IDEAL THEORY OF ORE EXTENSIONS

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Abstract

The aim of this note is to introduce the ideal theory of Ore extensions from the order theoretic point of view.

0. Introduction

Let R be any ring with identity 1, endomorphism σ and δ be a left σ-derivation, that is,
(i) δ is an endomorphism of R as an additive group and
(ii) $\delta(ab) = \sigma(a)\ \delta(b) + \delta(a)b$ for all a,$b \in R$.

Note that δ is a usual derivation if σ= 1.

The *Ore extension* (or the skew polynomial ring) over R in an indeterminate x is:

$R[x; \sigma, \delta] = \{f(x) = a_n x^n + ... + a_0 \mid a_i \in R\}$ with $xa = \sigma(a)x + \delta(a)$ for all $a \in R$.

This definition of non-commutative polynomial rings was first introduced by Ore [O], who combined earlier ideas of Hilbert (in the case $\delta = 0$) and Schlessinger (in the case $\sigma = I$). In the case R is a division ring, Ore made a firm foundation for the sudy of $R[x; \sigma, \delta]$ by establishing the unique factorization property of $R[x; \sigma, \delta]$, and using this, he studied, among other things, the problem of finding the greatest common divisors and the least common multiples of pairs of skew polynomials.

Ever since the appereance of Ore's fundamental paper [O], Ore extensions have played an important role in non-commutative ring theory and many non-commutative ring theorists have been investigated Ore extensions from different points of view such as; ideal theory, order theory, Galois theory, Homological algebras and so on.

In this note, we only discuss on ideal theory and order theory in Ore extensions.

1. A brief history

In the case when R is a division ring, Ore investigated the following basic properties [O]; principal ideal theory, Euclid algorithm, the greatest common divisors, the least common multiple and the decomposition into prime factors. He also defines the concept of transformations in order to get several decomposition theorems.

After Ore, in [J], Jacobson studied Ore extensions in the case when R is division ring and σ is an automorphism as follows:

Definition 1.1 (general case).

(1) A polynomial $p(x) \in R[x;\sigma,\delta]$ is called *semi-invariant* if $R[x;\sigma,\delta]p(x)$ is a right R-module. This is equivalent to; for any $a \in R$, there is a $b \in R$ such that $p(x)a = bp(x)$.

(2) A polynomial $p(x) \in R[x;\sigma,\delta]$ is called *invariant* if $R[x;\sigma,\delta]p(x)$ is an ideal (a two-sided ideal). This is equivalent to;

(i) it is semi-invariant and

(ii) $p(x)x = (cx + d)p(x)$ for some $c, d \in R$.

Definition 1.2 (general case). If $\sigma^n = I_u$, inner automorphism induced by u, a unit in R, then we say that the *inner order* is n if n is the smallest natural number for this property and we denote it by $\mathrm{o}(\sigma) = n$. If there are no such natural number, then we say that the inner order of σ is infinite and denote it by $\sigma = \infty$.

Under these notations, Jacobson obtained the following

Theorem 1.3 ([J]). Suppose that R is a division ring and σ is an automorphism of R.

(1) Let $I(R[x;\sigma,\delta])$ be the set of all invariant polynomials of $R[x;\sigma,\delta]$ and let $f(x) \in I(R[x;\sigma,\delta])$. Then

$f(x) = p_1(x) \ ... \ p_m(x)$,

where $p_i(x)$ are irreducible elements in $I(R[x;\sigma,\delta])$.

(2) Suppose that $\delta = 0$.

(a) If $\text{o}(\sigma) = \infty$, then $I(R[x;\sigma,\delta]) = \{x^n \mid n = 1, 2, ...\}$, that is, the set of all ideals is $x^n R[x;\sigma,\delta]$ (n = 1,2,...).

(b) If $\text{o}(\sigma) = n < \infty$, then any monic $f(x) \in I(R[x;\sigma,\delta])$ is of the form; $f(x) = x^l + a_{l-n}x^{l-n} + a_{l-2n}x^{l-2n} + ...$ with $\sigma(a_{l-jn}) = a_{l-jn}$ and $a_{l-jn}\sigma^{l-jn}(b) = \sigma^l(b)a_{l-jn}$ for all $b \in R$.

