ARITHMETIC HILBERT-SAMUEL FORMULA

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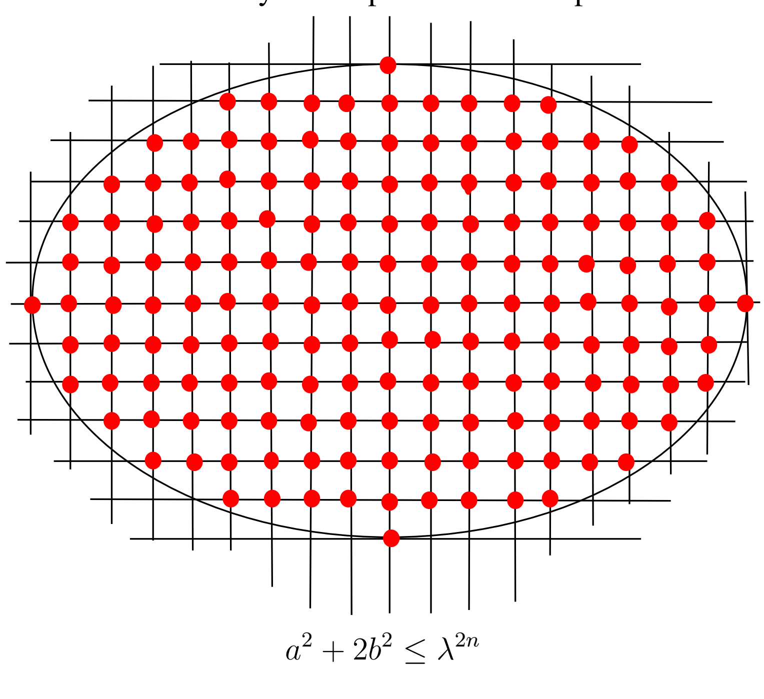
1. Introduction

Let us consider the following problem:

> For a real number $\lambda > 1$, find an asymptotic estimate of
> $$\log \# \left\{ (a, b) \in \mathbb{Z}^2 \mid a^2 + 2b^2 \leq \lambda^{2n} \right\}$$
> with respect to n.

How many lattice points in the ellipse?



$$a^2 + 2b^2 \leq \lambda^{2n}$$

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Considering a shrinking map $(x, y) \mapsto (\lambda^{-n}x, \lambda^{-n}y)$, we have

$$\# \left\{(a, b) \in \mathbb{Z}^2 \mid a^2 + 2b^2 \leq \lambda^{2n}\right\} = \# \left\{(a', b') \in \left(\mathbb{Z}\lambda^{-n}\right)^2 \mid a'^2 + 2b'^2 \leq 1\right\}.$$

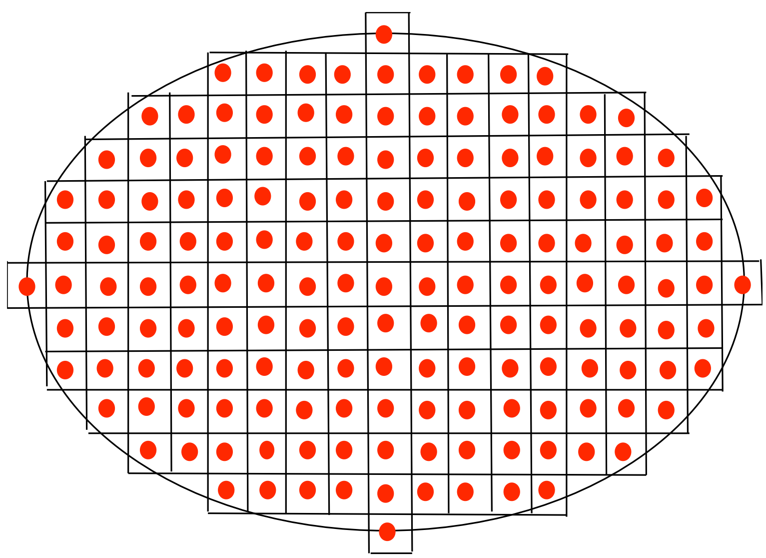
We assign a square

$$\left[a' - \frac{\lambda^{-n}}{2}, a' + \frac{\lambda^{-n}}{2}\right] \times \left[b' - \frac{\lambda^{-n}}{2}, b' + \frac{\lambda^{-n}}{2}\right]$$

