If this condition holds, we will also call R, or $I \subset S$, *k-Buchsbaum.* An ideal I is generelized CM if and only if it is k-Buchsbaum for some k. If I is k-Buchsbaum but not $(k-1)$-Buchsbaum, then we will call I *strict k-Buchsbaum.*

Proposition 2. *Let $I \subset S = K[X_1, \dots, X_n]$ be a generalized CM monomial ideal. Then $R = S/I$ is* $\left(\sum_{j=1}^n \rho_j - n + 1\right)$*-Buchsbaum.*

Proof. R is $\max\{a_i(R) - b_i(R) + 1 \mid i \neq d\}$-Buchsbaum. Then the required result follows immediately from Corollary 2 and Corollary 4. □

From Proposition 2, we immediately know that a Stanley-Reisner ideal is 1-Buchsbaum if it is generalized CM, which is a weaker version of the well-known result that a generalized CM Stanley-Reisner ideal is Buchsbaum.

The bound of k-Buchsbaumness given in Proposition 2 is best possible. In fact, we can construct strict $(\sum_{j=1}^n \rho_j - n+1)$-Buchsbaum ideals as in the following example.

Example 1. Let $I \subset S$ be a Stanley-Reisner Buchsbaum ideal. Notice that such ideals can be constructed with the method presented in [1] and $H^i_{\mathfrak{m}}(S/I)$ $(i \neq \dim R)$ is a K-vector space for $i \neq \dim R$.

Now consider a K-homomorphism

$$\varphi : S \longrightarrow S, \qquad X_i \longmapsto X_i^{a_i} \ (i = 1, \ldots, n)$$

where $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ with $a_i \geq 1$ for $i = 1, \ldots, n$. We define $\varphi(M) = M \otimes_S {}^{\varphi}S$ for a S-module, where a left-right S-module ${}^{\varphi}S$ is equal to S as a set, it is a right S-module in the ordinary sense and its left S-module structure is determined by φ. Then we have

1. $\varphi(S/I) = S/\varphi(I)S$,
2. φ is an exact functor.

Thus, for $i \neq \dim R$, we have $H^i_{\mathfrak{m}}(S/\varphi(I)S) \cong \varphi(H^i_{\mathfrak{m}}(S/I))$ and since $H^i_{\mathfrak{m}}(S/I)$ is a direct sum of $S/\mathfrak{m}$, $H^i_{\mathfrak{m}}(S/\varphi(I)S)$ is a direct sum of $S/(X_1^{a_1}, \ldots, X_n^{a_n})$. Then we know that $\mathfrak{m}^k H^i_{\mathfrak{m}}(S/I) = 0$ but $\mathfrak{m}^{k-1} H^i_{\mathfrak{m}}(S/I) \neq 0$ with $k = \sum_{j=1}^n \rho_j - n + 1 = \sum_{j=1}^n a_j - n + 1$.

Remark 1. Bresinsky and Hoa gave a bound for k-Buchsbaumness for ideals generated by monomials and binomials (Theorem 4.5 [2]). For monomial ideals, our bound is stronger than that of Bresinsky and Hoa. Also, as pointed out by K. Yanagawa, Cor 2 can also be deduced by using Taylor resolution of monomial ideals together with local duality, and Cor 4 can also be deduced with a similar discussion to Th. 2.6 and Cor. 2.7 in [10], by changing slightly the definition of squarefree modules.

Recall that Castelnuovo-Mumford regularity of the ring R is defined by

$$\operatorname{reg}(R) = \max\{i + j | H^i_{\mathfrak{m}}(R)_j \neq 0\}.$$

Let $r = \operatorname{reg}(R)$. Then we have $H^i_{\mathfrak{m}}(R)_j = 0$ for $j > r - i$. Then we have

Corollary 5. *Let $I \subset S$ be a generalized CM monomial ideal with $d = \dim R$ and $r = \operatorname{reg}(R)$. Then $H^i_{\mathfrak{m}}(R) = 0$ for $r + 1 \leq i < d$. In particular, if I has q-linear resolution, we have $H^i_{\mathfrak{m}}(R) = 0$ for $q \leq i < d$.*

Proof. First part is clear from Corollary 4. If R has q-linear resolution, we have $\operatorname{reg}(R) = q - 1$. Thus the last part also follows immediately. □

