Generalized Hecke Categories and their Representations —A Survey

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Completion of classification of finite simple groups!

Michael Aschbacher, Stephen D. Smith
"The Classification of Quasithin Groups (I, II)"
(Math. Surveys and Monographs) (2004/12/09) 496 p + 800 p

Daniel Gorenstein, Richard Lyons, Ronald Solomon
"The Classification of the Finite Simple Groups" (volume 1–5), AMS,

volume 1 (1994), 165 p
volume 2 (1996), 218 p
volume 3 (1998), 419 p
volume 4 (1999), 341 p
volume 5 (2002), 467 p

http://www.ams.org/online_bks/surv40-1/surv40-1-master.pdf

Finite group theory and other areas in math.

Transfer theorems and Dedekind sums

$x \in [G,G],\ |x| = 2 \ \Rightarrow\ |G|_2 = |C_G(x)|_2$ or $|C_G(x)|_2 \geq 16$.

$x \in [G,G],\ |x| = 3, x \not\sim x^{-1} \ \Rightarrow\ |C_G(x)|_3 \geq 27$

Came from **Dedekind sum** $12ms(m,n) \in \boldsymbol{Z}$.

$$s(m,n) = \sum_{k=1}^{n} \left(\left(\frac{km}{n}\right)\right)\left(\left(\frac{k}{n}\right)\right), \qquad ((x)) := \begin{cases} x - \lfloor x \rfloor & (x \in \boldsymbol{R} - \boldsymbol{Z}) \\ 0 & \text{else} \end{cases}$$

T.Y. Character-theoretic transfer (II), *J.Algebra*

Other area—Statistics and finite group theory.

Laurent Saloff-Coste, Random walks on finite groups, Encyclopaedian of Math.Sci. 110 (2004).

Y., Mathematical application to comparative linguistics. (Science topics, Hokkaaido Math. Fac. Sci. 2005). http://www.hokudai.ac.jp/science/science.htm

Graczyk, Letc, Massam, The complex Wishart distribution and the symmetric group, Ann.Stat.,31 (2003)

Satoshi Aoki, Analysis of contingency tables by Markov chain and Monte Carlo methods. (2005) http://www.stat.t.u-tokyo.ac.jp/~aoki/study.html

G finite group, $\boldsymbol{k} = \overline{\boldsymbol{k}}$, char$(\boldsymbol{k}) = p > 0$.
Weight is (P, V), P: p-subgrp, V : simple proj $\boldsymbol{k}[N_G(P)/P]$-module.

Conjecture(Alperin 1987). $\sharp\{$weight of $G\} = \sharp\{$simple $\boldsymbol{k}G$-module$\}$?

np$(G) := \sharp\{$non-projective simple $\boldsymbol{k}G$-modules$\}$.
$\Delta := \mathcal{S}_p(G) :=$ poset of p-subgroups $(\neq 1)$.

AC $\Leftrightarrow$ **(New AC)** np$(G) = \sum_{\sigma \in \Delta/G} (-1)^{\dim \sigma}np(G_\sigma)$ (Webb).

Conjecture (Qillen 1978). $\mathcal{S}_p(G)$ is contractible $\Leftrightarrow$ $1 \neq O_p(G) (\trianglelefteq G)$.

Hom-conjecture. $|\mathrm{Hom}(A,G)| \equiv 0 \bmod \gcd(|A/[A,A]|,|G|)$?
Asai-Y, $|\mathrm{Hom}(A,G)|$ (II), J.Alg.('93). (Reduced to p-groups!)

Hom-set $\mathrm{Hom}(A,G)$ rarely appears in finite group theory.

Dijkgraaf-Witten invariant M : cpt 3-mfd, $[\alpha] \in H^3(G,U(1))$

$$Z^{G,\alpha}(M) := \frac{1}{|G|} \sum_{\gamma:\pi_1(M)\to G} \langle \gamma^*(\alpha),[M]\rangle, \ Z_{G,1}(M) = \frac{|\mathrm{Hom}(\pi_1(M),G)|}{|G|}$$

No Conjecture. $|G|Z_{G,\alpha}(M) \equiv 0 \bmod \gcd(|H_1(M)|,|G|)$?

False for lens spaces! (Wakui)

Generating functions (Y '90, Y '91)

$s(A,G) := \sharp\{A' \le G \mid A' \cong A\}$, $h(A,G) := |\mathrm{Hom}(A,G)|$

$$S_G(x) := \sum_{n\ge 0} s({C_p}^n, G) p^{\binom{n}{2}} x^n \quad \text{(Zeta function?)}$$

$$S_G(-1) = 1 - \chi(\{p\text{-subgroups} \neq 1\}), \quad \text{RHS is related to } h({C_p}^n, G).$$

Geometric property of $\mathrm{Hom}(G, GL(n,F))$? GF of $\{|\mathrm{Hom}(G, \boldsymbol{F}_{q^r})|\}$?

$\exp(G) | q-1$, $r := \sharp\{$conj classes of $G\}$, then

$$1 + \sum_{n=1}^{\infty} \frac{h(G, GL(n,q))}{|GL(n,q)|} = \prod_{n\equiv 1,4(5)} \frac{1}{(1-q^{-n})^r}$$

Burnside ring. $B(G)$ is Grothendieck ring of finite G-sets.
Generators : $\{[X] \mid X \in \mathbf{set}^G\}$, Relation: $[X+Y] = [X] + [Y]$.