with sides of length λ^{-n} to each element of

$$\left\{(a', b') \in \left(\mathbb{Z}\lambda^{-n}\right)^2 \mid a'^2 + 2b'^2 \leq 1\right\}.$$

$$\sum(\text{the volume of each square}) \sim \text{the volume of the ellipse}$$



$x^2 + 2y^2 \leq 1$

Thus

$$\# \left\{(a, b) \in \mathbb{Z}^2 \mid a^2 + 2b^2 \leq \lambda^{2n}\right\} \times (\lambda^{-n})^2 \sim \text{the volume of } \left\{(x, y) \in \mathbb{R}^2 \mid x^2 + 2y^2 \leq 1\right\} = \frac{\pi}{\sqrt{2}}.$$

Therefore, we have the following answer:

$$\boxed{\log \# \left\{(a, b) \in \mathbb{Z}^2 \mid a^2 + 2b^2 \leq \lambda^{2n}\right\} \sim (2 \log \lambda)n.}$$

We would like to interpret the above problem in terms of Arakerov geometry.

Let K be a number field (i.e. finite extension field of $\mathbb{Q}$) and $K(\mathbb{C})$ the set of all embeddings $K \hookrightarrow \mathbb{C}$. Note that $\#(K(\mathbb{C})) = [K : \mathbb{Q}]$ and $K(\mathbb{C})$ is the set of $\mathbb{C}$-valued points of $\operatorname{Spec}(K)$. Let O_K be the ring of integers in K, that is,

$$O_K = \{x \in K \mid x \text{ is integral over } \mathbb{Z}\}.$$

Let L be a finitely generated and flat O_K-module with $\operatorname{rk}(L) = 1$. For each $\sigma \in K(\mathbb{C})$, let L_σ be the tensor product $L \otimes^{\sigma}_{O_K} \mathbb{C}$ with respect to $\sigma : K \hookrightarrow \mathbb{C}$. Let $|\cdot|_\sigma$ be a norm of L_σ. A collection

$$\overline{L} = \left(L, \{|\cdot|_\sigma\}_{\sigma \in K(\mathbb{C})}\right)$$

is called a *hermitian invertible O_K-module*.

Let s be a non-zero element of L. Then a quantity

$$\log \#(L/O_K s) - \sum_{\sigma \in K(\mathbb{C})} \log |s \otimes^{\sigma} 1|_\sigma$$

does not depend on the choice of s due to the product formula, so that it is denoted by $\widehat{\deg}(\overline{L})$ (the arithmetic degree of $\overline{L}$). Moreover, we set

$$\hat{h}^0(\overline{L}) = \log \#\{s \in L \mid \|s\|_{\sup} \leq 1\},$$

where $\|s\|_{\sup} = \max_{\sigma \in K(\mathbb{C})}\{|s \otimes^{\sigma} 1|_\sigma\}$.

Set $K = \mathbb{Q}(\sqrt{-2})$. Then $O_K = \mathbb{Z} + \mathbb{Z}\sqrt{-2}$ and $K(\mathbb{C}) = \{\sigma_1, \sigma_2\}$ given by $\sigma_1(\sqrt{-2}) = \sqrt{-2}$ and $\sigma_2(\sqrt{-2}) = -\sqrt{-2}$. Since $O_K \otimes^{\sigma_i} \mathbb{C}$ is naturally isomorphic to $\mathbb{C}$ via $1 \otimes^{\sigma_i} x \leftrightarrow x$, O_K has the canonical norm $|\cdot|_{can}$ arising from the absolute value of $\mathbb{C}$. Set $\overline{L} = (O_K, \lambda^{-1}|\cdot|_{can})$. Then $\overline{L}^{\otimes n} = (O_K, \lambda^{-n}|\cdot|_{can})$ because $O_K^{\otimes n}$ is isomorphic to O_K via $x \otimes 1 \otimes \cdots \otimes 1 \leftrightarrow x$. Thus

