SERRE DUALITY FOR NONCOMMUTATIVE ALGEBRAS

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Abstract. One of the major projects in noncommutative algebraic geometry is to classify noncommutative curves and noncommutative surfaces [17]. The first half of this article will briefly describe the classification of noncommutative projective curves due to Artin and Stafford [2] and the classification of quantum projective planes due to Artin, Tate and Van den Bergh [3]. In the second half, we will propose a new intersection theory in terms of Ext instead of Tor, so that we can define it over noncommutative schemes, and see that we can extend Riemann-Roch and Adjunction Formula to noncommutative surfaces satisfying Serre duality.

1. Quasi-schemes

Throughout, let k be an algebraically closed field. First, we extend the notion of scheme as follows.

Definition 1.1. [4], [18], [12] *A (triangulated) quasi-scheme over k is a pair*

$$X = (\operatorname{mod} X, \mathcal{O}_X)$$

where

- $\operatorname{mod} X$ *is a k-linear abelian (triangulated) category, and*
- $\mathcal{O}_X \in \operatorname{mod} X$ *is an object.*

Two (triangulated) quasi-schemes X and Y are isomorphic if there exists a k-linear (exact) equivalence functor

$$F : \operatorname{mod} X \to \operatorname{mod} Y$$

such that $F(\mathcal{O}_X) \cong \mathcal{O}_Y$.

The below are typical examples of a quasi-scheme.

Example 1.2. *A (usual) noetherian scheme X is a quasi-scheme*

$$X = (\operatorname{mod} X, \mathcal{O}_X)$$

where

- $\mathcal{O}_X$ *is the structure sheaf on X, and*
- $\operatorname{mod} X$ *is the category of coherent $\mathcal{O}_X$-modules.*

Example 1.3. *For a right noetherian algebra R,*

$$\operatorname{Spec} R := (\operatorname{mod} R, R)$$

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is a quasi-scheme where $\operatorname{mod} R$ *is the category of finitely generated right* R*-modules. If* R *is commutative and* $X = \operatorname{Spec} R$ *in the usual sense, then the global section functor*

$$\Gamma(X, -) : \operatorname{mod} X \to \operatorname{mod} R$$

induces an isomorphism $(\operatorname{mod} X, \mathcal{O}_X) \to (\operatorname{mod} R, R)$ *of quasi-schemes.*

Example 1.4. *For a Frobenius algebra* R*,*

$$(\underline{\operatorname{mod}} R, R/\mathfrak{r})$$

is a triangulated quasi-scheme where $\underline{\operatorname{mod}} R$ *is the stable category of* $\operatorname{mod} R$ *by projective modules, and* $\mathfrak{r}$ *is the radical of* R*.*

Example 1.5. *If* $X = (\operatorname{mod} X, \mathcal{O}_X)$ *is a quasi-scheme, then*

$$\mathcal{D}(X) := (\mathcal{D}^b(\operatorname{mod} X), \mathcal{O}_X)$$

is a triangulated quasi-scheme where $\mathcal{D}^b(\operatorname{mod} X)$ *is the bounded derived category of* $\operatorname{mod} X$*.*

Example 1.6. *For a right noetherian graded algebra* A*,*

$$\operatorname{GrSpec} A := (\operatorname{grmod} A, A)$$

is a quasi-scheme where $\operatorname{grmod} A$ *is the category of finitely generated graded right* A*-modules.*

Example 1.7. [4] *For a right noetherian graded algebra* A

$$\operatorname{Proj} A := (\operatorname{tails} A, \mathcal{A})$$

is a quasi-scheme where

$$\begin{aligned} &\operatorname{tails} A := \operatorname{grmod} A / \sim \\ &M \sim N \Leftrightarrow M_{\geq n} \cong N_{\geq n} \text{ for } n \gg 0, \end{aligned}$$

and $\mathcal{A}$ *is the image of* $A \in \operatorname{grmod} A$ *in* $\operatorname{tails} A$*.*