2. Generalized Cohen-Macaulay monomial ideals

2.1. **FLC property.** In this subsection, we give a combinatorial characterization of FLC (finite length cohomology) property for monomial ideals, as an application of Theorem 1. We prepare some notations. Let $I \subset S = K[X_1, \ldots, X_n]$ be a monomial ideal. If $X_i^a \in G(I)$ for some $1 \leq i \leq n$ and $a \in \mathbb{N}$, we easily know that a must be $\rho_i = \max\{\nu_i(u) \mid u \in G(I)\}$. Then, by changing the name of the variables if necessary, we can write without loss of generality that $G(I) = \{X_{m+1}^{\rho_{m+1}}, \ldots, X_n^{\rho_n}\} \cup G_0(I)$ with $m \leq n$, where $G_0(I) = \{u \in G(I) \mid |\operatorname{supp}(u)| \geq 2\}$. We denote by Δ the simplicial complex corresponding to a square-free monomial ideal $\sqrt{I}$, which is a complex over the vertex set $[m] = \{1, \ldots, m\}$. We regard $\mathbb{Z}^n$ as a partially ordered set by defining $a \leq b$, $a, b \in \mathbb{Z}^n$, to be $a_i \leq b_i$ for $i = 1, \ldots, n$. We denote $\rho - 1 = (\rho_1 - 1, \ldots, \rho_n - 1) \in \mathbb{Z}^n$. For $a \in \mathbb{Z}^n$ and a monomial $u \in S$, we define the upper segment $L(a, u) = \{i \in [n] \mid \nu_i(u) > a_i\}$. Also for $a \in \mathbb{Z}^n$ with $a \leq \rho - 1$ and $\sigma \subset [n]$, we define the masking $a(\sigma) \in \mathbb{Z}^n$ as follows:

$$a(\sigma)_i = \begin{cases} a_i & \text{if } i \notin \sigma \\ \rho_i & \text{if } i \in \sigma \end{cases}$$

We abbreviate $a(\{j\})$ as $a(j)$ for $j \in \mathbb{Z}$.

Now we introduce the following two notions.

Definition 1. *A pair $(\sigma, a) \in (\Delta, \mathbb{Z}^n)$ with $\mid \sigma \mid= j$ is called a jth* test pair *if*

(*a*) $0 \leq a \leq \rho - 1$*,*
(*b*) $L(a(\sigma), u) \neq \emptyset$ *for all* $u \in G_0(I)$*, and*
(*c*) $a(\sigma)$ *is maximal with the properties* (*a*) *and* (*b*) *with* σ *fixed.*

Notice that, in the square-free case, a test pair is nothing but a pair $(\sigma, (0, \dots, 0))$ with $\sigma \in \Delta$.

Definition 2. *Let $\sigma \subset [n]$ and $a \in \mathbb{Z}^n$. Then we define the* purity condition *$\mathcal{P}(\sigma, a, I)$ as follows: there exists $\ell \in [m]\backslash\sigma$ such that*

(*a*) $a_\ell = \rho_\ell - 1$*, and*
(*b*) *for all $u \in G_0(I)$ with $\nu_\ell(u) = \rho_\ell$ we have $L(a(\sigma \cup \{\ell\}), u) \neq \emptyset$.*

Notice that, in the square-free case, the purity condition $\mathcal{P}(\sigma, a, I)$ for a test pair $(\sigma, a) = (\sigma, 0)$ implies that $\{\ell\} \cup \sigma \in \Delta$ for some $\ell \notin \sigma$, namely $\sigma \in \Delta$ is contained in a strictly larger face in Δ. Now we prove

Theorem 2. *Let $I \subset S = K[X_1, \dots, X_n]$ be a monomial ideal. If $\ell(H^i_{\mathfrak{m}}(S/I)) < \infty$ ($i > 0$) then, for any ith test pair (σ, a) the purity condition $\mathcal{P}(sigma, a, I)$ is satisfied.*

Proof. Assume that $\ell(H^i_{\mathfrak{m}}(S/I)) < \infty$. Then, by Proposition 1, we have $\tilde{H}_{-1}(\Delta_a; K) = 0$ for all $a \in \mathbb{Z}^n$ with $a \leq \rho - 1$, $G_a \in \Delta$ and $|G_a| = i(> 0)$. This implies $\Delta_a \neq \emptyset$. Now for such $a \in \mathbb{Z}^n$ we set $\sigma = G_a$. Notice that $\sigma \subset [m]$ since $\sigma \in \Delta$. We also notice that, as far as the complex Δ_a is concerned, the values a_i for $i \in \sigma = G_a$ are irrelevant. Thus we will change the values a_i ($i \in \sigma$) and assume that $0 \leq a \leq \rho - 1$. Notice that Δ_a, for the new a, is the same as before we change the values a_i for $i \in \sigma$.

We know that $\Delta_a = \emptyset$ is equivalent to the condition that there exists $u \in G(I)$ such that $L(a(\sigma), u) = \emptyset$. Now we assume that $L(a(\sigma), u) \neq \emptyset$ for all $u \in G(I)$, namely $\Delta_a \neq \emptyset$.

The condition $\Delta_a \neq \{\emptyset\}$ is equivalent to the condition that $\{\ell\} \in \Delta_a$ for some $\ell \in [n]$, i.e., there exists $\ell \in [n]\backslash\sigma$ such that for all $u \in G(I)$ we can find $k \in [n]\backslash(\sigma \cup \{\ell\})$ satisfying $\nu_k(u) > a_k$. Namely,

(3) there exists $\ell \in [n]\backslash\sigma$ such that $L(a(\sigma \cup \{\ell\}), u) \neq \emptyset$ for all $u \in G(I)$.