$$\begin{aligned}
\hat{h}^0(\overline{L}^{\otimes n}) &= \log \#\{s \in O_K \mid \max\{\lambda^{-n}|s|, \lambda^{-n}|\bar{s}|\} \leq 1\} \\
&= \log \#\{s \in O_K \mid |s|^2 \leq \lambda^{2n}\} \\
&= \log \# \left\{(a, b) \in \mathbb{Z}^2 \mid a^2 + 2b^2 \leq \lambda^{2n}\right\}.
\end{aligned}$$

Moreover,

$$\widehat{\deg}(\overline{L}) = \log \#(O_K/O_K 1) - \sum_{i=1}^{2} \log(\lambda^{-1}|1|) = 2 \log \lambda.$$

Therefore, the answer of the first problem says that

$$\boxed{\hat{h}^0(\overline{L}^{\otimes n}) = \widehat{\deg}(\overline{L})n + o(n).}$$

This is a toy example of the arithmetic Hilbert-Samuel formula.

2. SMALL SECTIONS

Let X be a projective arithmetic variety (integral scheme, projective and flat over $\operatorname{Spec}(\mathbb{Z})$). Let L be an invertible sheaf on X and $|\cdot|$ a C^∞-hermitian norm of $L_{\mathbb{C}}$ on $X(\mathbb{C})$ (the set of $\mathbb{C}$-valued points of X). A pair $\overline{L} = (L, |\cdot|)$ is called a *C^∞-hermitian invertible sheaf* on X. Here we set

$$\begin{cases} \hat{h}^0(\overline{L}) = \{s \in \ H^0(X, L) \mid \|s\|_{\sup} \leq 1\}, \\ \hat{h}^0_{<1}(\overline{L}) = \{s \in \ H^0(X, L) \mid \|s\|_{\sup} < 1\}, \end{cases}$$

where $\|s\|_{\sup} = \sup_{x \in X(\mathbb{C})}\{|s|(x)\}$.

In order to understand why the above definition gives an arithmetic analogue of h^0, let us consider the following: Let V be a smooth projective variety over a finite field $\mathbb{F}_q$ and η the generic point of V. Let M be an invertible sheaf on V. For a rational section $s \in M_\eta$ and a prime divisor Γ, we set

$$\|s\|^M_\Gamma = \exp(-\operatorname{ord}_\Gamma(a_\Gamma)),$$

where $s = a_\Gamma m_\Gamma$ for a local basis m_Γ of M at Γ. Then $\|\cdot\|^M_\Gamma$ gives rise to a norm on M_η. Note that

$$\begin{cases} H^0(V, M) = \left\{s \in M_\eta \mid \|s\|^M_\Gamma \leq 1 \text{ for all } \Gamma\right\}, \\ h^0(V, M) = \log_q \#H^0(V, M). \end{cases}$$

Let us go back to the arithmetic case. We set $d = \dim X$. Here let us recall the history of the existence theorems of small sections, which are the main subjects of Arakerov geometry.

1. (Faltings) Assume that

$$\begin{cases} \text{(1) } X \text{ is regular,} \\ \text{(2) } d = 2, \\ \text{(3) } L \text{ is ample with respect to } X \to \operatorname{Spec}(\mathbb{Z}), \\ \text{(4) the first Chern form } c_1(\overline{L}) \text{ on } X(\mathbb{C}) \text{ is positive,} \\ \text{(5) } \widehat{\deg}(\widehat{c}_1(\overline{L})^2) > 0. \end{cases}$$

Then $\hat{h}^0_{<1}(\overline{L}^{\otimes n}) \neq 0$ for a sufficiently large n.