$\varphi : \boldsymbol{B}(G) \rightarrowtail \widetilde{\boldsymbol{B}}(G) := \boldsymbol{Z}^{C(G)}; [X] \longmapsto (|X^H|)_H$. $C(G) := \mathbf{Sub}(G)/\sim_G$.

cpi of $\boldsymbol{Q} \otimes \boldsymbol{B}(G)$: $e_H = |N_G(H)|^{-1} \sum_{D \leq H} |D|\mu(D,H)[G/D]$

cpi of $\boldsymbol{Z}_p \otimes \boldsymbol{B}(G)$: $e_Q^{(p)} := |N_G(Q)|^{-1} \sum_{H^p \sim_G Q} \sum_{D \leq H} |D|\mu(D,H)[G/D]$ (Y.'83).

Homological Sylow $\chi(\mathcal{S}_p(G)) \equiv 1 \pmod{|G|_p}$ (Brown '78).
Frobenius thm. $\sharp\{x^n = 1\} = |\mathrm{hom}(C_n, G)| \equiv 0 \bmod \gcd(n, |G|)$.

Hecke category $\mathbf{Hec}_{\boldsymbol{k}}(G)$. G : finite group, $\boldsymbol{k}$: commutative ring.
Objects : finite G-sets $X, Y, Z, \cdots$
$\mathrm{Hom}(Y, X) = \{G\text{-matrix }\} \ni A = (a_{xy})_{x\in X, y\in Y},\ a_{gx,gy} = a_{xy}\ (g \in G)$

Property (1) $\mathbf{Hec}_{\boldsymbol{k}}(G)$ is $\boldsymbol{k}$-additive cat.
(2) Full embedding $\mathbf{Hec}_{\boldsymbol{k}}(G) \hookrightarrow \mathbf{Mod}_{\boldsymbol{k}G}; X \longmapsto \boldsymbol{k}X$.

$$\mathrm{Hom}_{\mathbf{Hec}}(G/K, G/H) \cong \mathrm{Hom}_{\boldsymbol{k}G}(\boldsymbol{k}[G/H], \boldsymbol{k}[G/K]) \cong \boldsymbol{k}[H\backslash G/K]$$
$$\mathrm{End}_{\mathbf{Hec}}(G/H) \cong \mathrm{End}_{\boldsymbol{k}G}(\boldsymbol{k}[G/H]) \cong \boldsymbol{k}[H\backslash G/H] \quad \text{Hecke ring}$$
$$(HxK) \circ (KyL) = \sum_{(H^x\cap K)k(K\cap {}^yL)} (H^{xky} \cap K : H^{xky} \cap K^y \cap L)(HxkyL)$$

(3) $\mathbf{Hec}_{\boldsymbol{k}}(G) \longrightarrow \mathbf{Mod}_{\boldsymbol{k}}$ reflects iso. G-mat A is iso $\iff$ $\det A \in \boldsymbol{k}^{\times}$.
(4) $\mathbf{Hec}_{\boldsymbol{k}}(G)^{\mathrm{op}} \cong \mathbf{Hec}_{\boldsymbol{k}}(G)$ by transposition $A \longleftrightarrow {}^tA$.

Hecke functor $=$ Representation of $\mathbf{Hec}_{\boldsymbol{k}}(G)$, i.e., $\mathbf{Hec}_{\boldsymbol{k}}(G)^{\mathsf{op}} \longrightarrow \mathbf{Mod}_{\boldsymbol{k}}$.

First example (by Shimura) V : right $\boldsymbol{k}G$-module

$$\begin{aligned} H_V^* &: \mathbf{Hec}_{\boldsymbol{k}}(G)^{\mathsf{op}} \longrightarrow \mathbf{Mod}_{\boldsymbol{k}} : X \longmapsto \mathsf{Ext}^*_{\boldsymbol{k}G}(kX, V). \\ [HxK] &: H^*(H,V) \longrightarrow H^*(K,V) \cong \mathsf{Ext}^*_{\boldsymbol{k}G}(k[K\backslash G], V) \\ \alpha|[HxK] &= \mathsf{cor}^K \circ \mathsf{res}_{H^x \cap K}(\alpha^x) = \alpha^x{\downarrow_{H^x\cap K}}{\uparrow^K} \end{aligned}$$

$\mathsf{cor}_H^K = [H1K]$, $\mathsf{res}_H^K = [K1H]$, $\mathsf{con}_H^x = [HxH^x]$ $\quad (H \le K \le G, x \in G)$.

No action of Hecke ring on character ring! Why?