Theorem 1.8. [16] *If* A *is a commutative graded algebra finitely generated in degree 1 over* k *and* $X = \operatorname{Proj} A$ *in the usual sense, then the functor*

$$\begin{array}{ccc} \operatorname{mod} X & \longrightarrow & \operatorname{grmod} A \\ \mathcal{F} & \longrightarrow & \Gamma_*(X, \mathcal{F}) := \bigoplus_{n \in \mathbf{Z}} \Gamma(X, \mathcal{F}(n)) \end{array}$$

induces an isomorphism $(\operatorname{mod} X, \mathcal{O}_X) \to (\operatorname{tails} A, \mathcal{A})$ *of quasi-schemes.*

A noncommutative projective variety of dimension d is a quasi-scheme of the form $\operatorname{Proj} A$ where A is a graded domain finitely generated in degree 1 over k of $\operatorname{GKdim} A = d + 1$. One of the major projects in noncommutative algebraic geometry is to classify noncommutative projective curves and noncommutative projective surfaces (see [17] for more details).

2. Weak Divisors

In this section, we extend the notion of divisor, and construct a graded algebra analogous to a homogeneous coordinate ring.

Definition 2.1. [12] *Let X be a (triangulated) quasi-scheme. A weak divisor on X is a k-linear (exact) autoequivalence*

$$D : \operatorname{mod} X \to \operatorname{mod} X.$$

We denote by $\operatorname{WPic} X$ the group of weak divisors on X. For $D \in \operatorname{WPic} X$ and $n \in \mathbf{Z}$, we denote

$$\begin{aligned} D^n : \operatorname{mod} X &\to \operatorname{mod} X \\ \mathcal{M} &\mapsto \mathcal{M}(nD). \end{aligned}$$

If X is a (usual) noetherian scheme and D is a Cartier divisor on X, then

$$\begin{aligned} - \otimes_X \mathcal{O}_X(D) : \operatorname{mod} X &\to \operatorname{mod} X \\ \mathcal{F} &\mapsto \mathcal{F}(D) := \mathcal{F} \otimes_X \mathcal{O}_X(D) \end{aligned}$$

is a k-linear autoequivalence.

Theorem 2.2. [7] *If X is a smooth projective variety with an ample or anti-ample canonical divisor, then* $\operatorname{WPic} X$ *is a semi-direct product of* $\operatorname{Aut} X$ *and* $\operatorname{Pic} X$.

If $D = (\sigma, \mathcal{L}) \in \operatorname{WPic} X$ where $\sigma \in \operatorname{Aut} X, \mathcal{L} \in \operatorname{Pic} X$, then we define

$$\begin{aligned} D = (\sigma, \mathcal{L}) : \operatorname{mod} X &\to \operatorname{mod} X \\ \mathcal{F} &\mapsto \mathcal{F}(D) := \sigma_*(\mathcal{F} \otimes_X \mathcal{L}). \end{aligned}$$

If X is a (triangulated) quasi-scheme over k and $D \in \operatorname{WPic} X$, then we can construct a graded algebra over k by

$$B(X, D) := \bigoplus_{n \in \mathbf{Z}} \operatorname{Hom}_X(\mathcal{O}_X, \mathcal{O}_X(nD))$$

with the multiplication defined as follows: for

$$a \in \operatorname{Hom}_X(\mathcal{O}_X, \mathcal{O}_X(mD)), b \in \operatorname{Hom}_X(\mathcal{O}_X, \mathcal{O}_X(nD)),$$

$$ab := a(nD) \circ b \in \operatorname{Hom}_X(\mathcal{O}_X, \mathcal{O}_X((m+n)D))$$

where

$$\begin{tikzcd} & \mathcal{O}_X \arrow[r, "a"] \arrow[d, "nD"'] & \mathcal{O}_X(mD) \arrow[d, "nD"] \\ \mathcal{O}_X \arrow[r, "b"] & \mathcal{O}_X(nD) \arrow[r, "a(nD)"] & \mathcal{O}_X(mD)(nD) = \mathcal{O}_X((m+n)D). \end{tikzcd}$$

The below are a few examples of this construction.