2. (Gillet-Soulé) Assume that

$$\begin{cases} \text{(1) } X \text{ is regular,} \\ \text{(2) } L \text{ is ample with respect to } X \to \operatorname{Spec}(\mathbb{Z}), \\ \text{(3) the first Chern form } c_1(\overline{L}) \text{ on } X(\mathbb{C}) \text{ is positive,} \\ \text{(4) } \widehat{\deg}(\widehat{c}_1(\overline{L})^d) > 0. \end{cases}$$

Then $\hat{h}^0_{<1}(\overline{L}^{\otimes n}) \neq 0$ for a sufficiently large n.

3. (S. Zhang) Assume that

$$\begin{cases} \text{(1) } L \text{ is ample on the generic fiber of } X \to \operatorname{Spec}(\mathbb{Z}), \\ \text{(2) } L \text{ is nef on every fiber of } X \to \operatorname{Spec}(\mathbb{Z}), \\ \text{(3) the first Chern form } c_1(\overline{L}) \text{ on } X(\mathbb{C}) \text{ is semi-positive}, \\ \text{(4) } \widehat{\deg}(\widehat{c}_1(\overline{L})^d) > 0. \end{cases}$$

Then $\hat{h}^0_{<1}(\overline{L}^{\otimes n}) \neq 0$ for a sufficiently large n.

The last result due to Zhang has several applications, but the assumption "L is ample on the generic fiber of $X \to \operatorname{Spec}(\mathbb{Z})$" seems to be strong for birational Arakerov geometry.

3. VOLUME FUNCTION

Let X be a d-dimensional projective arithmetic variety and $\overline{L}$ a C^∞-hermitian invertible sheaf on X. Define

$$\widehat{\operatorname{vol}}(\overline{L}) = \limsup_{n\to\infty} \frac{\hat{h}^0(\overline{L}^{\otimes n})}{n^d/d!}.$$

It is an arithmetic analogue of the geometric volume function, so that it is called the *arithmetic volume* of $\overline{L}$. As elementary properties of the arithmetic volume function, the following are known:

(1) $\widehat{\operatorname{vol}}(\overline{L}) = \limsup_{n\to\infty} \dfrac{\hat{h}^0_{<1}(\overline{L}^{\otimes n})}{n^d/d!}$.

(2) If $\widehat{\operatorname{vol}}(\overline{L}) > 0$, then L is big on the generic fiber of $X \to \operatorname{Spec}(\mathbb{Z})$.

In terms of the arithmetic volume function, we have the following generalization of the results in the previous section.

Theorem 3.1 (Generalized Hodge index theorem)**.** *We assume that*

$$\begin{cases} \text{(1) } L \textit{ is nef on every fiber of } X \to \operatorname{Spec}(\mathbb{Z}), \\ \text{(2) } \textit{the first Chern form } c_1(\overline{L}) \textit{ on } X(\mathbb{C}) \textit{ is semi-positive.} \end{cases}$$

Then $\widehat{\operatorname{vol}}(\overline{L}) \geq \widehat{\deg}(\widehat{c}_1(\overline{L})^d)$.

Remark 3.2. The above theorem holds under the following weaker assumptions:

(1) L is nef on the generic fiber of $X \to \operatorname{Spec}(\mathbb{Z})$.

(2) $c_1(\overline{L})$ is semipositive on $X(\mathbb{C})$.

(3) There are a generic resolution of singularities $\mu : Y \to X$ and an ample invertible sheaf A on Y such that, for any positive integer n, we can find a positive integer m_0 with
$$\log \# \left(H^{2i}(Y, m(n\mu^*(L) + A))\right) = o(m^d)$$
for all $m \geq m_0$ and all $i > 0$.