Under this condition we have, for any $b \in \mathbb{Z}^n$ with $0 \leq b \leq \rho - 1$ and $b(\sigma) \leq a(\sigma)$, $L(b(\sigma \cup \{\ell\}), u) \supseteq L(a(\sigma \cup \{\ell\}), u) \neq \emptyset$ for all $u \in G(I)$. Thus we can assume that $a(\sigma)$ is maximal satisfying the condition that $L(a(\sigma), u) \neq \emptyset$ for all $u \in G(I)$ and $0 \leq a \leq \rho - 1$. Also, since $\sigma \subset [m]$ and $a \leq \rho - 1$, we have $L(a(\sigma), X_i^{\rho_i}) = \{i\} \neq \emptyset$

for all $m+1 \leq i \leq n$. Hence we can replace '$u \in G(I)$' by '$u \in G_0(I)$' in the maximality condition for $a(\sigma)$. Now we have only to show that the condition (3) is equivalent to (i) and (ii) in the statement.

Since we have $L(a(\sigma \cup \{\ell\}), X_j^{\rho_j}) = \emptyset$ for all $m+1 \leq j \leq n$, we can only find the index ℓ as in (3) in $[m]\backslash\sigma$. Now for $\ell \in [m]\backslash\sigma$, the existence of $k \in [n]\backslash(\sigma \cup \{\ell\})$ satisfying $\nu_k(u) > a_k$ is always assured for every $u \in \{X_{m+1}^{\rho_{m+1}}, \ldots, X_n^{\rho_n}\}$. Moreover, if $\ell \notin \operatorname{supp}(u)$ for $u \in G_0(I)$, then $L(a(\sigma \cup \{\ell\}), u) = L(a(\sigma), u)$ and this is $\neq \emptyset$ since $\Delta_a \neq \emptyset$. Thus (3) is equivalent to the existence of $\ell \in [m]\backslash\sigma$ such that

$$L(a(\sigma \cup \{\ell\}), u) \neq \emptyset \text{ for } u \in G_0(I) \text{ with } \ell \in \operatorname{supp}(u). \tag{4}$$

Assume that $a_\ell < \rho_\ell - 1$ and set $e \in \mathbb{Z}^n$ as $e_i = 0$ for $i \neq \ell$ and $e_\ell = \rho_\ell - 1 - a_\ell$. Then by the maximality of $a(\sigma)$ there exists $u \in G_0(u)$ such that $\emptyset = L(a(\sigma) + e, u) \supset L(a(\sigma \cup \ell), u)$, which contradicts the condition (3). Thus we must have $a_\ell = \rho_\ell - 1$ for ℓ as in (3). If $u \in G_0(I)$ is such that $0 < \nu_\ell(u) < \rho_\ell$, then $L(a(\sigma) \cup \{\ell\}, u) = L(a(\sigma), u) \neq \emptyset$ by assumption on $a(\sigma)$. Thus we can replace '$u \in G_0(I)$ with $\ell \in \operatorname{supp}(u)$' in the condition (4) by '$u \in G_0(I)$ with $\nu_\ell(u) = \rho_\ell$'. Consequently we know that (4) is equivalent to (i) and (ii). □

From Theorem 2, we can recover a weaker version of the well-known result as follows.

Corollary 6. *If $I \subset S$ is a generalized CM Stanley-Reisner ideal, i.e., Buchsbaum ideal, then Δ is pure, namely, every facet has the same dimension.*

Proof. Let $I \subset S$ be a generalized CM Stanley-Reisner ideal. Then by Theorem 2 we know that for every $0 < i < \dim S/I$ and for every $(i-1)$-face $\sigma \in \Delta$ there exists ℓ with $1 \leq \ell \leq n$ such that $\sigma \cup \{\ell\} \in \Delta$. From this we immediately know that Δ is pure. □

2.2. **generalized CM monomial ideals of** $\dim \leq 3$**.** If $\dim R \leq 1$, I is always (generalized) CM. For $dim R = 2, 3$, we can give combinatorial characterizations of generalized CM monomial ideals as follows. First we give the $\dim 2$ case.

