# Periods of automorphic forms and special values of $L$-functions

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# Plan

§1 Automorphic forms

§2 $L$-functions

§3 Periods

# §1 Automorphic forms

automorphic rep $\doteqdot$ rep of $G(\mathbb{A})$ with high symmetry

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$G$ : semisimple algebraic group over $\mathbb{Q}$

$\mathbb{A} = \prod_{p \le \infty}' \mathbb{Q}_p$ : adele ring of $\mathbb{Q}$, loc cpt top ring

$p$ : prime or $\infty$, $\mathbb{Q}_\infty = \mathbb{R}$

$\mathbb{Q} \hookrightarrow \mathbb{A}$ : diagonal embedding, discrete subring

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$\pi$ : irred automorphic rep of $G(\mathbb{A})$

$$G(\mathbb{A}) = \prod_{p \leq \infty}{}' G(\mathbb{Q}_p) \Rightarrow \pi = \bigotimes_{p \leq \infty}{}' \pi_p$$

$\pi_p$ : irred rep of $G(\mathbb{Q}_p)$

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$\rho$ : rep of $G(\mathbb{A})$ on the space

$$L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$$

where

$$[\rho(g)\phi](x) = \phi(xg)$$

for $g, x \in G(\mathbb{A})$, $\phi \in L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$

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Consider an irreducible subrepresentation

$$\pi \subset L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$$

and call it an automorphic rep.

Example

$1 \in L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$

(Fact: $G(\mathbb{Q})\backslash G(\mathbb{A})$ is not nec cpt, but finite volume.)

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"Building blocks" are cuspidal automorphic rep.

$$L^2(G(\mathbb{Q})\backslash G(\mathbb{A})) = L^2_{\text{disc}} \oplus L^2_{\text{cont}} \qquad L^2_{\text{disc}} = L^2_{\text{cusp}} \oplus L^2_{\text{res}}$$

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The trivial rep belongs to $L^2_{\text{res}}$ and is non-cuspidal.

# §2 $L$-functions

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$r$ : fin dim rep of ${}^L G$
$s \in \mathbb{C}$

$$L(s, \pi, r) := \prod_{p:\text{good}} \det \left[1 - p^{-s} \cdot r(c(\pi_p))\right]^{-1} \prod_{p:\text{bad}} \cdots$$

(Fact: almost all primes $p$ are good.)

$c(\pi_p) \in {}^L G$ : Satake parameter of $\pi_p$ at good $p$

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$L$-function is defined by an Euler product

**Theorem** (Langlands).
$L(s, \pi, r)$ *is absolutely convergent for* $\mathrm{Re}(s) \gg 0$.

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Problem

Show that automorphic $L$-functions are "nice".

- meromorphic continuation (MC) to $\mathbb{C}$
- functional equation (FE)

$$L(s, \pi, r) = \varepsilon(s, \pi, r) \cdot L(1 - s, \pi^\vee, r)$$

  ($\pi^\vee$ : contragredient of $\pi$)
- holomorphy, poles, non-vanishing . . .

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$s = 1$ is the center of FE ($s \leftrightarrow 2 - s$),
which is out of the range of convergence.

**Taniyama-Shimura conjecture** (proved after Wiles).

$E$ : *elliptic curve over* $\mathbb{Q}$

$\Rightarrow \exists\ \pi_E$ : *irred cuspidal automorphic rep of* $\mathrm{GL}_2(\mathbb{A})$ *s.t.*

$$L(s, E) = L(s + \tfrac{1}{2}, \pi_E, \mathrm{st})$$

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RHS is abs conv for all $s \in \mathbb{C}$, so $\operatorname*{ord}_{s=1} L(s,E)$ is well-def.

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I'm very shocked.

# §3 Periods

$G_0 \subset G_1$ : both semisimple over $\mathbb{Q}$

$\pi_i$ : irred aut rep of $G_i(\mathbb{A})$ $\quad (i = 0, 1)$

$\phi_i \in \pi_i$ : aut form

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Recall $\phi_i$ is a left $G_i(\mathbb{Q})$-invariant function on $G_i(\mathbb{A})$.

Consider an integral

$$\langle \phi_1|_{G_0}, \phi_0 \rangle := \int_{G_0(\mathbb{Q})\backslash G_0(\mathbb{A})} \phi_1(g)\overline{\phi_0(g)}\, dg \in \mathbb{C}$$

(if it converges) and call it a period.