Example 2.3. *If $X = \operatorname{GrSpec} A$ where A is a graded algebra, then*

$$
\begin{aligned}
(1) : \operatorname{grmod} A &\to \operatorname{grmod} A \\
M &\mapsto M(1)
\end{aligned}
$$

where $M(1)_i = M_{i+1}$ is a weak divisor on X, and

$$B(X, (1)) = \bigoplus_{n \in \mathbf{Z}} \operatorname{Hom}_{\operatorname{grmod} A}(A, A(n)) \cong A.$$

Example 2.4. *(Yoneda Ext-algebra) If $X = (\mathcal{D}^b(\operatorname{mod} R), R/\mathfrak{r})$ is a triangulated quasi-scheme where R is an algebra and $\mathfrak{r}$ is the radical of R, then*

$$
\begin{aligned}
[1] : \mathcal{D}^b(\operatorname{mod} R) &\to \mathcal{D}^b(\operatorname{mod} R) \\
M &\mapsto M[1]
\end{aligned}
$$

where $M[1]^i = M^{i+1}$ is a weak divisor on X, and

$$B(X, [1]) = \bigoplus_{n \in \mathbf{Z}} \operatorname{Hom}_{\mathcal{D}^b(\operatorname{mod} R)}(R/\mathfrak{r}, R/\mathfrak{r}[n]) \cong \bigoplus_{n \in \mathbf{Z}} \operatorname{Ext}^n_R(R/\mathfrak{r}, R/\mathfrak{r})$$

is the Yoneda Ext-algebra of R.

Example 2.5. *(Homogeneous coordinate ring) Let X be a variety. Then X is projective if and only if X has an ample divisor. In fact, if X has an ample divisor, then X has a very ample divisor $D \in \operatorname{Pic} X$, so that $B(X, D)$ is a homogeneous coordinate ring of X and $X \cong \operatorname{Proj} B(X, D)$.*

3. Classification of Noncommutative Projective Curves and Quantum Projective Planes

In this section, we will describe the classification of noncommutative projective curves and quantum projective planes. We first characterize a quasi-scheme to be a noncommutative projective scheme.

Definition 3.1. [4] *Let X be a quasi-scheme and $D \in \operatorname{WPic} X$ a weak divisor. We say that D is ample if*

- *$\{\mathcal{O}_X(-nD)\}_{n \in \mathbf{N}}$ is a set of generators for $\operatorname{mod} X$, and*
- *for every epimorphism $\mathcal{M} \to \mathcal{N}$,*

  $$\operatorname{Hom}(\mathcal{O}_X(-nD), \mathcal{M}) \to \operatorname{Hom}_X(\mathcal{O}_X(-nD), \mathcal{N})$$

  is surjective for all $n \gg 0$.

Definition 3.2. *We say that a right noetherian graded algebra A satisfies χ_1 if $\dim_k \operatorname{Ext}^1_A(A/A_{\geq 1}, M) < \infty$ for all $M \in \operatorname{grmod} A$.*

Theorem 3.3. [4] *Let X be a noetherian Hom-finite quasi-scheme. Then $X \cong \operatorname{Proj} A$ for some right noetherian graded algebra A satisfying χ_1 if and only if X has an ample weak divisor. In particular, if D is an ample weak divisor on X, then $X \cong \operatorname{Proj} B(X, D)$.*

The following theorem due to Artin and Stafford says that every noncommutative projective curve is isomorphic to a commutative projective curve.