Mackey decomposition and $\mathsf{cor}_H^K \circ \mathsf{res}_H^K = (K : H)\mathsf{id}$

At degree 0. V : $\boldsymbol{k}G$-module. $H^0(H,V) = \mathrm{Hom}_{\boldsymbol{k}G}(\boldsymbol{k}[G/H],V)$.

$$\boldsymbol{c} : \mathbf{Mod}_{\boldsymbol{k}G} \longrightarrow [\mathbf{Hec}_{\boldsymbol{k}}(G)^{\mathrm{op}}, \mathbf{Mod}_{\boldsymbol{k}}] : V \longmapsto \boldsymbol{c}_V$$
$$\boldsymbol{c}_V(H) = V^H := \{v \in V \mid hv = v \ \forall h \in H\}, \ c_V(X) = \mathrm{Hom}_{\boldsymbol{k}G}(\boldsymbol{k}X, V)$$
$$\mathrm{cor}_H^K(u) = \sum_{kH \in K/H} ku, \quad \mathrm{res}_H^K(v) = v, \ \mathrm{con}_H^g(u) = gu$$

Problem. Construct theory of Hecke categories and Hecke functors.
Even if V is irreducible, $\boldsymbol{c}_V$ is not in general!

Submodules of $\boldsymbol{c}_{\boldsymbol{k}}$ $\longleftrightarrow$ Ideal of poset {p−subgroups}/ $\sim_G$.
Webb conjecture: {p−subgroups $\neq 1$}/$\sim_G$ is contractible?

Center of category, Centralizer of functor.

$$
\begin{aligned}
Z(\mathcal{C}) &= \mathsf{EndNat}(\mathsf{Id} : \mathcal{C} \to \mathcal{C}) \\
&= \{(\omega(X) \in \mathsf{End}(X))_X \mid f \circ \omega(X) = \omega(Y) \circ f \quad (\forall f : X \to Y)\} \\
C(F) &= \mathsf{EndNat}(F : \mathcal{C} \to \mathcal{D}).
\end{aligned}
$$

Example. R ring viewd as a cat, then $Z(R)$ is usual center.
$Z(R) \longrightarrow Z(\mathbf{Mod}_R); z \longmapsto (z \cdot \mathsf{id}_M)_M$ is isomorphism.

$\mathcal{C}$ is $\boldsymbol{k}$-additive, then $Z(\mathcal{C})$ is commutative $\boldsymbol{k}$-algebra.
cpi(central primitive idempotent) or **block** of $\mathcal{C}$ is $0 \neq e^2 = e \in Z(\mathcal{C})$ (not proper orthogonal sum).
$\{e_1, \cdots, e_n\}$ cpi's, then $1 = e_1 + \cdots + e_n$, $e_i e_j = \delta_{ij} e_i$.

$\boldsymbol{M} : \mathcal{C}^{\mathsf{op}} \longrightarrow \mathbf{Mod}_{\boldsymbol{k}}$, then $\boldsymbol{M} = \bigoplus e_i \boldsymbol{M}$, $e_i \boldsymbol{M}(X) := \mathsf{Im}(\boldsymbol{M}(e_i(X)))$.
$\therefore$ indecomposable $\boldsymbol{M}$ **belongs** to a unique block e, i.e., $\boldsymbol{M} = e\boldsymbol{M}$.

Block theory of $\mathbf{Hec}_{\boldsymbol{k}}(G)$. C conjugacy class, $\overline{C} := \sum_{c \in C} c$ (class sum)

$$Z(\boldsymbol{k}G) \overset{\omega}{\cong} Z(\mathbf{Hec}_{\boldsymbol{k}}(G)) \quad ; \quad \overline{C} \longmapsto (\omega(\overline{C})(X) : X \to X)_X,$$
$$\omega(\overline{C})(X)_{xy} = \sharp\{c \in C \mid cy = x\}.$$

Krull-Schmidt cat. $\mathbf{Hec}_{\boldsymbol{k}}(G)^{+} := \{(X, e) \mid e^2 = e \in \mathrm{End}_{\mathbf{Hec}}(X)\}$.

Brauer functor. $\mathrm{char}(\boldsymbol{k}) = p > 0$, $P \leq G$ p-subgroup

$$\mathrm{Br}_P \ : \ \mathbf{Hec}_{\boldsymbol{k}}(G) \longrightarrow \mathbf{Hec}_{\boldsymbol{k}}(N_G(P))\,;\ X \longmapsto X^P \quad (P\text{-fixed points})$$
$$(a_{xy})_{x \in X, y \in Y} \longmapsto (a_{xy})_{x \in X^P, y \in Y^P}$$

$$\beta_P \ : \ Z(\boldsymbol{k}G) \longrightarrow Z(\boldsymbol{k}N_G(P));\ \sum_{g \in G} a_g g \longmapsto \sum_{g \in N_G(P)} a_g g$$

$$\mathrm{Br}_P^{*} \ : \ [\mathbf{Hec}_{\boldsymbol{k}}^{\mathrm{op}}(N_G(P)), \mathbf{Mod}_{\boldsymbol{k}}] \longrightarrow [\mathbf{Hec}_{\boldsymbol{k}}(G)^{\mathrm{op}}, \mathbf{Mod}_{\boldsymbol{k}}]$$

Example of Block Designs(BIBD). $|X| = v,\ |B| = b,\ B \subset 2^X$.

incidence matrix : $A = (a_{x,\beta})_{x\in X,\beta\in B}, \quad a_{x,\beta} := \begin{cases} 1 & x \in \beta \\ 0 & \text{else} \end{cases}$

(v, b, r, k, λ)-design (X, B) is defined by

$AJ = rJ,\ JA = kJ,\ A\,{}^tA = (r - \lambda)I + J \quad$ (J all-one matrix).