Definition 1.4 (general case). δ is called σ-*inner* if there is a $b \in R$ such that

$$\delta(a) = ba - \sigma(a)b \text{ for all } a \in R.$$

Otherwise, δ is called σ-*outer.*

If δ is σ- inner, then $R[x;\sigma,\delta] = R[y;\sigma,0]$, where $y = x - b$. This shows that Ore extensions become an automorphism type (or an endomorphism type) if δ is σ- inner.

If $\sigma = 1$, then we have

$$x^j a = ax^j + \begin{pmatrix} j \\ 1 \end{pmatrix} \delta(a)x^{j-1} + ... + \delta^j(a)$$

for any $a \in R$ and any natural number j.

With this property, Jacobson obtained the following

Theorem 1.5 ([J]). Suppose that R is a division ring, $\sigma = 1$ and δ is 1-outer. If char $R = 0$, then $R[x;1,\delta]$ is a simple ring.

However, he could not get any result in the case when char $R = p > 0$ and δ is 1-outer. He said; the conditions that $f(x) \in I(R[x;1,\delta])$ are somewhat complicated but may be satisfied non-trivially. We hope to discuss this case in a later paper.

In 1957, Amitsur investigated the ideal theory in Ore extensions in the case when R is a division ring and $\sigma = 1$ (he does not assume that R has an identity element). However, in the case when R has an identity element, he obtained the following

Theorem 1.6 ([A_2]). Suppose that R is a division ring and $\delta \neq 0$.

(1) $\mathbb{Z}(R[x;1,\delta])$, the center of $R[x;1,\delta]$, is the set of all invariant polynomials.

(2) If $\mathbb{Z}(R[x;1,\delta]) \subseteq R$, then $\mathbb{Z}(R[x;1,\delta]) = R_{1,\delta} = \{a \in \mathbb{Z}(R) \mid \delta(a) = 0\ \}$.
(3) If $\mathbb{Z}(R[x;1,\delta]) \not\subseteq R$, then $\mathbb{Z}(R[x;1,\delta]) = R_{1,\delta}[p(x)]$, where $p(x)$ is the central polynomial of minimal non-zero degree.

Those works by Jacobson and Amitsur had been extended to the general case by Cauchon [C], Carcanaque [Ca], Lemonnier [Le], Lam, Leroy and Matczuk ([LL], [LLM]).

In their works, the following concept plays an important role in order to study the ideal theory in Ore extensions:

Definition 1.7. δ is a *quasi-algebraic σ-derivation* if there is a θ, an endomorphism of R, $0 \neq a_n, a_{n-1}, ..., a_0, b \in R$ such that
$a_n\delta^n(y) + a_{n-1}\delta^{n-1}(y) + ... + a_1\delta(y) + b\delta_{a_0,\theta}(y) = 0$
for all $y \in R$, where $\delta_{a_0,\theta}(y) = a_0y - \theta(y)a_0$.

This concept was more or less created by Amitsur to study differrential equations ([A_1]).

In 1991, Leroy and Matczuk generalized the results in the case R is a division ring (or simple Artinian ring) to the case R is a prime ring as follows:

Definition 1.8. An ideal I of $R[x;\sigma,\delta]$ is called *R-disjoint* if $I \cap R = \{0\}$.

Theorem 1.9 ([LM]). Suppose that R is a prime ring and $Q_s(R)$ is its symmetric Martindale ring of quotients (see [P]). The following conditions are equivalent:
(1) There exists a non-zero R-disjoint ideal of $R[x;\sigma,\delta]$.
(2) There exists a non-zero $Q_s(R)$-disjoint ideal of $Q_s[x;\sigma,\delta]$.
(3) There exists a monic non-constant invariant polynomial in $Q_s(R)[x;\sigma,\delta]$.
(4) There exists a monic non-constant semi-invariant polynomial in $Q_s(R)[x;\sigma,\delta]$.
(5) δ is a quasi-algebraic σ-derivation of $Q_s(R)$.