Corollary 7. *A monomial ideal $I \subset S$ is generalized CM with $\dim S/I = 2$ if and only if*

(i) $\dim \Delta = 1$, *and*
(ii) *every 1st test pair $(\{j\}, a)$ satisfies the purity condition $\mathcal{P}(\{j\}, a, I)$.*

Proof. As is well-known, $\dim S/I = \dim S/\sqrt{I} = 2$ if and only if $\dim \Delta = 1$. Then S/I is generalized CM if and only if $\ell(H^i_{\mathfrak{m}}(S/I)) < \infty$ for $i = 0, 1$. $H^0_{\mathfrak{m}}(S/I)$ is always of finite length and $H^1_{\mathfrak{m}}(S/I)$ is of finite length if and only if $\tilde{H}_{-|G_a|}(\Delta_a; K) = 0$ for all $a \in \mathbb{Z}^n$ with $a \leq \rho - 1$ and $\emptyset \neq G_a \in \Delta$ by Proposition 1. If $|G_a| \geq 2$, we always have $\tilde{H}_{-|G_a|}(\Delta_a; K) = 0$. Now let $G_a = \{j\}$. Since $G_a \in \Delta$ we must have $j \in [m]$. Also $\tilde{H}_{-1}(\Delta_a; K) = 0$ if and only if $\Delta_a \neq \emptyset$, which is equivalent to the condition in the statement by the proof of Theorem 2. □

In $\dim 3$ case, we need to give a combinatorial criterion for connectedness of simplicial complexes. Notice that a simplicial complex Δ over the vertex set $[m]$ is *not* connected if and only if there exists non-empty disjoint subsets $P, Q \subset [m]$ such that $P \cup Q = [m]$ and for all $p \in P$ and all $q \in Q$ there is no 1-face $\{p, q\} \in \Delta$.

Lemma 3. *Assume that $G_a = \{j\}$, $j \in [m]$, and $\Delta_a \neq \emptyset$. Then the set of vertices of Δ_a is $\{\ell \in [m]\backslash\{j\} \mid$ for all $u \in G_0(I)$ we have $L(a(\{\ell, j\}), u) \neq \emptyset\}$.*

Proof. The 0th skeleton of Δ_a is

$$\begin{aligned}
&\{\{\ell\} \mid \ell \neq j, \text{ for all } u \in G(I) \text{ there exists } k \notin \{\ell, j\} \text{ such that } \nu_k(u) > a_k\} \\
&= \{\{\ell\} \mid \ell \in [m]\backslash\{j\}, \text{ for all } u \in G_0(I) \text{ there exists } k \notin \{\ell, j\} \text{ such that } \nu_k(u) > a_k\} \\
&= \{\{\ell\} \mid \ell \in [m]\backslash\{j\}, \text{ for all } u \in G_0(I) \text{ we have } L(a(\{\ell, j\}), u) \neq \emptyset\},
\end{aligned}$$

where the first equation is because if $m + 1 \leq \ell \leq n$ there is no $k \notin \{\ell, j\}$ such that $\nu_k(X_\ell^{\rho_\ell}) > a_k$ and if $\ell \in [m]$ we always have the index $k \notin \{\ell, j\}$ such that $\nu_k(X_i^{\rho_i}) > a_i$ for $i = m + 1, \dots, n$, which is actually $k = i$. □

Now we show a combinatorial characterization of dim 3 generalized CM monomial ideals.

Theorem 3. *A monomial ideal $I \subset S$ is generalized CM with $\dim S/I = 3$ if and only if*

- (i) $\dim \Delta = 2$, *and*
- (ii) *for every 1st test pair $(\{j\}, a)$*
  - (1) *the purity condition $\mathcal{P}(\{j\}, a, I)$ is satisfied, and*
  - (2) *$\Delta_{a(j)}$ is continuous, in other words, there are no non-empty disjoint subsets $P, Q \subset [m]$ satisfying the following property:*
    1. *$P \cup Q = [m]\backslash\{j\} - L_a$ where $L_a = \{\ell \mid L(a(j), u) = \{\ell\}$ for some $u \in G_0(U)\}$, and*
    2. *for all $x \in P$ and all $y \in Q$ there exists $u \in G_0(I)$ such that $L(a(j), u) = \{x, y\}$*
- (iii) *every 2nd test pair $(\{i, j\}, a)$ satisfies the purity condition $\mathcal{P}(\{i, j\}, a, I)$.*

Proof. $\dim S/I = \dim S/\sqrt{I} = 3$ if and only if $\dim \Delta = 2$. Now assume that $\dim S/I = 3$. Then S/I is generalized CM if and only if $\ell(H_{\mathfrak{m}}^1(S/I)) < \infty$ and $\ell(H_{\mathfrak{m}}^2(S/I)) < \infty$, which is equivalent to

$$\tilde{H}_{-|G_a|}(\Delta_a; K) = 0 \tag{5}$$

and

$$\tilde{H}_{1-|G_a|}(\Delta_a; K) = 0 \tag{6}$$

for all $a \in \mathbb{Z}^n$ with $a \leq \rho - 1$ and $\emptyset \neq G_a \in \Delta$ by Proposition 1. The condition (5) is equivalent to (ii)(1) by Corollary 7 and the condition (6) is equivanent to