Theorem 3.4. [2] *If A is a graded domain finitely generated in degree 1 over k of* $\mathrm{GKdim}\, A = 2$*, so that* $\operatorname{Proj} A$ *is a noncommutative projective curve, then there exist a commutative projective curve X and an ample weak divisor D on X such that $A \sim B(X, D)$. In particular,* $\operatorname{Proj} A \cong \operatorname{Proj} B(X, D) \cong X$.

We now define a quantum projective plane which is a noncommutative analogue of the projective plane.

Definition 3.5. [1] *A connected graded algebra A is called AS-regular (resp. AS-Gorenstein) if*

- $\operatorname{gldim} A = d < \infty$ *(resp.* $\mathrm{id}(A) = d < \infty$*), and*
- $\operatorname{Ext}^i_A(k, A) = \begin{cases} k & \text{if } i = d \\ 0 & \text{if } i \neq d. \end{cases}$

A quantum polynomial algebra is a noetherian AS-regular algebra A such that

$$H_A(t) := \sum_{i \in \mathbf{N}} (\dim_k A_i) t^i = (1 - t)^{-d}.$$

A quantum projective space is a quasi-scheme of the form $\operatorname{Proj} A$ *where A is a quantum polynomial algebra.*

We will construct another graded algebra from geometry. If $X \subset \mathbf{P}(V)$ is a projective scheme and $\sigma \in \operatorname{Aut} X$ is an automorphism, then we can construct a quadratic algebra

$$A(X, \sigma) = T(V^*)/(\{f \in V^* \otimes_k V^* \mid f(p, \sigma(p)) = 0 \text{ for all } p \in X\})$$

where V^* is the vector space dual of V. There is a natural graded algebra homomorphism (often surjective)

$$A(X, \sigma) \to B(X, D)$$

which induces a map (often an embedding)

$$\operatorname{Proj} B(X, D) \to \operatorname{Proj} A(X, \sigma)$$

where $D = (\sigma, \mathcal{O}_X(1)) \in \operatorname{WPic} X$.

The following theorem due to Artin, Tate and Van den Bergh says that quantum projective planes can be classified by geometric pairs (X, σ).

Theorem 3.6. [3] *If A is a quantum polynomial algebra of* $\operatorname{gldim} A = 3$*, then there exist a commutative projective scheme X and an automorphism $\sigma \in \operatorname{Aut} X$ such that $A \cong A(X, \sigma)$.*

Example 3.7. *(3-dimensional Sklyanin algebra) For a generic choice of $a, b, c \in k$,*

$$A = k\langle x, y, z\rangle/(cx^2 + bzy + ayz, azx + cy^2 + bxz, byx + axy + cz^2) = A(X, \sigma)$$

is a quantum polynomial algebra of $\operatorname{gldim} A = 3$ *where*

$$X = \mathcal{V}((a^3 + b^3 + c^3)xyz - abc(x^3 + y^3 + z^3)) \subset \mathbf{P}^2$$

is a smooth elliptic curve and $\sigma \in \operatorname{Aut} X$ is the translation by the point $(a, b, c) \in X$.

4. Serre Duality

It is impractical to classify all surfaces. We would like to classify only nice ones. Unfortunately, we do not have a notion of smoothness for noncommutative schemes yet, but one possible condition we may impose on noncommutative schemes to be nice is existence of a Serre functor defined below.

Definition 4.1. [6] *Let X be a* Hom*-finite quasi-scheme over k. A Serre functor on X is a weak divisor $K \in \operatorname{WPic} X$ such that*

$$D\operatorname{Hom}_X(\mathcal{M},\mathcal{N}) \cong \operatorname{Hom}_X(\mathcal{N},\mathcal{M}(K))$$

for all $\mathcal{M},\mathcal{N} \in \operatorname{mod} X$, where

$$D : \operatorname{mod} k \to \operatorname{mod} k$$

is the autoequivalence taking the k-vector space dual.

A Serre functor induces both Serre duality and Auslander-Reiten Formula.