Here, (4)$\leftrightarrow$(5) and (1)$\leftrightarrow$(3) are not difficult. The equivalence of (4), (5) with (2) is much harder. Furthermore, they could obtain the complete structure of the center $\mathbb{Z}(Q_s(R)[x;\sigma,\delta])$.

Theorem 1.10 [LM]. Suppose that R is a prime ring and Q_sR is its symmetric Martindale ring of quotients.
(1) $\mathbb{Z}(Q_s(R)[x;\sigma,\delta]) \not\subseteq Q_s(R)$ if and only if

(i) $o(\sigma) < \infty$ and

(ii) δ is a quasi-algebraic σ-derivation.

(2) If $\mathbb{Z}(Q_s(R)[x;\sigma,\delta]) \subseteq Q_s(R)$, then

$\mathbb{Z}(Q_s(R)[x;\sigma,\delta]) = \mathbb{Z}(Q_s(R))_{\sigma,\delta} = \{a \in \mathbb{Z}(Q_s(R)) \mid \sigma(a) = a$ and $\delta(a) = 0\}$.

(3) If $\mathbb{Z}(Q_s(R)[x;\sigma,\delta]) \not\subseteq Q_s(R)$, then

(i) $\mathbb{Z}(Q_s(R)[x;\sigma,\delta]) = \mathbb{Z}(Q_s(R))_{\sigma,\delta}[h(x)]$, where $h(x)$ is the central polynomial of minimal non-zero degree.

(ii) Let $m(x)$ be the monic invariant polynomial of minimal non-zero degree. Then $h(x) = am(x)^l + c$, where $a \in Q_s(R)$ with $aR = Ra$ and $c \in \mathbb{Z}(Q_s(R))_{\sigma,\delta}$.

(iii) Let $p(x)$ be the monic semi-invariant polynomial of minimal non-zero degree, say, deg $p(x) = n$. Then any $f(x) \in \mathbb{Z}(Q_s(R)[x;\sigma,\delta])$ is of the form;

$f(x) = c_r p(x)^r + ... + c_0$,

where c_i is invertible if $c_i \neq 0$ and $\sigma^{-in} = I_{c_i}$, the inner automorphism induced by c_i.

In the case when R is a Noetherian ring (not necessarily prime), the ideal theory of Ore extensions have been investigated by Irving, Goodearl and Goodearl-Letzter ([I_1], [I_2], [G] and [GL]).

2. Ore extensions which are unique factorization rings (UFRs)

Let D be a commutative principal ideal domain. Then $D[x]$ is not necessarily a principal ideal domain. In fact $D[x]$ is a UFD. Moreover if D is a UFD, then $D[x]$ is a UFD. This fact suggests that it is natural to define non-commutative UFRs and to investigate UFRs in Ore extensions.

P.M. Cohn defined non-commutative UFRs by using elements as in commutative case. However, he could not get fruitful results ([Co_1],[Co_2]). In [CJ], Chatters and Jordan defined UFRs by using ideals and they studied the structure of UFRs and also Ore extensions which are UFRs.

In this section, for simplicity, let R be a Noetherian prime ring with its quotient ring Q which is a simple Artinian ring. Furthermore, let σ be an automorphism of R and δ is a left σ-derivation.

Definition 2.1. An ideal I of R is called (σ,δ)-*stable* if $\sigma(I) \subseteq I$ and $\delta(I) \subseteq I$.

Note that I is (σ, δ)-stable if and only if $I[x; \sigma, \delta]$ is an ideal of $R[x; \sigma, \delta]$.
Chatters-Jordan defined UFRs by adopting the definition in [K] as follows:

Definition 2.2 [CJ]. R is called a (σ, δ)-*UFR* if for any non-zero (σ, δ)-prime ideal P, there is a non-zero (σ, δ)-prime ideal P_0 such that $P \supseteq P_0$ and $P_0 = pR = Rp$ for some $p \in P_0$. In particular, R is a *UFR* if R is a (1,0)-UFR.