$$\tilde{H}_0(\Delta_a; K) = 0 \text{ for all } a \in \mathbb{Z}^n \text{ with } a \leq \rho - 1 \text{ and } G_a = \{j\} \in [m], \tag{7}$$

and

$$\tilde{H}_{-1}(\Delta_a; K) = 0 \text{ for all } a \in \mathbb{Z}^n \text{ with } a \leq \rho - 1 \text{ and } G_a = \{i, j\} \in [m] \tag{8}$$

since $\tilde{H}_{-k}(\Delta_a; K) = 0$ for $k \geq 2$. The condition (7) exactly means the connectedness of the simplicial complex Δ_a. Let $\mathcal{V}_a$ be the set of vertices of Δ_a. By what we noticed just before Lemma 3, this is equivalent to the condition that there exist disjoint no non-empty subsets $P, Q \subset \mathcal{V}_a$ such that $P \cup Q = \mathcal{V}_a$ and for all $x \in P$ and all $y \in Q$ we have $\{x, y\} \notin \Delta_a$. By Lemma 3 we have $\mathcal{V}_a = [m] \backslash \{j\} - L_a$ where $L_a = \{\ell \mid \ell \neq j,\ L(a(\{j, \ell\}), u) = \emptyset$ for some $u \in G_0(I)\}$. Since we have $L(a(j), u) \neq \emptyset$, $L(a(\{j, \ell\}), u) = \emptyset$ implies $L(a(j), u) = \{\ell\}$. Thus $L_a = \{\ell \mid L(a(j), u) = \{\ell\}$ for some $u \in G_0(I)\}$. The condition $\{x, y\} \notin \Delta_a$ is equivalent to the condition that $L(a(\{x, y, j\}), u) = \emptyset$ for some $u \in G(I)$. This can also be refined to the condition that $L(a(j), u) = \{x, y\}$ for some $u \in G_0(I)$. In fact, first of all we have $L(a(\{x, y, j\}), X_i^{\rho_i}) = \{i\} \neq \emptyset$, for all $i = m + 1, \ldots, n$, since $x, y, j \in [m]$ and $a_i \leq \rho_i - 1$. Thus we can replace '$u \in G(I)$' in the above condition by '$u \in G_0(I)$'. Also, since $x, y \in \mathcal{V}_a$, $L(a(\{x, j\}), u) \neq \emptyset$ and $L(a(\{y, j\}), u) \neq \emptyset$ for all $u \in G_0(I)$. Thus $L(a(\{x, y, j\}), u) = \emptyset$ is equivalent to $L(a(j), u) = \{x, y\}$ as required. This is the condition $(ii)(2)$.

Now we will show that if $b \in \mathbb{Z}^n$ is such that $b \leq \rho - 1$, $G_b = \{j\}$, $L(b(j), u) \neq \emptyset$ for all $u \in G_0(I)$ and $b(j) \leq a(j)$, then Δ_b is also connected. We prove the contrapositon: if Δ_b is disjoint then Δ_a is disjoint too. Assume that there exist disjoint non-empty subsets $P, Q \subset \mathcal{V}_b$ such that $P \cup Q = \mathcal{V}_b$ and for all $x \in P$ and all $y \in Q$ we have $L(b(j), u) = \{x, y\}$ for some $u \in G_0(I)$. First of all, for $u \in G_0(I)$ we have $L(b(\{j, \ell\}), u) \supseteq L(a\{j, \ell\}, u)$ so that $L_b \subseteq L_a$ and thus $\mathcal{V}_a \subseteq \mathcal{V}_b$. Now for all $x \in P \cap \mathcal{V}_a$ and $y \in Q \cap \mathcal{V}_a$, $L(a(j), u) \subset L(b(j), u) = \{x, y\}$ for some $u \in G_0(I)$. Also since $x, y \in \mathcal{V}_a$ we must have $L(a(\{j, x\}), u) \neq \emptyset$ and $L(a(\{j, y\}), u) \neq \emptyset$. Then we know that we must have $L(a(j), u) = \{x, y\}$. Thus, by setting $P' = P \cap \mathcal{V}_a$ and $Q' = Q \cap \mathcal{V}_a$, we obtain the non-empty disjoint subsets $P', Q' \subset \mathcal{V}_a$ showing the disjointness of Δ_a. Consequently, we can assume $a(j)$ to be maximal as in the statement $(ii)(c)$. Finally, by the proof of Theorem 2, we know that the condition (8) is equivalent to (iii). □

The square-free version of Theorem 3 is the following well-known fact.

Corollary 8. *Consider the Stanley-Reisner ring $k[\Delta]$ of dimension 3. Then $k[\Delta]$ is Buchsbaum if and only if Δ is pure and $\mathrm{lk}_\Delta(\{i\})$ is connected for $i = 1, \ldots, n$.*

Remark 2. Unfortunately we do not know a good combinatorial characterization for $\tilde{H}_j(\Delta_a; K) = 0$ for $j \geq 1$, which is needed to obtain similar results to Theorem 3 for $\dim R \geq 4$.