Let $G \leq \mathrm{Sym}(X)$ st. $x \in \beta \Rightarrow gx \in g\beta$.

$\det(A\,{}^tA) = \det(nI + \lambda J) = n^{v-1}kr$ ($n := r - \lambda$)
Thus if $(nrk)^{-1} \in \boldsymbol{k}$, then ${}^tA : X \to B$ is split mono ($\therefore\ |X| \leq |B|$).

$\mathrm{Ext}^*_{\boldsymbol{k}G}(\boldsymbol{k}X, V) \,|\, \mathrm{Ext}^*_{\boldsymbol{k}G}(\boldsymbol{k}B, V) \quad$ (V $\boldsymbol{k}G$-module).

$G = M_{24}$ (Mathieu group) acts on a (24, 759, 253, 8, 77)-design.
If $2^{-1}, 11^{-1}, 23^{-1} \in \boldsymbol{k}$, then $H^*(M_{23}, V) \,|\, H^*(2^4 \rtimes A_8, V)$.

Induction-transfer theorems. G finite group, $\boldsymbol{k}$ com. ring

$\mathrm{Ind}_H^G : \mathbf{Mod}_{\boldsymbol{k}H} \longrightarrow \mathbf{Mod}_{\boldsymbol{k}G}$; $U \longmapsto V{\uparrow}^G$ (induction functor)
$\mathrm{Res}_H^G : \mathbf{Mod}_{\boldsymbol{k}G} \longrightarrow \mathbf{Mod}_{\boldsymbol{k}H}$; $V \longmapsto V{\downarrow}_H$ (restriction functor)

$\mathrm{ind}_H^G : R(H) \longrightarrow R(G)$ (induciton map)
$\mathrm{res}_H^G : R(G) \longrightarrow R(H)$ (restriction map) $\qquad R(G)$ character ring

Bi-adjunction $\boxed{\mathrm{Ind}_H^G \dashv \mathrm{Res}_H^G, \quad \mathrm{Res}_H^G \dashv \mathrm{Ind}_H^G}$ induce
$\mathrm{cor}_H^G : H^n(H, V) \longrightarrow H^n(G, V)$ (transfer map)
$\mathrm{res}_H^G : H^n(G, V) \longrightarrow H^n(H, V)$ (restriction map)

$\mathrm{tr} : \mathrm{Hom}_{\boldsymbol{k}H}(U{\downarrow}_H, V{\downarrow}_H) \xrightarrow{\cong} \mathrm{Hom}_{\boldsymbol{k}G}(U, V{\downarrow}_H{\uparrow}^G) \xrightarrow{\epsilon_*} \mathrm{Hom}_{\boldsymbol{k}G}(U, V)$
$\epsilon : V{\downarrow}_H{\uparrow}^G \longrightarrow V$ the counit

Induction theorems.

– $\boldsymbol{Q} \otimes R(G) = \sum_C \boldsymbol{Q} \otimes \mathrm{Ind}_C^G(R(C))$ (C cyclic subgroups) (by Artin)

– $R(G) = \sum_E \mathrm{Ind}_E^G(R(E))$ (E elementary subgroups) (by Brauer)

Transfer Theorems. $P \in \mathrm{Syl}_p(G)$, $N := N_G(P)$, $(G:P)^{-1} \in \boldsymbol{k}$

– $H^n(G,k) \cong \{\alpha \in H^n(P,k) \mid \alpha|[PxP] = \deg([PxP])\alpha \ \forall x \in G\}$.

– $P \cap [G,G] = P \cap [N,N]$ if $P/K \ncong (\boldsymbol{Z}/p\boldsymbol{Z})\mathrm{wr}(\boldsymbol{Z}/p\boldsymbol{Z})$ $(\forall K \trianglelefteq P)$.

Q. Characterize Sylow 2-subgroups of perfect groups.

Want a unified theory of transfer theory and induction theory.

Generalized Hecke operator. $H, K \leq G,\ x \in G,\ \theta \in R(H)$

(1) $\theta|[H,x,A,K] := \theta^x \downarrow_A \uparrow^K \in R(K), \quad A \leq H^x \cap K.$

(2) $\theta|[H,x,\alpha,K] := (\theta^x \downarrow_{H^x \cap K} \alpha) \uparrow^K, \quad \alpha \in R(H^x \cap K).$

$$[H,x,\alpha,K] \circ [K,y,\beta,L] = \sum_{(H^x \cap K)k(K \cap {}^yL)} [H, xky, (\alpha^{ky} \downarrow_{H^{xky} \cap K^y \cap L} \beta \downarrow_{H^{xky} \cap K^y \cap L}) \uparrow^{H^{xky} \cap L}, K]$$

Similar action on group cohomology.

Abstract induction-transfer theory.
J.A.Green, Axiomatic rep theory of f.grps (JPAA, 1971)
A.Dress, Contributions to theory of induced reps (in SLNS 342, 1973),

Mackey functor. cat with sum and pullback, $\mathcal{S}$: cat.

$M = (M^*, M_*) : \mathcal{E} \longrightarrow \mathcal{S}$ with $M^*(X) = M_*(X) =: M(X)$.