In the case when either $\sigma = 1$ or $\delta = 0$, they obtained the following

Theorem 2.3 [CJ].
(1) R is a $(\sigma,\ 0)$-UFR if and only if $R[x; \sigma, 0]$ is a UFR.
(2) If R is a $(1,\ \delta)$-UFR, then $R[x; \sigma, 0]$ is a UFR. Furthermore if R is a domain, then the converse is also true.

However, they gave the following examples:
(i) R is a UFR but $R[x; \sigma, 0]$ is not a UFR.
(ii) R is a UFR but $R[x; 1, \delta]$ is not a UFR.
It is natural that the following property is held:

If R is a UFR, then $R[x; \sigma, \delta]$ is a UFR.

This is one of the reasons that I defined another UFRs by using "v-ideals".

Notation. For any R-ideal I, we write
$(R : I)_l = \{q \in Q \mid qI \subseteq R\}$;
$(R : I)_r = \{q \in Q \mid Iq \subseteq R\}$;
$I_v = (R : (R : I)_l)_r$, a v-ideal containing I.
If $I_v = I$, then it is called a *right v-ideal*. Similarly, ${}_vI = (R : (R : I)_r)_l$ and if ${}_vI = I$, then it is called a *left v-ideal*. If $I_v = I = {}_vI$, then we simply say that I is a *v-ideal*.

Definition 2.4 ([AKM]). R is called a (σ, δ)-*UFR* if any (σ, δ)-prime ideal P with $P = P_v$ (or $P = {}_vP$) is principal, namely, $P = pR = Rp$ for some $p \in P$. In particular, R is a *UFR* if R is a (1, 0)-UFR.

Theorem 2.5 ([AKM]).
(1) If R is a UFR in the sense of Chatters-Jordan, then R is a UFR in the

sense of ours.
(2) If R is a UFR, then R is a (σ, δ)-UFR.
(3) R is a $(\sigma, 0)$-UFR if and only if $R[x; \sigma, 0]$ is a UFR.
(4) R is a $(1, \delta)$-UFR if and only if $R[x; 1, \delta]$ is a UFR.

In the case when either $\sigma = 1$ or $\delta = 0$, we could obtain a necessary and sufficient conditions for $R[x; \sigma, \delta]$ to be a UFR. So, in what follows, we assume that $\sigma \neq 1$ and $\delta \neq 0$.

In the case when σ and δ are both non-trivial, we have some obstructions from the ideal theoretical's point of view:

(i) Let I be a non-zero ideal of $R[x; \sigma, \delta]$ with $\mathfrak{A} = R \cap I \neq (0)$. Then

(a) If $\sigma = 1$, then $\mathfrak{A}$ is a $(1, \delta)$-stable ideal.

(b) If $\delta = 0$ and I is a prime ideal with $I \not\ni x$, then $\mathfrak{A}$ is a $(\sigma, 0)$-stable ideal.

If $I \ni x$, then $I \supseteq P = xR[x; \sigma, \delta]$, a prime v-ideal so that we do not need to consider a prime ideal I properly containing P.

In general, $\mathfrak{A}$ is not necessarily a (σ, δ)-stable ideal.

(ii) Let I be a non-zero ideal of $R[x; \sigma, \delta]$. Set $L(I) = \{a_n \in R \mid f(x) = a_n x^n + \ldots + a_0 \in I\}$, *the leading coefficient ideal.*

(a) If $\delta = 0$, then $L(I)$ is a $(\sigma, 0)$-stable ideal.

(b) If $\sigma = 1$, then $L(I)$ is a $(1, \delta)$-stable ideal.

However, in general, $L(I)$ is not necessarily a (σ, δ)-stable ideal.

The following lemma is easily proved by the classical method.

Lemma 2.6. R is a (σ, δ)-UFR if and only if any non-zero (σ, δ)-stable ideal $\mathfrak{A}$ of R with $\mathfrak{A} = \mathfrak{A}_v$ (or $\mathfrak{A} = {}_v\mathfrak{A}$) is principal.

Now we will explore some properties of $R[x; \sigma, \delta]$ under the assumption that it is a UFR:

Lemma 2.7. Suppose that $R[x; \sigma, \delta]$ is a UFR.