2.3. **Construction from Buchsbaum Stanley-Reisner ideals.** In this subsection, we compare local cohomologies of monomial ideals $I \subset S$ and $\sqrt{I}$. It is well known that $H^i_{\mathfrak{m}}(S/\sqrt{I})_a = 0$ for all $a \in \mathbb{Z}^n$ with $H_a \neq \emptyset$, which is an immediate consequence from the original version of Hochster's formula for Stanley-Reisner ideals. On the other hand, we may have $H^i_{\mathfrak{m}}(S/I)_a \neq 0$ for such $a \in \mathbb{Z}^n$. But for multi-degrees $a \in \mathbb{Z}^n$ with $H_a = \emptyset$, we have an isomorphism.

Proposition 3 (Herzog-Takayama-Terai [4]). *Let $I \subset S$ be a monomial ideal. Then we have the following isomorphisms of K-vector spaces*

$$H^i_{\mathfrak{m}}(S/I)_a \cong H^i_{\mathfrak{m}}(S/\sqrt{I})_a$$

for all $a \in \mathbb{Z}^n$ with $H_a = \emptyset$.

Proposition 4 (Herzog-Takayama-Terai [4]). *Let $I \subset S$ be a monomial ideal. Then $\ell(H^i_{\mathfrak{m}}(S/I)) < \infty$ implies $\ell(H^i_{\mathfrak{m}}(S/\sqrt{I})) < \infty$. In particular, if I is generalized CM, then $\sqrt{I}$ is also generalized CM (Buchsbaum).*

Proof. We will give here a new proof, which is different from that in [4]. Assume that $\ell(H^i_{\mathfrak{m}}(S/I)) < \infty$. Then by Proposition 1 $H^i_{\mathfrak{m}}(S/I)_a = 0$ for all $a \in \mathbb{Z}^n$ with $G_a \neq \emptyset$. Thus if $H^i_{\mathfrak{m}}(S/I)_a \neq 0$ and $H_a = \emptyset$, we must have $a = (0, \dots, 0)$. Now by Proposition 3 we have $H^i_{\mathfrak{m}}(S/\sqrt{I})_a \cong H^i_{\mathfrak{m}}(S/I)_a$ for all $a \in \mathbb{Z}^n$ with $H_a = \emptyset$, and this is non-zero if and only if $a = (0, \dots, 0)$. Thus we have $\ell(H^i_{\mathfrak{m}}(S/\sqrt{I})) < \infty$. □

Corollary 9. *Let $I \subset S$ be a generalized CM monimial ideal and assume that $\sqrt{I}$ is not Cohen-Macaulay (but generalized CM by Proposition 4). Then I is not Cohen-Macaulay.*

Proof. Assume that I is Cohen-Macaulay. Then, by Proposition 3 and the comment before Proposition 3, we have $H^i_{\mathfrak{m}}(S/\sqrt{I})_a = 0$ for all $i < \dim S/\sqrt{I}$ and for all $a \in \mathbb{Z}^n$, namely $H^i_{\mathfrak{m}}(S/\sqrt{I}) = 0$ for all $i < \dim S/\sqrt{I}$ and $\sqrt{I}$ is Cohen-Macaulay. □

These results suggests a method for constructing (non-CM) generalized CM monomial ideals from Buchsbaum Stanley-Reisner ideals: given a Buchsbaum Stanley-Reisner ideal J, make monomials $X_{j_1}^{e_1} \cdots X_{j_p}^{e_p}$, $(e_i \geq 1, i = 1, \dots, n)$, for each generator $X_{j_1} \cdots X_{j_p} \in G(J)$. In general, one can make more than one monomial generators from a single square-free generator. If we choose suitable exponents e_i, the ideal generated by the monomials is (non-CM) generalized CM. Theorem 2, Corollary 7 and Theorem 3 give the criteria for suitable exponents.

Example 2. Let $J = (X_1, \dots, X_n)(X_{n+1}, \dots, X_{2n}) \subset S = K[X_1, \dots, X_{2n}]$, $(n \geq 2)$. It is easy to check that S/I is Buchsbaum of dimension n and depth 1. Let $I = (X_i^{\alpha_{ij}} X_j^{\beta_{ij}} \mid 1 \leq i \leq n,\ n+1 \leq j \leq 2n)$ for some $\alpha_{ij}, \beta_{ij} \in \mathbb{N}$. Then I is generalized CM ideal if and only if $\alpha_{i,n+1} = \cdots = \alpha_{i,2n}$ for all $1 \leq i \leq n$ and $\beta_{1,j} = \cdots = \beta_{n,j}$ for all $n+1 \leq j \leq 2n$, namely I is an image of Frobenius map in the sense of Example 1.

