DERIVED CATEGORIES AND THE REPRESENTATION THEORY OF ALGEBRAS

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We survey equivalences between derived categories which were studied in the representation theory of algebras. We begin to review properties of compact objects and Brown representability theorem in triangulated categories.

1. Triangulated categories and ∂-functors

Definition 1.1. *A triangulated category $\mathcal{D}$ is an additive category together with (1) an autofunctor $\Sigma : \mathcal{D} \xrightarrow{\sim} \mathcal{D}$ (i.e. there is Σ^{-1} such that $\Sigma \circ \Sigma^{-1} = \Sigma^{-1} \circ \Sigma = \mathbf{1}_{\mathcal{D}}$) called the translation (or suspension), and (2) a collection $\mathcal{T}$ of sextuples (X, Y, Z, u, v, w):*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma(X)$$

called (distinguished) triangles. These data are subject to the following four axioms:

(TR1) *(1) Every sextuple (X, Y, Z, u, v, w) which is isomorphic to a triangle is a triangle.*
(2) Every morphism $u : X \to Y$ is embedded in a triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma(X)$$

(3) For any $X \in \mathcal{D}$, $X \xrightarrow{1} X \to 0 \to \Sigma(X)$ is a triangle

(TR2) *A sextuple*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma(X)$$

is a triangle if and only if

$$Y \xrightarrow{v} Z \xrightarrow{w} \Sigma(X) \xrightarrow{-\Sigma(u)} \Sigma(Y)$$

is a triangle.

(TR3) *For any triangles* $(X,\ Y,\ Z,\ u,\ v,\ w)$, (X', Y', Z', u', v', w') *and a commutative diagram*

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma(X) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & & & \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma(X') \end{array}$$

there exists $h : Z \to Z'$ *which makes a commutative diagram*

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma(X) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \Sigma(f)} \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma(X') \end{array}$$

(TR4) *(Octahedral axiom) For any two consecutive morphisms* $u : X \to Y$ *and* $v : Y \to Z$, *if we embed* u, vu *and* v *in triangles* (X, Y, Z', u, i, i'), (X, Z, Y', vu, k, k') *and* (Y, Z, X', v, j, j'), *respectively, then there exist morphisms* $f : Z' \to Y'$, $g : Y' \to X'$ *such that the following diagram commutes*

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{i} & Z' & \xrightarrow{i'} & \Sigma(X) \\ \| & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle f} & & \| \\ X & \xrightarrow{vu} & Z & \xrightarrow{k} & Y' & \xrightarrow{k'} & \Sigma(X) \\ & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \Sigma(u)} \\ & & X' & = & X' & \xrightarrow{j'} & \Sigma(Y) \\ & & \downarrow{\scriptstyle j'} & & \downarrow{\scriptstyle \Sigma(i)j'} & & \\ & & \Sigma(Y) & \xrightarrow{\Sigma(i)} & \Sigma(Z') & & \end{array}$$

and the third column is a triangle.

Sometimes, we write $X[i]$ *for* $\Sigma^i(X)$.

Definition 1.2 (∂-functor)**.** *Let* $\mathcal{D}$, $\mathcal{D}'$ *be triangulated categories. An additive functor* $F : \mathcal{D} \to \mathcal{D}'$ *is called* ∂*-functor (sometimes exact functor) provided that there is a functorial isomorphism* $\alpha : F\Sigma_{\mathcal{D}} \xrightarrow{\sim} \Sigma_{\mathcal{D}'}F$ *such that*

$$F(X) \xrightarrow{F(u)} F(Y) \xrightarrow{F(v)} F(Z) \xrightarrow{\alpha_X \circ F(w)} \Sigma_{\mathcal{D}'}(F(X))$$

is a triangle in $\mathcal{D}'$ *whenever*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma_{\mathcal{D}}(X)$$

is a triangle in $\mathcal{D}$. Moreover, if a ∂-functor F is an equivalence, then F is called a triangulated equivalence. In this case, we denote by $\mathcal{D} \overset{\Delta}{\cong} \mathcal{D}'$.

For $(F, \alpha), (G, \beta) : \mathcal{D} \to \mathcal{D}'$ ∂-functors, a functorial morphism $\phi : F \to G$ is called a ∂-functorial morphism if $(\Sigma_{\mathcal{D}'}\phi) \circ \alpha = \beta \circ \phi\Sigma_{\mathcal{D}}$.

Proposition 1.3. *Let $F : \mathcal{D} \to \mathcal{D}'$ be a ∂-functor between triangulated categories. If $G : \mathcal{D}' \to \mathcal{D}$ is a right (or left) adjoint of F, then G is also a ∂-functor.*

Definition 1.4. *A contravariant (resp., covariant) additive functor $H : \mathcal{D} \to \mathcal{A}$ from a triangulated category $\mathcal{D}$ to an abelian category $\mathcal{A}$ is called a homological functor (resp., cohomological functor), if for any triangle (X, Y, Z, u, v, w) in $\mathcal{D}$ the sequence*

$$H(\Sigma(X)) \to H(Z) \to H(Y) \to H(X)$$
$$\text{(resp., } H(X) \to H(Y) \to H(Z) \to H(\Sigma(X))\text{)}$$

is exact. Taking $H(\Sigma^i(X)) = H^i(X)$, we have the long exact sequence:

$$\cdots \to H^{i-1}(X) \to H^i(Z) \to H^i(Y) \to H^i(X) \to \cdots$$

Proposition 1.5. *The following hold.*

1. *If (X, Y, Z, u, v, w) is a triangle, then $vu = 0$, $wv = 0$ and $\Sigma(u)w = 0$.*
2. *For any $X \in \mathcal{D}$, $\mathrm{Hom}_{\mathcal{D}}(-, X) : \mathcal{D} \to \mathfrak{Ab}$ (resp., $\mathrm{Hom}_{\mathcal{D}}(X, -) : \mathcal{D} \to \mathfrak{Ab}$) is a homological functor (resp., cohomological functor).*
3. *For any homomorphism of triangles*

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma(X) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \Sigma(f)} \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma(X') \end{array}$$

if two of f, g and h are isomorphisms, then the rest is also an isomorphism.