Example 4.2. *(Serre Duality) If X is a (usual) noetherian smooth projective scheme of $\dim X = d$, then*

$$K = - \otimes_X \omega_X[d] : \mathcal{D}^b(\operatorname{mod} X) \to \mathcal{D}^b(\operatorname{mod} X)$$

is a Serre functor where ω_X is the canonical sheaf on X. In particular, for any $\mathcal{F} \in \operatorname{mod} X$,

$$\begin{aligned} D\operatorname{H}^i(X,\mathcal{F}) &\cong D\operatorname{Ext}^i_X(\mathcal{O}_X,\mathcal{F}) \cong D\operatorname{Hom}_X(\mathcal{O}_X,\mathcal{F}[i]) \\ &\cong \operatorname{Hom}_X(\mathcal{F}[i],\mathcal{O}_X \otimes_X \omega_X[d]) \cong \operatorname{Hom}_X(\mathcal{F},\omega_X[d-i]) \\ &\cong \operatorname{Ext}^{d-i}_X(\mathcal{F},\omega_X). \end{aligned}$$

Example 4.3. *(Auslander-Reiten Formula) If R is a Frobenius algebra, then*

$$K = \mathcal{N}\Omega : \underline{\operatorname{mod}} R \to \underline{\operatorname{mod}} R$$

is a Serre functor where

$$\mathcal{N}(-) = D\operatorname{Hom}_R(-,R) : \operatorname{mod} R \to \operatorname{mod} R$$

is the Nakayama functor. In particular, for any pair $M,N \in \operatorname{mod} R$,

$$\begin{aligned} D\operatorname{Ext}^1_R(M,N) &\cong D\underline{\operatorname{Hom}}_R(\Omega M,N) \\ &\cong \underline{\operatorname{Hom}}_R(N,\mathcal{N}\Omega^2 M) \cong \underline{\operatorname{Hom}}_R(N,D\operatorname{Tr} M). \end{aligned}$$

We will see that the above two examples of a Serre functor are somehow related. Let A be a connected graded algebra and $M \in \operatorname{grmod} A$. We define the i-th local cohomology of M by

$$\operatorname{H}^i_{\mathfrak{m}}(M) := \lim_{n\to\infty} \operatorname{Ext}^i_A(A/A_{\geq n},M).$$

If A is a noetherian AS-Gorenstein algebra of $\operatorname{id}(A) = d$, then the canonical module of A is a graded A-A bimodule defined by

$$\omega_A := D\operatorname{H}^d_{\mathfrak{m}}(A),$$

and the (generalized) Nakayama functor of A is defined by

$$\mathcal{N} = - \otimes_A \omega_A : \operatorname{grmod} A \to \operatorname{grmod} A.$$

In fact, if A is Frobenius, then $\operatorname{id}(A) = 0$, so

$$\omega_A = D \operatorname{H}^0_{\mathfrak{m}}(A) = D \lim_{n\to\infty} \operatorname{Hom}_A(A/A_{\geq n}, A) \cong D(A)$$

hence

$$\mathcal{N} = - \otimes_A \omega_A \cong - \otimes_A D(A) : \operatorname{grmod} A \to \operatorname{grmod} A$$

is the usual Nakayama functor.

Example 4.4. [10], [13] *If A is a noetherian AS-regular algebra of $\operatorname{gldim} A = d+1$ so that $X = \operatorname{Proj} A$ is a noncommutative (smooth) projective scheme of $\dim X = d$, and $\mathcal{N}$ is the (generalized) Nakayama functor of A, then*

$$K = \mathcal{N}(-)[d] \cong - \otimes_A \omega_A[d] : \mathcal{D}^b(\operatorname{mod} X) \to \mathcal{D}^b(\operatorname{mod} X)$$

is a Serre functor.

5. Intersection Theory

Since intersection theory plays an essential role in classifying commutative surfaces, we should extend it to noncommutative schemes. Since there is no geometry on noncommutative schemes, we should find a way of defining intersection theory using homological algebra rather than geometry. Serre defined the intersection multiplicity in terms of Tor, but, in general, we cannot define Tor for noncommutative schemes. Fortunately, we can always define Ext for noncommutative schemes.