M^* contravariant, $f^* := M^*(f) : M(Y) \longrightarrow M(X)$
M_* covariant, $f_* := M_*(f) : M(X) \longrightarrow M(Y)$ $\quad (\forall f : X \longrightarrow Y)$

(M1) $\boldsymbol{M}(\emptyset) = \{0\}, \boldsymbol{M}(X + Y) \cong \boldsymbol{M}(X) \times \boldsymbol{M}(Y)$ via $\boldsymbol{M}^*$.

(M2)

$$\begin{array}{ccc} W & \xrightarrow{p} & X \\ {\scriptstyle q}\downarrow & \text{P.B.} & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & Z \end{array} \quad \Rightarrow \quad \begin{array}{ccc} \boldsymbol{M}(W) & \xrightarrow{p_*} & \boldsymbol{M}(X) \\ {\scriptstyle q^*}\uparrow & \text{C} & \uparrow{\scriptstyle f^*} \\ \boldsymbol{M}(Y) & \xrightarrow{g_*} & \boldsymbol{M}(Z) \end{array}$$

Addition on $+ : \boldsymbol{M}(X) \times \boldsymbol{M}(X) \cong \boldsymbol{M}(X + X) \xrightarrow{\nabla_*} \boldsymbol{M}$

Example (1) $\boldsymbol{c}_V : X \mapsto \mathsf{Map}_G(X,V) = \mathsf{Hom}_{\boldsymbol{k}G}(\boldsymbol{k}X, V)$ $(V \in \mathbf{Mod}_{\boldsymbol{k}G})$.

$$(X \xrightarrow{f} Y) \longmapsto (\boldsymbol{k}X \rightleftarrows \boldsymbol{k}Y) \longmapsto (\boldsymbol{c}_V(X) \underset{f_*}{\overset{f^*}{\rightleftarrows}} \boldsymbol{c}_V(Y))$$

$$f^*(\beta : Y \to V)(x) = \beta \circ f(x), \quad f_*(\alpha : X \to V)(y) = \sum_{x \in f^{-1}(y)} \alpha(x)$$

(2) $\boldsymbol{E}_V^* : X \mapsto \mathsf{Ext}^*_{\boldsymbol{k}G}(\boldsymbol{k}X, V)$. $\boldsymbol{E}_V^*(G/H) \cong H^*(H,V)$. $\boldsymbol{c}_V(G/H) \cong V^H$.

(3) $X \longmapsto R(X)$ (Grothendienck ring of $\boldsymbol{C}G$-modules over X)
$\pi : A \to X$ is G-map and $\pi^{-1}(x)$ is $\boldsymbol{C}G_x$-module. $R(G/H) \cong R(H)$.

(4) **Burnside ring functor.** $X \mapsto \boldsymbol{B}(X)$ (Gro ring of f-G-sets over X)
(5) $X \mapsto \mathbf{Sub}_G(X)$, where $f^*(B) := f^{-1}(B)$, $f_*(A) := f(A)$.

Pairing $\rho : \boldsymbol{M} \times \boldsymbol{N} \longrightarrow \boldsymbol{L}$ is a family of biadditive maps

$$\rho_{X,Y} : \boldsymbol{M}(X) \times \boldsymbol{N}(Y) \longrightarrow \boldsymbol{L}(X \times Y)$$

$$\begin{array}{l} \rho_{X,Y} \circ (f^* \times g^*) = (f \times g)^* \circ \rho_{X',Y'} \\ \rho_{X',Y'} \circ (f_* \times g_*) = (f \times g)_* \circ \rho_{X,Y} \end{array} \qquad (\forall f : X \to X', g : Y \to Y')$$

Pairing induces biadditive maps $\rho' = (\rho'_X)$

$$\rho'_X : \boldsymbol{M}(X) \times \boldsymbol{N}(X) \xrightarrow{\rho_{XX}} \boldsymbol{L}(X \times X) \xrightarrow{\Delta^*} \boldsymbol{L}(X); (\alpha, \beta) \longmapsto \alpha \cdot \beta$$

$$f_*(\alpha) \cdot \beta' = f_*(\alpha \cdot f^*(\beta')),\ \alpha' \cdot f_*(\beta) = f_*(f^*(\alpha') \cdot \beta)$$

$\boldsymbol{R}$ is ring if $\boldsymbol{R}(X)$ is a "ring" by a pairing with ring homs f^*.
$\boldsymbol{M}$ is $\boldsymbol{R}$-module if $\boldsymbol{M}(X)$ is a "$\boldsymbol{R}(X)$-module".

Generalized Hecke category $\mathbf{Hec}(\mathcal{E}, \boldsymbol{A})$ for "ring" $\boldsymbol{A} : \mathcal{E} \to \mathbf{Mod}_{\boldsymbol{k}}$.

$\mathrm{Obj}(\mathbf{Hec}(\mathcal{E}, \boldsymbol{A})) = \mathrm{Obj}(\mathcal{E})$.
$\mathrm{Hom}_{\mathbf{Hec}}(Y, X) = \boldsymbol{A}(XY)$ ($XY := X \times Y$, etc.).

$$\boldsymbol{A}(XY) \times \boldsymbol{A}(YZ) \xrightarrow{\pi_{12}^* \times \pi_{23}^*} \boldsymbol{A}(XYZ) \times \boldsymbol{A}(XYZ) \xrightarrow{\mu} \boldsymbol{A}(XYZ) \xrightarrow{(\pi_{13})_*} \boldsymbol{A}(XZ)$$

Lemma. Cat of "$\boldsymbol{A}$-module" $\cong$ $\mathrm{Add}[\mathbf{Hec}(\mathcal{E}, \boldsymbol{A})^{\mathrm{op}}, \mathbf{Mod}_{\boldsymbol{k}}]$.