Definition 1.6 (stable t-structure [Mi1])**.** *For full subcategories $\mathcal{U}$ and $\mathcal{V}$ of a triangulated category $\mathcal{D}$, $(\mathcal{U}, \mathcal{V})$ is called a stable t-structure in $\mathcal{D}$ provided that*

1. *$\Sigma(\mathcal{U}) = \mathcal{U}$ and $\Sigma(\mathcal{V}) = \mathcal{V}$.*
2. $\mathrm{Hom}_{\mathcal{D}}(\mathcal{U}, \mathcal{V}) = 0$.
3. *For every $X \in \mathcal{D}$, there exists a triangle $U \to X \to V \to \Sigma(U)$ with $U \in \mathcal{U}$ and $V \in \mathcal{V}$.*

Proposition 1.7 ([BBD], c.f. [Mi1]). *Let $\mathcal{D}$ be a triangulated category, $(\mathcal{U}, \mathcal{V})$ a stable t-structure in $\mathcal{D}$, and $i_* : \mathcal{U} \to \mathcal{D}, j_* : \mathcal{V} \to \mathcal{D}$ the canonical embeddings. Then the following hold.*

(1) *$\mathcal{U}$ and $\mathcal{V}$ is épaisse subcategories of $\mathcal{D}$.*
(2) *i_* (resp., j_*) has a right adjoint $i^!$ (resp., a left adjoint j^*).*
(3) *The adjunction arrows induce a triangle*

$$i_*i^!X \xrightarrow{\alpha_X} X \xrightarrow{\beta_X} j_*j^*X \to i_*i^!X[1]$$

for any $X \in \mathcal{D}$.

(4) *The quotient category $\mathcal{D}/\mathcal{U}$ (resp., $\mathcal{D}/\mathcal{V}$) exists , and it is triangulated equivalent to $\mathcal{V}$ (resp., $\mathcal{U}$).*

$$\begin{array}{ccccc} \mathcal{D}/\mathcal{V} & & & & \mathcal{D}/\mathcal{U} \\ \wr\uparrow & \nwarrow & & \nearrow & \uparrow\wr \\ \mathcal{U} & \underset{i^!}{\overset{i_*}{\rightleftarrows}} & \mathcal{D} & \underset{j_*}{\overset{j^*}{\rightleftarrows}} & \mathcal{V} \end{array}$$

Remark 1.8. *In the above, the right adjoint j_* is often called the Bousfield localization functor of j^*. The quotient category $\mathcal{D}/\mathcal{U}$ has the same objects as $\mathcal{D}$ and that morphisms in $\mathcal{D}/\mathcal{U}$ from X to Y are given by equivalence classes $s^{-1}f$ of diagrams*

$$\begin{array}{ccc} X & & Y \\ & \searrow^{f} & \downarrow s \\ & & Y' \end{array}$$

where $Y \xrightarrow{s} Y' \to Z \to \Sigma(Y)$ is a triangle with $Z \in \mathcal{U}$.

Definition 1.9 (Compact Object). *Let $\mathcal{D}$ be a triangulated category. An object $C \in \mathcal{D}$ is called a compact object in $\mathcal{D}$ if the canonical morphism*

$$\coprod_{i \in I} \mathrm{Hom}_{\mathcal{D}}(C, X_i) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}}(C, \coprod_{i \in I} X_i)$$

is an isomorphism for any set $\{X_i\}_{i \in I}$ of objects (if $\coprod_{i \in I} X_i$ exists in $\mathcal{D}$).

A triangulated category $\mathcal{D}$ is compactly generated if $\mathcal{D}$ contains arbitrary coproducts, and if there is a set S of compact objects such that

$$\mathrm{Hom}_{\mathcal{D}}(S, X) = 0 \Rightarrow X = 0$$

For a compactly generated triangulated category $\mathcal{D}$, a set S of compact objects is called a generating set if

(1) $\mathrm{Hom}_{\mathcal{D}}(S, X) = 0 \Rightarrow X = 0$,
(2) $\Sigma(S) = S$.

Definition 1.10 (Homotopy Limit)**.** *Let $\mathcal{D}$ be a triangulated category which contains arbitrary coproducts (resp., products). For a sequence $\{X_i \to X_{i+1}\}_{i\in\mathbb{N}}$ (resp., $\{X_{i+1} \to X_i\}_{i\in\mathbb{N}}$) of morphisms in $\mathcal{D}$, the homotopy colimit (resp., homotopy limit) of the sequence is the third (resp., second) term of the triangle*

$$\coprod_i X_i \xrightarrow{1-\ shift} \coprod_i X_i \to \underset{\longrightarrow}{\mathsf{hocolim}}\ X_i \to \Sigma(\coprod_i X_i)$$
$$(resp.,\ \Sigma^{-1}(\prod_i X_i) \to \underset{\longleftarrow}{\mathsf{holim}}\ X_i \to \prod_i X_i \xrightarrow{1-\ shift} \prod_i X_i)$$

where the above shift morphism is the coproduct (resp., product) of $X_i \xrightarrow{f_i} X_{i+1}$ (resp., $X_{i+1} \xrightarrow{f_i} X_i$) ($i \in \mathbb{N}$).

The next lemma is the key to proving Theorems 1.14 and 1.15.