Remark 3. Notice that if $n = 1$ then both J and $I = (X_1^{\alpha_{11}} X_2^{\beta_{11}})$ are Cohen-Macaulay for all $\alpha_{11}, \beta_{11} \in \mathbb{N}$.

Proof of Example 2. Assume in the following that I is generalized CM. Then I must satisfy the conditions in Theorem 2, in particular the condition for $\ell(H^{n-1}_{\mathfrak{m}}(S/I)) < \infty$.

Now we set $\rho_i = \max\{\alpha_{ij} \mid n+1 \leq j \leq 2n\}$ and $\varepsilon_i = \min\{\alpha_{ij} \mid n+1 \leq j \leq 2n\}$ for $1 \leq i \leq n$ and $\rho_j = \max\{\beta_{ij} \mid 1 \leq i \leq n\}$ and $\varepsilon_j = \min\{\beta_{ij} \mid 1 \leq i \leq n\}$ for $n+1 \leq j \leq 2n$. Notice that $\rho_k (\geq 1)$ and $\varepsilon_k (\geq 1)$ denote the maximal and the minimal exponents of the variable X_k, $k = 1, \dots, 2n$. Let Δ be the simplicial complex corresponding to J, which is the disjoint union of two $(n-1)$-simplices over the vertex set $[n]$ and $[2n]\backslash[n]$. Then an $(n-2)$-face $\sigma \in \Delta$ is either $\sigma = \{1, \dots, n\}\backslash\{k\}$ for $k = 1, \dots, n$ or $\sigma = \{n+1, \dots, 2n\}\backslash\{k\}$ for $k = n+1, \dots, 2n$.

We know that the condition for $a \in \mathbb{Z}^{2n}$ such that $0 \le a \le \rho - 1$ to be $L(a(\sigma), u) \neq \emptyset$ for all $u \in G(I) = G_0(I)$ is as follows:

Case (1) $\sigma = \{1, \dots, n\}\backslash\{k\}$ **for** $1 \le k \le n$**:** Since

$$
\begin{aligned}
&L(a(\sigma), X_i^{\alpha_{ij}} X_j^{\beta_{ij}}) \\
&= \begin{cases} \{k, j\} & \text{if } i = k,\ a_k < \alpha_{kj}, a_j < \beta_{kj} \\ \{j\} & \text{if } i = k,\ a_k \ge \alpha_{kj}, a_j < \beta_{kj}, \text{ or if } i \neq k, a_j < \beta_{ij} \\ \{k\} & \text{if } i = k,\ a_k < \alpha_{kj}, a_j \ge \beta_{kj} \\ \emptyset & \text{otherwise,} \end{cases}
\end{aligned}
$$

we must have

1. $a_j < \min\{\beta_{ij} \mid 1 \le i \le n, i \neq k\}$ for all $n+1 \le j \le 2n$, and
2. for every $n+1 \le j \le 2n$ we have at least one of the followings: (a) $a_k < \alpha_{kj}$, (b) $a_j < \beta_{kj}$.

Case (2) $\sigma = \{n+1, \dots, 2n\}\backslash\{k\}$ **for** $n+1 \le k \le 2n$**:** Since

$$
\begin{aligned}
&L(a(\sigma), X_i^{\alpha_{ij}} X_j^{\beta_{ij}}) \\
&= \begin{cases} \{k, i\} & \text{if } j = k,\ a_k < \beta_{ik}, a_i < \alpha_{ik} \\ \{i\} & \text{if } j = k,\ a_k \ge \beta_{ik}, a_i < \alpha_{ik}, \text{ or if } j \neq k, a_i < \alpha_{ij} \\ \{k\} & \text{if } j = k,\ a_k < \beta_{ik}, a_i \ge \alpha_{ik} \\ \emptyset & \text{otherwise,} \end{cases}
\end{aligned}
$$

we must have

1. $a_i < \min\{\alpha_{ij} \mid n+1 \le j \le 2n, j \neq k\}$ for all $1 \le i \le n$, and
2. for every $1 \le i \le n$ we have at least one of the followings: (a) $a_k < \beta_{ik}$, (b) $a_i < \alpha_{ik}$.

According to Theorem 2, for a maximal $a(\sigma)$, where a and σ are as above, there must exist an index $\ell \in [2n]\backslash\sigma$ such that (i) $a_\ell = \rho_\ell - 1$ and $(ii)'$ for every $u = X_i^{\alpha_{ij}} X_j^{\beta_{ij}} \in G(I)$ with $\nu_\ell(u) = \rho_\ell$ we have $L(a(\sigma \cup \{\ell\}), u) \neq \emptyset$. Moreover, by the proof of the Theorem 2 we know that $(ii)'$ can be replaced by

(ii) for every $u \in G(I)$ with $\ell \in \operatorname{supp}(u)$ we have $L(a(\sigma \cup \{\ell\}), u) \neq \emptyset$.