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$\phi \in \pi_E$ : suitably normalized

$\omega$ : non-zero diff form on $E$ over $\mathbb{Q}$

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Forget geometry and consider only automorphic rep.

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So far, there is no method to study problems in general.

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Multiplicity free

We expect that

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G_0(\mathbb{Q}_p)}(\pi_{1,p} \otimes \bar{\pi}_{0,p}, \mathbb{C}) \leq 1$$

for all $p \leq \infty$.

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**Assumption**

- $\pi_i$ is tempered ($i = 0, 1$).
- No local obstruction:

$$\mathrm{Hom}_{G_0(\mathbb{Q}_p)}(\pi_{1,p} \otimes \bar{\pi}_{0,p}, \mathbb{C}) \neq 0 \qquad \forall p \leq \infty$$

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**Gross-Prasad conjecture** ('92).

$$\langle \phi_1|_{G_0}, \phi_0 \rangle \neq 0 \textit{ for some } \phi_i \in \pi_i \Leftrightarrow L(\tfrac{1}{2}, \pi_1 \boxtimes \pi_0) \neq 0$$

$L(s, \pi_1 \boxtimes \pi_0)$ : associated to the tensor product of the standard rep of ${}^L G_1$ and ${}^L G_0$

$s = \frac{1}{2}$ is the center of FE ($s \leftrightarrow 1 - s$).

Difficulty:
$L(\frac{1}{2}, \pi_1 \boxtimes \pi_0)$ has not been defined in general.

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Not only the non-vanishing criterion,
we want a formula for $\langle \phi_1|_{G_0}, \phi_0 \rangle$, at least conjecturally.

**Conjecture** (with Tamotsu Ikeda).

$$\frac{|\langle \phi_1|_{G_0}, \phi_0\rangle|^2}{\|\phi_1\|^2 \cdot \|\phi_0\|^2} = 2^{\beta} \cdot C_0 \cdot L^S(M_1^\vee(1)) \times \frac{L^S(\frac{1}{2}, \pi_1 \boxtimes \pi_0)}{L^S(1, \pi_1, \mathrm{Ad}) L^S(1, \pi_0, \mathrm{Ad})} \times \prod_{p \in S} \frac{I_p(\phi_{1,p}, \phi_{0,p})}{\|\phi_{1,p}\|_p^2 \cdot \|\phi_{0,p}\|_p^2}$$

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$$\times \prod_{p \in S} \frac{I_p(\phi_{1,p}, \phi_{0,p})}{\|\phi_{1,p}\|_p^2 \cdot \|\phi_{0,p}\|_p^2}$$

Our period and $L$-value are $\langle \phi_1|_{G_0}, \phi_0 \rangle$ and $L^S(\frac{1}{2}, \pi_1 \boxtimes \pi_0)$.

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$S$ : fin set of bad primes

$L^S$ : Euler product without local factor at $p \in S$

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Ad : adjoint rep of ${}^L G_i$ on its Lie algebra

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$C_0$ : constant dep on normalization of Haar measures

$M_1$ : Gross' Artin motive attached to $G_1$

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$$\frac{I_p(\phi_{1,p}, \phi_{0,p})}{\|\phi_{1,p}\|_p^2 \cdot \|\phi_{0,p}\|_p^2} \geq 0 : \text{ local object dep only on } \phi_{i,p} \in \pi_{i,p}$$

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We believe that $\beta$ is related to Arthur's conjecture (multiplicity of rep in the space of automorphic forms).

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$\exists$ non-tempered cusp aut rep of $\mathrm{SO}_n(\mathbb{A})$

Known results

- $\mathrm{SO}_2 \subset \mathrm{SO}_3$ : Waldspurger '85
- $\mathrm{SO}_3 \subset \mathrm{SO}_4$ : Garrett '87, Harris-Kudla '91 . . . I.

When $n \geq 4$, our $L$-values are not well-def in general.
But, for $\mathrm{SO}_4 \subset \mathrm{SO}_5$,
$\exists$ example of Böcherer-Furusawa-Schulze-Pillot '04.

$\exists$ non-tempered cusp aut rep of $\mathrm{SO}_n(\mathbb{A})$

Some non-tempered examples

- $\mathrm{SO}_4 \subset \mathrm{SO}_5$ : I. '05
- $\mathrm{SO}_5 \subset \mathrm{SO}_6$ : I.-Ikeda

But $\exists$ more difficulty to formulate a conjecture.

Thank you!