Lemma 1.11. *Let $\mathcal{D}$ be a triangulated category which contains arbitrary coproducts, $\{X_i \to X_{i+1}\}_{i\in\mathbb{N}}$ a sequence of morphisms in $\mathcal{D}$. For a compact object C in $\mathcal{D}$, we have*

$$\mathrm{Hom}(C, \underset{\longrightarrow}{\mathsf{hocolim}} X_i) \cong \varinjlim \mathrm{Hom}(C, X_i)$$

Proof. We have an exact sequence

$$0 \to \coprod_i \mathrm{Hom}(C, X_i) \to \coprod_i \mathrm{Hom}(C, X_i) \to \mathrm{Hom}(C, \underset{\longrightarrow}{\mathsf{hocolim}}\ X_i) \to 0$$

□

Definition 1.12 (Épaisse Subcategory & Localizing Subcategory)**.** *Let $\mathcal{D}$ be a triangulated category. A triangulated full subcategory $\mathcal{E}$ of $\mathcal{D}$ is called an épaisse subcategory of $\mathcal{D}$ if $\mathcal{E}$ is closed under direct summands. A triangulated full subcategory $\mathcal{L}$ of $\mathcal{D}$ is called a localizing subcategory if $\mathcal{L}$ is closed under coproducts.*

Proposition 1.13 (Bökstedt-Neeman [BN])**.** *Let $\mathcal{D}$ be a triangulated category with coproducts. Any localizing subcategory is an épaisse subcategory.*

Theorem 1.14 (Adams, Bousefield, Neeman [Ne1])**.** *Let $\mathcal{D}$ be a triangulated category with coproducts. Let S be a set of compact objects of $\mathcal{D}$ with $\Sigma(S) = S$. Let $\mathcal{S}$ be the smallest localizing subcategory containing all of S.*

1. *The canonical embedding $\mathcal{S} \hookrightarrow \mathcal{D}$ has a right adjoint.*
2. *Any compact object of $\mathcal{S}$ is a compact object of $\mathcal{D}$.*

Theorem 1.15 (Brown Representability Theorem [Ne2])**.** *Let $\mathcal{D}$ be a compactly generated triangulated category. If a homological functor*

$H : \mathcal{D} \to \mathfrak{Ab}$ sends coproducts to products, then it is representable, that is, there is an object $X \in \mathcal{D}$ such that $H \cong \mathrm{Hom}_{\mathcal{D}}(-, X)$.

Corollary 1.16 ([Kr]). *Let $\mathcal{D}$ be a compactly generated triangulated category which contains arbitrary coproducts. Then $\mathcal{D}$ contains arbitrary products.*

Sketch of proof. For a collection $\{X_i\}_{i\in I}$ of objects, a homological functor $\prod_i \mathrm{Hom}_{\mathcal{D}}(-, X_i)$ is represented by $\mathrm{Hom}_{\mathcal{D}}(-, X)$. □

Theorem 1.17 (Dual Brown Representability Theorem [Ne2], [Kr]). *Let $\mathcal{D}$ be a compactly generated triangulated category. If a cohomological functor $H : \mathcal{D} \to \mathfrak{Ab}$ preserves products, then it is representable, that is, there is an object $X \in \mathcal{D}$ such that $H \cong \mathrm{Hom}_{\mathcal{D}}(X, -)$.*

Corollary 1.18 (Adjoint Functor Theorem [Ne2], [Kr]). *Let $\mathcal{D}$ be a compactly generated triangulated category. If a ∂-functor $F : \mathcal{D} \to \mathcal{D}$ commutes with arbitrary coproducts (resp., products), then there exists a ∂-functor $G : \mathcal{D} \to \mathcal{D}$ which is a right (resp., left) adjoint of F.*

Proof. Since $\mathrm{Hom}_{\mathcal{D}}(F(-), Y) : \mathcal{D} \to \mathfrak{Ab}$ (resp., $\mathrm{Hom}_{\mathcal{D}}(Y, F(-)) : \mathcal{D} \to \mathfrak{Ab}$) is a homological (resp., cohomological) functor, there is an object $GY \in \mathcal{D}$ such that $\mathrm{Hom}_{\mathcal{D}}(F(-), Y) \cong \mathrm{Hom}_{\mathcal{D}}(-, G(Y))$ (resp., $\mathrm{Hom}_{\mathcal{D}}(Y, F(-)) \cong \mathrm{Hom}_{\mathcal{D}}(G(Y), -)$) □

2. Derived Categories

Throughout this section, $\mathcal{A}$ is an abelian category and $\mathcal{B}$, $\mathcal{C}$ are additive subcategories of $\mathcal{A}$.

Definition 2.1 (Complex). *A (cochain) complex is a collection $X^{\cdot} = (X^n, d_X^n : X^n \to X^{n+1})_{n\in\mathbb{Z}}$ of objects and morphisms of $\mathcal{B}$ such that $d_X^{n+1} d_X^n = 0$. A complex $X^{\cdot} = (X^n, d_X^n : X^n \to X^{n+1})_{n\in\mathbb{Z}}$ is called bounded below (resp., bounded above, bounded) if $X^n = 0$ for $n \ll 0$ (resp., $n \gg 0$, $n \ll 0$ and $n \gg 0$).*

A morphism $f : X^{\cdot} \to Y^{\cdot}$ of complexes is a collection of morphisms $f^n : X^n \to Y^n$ satisfying $d_Y^n f^n = f^{n+1} d_X^n$ for any $n \in \mathbb{Z}$.