$\boldsymbol{B}$ Burnside ring functor. $\boldsymbol{B}(X) = \mathrm{Gro}(\mathcal{E}/X)$.

$$[X \xleftarrow{l} A \xrightarrow{r} Y] \circ [Y \xleftarrow{l} B \xrightarrow{r} Z] = [X \longleftarrow A \times_Y B \longrightarrow Z]$$

$\mathrm{Sp}(\mathcal{E})$ (cat of spans) is rep cat of McF's (Lindner 1976).

J.A.Green (1973): $H(\leq G) \longmapsto \boldsymbol{M}(H)$. Subgroup form.
A.Dress(1974): $X(\in \mathbf{set}^G) \longmapsto \boldsymbol{M}(X)$. G-set form.
H.Lindner(1976): McF as rep of cat of spans.
Y(1983): Hecke functor as rep of Hecke category.
P.Webb(1991): representation of Mackey algebra(=path alg of spans).
Bouc(1997): multiplicative induction of McF.
Tambara(1993) : Tambara functor (McF with multiplicative transfer)

Alperin conjecture $\Leftrightarrow$ $\exists \boldsymbol{M}_1, \boldsymbol{M}_2$ (McF) st.
(i) $\forall H \leq G$, $\mathrm{Res}^G_H(M_i)$ are projective relative to p-local subgrps of H.
(ii) $\forall H \leq G$, $\dim \boldsymbol{M}_1(H) - \dim \boldsymbol{M}_2(H) = \mathrm{np}(H)$.

Can not take as $\boldsymbol{M}_2 = 0$.

$(\boldsymbol{K},\boldsymbol{\mathcal{O}},\boldsymbol{F})$. char$(\boldsymbol{K}) = 0$, char$(\boldsymbol{F}) = p > 0$, $\boldsymbol{K} = (\boldsymbol{\mathcal{O}})$, $\boldsymbol{F} = \boldsymbol{\mathcal{O}}/J(\boldsymbol{\mathcal{O}})$

gen.Hecke cat $\mathbf{Hec}(G,\boldsymbol{R})$ ($\boldsymbol{R}$ ordinary character ring functor)

$\boldsymbol{R}(X \times Y) = \{(\alpha_{xy}) \mid \alpha_{xy} \in R(G_{xy}), \alpha_{gx,gy} = {}^{g}\alpha_{xy}\}$.
$[H,u,\alpha,K] \longleftrightarrow (\alpha_{xy})$, $\alpha_{ux,u} = {}^{u}\alpha$.

$$\Phi = (\Phi_t) : \mathbf{Hec}(G,\boldsymbol{R}) \rightarrowtail \prod_t \mathbf{Hec}(C_G(t)/\langle t\rangle,\boldsymbol{R}); X \longmapsto (X^{\langle t\rangle})$$
$1_K \otimes \Phi$ is full.

Theorem. $Z(\Phi) : Z(\mathbf{Hec}(G,\boldsymbol{\mathcal{O}})) \rightarrowtail \prod_t Z(\boldsymbol{\mathcal{O}}[C_G(t)/\langle t\rangle])$.
$$Z(\Phi) : Z(\mathbf{Hec}(G,\boldsymbol{K})) \cong \prod_t Z(K[C_G(t)/\langle t\rangle]).$$

cpi formulas of gen Hecke cats. $\boldsymbol{K} \otimes \mathbf{Hec}(G, \boldsymbol{R})$.

cpi of $\boldsymbol{K} \otimes \mathbf{Hec}(G, \boldsymbol{R})$ has the form $E_{t,\lambda} := (E_{t,\lambda}(X))_X$,
where t representative of conj class of G and $\lambda \in \mathsf{Irr}(C_G(t)/\langle t \rangle)$.

$$E_{t,\lambda}(X) = (\epsilon_{x,x'})_{x,x' \in X}, \ \epsilon_{x,x'} \in \boldsymbol{K} \otimes \boldsymbol{R}(G_{x,x'}),$$
$$\epsilon_{x,x'}(s) = 0 \quad \text{if } s \not\sim_G t$$
$$\epsilon_{xx'}(gtg^{-1}) = \frac{\lambda(1)}{|C_G(t)|} \sum_{c : x'g = xgc} \lambda(c^{-1})$$

$$E_{t,\lambda}(H) = \frac{\lambda(1)}{|H| \cdot |C_G(t)|^2} \sum_{\substack{c \in C_G(t) \\ g \in G : t^g \in H}} \sum_{\alpha \in \mathsf{Irr}(H \cap H^{c^g})} \lambda(c^{-1}) \alpha(t^g) [H, c^g, \alpha, H].$$

p-**local cpi formulas** for $\mathcal{O}\mathbf{Hec}(G,\boldsymbol{R})$. E_1, E_2 cpi of $\boldsymbol{K}\mathbf{Hec}(G,\boldsymbol{R})$.
$E_1 \sim E_2 \Leftrightarrow \exists E$(cpi of $\boldsymbol{\mathcal{O}}\mathbf{Hec}(G,\boldsymbol{R})$) st. $E_i = EE_i$

$$E^{(p)}_{t,B} = \sum_{\overset{(s)}{:s_{p'}=t}} \sum_{b:b^{C_G(t)}=B} \sum_{\lambda\in b} E_{s,\lambda}.$$

B is a p-block of $C_G(t)$, b is a p-block of $C_G(s)$.