(1) R is a (σ, δ)-UFR.

(2) Let I be a non-zero ideal of $R[x; \sigma, \delta]$ such that $I = I_v$ (or $I = {}_vI$) and $\mathfrak{A} = I \cap R \neq (0)$. Then $\mathfrak{A}$ is a (σ, δ)-stable ideal with $\mathfrak{A} = \mathfrak{A}_v$ (or $\mathfrak{A} = {}_v\mathfrak{A}$).

Because of the property (2) in Lemma 2.7, we have the following definition.

Definition 2.8. A non-zero ideal $\mathfrak{A}$ of R with $\mathfrak{A} = \mathfrak{A}_v$ (or $\mathfrak{A} = {}_v\mathfrak{A}$) is called a *$v$-contracted ideal* if $\mathfrak{A} = I \cap R$ for some ideal I of $R[x;\sigma,\delta]$ with $I = I_v$ (or $I = {}_vI$).

The following lemma is crucial, in which we got some ideas from [G].

Lemma 2.9. Suppose that R is a (σ,δ)-UFR and that any v-contracted ideal is (σ,δ)-stable. If I is a non-zero ideal with $I = I_v$ (or $I = {}_vI$) and $\mathfrak{A} = I \cap R \neq (0)$, then $I = \mathfrak{A}[x;\sigma,\delta]$ and is principal.

The properties (1) and (2) in Lemma 2.7 are enough information from the coefficient ring R. Now we need some information from Q (or $Q[x;\sigma,\delta]$). As it has been shown in §1, $Q[x;\sigma,\delta]$ is a simple ring if and only if δ is not a quasi-algebraic σ-derivation. Hence we have the following

Theorem 2.10. Suppose that δ is not a quasi-algebraic σ-derivation of Q. $R[x,\sigma,\delta]$ is a UFR if and only if the following two conditions are satisfied:
(1) R is a (σ,δ)-UFR.
(2) Any v-contracted ideal is (σ,δ)-stable.

In what follows, suppose that δ is a quasi-algebraic σ-derivation of Q, that is, $Q[x,\sigma,\delta]$ is not a simple ring. So there exists
$p(x) = x^n + p_{n-1}x^{n-1} + ... + p_0 = x^n + x^{n-1}q_{n-1} + ... + q_0$
, the invariant polynomial of minimal non-zero degree.
Set $M' = Q[x;\sigma,\delta]p(x)$, a maximal ideal and $M = M' \cap R[x;\sigma,\delta]$. It is easy to see that M is a prime v-ideal. So if $R[x;\sigma,\delta]$ is a UFR, then M must be principal.
The following is a characterization for M to be principal.

Lemma 2.11. M is principal if and only if the following two conditions are satisfied:
(1) $\bigcap_{i=0}^n R.p_i^{-1} = L(M) = \bigcap_{i=0}^n q_i^{-1}.R$.
(2) $L(M)$ is principal.
Here $q^{-1}.R = \{r \in R \mid qr \in R\}$ and $R.q^{-1} = \{r \in R \mid qr \in R\}$ for any $q \in Q$.

Now we are ready to state the main theorem:

Theorem 2.12. Suppose that δ is a quasi-algebraic σ-derivation of Q. Then
$R[x;\sigma,\delta]$ is a UFR if and only if the following three conditions are satisfied:
(1) R is a (σ,δ)-UFR.
(2) Any v-contracted ideal is a (σ,δ)-stable ideal.
(3) $\bigcap_{i=0}^{n} R.p_i^{-1} = L(M) = \bigcap_{i=0}^{n} q_i^{-1}.R$ and is principal.

We will give some examples of $R[x;\sigma,\delta]$ which are UFRs in the case when $\sigma \neq 1$ and $\delta \neq 0$.

Example 1. Suppose that δ is σ-inner, that is, there is a non-zero $b \in R$ with $\delta(a) = ba - \sigma(a)b$ for all $a \in R$. Then $R[x;\sigma,\delta] = R[y;\sigma,0]$, where $y = x - b$. Hence $R[x;\sigma,\delta]$ is a UFR if and only if R is a $(\sigma,0)$-UFR.