We denote by $\mathsf{C}(\mathcal{B})$ (resp., $\mathsf{C}^+(\mathcal{B})$, $\mathsf{C}^-(\mathcal{B})$, $\mathsf{C}^{\mathrm{b}}(\mathcal{B})$) the category of complexes (resp., bounded below complexes, bounded above complexes, bounded complexes) of $\mathcal{B}$. An autofunctor $\Sigma : \mathsf{C}(\mathcal{B}) \to \mathsf{C}(\mathcal{B})$ is called translation if $(\Sigma(X^{\cdot}))^n = X^{n+1}$ and $(\Sigma(d_X))^n = -d_X^{n+1}$ for any complex $X^{\cdot} = (X^n, d_X^n)$.

In $\mathsf{C}(\mathcal{A})$, a morphism $u : X^{\cdot} \to Y^{\cdot}$ is called a quasi-isomorphism if $\mathrm{H}^n(u)$ is an isomorphism for any n.

In this section, "$*$" means "nothing", "$+$", "$-$" or "b".

Definition 2.2 (Mapping Cone)**.** *For $u \in \mathrm{Hom}_{\mathsf{C}(\mathcal{B})}(X^{\bullet}, Y^{\bullet})$, the mapping cone of u is a complex $\mathrm{M}^{\bullet}(u)$ with*

$$\mathrm{M}^n(u) = X^{n+1} \oplus Y^n,$$
$$d^n_{\mathrm{M}^{\bullet}(u)} = \begin{bmatrix} -d_X^{n+1} & 0 \\ u^{n+1} & d_X^n \end{bmatrix} : X^{n+1} \oplus Y^n \to X^{n+2} \oplus Y^{n+1}.$$

$$\begin{array}{ccccccccc}
X^{\cdot} & & \cdots & \longrightarrow & X^n & \xrightarrow{d_X^n} & X^{n+1} & \longrightarrow & \cdots \\
\downarrow u & & & & \downarrow u^n & & \downarrow u^{n+1} & & \\
Y^{\cdot} & & \cdots & \longrightarrow & Y^n & \xrightarrow{d_Y^n} & Y^{n+1} & \longrightarrow & \cdots \\
\downarrow v & & & & \downarrow \binom{0}{1} & & \downarrow \binom{0}{1} & & \\
\mathrm{M}^{\cdot}(u) & & \cdots & \longrightarrow & X^{n+1} \oplus Y^n & \xrightarrow{d^n_{\mathrm{M}^{\bullet}(u)}} & X^{n+2} \oplus Y^{n+1} & \longrightarrow & \cdots \\
\downarrow w & & & & \downarrow (1\ 0) & & \downarrow (1\ 0) & & \\
\Sigma(X^{\cdot}) & & \cdots & \longrightarrow & X^{n+1} & \xrightarrow{-d_X^{n+1}} & X^{n+2} & \longrightarrow & \cdots
\end{array}$$

Definition 2.3 (Homotopy Category)**.** *The homotopy category $\mathsf{K}^*(\mathcal{B})$ of $\mathcal{B}$ is defined by*

1. $\mathrm{Ob}(\mathsf{K}^*(\mathcal{B})) = \mathrm{Ob}(\mathsf{C}^*(\mathcal{B}))$,
2. $\mathrm{Hom}_{\mathsf{K}^*(\mathcal{B})}(X^{\cdot}, Y^{\cdot}) = \mathrm{Hom}_{\mathsf{C}^*(\mathcal{B})}(X^{\cdot}, Y^{\cdot})/\{$*homotopy relation*$\}$ *for* $X^{\cdot}, Y^{\cdot} \in \mathrm{Ob}(\mathsf{K}^*(\mathcal{B}))$.

Proposition 2.4. *A category $\mathsf{K}^*(\mathcal{B})$ is a triangulated category whose distinguished triangles are defined to be isomorphic to*

$$X^{\cdot} \xrightarrow{u} Y^{\cdot} \xrightarrow{v} \mathrm{M}^{\cdot}(u) \xrightarrow{w} \Sigma(X^{\cdot})$$

for any $u : X^{\cdot} \to Y^{\cdot}$ in $\mathsf{K}^*(\mathcal{B})$.

Definition 2.5 (Derived Category)**.** *The derived category $\mathsf{D}^*(\mathcal{A})$ of an abelian category $\mathcal{A}$ is the quotient category $\mathsf{K}^*(\mathcal{A})/\,\mathsf{K}^{*,\phi}(\mathcal{A})$, where $\mathsf{K}^{*,\phi}(\mathcal{A})$ is the full subcategory of $\mathsf{K}^*(\mathcal{A})$ consisting of null complexes, that is, complexes whose all cohomologies are 0.*

Proposition 2.6. *The following hold.*

1. *$\mathsf{D}^*(\mathcal{A})$ is a triangulated category, and the canonical functor $Q : \mathsf{K}^*(\mathcal{A}) \to \mathsf{D}^*(\mathcal{A})$ is a ∂-functor.*
2. *The i-th cohomology of complexes is a cohomological functor in the sense of Definition 1.4.*

3. Equivalences between derived categories

For a ring A, $\mathsf{Mod}\,A$ (resp., $\mathsf{mod}\,A$) is the category of right (resp., finitely presented right) A-modules, $\mathsf{Proj}\,A$ (resp., $\mathsf{proj}\,A$) is the full subcategory of $\mathsf{Mod}\,A$ consisting of projective (resp., finitely generated projective) A-modules, and $\mathsf{Inj}\,A$ is the full subcategory of $\mathsf{Mod}\,A$ consisting of injective A-modules. Similarly, for an abelian category $\mathcal{A}$ $\mathsf{Proj}\,\mathcal{A}$ (resp., $\mathsf{Inj}\,\mathcal{A}$) is the full subcategory of $\mathcal{A}$ consisting of projective (resp., injective) objects.