Brauer functor $\mathrm{Br}_P : \boldsymbol{\mathcal{O}}\mathbf{Hec}(G,\boldsymbol{R}) \longrightarrow \boldsymbol{F}\mathbf{Hec}(N_G(P)/P,\boldsymbol{R}); X \longmapsto X^P$.

Transfer theorem. $\epsilon^{(p)}_t$: cpi of $\boldsymbol{\mathcal{O}}\boldsymbol{R}(G)$.
$\epsilon^{(p)}_t \boldsymbol{\mathcal{O}}\mathbf{Hec}(G,\boldsymbol{R}) \cong \mathrm{res}_{C_G(t)}(\epsilon^{(p)}_t)\boldsymbol{\mathcal{O}}\mathbf{Hec}(C_G(t),\boldsymbol{R})$.

Coefficient of $[H, x, \alpha, H]$ is given by

$$\sum_{(s):s_{p'}=t} \sum_{b^C=B} \sum_{\lambda\in b} \sum_{g\in G:s^g\in H} \sum_{h\in H\cap C_G(s)^g x^{-1}} \frac{\lambda(1)\lambda(gx^{-1}h^{-1}g^{-1})\alpha(s^g)}{|H^x \cap H| \cdot |C_G(s)|^2}.$$

Special case : $t = 1, B = \{\lambda\} \Rightarrow \frac{1}{|H|} \sum_{y\in HxH} \lambda(y) \in \mathcal{O}$ (Nagao).

$$\boldsymbol{\mathcal{O}R}(H) = \bigoplus_{t,B} E_{t,B}^{(p)}(H)\boldsymbol{\mathcal{O}R}(H).$$

pi of $\boldsymbol{KR}(G)$ is $\epsilon_\chi := \frac{\chi(1)}{|G|} \sum_{g\in G} \chi(g^{-1})\chi.$

pi of $\boldsymbol{\mathcal{O}R}(G)$ is $\sum_{\chi\in B} \epsilon_\chi = |G|^{-1} \sum_{g\in G} \sum_{\chi\in B} \chi(g^{-1})\chi.$

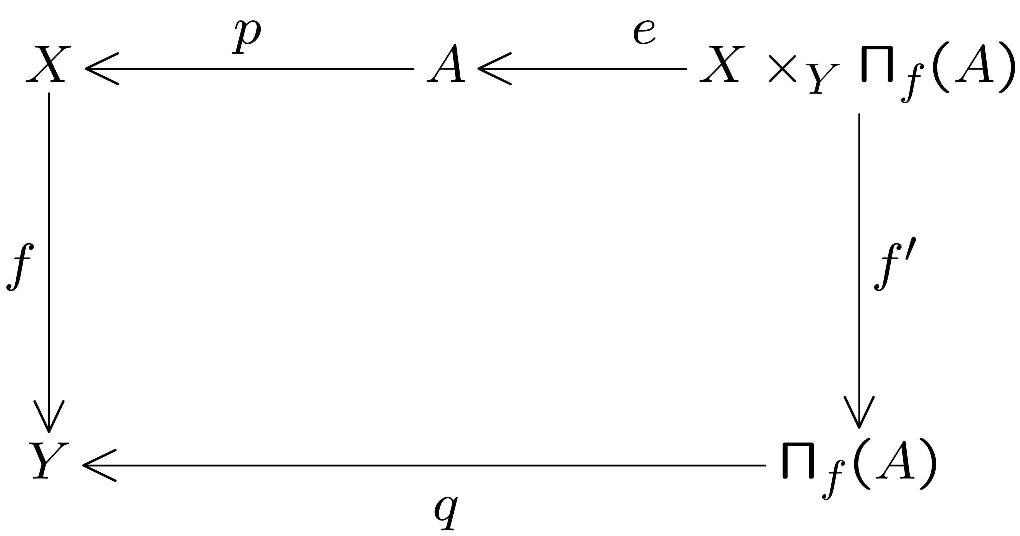
Further generalizations. $\mathcal{E}$: **locally finite topos** (LFT),
$\mathcal{E} = \mathbf{set}, \mathbf{set}^G, [\Gamma^{\mathrm{op}}, \mathbf{set}], (\text{Forests})$
$\mathcal{S} =$ (Bimodule), Cat of cats, (grp) (2-)cats.
Study $\mathbf{Hec}(\mathcal{E}, \boldsymbol{A})$ of LFT $\mathcal{E}$ for $\boldsymbol{A}$: Sp$(\mathcal{E})^{\mathbf{op}} \to \mathcal{S}$.