We now consider the condition for a, α_{ij} and β_{ij} satisfying (i) and (ii).

Case (3) $1 \le \ell \le n$ **with** $\ell \notin \sigma$**:** By (ii) we must have $\emptyset \neq L(a(\sigma \cup \{\ell\}), X_\ell^{\alpha_{\ell j}} X_j^{\beta_{\ell j}}) (\subset \{j\})$ for all $n+1 \le j \le 2n$. This holds if and only if $j \notin \sigma$ and $a_j < \beta_{\ell,j}$.

Case (4) $n+1 \le \ell \le 2n$ **with** $\ell \notin \sigma$**:** Similarly, we have and $a_i < \alpha_{i,\ell}$ for any $1 \le i \le n$ with $i \notin \sigma$.

Now in case (1), an $\ell \in [2n]\backslash\sigma$ satisfying (i) and (ii) must be $\ell = k$, $(1 \le k \le n)$ or $\ell \in \{n+1, \dots, 2n\}$. Assume that $\ell \in \{n+1, \dots, 2n\}$. Then the condition of case

(1) must imply the condition of case (4). Since

$$\begin{aligned}\max\{\beta_{i\ell} \mid 1 \le i \le n, i \ne k\} - 1 &\le \max\{\beta_{i\ell} \mid 1 \le i \le n\} - 1 = \rho_\ell - 1 = a_\ell \\ &< \min\{\beta_{i\ell} \mid 1 \le i \le n, i \ne k\},\end{aligned}$$

we have $\max\{\beta_{i\ell} \mid 1 \le i \le n, i \ne k\} = \min\{\beta_{i\ell} \mid 1 \le i \le n, i \ne k\}$ so that $\beta_{i\ell}$ is constant for all $0 \le i \le n$ with $i \ne k$. Now if $\beta_{k\ell} > \beta_{i\ell}$ for some, and equivalently all, $i(\ne k)$, then we have $\rho_\ell = \max\{\beta_{i\ell} \mid 1 \le i \le n\} = \beta_{k\ell}$ so that $\beta_{k\ell} - 1 = \rho_\ell - 1 = a_\ell < \min\{\beta_{i\ell} \mid 1 \le i \le n, i \ne k\} = \beta_{i\ell}$. Then we have $\beta_{k\ell} \le \beta_{i\ell}$, a contradiction. Thus we know $\beta_{k\ell} \le \beta_{i\ell}$ for all $i \ne k$, and the index i as in the condition of case (4) can be at least $i = 1, \ldots, n$ with $i \ne k$ but this contradicts the condition $i \notin \sigma = \{1, 2, \ldots, n\} \backslash \{k\}$. Consequently, an $\ell \in [2n] \backslash \sigma$ satisfying (i) and (ii) cannot be from $\{n+1, \ldots, 2n\}$ and we must have $\ell = k$ with $1 \le k \le n$. Now the condition of **Case (1)** must imply the condition of **Case (3)**. Comparing **Case (1)** 1 with the condition of **Case (3)**, we know that we must have

$$\min\{\beta_{ij} \mid 1 \le i \le n, i \ne k\} \le \beta_{kj} \tag{9}$$

for every j with $n+1 \le j \le 2n$. Now we consider similarly with the **Case (2)** and obtain the condition

$$\min\{\alpha_{ij} \mid n+1 \le j \le 2n, j \ne k\} \le \alpha_{ik} \tag{10}$$

for every i with $1 \le i \le n$. Finally, the condition (9) for $k = 1, \ldots, n$ together with the condition (10) for $k = n+1, \ldots, 2n$ entails β_{ij} are constant for all $1 \le i \le n$ and α_{ij} are constant for all $n+1 \le j \le 2n$, i.e., I is obtained from J by Frobenius transformation in the sense of Example 1. □

Example 3. If we allow to make more than two monomial generators from a single square-free generator, we can construct more generalized CM monomial ideals from the same Stanley-Reisner ideals as in Example 2, For example, from $J_1 = (X_1, X_2)(X_3, X_4) \subset K[X_1, X_2, X_3, X_4]$ we make

$$I_1 = (X_1X_3, X_1^2X_4, X_1X_4^2, X_2^2X_3, X_2X_3^2, X_2X_4).$$

Also from $J_2 = (X_1, X_2, X_3)(X_4, X_5, X_6) \subset K[X_1, \ldots, X_6]$ we make

$$I_2 = (X_1^3X_4, X_1X_4^5, X_1X_5, X_1X_6, X_2X_4, X_2X_5, X_2X_6, X_3X_4, X_3X_5, X_3X_6).$$

I_1 and I_2 are both generalized CM, but for example

$$I_3 = (X_1^3X_4, X_1^2X_4^2, X_1X_4^3, X_1X_5, X_1X_6, X_2X_4, X_2X_5, X_2X_6, X_3X_4, X_3X_5, X_3X_6)$$

is not generalized CM.

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