Definition 3.1. *A complex $X^{\cdot}$ of $\mathsf{K}(\mathcal{B})$ is called K-injective (resp., K-projective) if*

$$\mathrm{Hom}_{\mathsf{K}(\mathcal{B})}(N^{\cdot}, X^{\cdot}) = 0 \text{ (resp., } \mathrm{Hom}_{\mathsf{K}(\mathcal{B})}(X^{\cdot}, N^{\cdot}) = 0 \text{)}$$

for any null complex $N^{\cdot}$.

Example 3.2. *For a ring A, any complex $I^{\cdot} \in \mathsf{K}^{+}(\mathsf{Inj}\,A)$ (resp., $P^{\cdot} \in \mathsf{K}^{-}(\mathsf{Proj}\,A)$) is a K-injective (resp., K-projective) complex in $\mathsf{K}(\mathsf{Mod}\,A)$. Moreover, $(\mathsf{K}^{+,\phi}(\mathsf{Mod}\,A), \mathsf{K}^{+}(\mathsf{Inj}\,A))$ is a stable t-structure in $\mathsf{K}^{+}(\mathsf{Mod}\,A)$, and hence $\mathsf{D}^{+}(\mathsf{Mod}\,A) \overset{\triangle}{\cong} \mathsf{K}^{+}(\mathsf{Inj}\,A)$. Similarly, we have $\mathsf{D}^{-}(\mathsf{Mod}\,A) \overset{\triangle}{\cong} \mathsf{K}^{-}(\mathsf{Proj}\,A)$.*

Theorem 3.3 ([Sp], [Ne2], [LAM], [Fr])**.** *Let $\mathsf{K}^{inj}(\mathsf{Mod}\,A)$ (resp., $\mathsf{K}^{proj}(\mathsf{Mod}\,A)$) be the homotopy category of K-injective (resp., K-projective) complexes, then the following hold.*

1. *$(\mathsf{K}^{proj}(\mathsf{Mod}\,A), \mathsf{K}^{\phi}(\mathsf{Mod}\,A))$ is a stable t-structure in $\mathsf{K}(\mathsf{Mod}\,A)$, and hence $\mathsf{D}(\mathsf{Mod}\,A) \overset{\triangle}{\cong} \mathsf{K}^{proj}(\mathsf{Mod}\,A)$.*
2. *$(\mathsf{K}^{\phi}(\mathsf{Mod}\,A), \mathsf{K}^{inj}(\mathsf{Mod}\,A))$ is a stable t-structure in $\mathsf{K}(\mathsf{Mod}\,A)$, and hence $\mathsf{D}(\mathsf{Mod}\,A) \overset{\triangle}{\cong} \mathsf{K}^{inj}(\mathsf{Mod}\,A)$.*
3. *For a Grothendieck category $\mathcal{A}$, $(\mathsf{K}^{\phi}(\mathcal{A}), \mathsf{K}^{inj}(\mathcal{A}))$ is a stable t-structure in $\mathsf{K}(\mathcal{A})$, and hence $\mathsf{D}(\mathcal{A}) \overset{\triangle}{\cong} \mathsf{K}^{inj}(\mathcal{A})$.*

Definition 3.4. *Let $\mathcal{C}$ be an additive category. For $M \in \mathcal{C}$, We define $\mathsf{Add}\,M$ (resp., $\mathsf{add}\,M$) the full subcategory of $\mathcal{C}$ consisting of objects which are direct summands of coproducts (resp., finite coproducts) of copies of M.*

Proposition 3.5 (cf. [Ha])**.** *Let $\mathcal{A}$ be an abelian category, M an object of $\mathcal{A}$ with $\mathrm{Ext}^{i}_{\mathcal{A}}(M, M) = 0$ for any $i \neq 0$, and $B = \mathrm{End}_{\mathcal{A}}(M)$.*

1. *The canonical functor $\mathsf{K}^{\mathrm{b}}(\mathsf{add}\,M) \to \mathsf{D}^{\mathrm{b}}(\mathcal{A})$ is fully faithful.*
2. *We have $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,B) \overset{\triangle}{\cong} \mathsf{K}^{\mathrm{b}}(\mathsf{add}\,M)$, and then fully faithful ∂-functor $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,B) \to \mathsf{D}^{\mathrm{b}}(\mathcal{A})$.*

Sketch of proof. Let X, Y^i be objects of $\mathsf{add}\, M$. Consider complexes

$$\begin{array}{ccccccccccc}
\Sigma^{n-1}(Y^{-n}) & & & & Y^{-n} & & & & & & \\
\downarrow & & & & \downarrow & & & & & & \\
Y^{\cdot}_{n-1} & & & & Y^{-n+1} & \longrightarrow & \cdots & \longrightarrow & Y^{-1} & \longrightarrow & Y^0 \\
\downarrow & & & & \| & & & & \| & & \| \\
Y^{\cdot}_{n} & & Y^{-n} & \longrightarrow & Y^{-n+1} & \longrightarrow & \cdots & \longrightarrow & Y^{-1} & \longrightarrow & Y^0 \\
\downarrow & & \| & & & & & & & & \\
\Sigma^{n}(Y^{-n}) & & Y^{-n} & & & & & & & &
\end{array}$$

Then we have a morphism between exact sequences

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{add}\, M)}(X, \Sigma^{i+n-1}(Y^{-n})) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathcal{A})}(X, \Sigma^{i+n-1}(Y^{-n})) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{add}\, M)}(X, \Sigma^{i}(Y^{\cdot}_{n-1})) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathcal{A})}(X, \Sigma^{i}(Y^{\cdot}_{n-1})) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{add}\, M)}(X, \Sigma^{i}(Y^{\cdot}_{n})) & \longrightarrow & \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathcal{A})}(X, \Sigma^{i}(Y^{\cdot}_{n})) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{add}\, M)}(X, \Sigma^{i+n}(Y^{-n})) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathcal{A})}(X, \Sigma^{i+n}(Y^{-n})) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{add}\, M)}(X, \Sigma^{i+1}(Y^{\cdot}_{n-1})) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathcal{A})}(X, \Sigma^{i+1}(Y^{\cdot}_{n-1}))
\end{array}$$