Definition 5.1. [15], [12] *Let X be a (triangulated) quasi-scheme over k. We define the Euler form of $\mathcal{M}, \mathcal{N} \in \operatorname{mod} X$ by*

$$\xi(\mathcal{M}, \mathcal{N}) := \sum_{i\in\mathbf{Z}} (-1)^i \dim_k \operatorname{Ext}^i_X(\mathcal{M}, \mathcal{N}),$$

and the Euler characteristic of $\mathcal{M} \in \operatorname{mod} X$ by

$$\chi(\mathcal{M}) := \xi(\mathcal{O}_X, \mathcal{M}) = \sum_{i\in\mathbf{Z}} (-1)^i \dim_k \operatorname{Ext}^i_X(\mathcal{O}_X, \mathcal{M}).$$

We define the intersection multiplicity of $\mathcal{M}$ and $\mathcal{N}$ by

$$\mathcal{M} \cdot \mathcal{N} := (-1)^{\operatorname{codim} \mathcal{M}} \xi(\mathcal{M}, \mathcal{N}).$$

Fortunately, this new definition of intersection multiplicity in terms of Ext agrees with the classical one in terms of Tor over a (usual) smooth scheme.

Theorem 5.2. [8] *If X is a (usual) noetherian smooth scheme over k, and C, D are closed subschemes of X such that $\dim C + \dim D \leq \dim X$, then*

$$C \cdot D = \mathcal{O}_C \cdot \mathcal{O}_D$$

(where the left-hand side is defined in term of Tor *and the right-hand side is defined in term of* Ext*).*

Definition 5.3. [12] *Let X be a noetherian (triangulated) quasi-scheme. The Grothendieck group of X is defined by*

$$K_0(X) := K_0(\operatorname{mod} X).$$

The structure sheaf of a weak divisor $D \in \operatorname{WPic} X$ is defined by

$$\mathcal{O}_D := [\mathcal{O}_X] - [\mathcal{O}_X(-D)] \in K_0(X).$$

If $C, D \in \operatorname{WPic} X$, then we define

$$C \cdot D := \mathcal{O}_C \cdot \mathcal{O}_D = -\xi(\mathcal{O}_C, \mathcal{O}_D).$$

If X is a noetherian smooth scheme, D is an effective Cartier divisor, and $\mathcal{O}_D$ is the structure sheaf of D in the usual sense, then there is an exact sequence

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$$

in $\operatorname{mod} X$, so

$$[\mathcal{O}_D] = [\mathcal{O}_X] - [\mathcal{O}_X(-D)] \in K_0(X).$$

Theorem 5.4. [12] *Let X be a (triangulated) quasi-scheme over k having a Serre functor $K \in \operatorname{WPic} D(X)$ ($K \in \operatorname{WPic} X$), and $D \in \operatorname{WPic} X$ any weak divisor on X. Then*

(1) *(Riemann-Roch)*

$$\chi(\mathcal{O}_X(D)) = \frac{1}{2}(D \cdot D - K \cdot D) + 1 + p_a,$$

where $p_a = \chi(\mathcal{O}_X) - 1$ is the arithmetic genus of X, and

(2) *(Adjunction Formula)*

$$2g - 2 = D \cdot D + K \cdot D,$$

where $g = 1 - \chi(\mathcal{O}_D)$ is the genus of D.

If X is a (usual) noetherian smooth projective surface, then the above formulas agree with the usual Riemann-Roch and Adjunction Formula ([9, Chapter V Theorem 1.6, Proposition 1.5]). However, in the above formulas:

(1) D does not have to be an effective divisor. Formulas hold for any weak divisor $D \in \operatorname{WPic} X$.
(2) $\mathcal{O}_X$ does not have to be the structure sheaf of X. Formulas hold for any choice of an object $\mathcal{O}_X \in \operatorname{mod} X$.
(3) X does not have to be a surface. Formulas hold for a noetherian smooth projective scheme of any dimension.
(4) X does not have to be smooth. Formulas hold for any noetherian Gorenstein projective scheme.