Sp($\mathcal{E}$) $((X \longleftarrow A \longrightarrow Y) \in \mathrm{Hom}(Y, X) = \mathcal{E}/X \times Y)$ is 2-cat.
Study 2-Mackey functor $\boldsymbol{M}$: Sp($\mathcal{E}$) $\to \mathcal{S}$.
$H \longmapsto \mathbf{Mod}_{\boldsymbol{k}H}$, Ind, Res, Con. $X \longmapsto \mathcal{E}/X$(2-Tambara functor).
2-Mackey functor from $\mathbf{set}$ to (Bimodule) ??
span is a bipartite graph. association schemes??

Non-module valued McF. $X(\in \mathcal{E}) \longmapsto \mathbf{Sub}(X)$ (Heyting algebra).

grp :cat of f-groups. **Finite group theory** in $\mathcal{E} \subseteq [\mathrm{grp}^{\mathrm{op}}, \mathbf{set}]$!?

Exponential diagram



$$\Pi_f A := \{(y,\sigma) \mid y \in Y, \sigma : q^{-1}(y) \longrightarrow A, p\sigma = \mathsf{id}\},$$
$$q : (y,\sigma) \longmapsto y, \quad f' : (x,y,\sigma) \longmapsto (y,\sigma)$$
$$e : (x,y,\sigma) \longmapsto \sigma(x)$$

Tambara functor. $\boldsymbol{T} = (T_!, T^*, T_\star) : \mathbf{set}^G \longrightarrow \mathbf{Set}$.

(T1) $(T_!, T^*), (T_\star, T^*)$ are both Mackey functors,

(T2)

$$
\begin{array}{ccccc}
\cdot & \xleftarrow{\ p\ } & \cdot & \xleftarrow{\ e\ } & \cdot \\
{\scriptstyle f}\downarrow & & \text{EX} & & \downarrow{\scriptstyle f'} \\
\cdot & & \xleftarrow[\ q\ ]{} & & \cdot
\end{array}
\quad \Rightarrow \quad f_\star \circ p_* = q_* \circ f'_\star \circ e^*.
$$

f_* :additive transfer, $f_\star$: multiplicative transfer.

$H^{**}(\cdot, \boldsymbol{k})$, $\boldsymbol{R}$

Polynomials and power series. $\mathcal{E}$ locally finite topos(LFT).
$\boldsymbol{T} : \mathcal{E} \longrightarrow \mathbf{Set}$ Tambara functor.

$i : N \rightarrowtail N'$ induces $i_! : \boldsymbol{T}(\Omega^N) \longrightarrow \boldsymbol{T}(\Omega^{N'})$, $i^* : \boldsymbol{T}(\Omega^{N'}) \longrightarrow \boldsymbol{T}(\Omega^N)$.
$\boldsymbol{T}[\cdot] := \varinjlim \boldsymbol{T}(\Omega^N), \quad \boldsymbol{T}[[\cdot]] := \varprojlim \boldsymbol{T}(\Omega^N)$

addition, multiplication, composition, derivation, substitution, ...

Example. in set,

$$[A \xrightarrow{\delta} 2^N] \;\longleftrightarrow\; f(t) = \sum_{a \in A} t^{|\delta(a)|} = \sum_{n \geq 0} \sharp\{a \in A \mid |\delta(a)| = n\} t^n$$

Example from error correcting codes $F := \boldsymbol{F}_q,\ N := \{1, 2, \cdots, n\}$
code $C \leq V := F^N := \{v = (v_1, \cdots, v_n) \mid v_i \in F\}$.
$\mathsf{supp}(v) := \{i \in N \mid v_i \neq 0\} \subseteq N,\ |v| := |\mathsf{supp}(v)|$.

$$\begin{array}{ccc} W_C[X,Y] & \longrightarrow & C \\ \downarrow & \text{P.B.} & \downarrow \\ (X+Y)^N & \longrightarrow & 2^N \end{array}$$

$$[C \hookrightarrow F^N \xrightarrow{\mathsf{supp}} 2^N] \longleftrightarrow \sum_{u \in C} t^{\mathsf{supp}(u)} = W_C(t)$$

(weight enumerator)

$$|W_C[X,Y]| = \sum_{u \in C} |X|^{n-|u|} |Y|^{|u|}$$

$$W_C[X,Y] = \{(u, \lambda) \mid u \in C, \lambda : N \to X + Y, \lambda^{-1}(Y) = \mathsf{supp}(u)\}$$

Let $G \leq \mathsf{Aut}(C) \leq \mathsf{Sym}(N)$, X, Y finite G-sets. $W_C[X,Y]$ is G-set.

$$
\begin{aligned}
|W_C[X,Y]^H| &= \sum_{u \in C^H} |\mathsf{Map}_H(N - \mathsf{supp}(u), X)| \cdot |\mathsf{Map}_H(\mathsf{supp}(u), Y)| \\
&= \sum_{(r_i)} \sharp\{u \in C^H \mid \mathsf{supp}(u) \cong_H \coprod_i r_i(H/D_i)\} \prod_i x_i^{n_i - r_i} y_i^{r_i}
\end{aligned}
$$

MacWilliams identity. $C \times W_C[X+Y, Y] \cong_G W_C[X + F \times Y, X]$.

$C(R) := \{u \in C \mid \mathsf{supp}(u) \subseteq R\} \quad (R \in 2^N)$.

GF of $C(\cdot)$ is $W_C[X+Y,Y]$. Y(1993)