Moreover, it is easy to see that $\mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{add}\, M)}(X^{\cdot}, Y^{\cdot}) \cong \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathcal{A})}(X^{\cdot}, Y^{\cdot})$ for any $X^{\cdot}, Y^{\cdot} \in \mathsf{K}^{\mathrm{b}}(\mathsf{add}\, M)$. By $\mathsf{proj}\, B \cong \mathsf{add}\, M$, (2) holds. □

For an object M of a triangulated category $\mathcal{D}$, we say that M generates $\mathcal{D}$ if $\mathcal{D}$ is the smallest full triangulated subcategory of $\mathcal{D}$ containing $\mathsf{add}\, M$ which is closed under isomorphisms. A ring R is called right coherent if $\mathsf{mod}\, R$ is an abelian category.

Definition 3.6. *Let $\mathcal{A}$ be an abelian category. An object $M \in \mathcal{A}$ is called a tilting object if*

(a) $\mathrm{Ext}^i_{\mathcal{A}}(M, M) = 0$ *for all $i > 0$.*
(b) *M generates $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$.*
(c) $\mathrm{End}_{\mathcal{A}}(M)$ *is a right coherent ring of which the right global dimension is finite.*

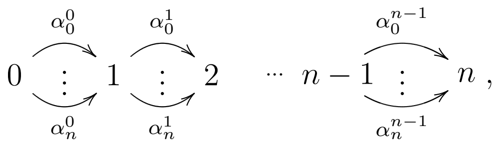
Corollary 3.7. *Let $\mathcal{A}$ be an abelian category, M an tilting object of $\mathcal{A}$ with $B = \mathrm{End}_{\mathcal{A}}(M)$. Then we have $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, B) \overset{\triangle}{\cong} \mathsf{D}^{\mathrm{b}}(\mathcal{A})$.*

Sketch of proof. Since the right global dimension of B is finite, we have $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, B) \overset{\triangle}{\cong} \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, B)$. □

Theorem 3.8 (Beilinson [Be])**.** *Let $\mathbf{P} = \mathbf{P}^n_k$ be the n-dimensional projective space over a field k, and let $\mathcal{T}_1 = \bigoplus_{i=0}^n \mathcal{O}(-i)$, $\mathcal{T}_2 = \bigoplus_{i=0}^n \Omega^i(i)$, and $B_1 = \mathrm{End}_{\mathbf{P}}(\mathcal{T}_1)$, $B_2 = \mathrm{End}_{\mathbf{P}}(\mathcal{T}_2)$. Then $\mathcal{T}_1$ and $\mathcal{T}_2$ are tilting objects, and*

$$\mathsf{D}^{\mathrm{b}}(\mathsf{coh}\, \mathbf{P}) \overset{\triangle}{\cong} \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, B_1) \overset{\triangle}{\cong} \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, B_2)$$

The algebra $B_i \cong k(\vec{Q}, \rho_i)$, where $(\vec{Q}, \rho_i)$ is the following quiver with relations:



and ρ_i is the set of relations over k:

$\rho_1 :\quad \alpha_i^{l+1}\alpha_j^l = \alpha_j^{l+1}\alpha_i^l$ for $0 \le i < j \le n, 0 \le l < n-1$.

$\rho_2 :\quad \alpha_i^{l+1}\alpha_i^l = 0$ for $0 \le i \le n, 0 \le l < n-1$,

$\qquad\;\; \alpha_i^{l+1}\alpha_j^l + \alpha_j^{l+1}\alpha_i^l = 0$ for $0 \le i < j \le n, 0 \le l < n-1$.

Sketch of proof. Let V be an $(n+1)$-dimensional k-vector space. Since we have quasi-isomorphisms

$$\begin{array}{l}
\mathcal{O}_{\mathbf{P}}(-n-1) \\
\quad\downarrow \\
\wedge^n V \otimes \mathcal{O}_{\mathbf{P}}(-n) \longrightarrow \ldots \longrightarrow \wedge^1 V \otimes \mathcal{O}_{\mathbf{P}}(-1) \longrightarrow \mathcal{O}_{\mathbf{P}}
\end{array}$$

$$\begin{array}{r}
\mathcal{O}_{\mathbf{P}}(-n) \longrightarrow \ldots \longrightarrow \wedge^2 V \otimes \mathcal{O}_{\mathbf{P}}(-1) \longrightarrow \wedge^1 V \otimes \mathcal{O}_{\mathbf{P}} \\
\downarrow\quad \\
\mathcal{O}_{\mathbf{P}}(1)
\end{array}$$

$$\begin{array}{l}
\Omega^i(i) \\
\quad\downarrow \\
\wedge^i V \otimes \mathcal{O}_{\mathbf{P}} \longrightarrow \ldots \longrightarrow \wedge^1 V \otimes \mathcal{O}_{\mathbf{P}}(i-1) \longrightarrow \mathcal{O}_{\mathbf{P}}(i)
\end{array}$$

$\mathcal{T}_1$ (resp., $\mathcal{T}_2$) generates $\mathsf{D}^{\mathrm{b}}(\mathsf{coh}\, \mathbf{P})$. □

Remark 3.9. *On the derived categories of coherent sheaves on weighted projective lines, weighted projective spaces, Grassmann varieties, flag varieties, some toric varieties, similar results were obtained (e.g. Baer* [Ba]*, Kapranov* [Kp1], [Kp2], [Kp3]*, Geigle-Lenzing* [GL]*, Kawamata* [Kw]*).*