6. Symmetry in the Vanishing of Ext

Let R be an algebra and $M, N \in \operatorname{mod} R$. Then

$$M \cdot N = (-1)^{\operatorname{codim} M} \sum_{i \in \mathbf{N}} (-1)^i \dim_k \operatorname{Ext}^i_R(M, N)$$

is well-defined if and only if

- $\dim_k \operatorname{Ext}^i_R(M, N) < \infty$ for all $i \in \mathbf{N}$, and
- $\operatorname{Ext}^i_R(M, N) = 0$ for all $i \gg 0$.

Question 6.1. *$M \cdot N$ is well-defined if and only if $N \cdot M$ is well-defined?*

The first condition is symmetric over a commutative local ring.

Theorem 6.2. [11] *If R is a noetherian commutative local ring, then, for any pair $M, N \in \operatorname{mod} R$,*

$$\lambda(\operatorname{Ext}^i_R(M, N)) < \infty \text{ for all } i \in \mathbf{N} \Longleftrightarrow \lambda(\operatorname{Ext}^i_R(N, M)) < \infty \text{ for all } i \in \mathbf{N},$$

where $\lambda(-)$ is the length of a module.

Definition 6.3. *We say that a ring R satisfies (**ee**) if, for any pair $M, N \in \operatorname{mod} R$,*

$$\operatorname{Ext}^i_R(M, N) = 0 \text{ for all } i \gg 0 \Longleftrightarrow \operatorname{Ext}^i_R(N, M) = 0 \text{ for all } i \gg 0.$$

It is easy to see that R must be Gorenstein in order to satisfy (**ee**).

Theorem 6.4. [5] *If R is a noetherian commutative complete intersection ring, then R satisfies (**ee**).*

We will show an analogous result for noncommutative rings. Let R be a ring and $\underline{\operatorname{mod}} R$ the stable category of $\operatorname{mod} R$ modulo projective modules. We define the category $\mathcal{S}(\underline{\operatorname{mod}} R)$ by

$$\mathcal{S}(\underline{\operatorname{mod}} R) = \{\Omega^n M \mid M \in \underline{\operatorname{mod}} R, n \in \mathbf{Z}\}/\sim;$$
$$M \sim N \Leftrightarrow \Omega^n M \cong \Omega^n N \text{ for } n \gg 0.$$

Then $\mathcal{S}(\underline{\operatorname{mod}} R)$ is a triangulated category with the translation functor

$$\Omega^{-1} : \mathcal{S}(\underline{\operatorname{mod}} R) \to \mathcal{S}(\underline{\operatorname{mod}} R)$$

so that $(\mathcal{S}(\underline{\operatorname{mod}} R), R/\mathfrak{r})$ is a triangulated quasi-scheme. If R is Frobenius, then $\mathcal{S}(\underline{\operatorname{mod}} R) \cong \underline{\operatorname{mod}} R$.

Theorem 6.5. [14] *If R is a noetherian algebra such that*

- *$R \cong S/(x_1, \dots, x_n)$ where S is a regular algebra and $\{x_1, \dots, x_n\}$ is a regular central sequence of S (cf. a complete intersection), and*
- *$\Omega^d : \mathcal{S}(\underline{\operatorname{mod}} R) \to \mathcal{S}(\underline{\operatorname{mod}} R)$ is a Serre functor for some $d \in \mathbf{Z}$ (e.g. a symmetric algebra, cf. Calabi-Yau type),*

*then R satisfies (**ee**).*

Example 6.6. *If n is odd, then the exterior algebra $\Lambda(k^n)$ satisfies (**ee**).*

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