Next we will give two examples in the case when δ is σ-outer.

Example 2 (essentially [L]). Let D be a principal ideal ring of a division ring F with char $F \neq 2$ and let $K = F(x_i \mid i \in \mathbb{N})$ be a rational function division ring over F in indeterminates x_i ($\mathbb{N}$ is the set of natural numbers) with

$ax_i = x_i a$ for all $a \in F$ and $x_i x_j = -x_j x_i$ if $i > j$.

Set $R = D[x_i \mid i \in \mathbb{N}]$ and define the automorphism σ of R as follows; $\sigma(a) = a$ for all $a \in F$ and $\sigma(x_i) = -x_i$ so that $\sigma^2 = 1$. We inductively define a left σ-derivation; $K_1 = F(x_1)$, ..., $K_i = K_{i-1}(x_i,\sigma_i)$, where $\sigma_i = \sigma \mid K_{i-1}$ and for any $\alpha \in K_i$, $\delta_i(\alpha) = (x_1 + ... + x_i)\alpha - \sigma(\alpha)(x_1 + ... + x_i)$. Set $\delta = \cup\delta_i$. Then we have the following properties:
(i) δ is a left σ-derivation, $\delta^2 = 0$, δ is a σ-outer and $\sigma\delta \neq \delta\sigma$.
(ii) R is a UFR so that R is a (σ,δ)-UFR.
(iii) Any ideal $\mathfrak{A}$ of R with $\mathfrak{A} = \mathfrak{A}_v$ (or $\mathfrak{A} = {}_v\mathfrak{A}$) is a (σ,δ)-stable ideal.
(iv) $p(x) = x^2$ is the invariant polynomial of minimal non-zero degree.
Hence $R[x;\sigma,\delta]$ is a UFR.

Example 3 ([XM]). Let F be a division ring with an automorphism σ, $\mathrm{o}(\sigma) = \infty$ and $\sigma(a) \neq a - 1$ for all $a \in F$ and let $R = F[[t,\sigma]]$ be the skew formal power series ring with its quotient ring $K = F((t,\sigma))$. We extend σ to an automorphism of K and define a left σ-derivation as follows;

$$\sigma(\sum a_n t^n) = \sum \sigma(a_n) t^n \text{ and } \delta(\sum a_n t^n) = \sum n\sigma(a_n) t^{n+1}.$$

Then we have the following properties:
(i) δ is a left σ-derivation with $\sigma\delta = \delta\sigma$.
(ii) Any ideal of R is a (σ, δ)-stable ideal.
(iii) R is a discrete rank one valuation ring.
(iv) If char $K = 0$, then δ is not a quasi-algebraic σ-derivation.
(v) If char $K = p > 0$, then $\delta^p = 0$ and $p(x) = x^p$ is the invariant polynomial of minimal non-zero degree.

Hence in any cases, $R[x; \sigma, \delta]$ is a UFR by Theorems 2.10 or 2.12.

The following are other examples of UFRs.

(1) Let g be a solvable Lie algebra. Then the enveloping algebra $U(g)$ is a UFR ([C_2]).
(2) Let G be a polycyclic-by-finite group and let R be a UFR. Then the group ring $R[G]$ is a UFR if and only if the following two conditions are satisfied
(i) $\triangle^+(G) = \{g \in G \mid \mid G : C_G(g) \mid < \infty\} = < 1 >$.
(ii) G is dihedral free ([AKM]).
(3) Quantum algebras with certain conditions([LLR]).

We only consider Ore extensions which are UFRs. There are important classes in Ore extensions which are Krull rings or Krull type generalization of hereditary Noetherian rings (they are called v-HC orders). We refer the readers to [C_1], [C_2] and [MR] for Ore extensions which are Krull rings and to [M] and [KMU] for Ore extensions which are v-HC orders. Concerning UFRs in the sense of Chatters and Jordan, we refer the readers to [Ch], [CC], [CJ], [CGW] and [GS].

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