Theorem 3.10 (Rickard [Rd1])**.** *Let A be a ring. Let $T^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$ with $\mathrm{Hom}_{\mathsf{K}(\mathsf{Mod}\, A)}(T^\bullet, T^\bullet[i]) = 0$ for $i \neq 0$, and $B = \mathrm{End}_{\mathsf{K}(\mathsf{Mod}\, A)}(T)$. Then there exists a fully faithful ∂-functor $F : \mathsf{K}^-(\mathsf{Proj}\, B) \to \mathsf{K}^-(\mathsf{Proj}\, A)$ such that*

1. *$FB \cong T^\bullet$.*
2. *F preserves coproducts.*
3. *F has a right adjoint $G : \mathsf{K}^-(\mathsf{Proj}\, A) \to \mathsf{K}^-(\mathsf{Proj}\, B)$.*

Let $T^{0\,\bullet} \to T^{1\,\bullet} \to T^{2\,\bullet}$ be a complex in $\mathsf{K}^-(\mathsf{Add}\, T^{\cdot})$. Then $T^{0\,\bullet} \to T^{1\,\bullet} \to T^{2\,\bullet}$ is homotopic to 0. Therefore the above theorem cannot be directly derived from the method of Proposition 3.5. Compare Theorem 1.14 concerning an existence of a right adjoint.

Definition 3.11 (Perfect Complex)**.** *Let A be a ring. A complex $X^{\cdot} \in \mathsf{D}(\mathsf{Mod}\, A)$ is called a perfect complex if $X^{\cdot}$ is quasi-isomorphic to a bounded complex of finitely generated projective A-modules.*

Let X be a scheme, $\mathsf{D}(X)$ the derived category of sheaves of $\mathcal{O}_X$-modules. We denote by $\mathsf{D}_{qc}(X)$ the full subcategory of $\mathsf{D}(X)$ consisting of complexes whose cohomologies are quasi-coherent sheaves. A complex $X^{\cdot} \in \mathsf{D}_{qc}(X)$ is called a perfect complex if $X^{\cdot}$ is locally quasi-isomorphic to a bounded complex of vector bundles.

Proposition 3.12 ([Rd1], [Ne2])**.** *For a ring A, the following hold.*

1. *A complex $X^{\cdot} \in \mathsf{D}(\mathsf{Mod}\, A)$ is perfect if and only if it is a compact object in $\mathsf{D}(\mathsf{Mod}\, A)$.*
2. *$\mathsf{D}(\mathsf{Mod}\, A)$ is compactly generated.*

Theorem 3.13 (Bondal-Van den Bergh [BV])**.** *Let X be a quasi-compact quasi-separated scheme, then the following hold.*

1. *A complex $X^{\cdot} \in \mathsf{D}_{qc}(X)$ is perfect if and only if it is a compact object in $\mathsf{D}_{qc}(X)$.*
2. *$\mathsf{D}_{qc}(X)$ is compactly generated.*

Theorem 3.14 ([Rd1], [Rd2])**.** *Let A, B be algebras over a field k. The following are equivalent.*

1. *$\mathsf{D}(\mathsf{Mod}\, A) \overset{\triangle}{\cong} \mathsf{D}(\mathsf{Mod}\, B)$.*
2. *$\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A) \overset{\triangle}{\cong} \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, B)$.*

(3) *There is a perfect complex* $T^{\boldsymbol{\cdot}} \in \mathsf{D}(\mathsf{Mod}\, A)$ *such that*
 (a) $B \cong \mathrm{End}_{\mathsf{D}(\mathsf{Mod}\, A)}(T^{\boldsymbol{\cdot}})$,
 (b) $\mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\, A)}(T^{\boldsymbol{\cdot}}, T^{\boldsymbol{\cdot}}[i]) = 0$ *for* $i \neq 0$,
 (c) $\{T^{\cdot}[i] | i \in \mathbb{Z}\}$ *is a generating set in* $\mathsf{D}(\mathsf{Mod}\, A)$.

(4) *There is a complex* $V^{\cdot}$ *of* B-A-*bimodules such that*

$$\boldsymbol{R}\,\mathrm{Hom}^{\boldsymbol{\cdot}}_A(V^{\cdot}, -) : \mathsf{D}(\mathsf{Mod}\, A) \to \mathsf{D}(\mathsf{Mod}\, B)$$

is an equivalence.

In this case, $T^{\cdot}$ *is called a tilting complex for* A, $V^{\cdot}$ *is called two-sided tilting complex, and* $\boldsymbol{R}\,\mathrm{Hom}^{\boldsymbol{\cdot}}_A(V^{\cdot}, -)$ *is called a standard equivalence.*

Definition 3.15. *Let* A *be an algebra over a field* k. *The derived Picard group of* A (*relative to* k) *is*

$$\mathrm{DPic}_k(A) := \frac{\{\textit{tilting complexes } T \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A^{\mathrm{op}} \otimes A)\}}{\textit{isomorphism}}$$

with identity element A, *product* $(T_1, T_2) \mapsto T_1 \otimes^{\mathrm{L}}_A T_2$ *and inverse* $T \mapsto T^{\vee} := \mathrm{R}\,\mathrm{Hom}_A(T, A)$. *Given any* k-*linear triangulated category* $\mathcal{D}$ *we let*

$$\mathrm{Out}^{\triangle}_k(\mathcal{D}) := \frac{\{k\textit{-linear triangulated self-equivalences of } \mathcal{D}\}}{\partial\textit{-functorial isomorphism}}. \tag{3.16}$$

Theorem 3.17 ([MY]). *Let* k *be an algebraically closed field, and* A *a finite dimensional hereditary* k-*algebra. Then we have*

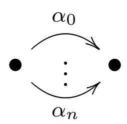
$$\mathrm{DPic}_k(A) = \mathrm{Out}^{\triangle}_k(\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A)) = \mathrm{Out}^{\triangle}_k(\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A))$$

M. Kontsevich and A. Rosenberg introduced the notion of non-commutative projective spaces $\mathbf{NP}^n$ [KR], and showed that

$$\mathsf{D}^{\mathrm{b}}(\mathrm{Qcoh}\,\mathbf{NP}^n) \overset{\triangle}{\cong} \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, kQ_n)$$

$$\mathsf{D}^{\mathrm{b}}(\mathrm{coh}\,\mathbf{NP}^n) \overset{\triangle}{\cong} \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, kQ_n)$$

where Q_n is the quiver



Corollary 3.18 ([MY]). *For a non-commutative projective spaces* $\mathbf{NP}^n$, *we have*

$$\begin{aligned}\mathrm{Out}^{\triangle}_k(\mathsf{D}^{\mathrm{b}}(\mathrm{Qcoh}\,\mathbf{NP}^n)) &\cong \mathrm{Out}^{\triangle}_k(\mathsf{D}^{\mathrm{b}}(\mathrm{coh}\,\mathbf{NP}^n)) \\ &\cong \mathbb{Z} \times \big(\mathbb{Z} \ltimes \mathrm{PGL}_{n+1}(k)\big)\end{aligned}$$

Theorem 3.19 (Bondal-Orlov [BO]). *Let X be a smooth irreducible projective variety with ample canonical or anticanonical sheaf. Then* $\mathrm{Out}_k^{\triangle}(\mathsf{D}^{\mathrm{b}}(\mathrm{coh}\, X))$ *is generated by the automorphisms of variety, the twists by invertible sheaves and the translations, and hence* $\mathrm{Out}_k^{\triangle}(\mathsf{D}^{\mathrm{b}}(\mathrm{coh}\, X)) \cong (\mathrm{Aut}_k X \ltimes \mathrm{Pic}\, X) \times \mathbb{Z}$.

Definition 3.20. *Let $\mathcal{B}$ be an additive category.*

$$\mathrm{Out}_k(\mathcal{B}) := \frac{\{\textit{auto-equivalences of } \mathcal{B}\}}{\textit{isomorphism}}.$$

Let $\mathcal{D}$ be a triangulated category, $\mathcal{E}$ its full subcategory which is closed under isomorphisms. We define the isotropy group

$$\mathrm{Out}_k^{\triangle}(\mathcal{D})_{\mathcal{E}} = \{[F] \in \mathrm{Out}_k^{\triangle}(\mathcal{D}) | F|_{\mathcal{E}} \textit{ is an auto-equivalence of } \mathcal{E}\}$$

Then we have the canonical morphism

$$\pi_{\mathcal{E}} : \mathrm{Out}_k^{\triangle}(\mathcal{D})_{\mathcal{E}} \to \mathrm{Out}_k(\mathcal{E})$$

For a k-algebra B, we have

$$\mathrm{Out}_k^{\triangle}(\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, B))_{\mathcal{P}_B} = \{[F] \in \mathrm{Out}_k^{\triangle}(\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, B)) | FB \in \mathcal{P}_B\},$$

and we have $\mathrm{Out}_k^{\triangle}(\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, B))_{\mathcal{P}_B} \cong \mathrm{Out}_k(\mathcal{P}_B) \ltimes \mathrm{Ker}\, \pi_{\mathcal{P}_B}$.

It is easy to see that

$$\mathrm{DPic}_k(B) = \mathrm{Out}_k^{\triangle}(\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, B)) \Leftrightarrow \mathrm{Ker}\, \pi_{\mathcal{P}_B} = 1$$

Moreover, by replacing $\mathsf{mod}\, B$ with $\mathsf{Mod}\, B$ the similar result holds.

Proposition 3.21. *Let $\mathcal{A}$ be a k-linear abelian category, M a tilting object. If the canonical morphism*

$$\pi_{\mathsf{add}\, M} : \mathrm{Out}_k^{\triangle}(\mathsf{D}^{\mathrm{b}}(\mathcal{A}))_{\mathsf{add}\, M} \to \mathrm{Out}_k(\mathsf{add}\, M)$$

is an isomorphism, then

$$\mathrm{DPic}_k(B) = \mathrm{Out}_k^{\triangle}(\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, B)) \cong \mathrm{Out}_k^{\triangle}(\mathsf{D}^{\mathrm{b}}(\mathcal{A})),$$

where $B = \mathrm{End}_{\mathcal{A}}(M)$.

Example 3.22. *According to a result of Bondal-Orlov,* $\mathrm{Out}_k^{\triangle}(\mathsf{D}^{\mathrm{b}}(\mathsf{coh}\, \mathbf{P}_k^n))$ *is generated by translations, twists and $\mathrm{Aut}_k(\mathbf{P}_k^n)$, is isomorphic to $\mathbb{Z}^2 \times \mathrm{PGL}_{n+1}(k)$. By Corollary 3.8, we have* $\mathrm{Out}_k^{\triangle}(\mathsf{D}^{\mathrm{b}}(\mathsf{coh}\, \mathbf{P}_k^n))_{\mathsf{add}\, \mathcal{T}_i} = \mathrm{Out}_k(\mathsf{add}\, \mathcal{T}_i)$. *Hence we have*

$$\begin{aligned} \mathrm{DPic}_k(B_i) &= \mathrm{Out}_k^{\triangle}(\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, B_i)) \\ &\cong \mathbb{Z}^2 \times \mathrm{PGL}_{n+1}(k). \end{aligned}$$

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