Corollary 3.3. *Under the assumption of the theorem above* (*or the weaker assumptions* (1), (2) *and* (3) *of the remark above*),

(1) *If* $\widehat{\deg}(\widehat{c}_1(\overline{L})^d) > 0$*, then* $\hat{h}^0_{<1}(\overline{L}^{\otimes n}) \neq 0$ *for* $n \gg 1$.
(2) *If* L *is not big on the generic fiber of* $X \to \operatorname{Spec}(\mathbb{Z})$*, then* $\widehat{\deg}(\widehat{c}_1(\overline{L})^d) \leq 0$.

In the case $d = 2$, (2) is nothing more than the arithmetic Hodge index theorem (due to Faltings-Hriljac) on an arithmetic surface. Moreover, (2) implies the arithmetic Bogomolov's inequality.

4. Arithmetic Hilbert-Samuel formula

Let X be a d-dimensional projective arithmetic variety and $\overline{L}$ a C^∞-hermitian invertible sheaf on X. We assume that
$$\begin{cases} (1)\ L \text{ is ample } X, \\ (2) \text{ the first Chern form } c_1(\overline{L}) \text{ on } X(\mathbb{C}) \text{ is positive}, \\ (3)\ H^0(X, L^{\otimes n}) \text{ is generated by sections with } \|s\|_{\sup} < 1 \text{ for } n \gg 0. \end{cases}$$
Then the arithmetic Hilbert-Samuel formula is the following:
$$\hat{h}^0(\overline{L}^{\otimes n}) = \frac{\widehat{\deg}(\widehat{c}_1(\overline{L})^d)}{d!} n^d + o(n^d).$$
This is the main application of Gillet-Soulé Riemann-Roch for arithmetic varieties using an asymptotic estimate of analytic torsions due to Bismut-Vasserot. Later Abbes-Bouche found the proof of the arithmetic Hilbert-Samuel formula without using Gillet-Soulé Riemann-Roch. Here we have the following generalization:

Theorem 4.1 (Arithmetic Hilbert-Samuel formula)**.** *We assume that* $\overline{L}$ *is nef, that is,*
$$\begin{cases} (1)\ L \text{ is nef on every fiber of } X \to \operatorname{Spec}(\mathbb{Z}), \\ (2) \text{ the first Chern form } c_1(\overline{L}) \text{ on } X(\mathbb{C}) \text{ is semi-positive}, \\ (3)\ \widehat{\deg}(\overline{L}\big|_Z) \geq 0 \text{ for all integral } 1\text{-dimensional subschemes } Z. \end{cases}$$
Then
$$\hat{h}^0(\overline{L}^{\otimes n}) = \frac{\widehat{\deg}(\widehat{c}_1(\overline{L})^d)}{d!} n^d + o(n^d).$$

In particular, $\overline{L}$ *big* (*i.e.* $\widehat{\mathrm{vol}}(\overline{L}) > 0$) *if and only if* $\widehat{\deg}(\widehat{c}_1(\overline{L})^d) > 0$.

In the geometric case, for the proof of the above theorem, we use Fujita's vanishing theorem, but we need more sophisticated way for the arithmetic case.

5. Continuity of arithmetic volumes

Let X be a d-dimensional projective arithmetic variety. From now on, the tensor product on $\widehat{\mathrm{Pic}}(X)$ (the group of isomorphism classes of C^∞-hermitian invertible sheaves on X) is denoted by the additive way. It is easy to see that $\widehat{\mathrm{vol}}(n\overline{L}) = n^d \widehat{\mathrm{vol}}(\overline{L})$. This means that $\widehat{\mathrm{vol}} : \widehat{\mathrm{Pic}}(X) \to \mathbb{R}$ extends to $\widehat{\mathrm{Pic}}(X) \otimes \mathbb{Q} \to \mathbb{R}$. The most important property of the volume function is the following continuity, which is the main tool for the previous results.

Theorem 5.1 (Continuity of arithmetic volumes)**.** *Let* $\overline{L}, \overline{A}_1, \ldots, \overline{A}_r$ *be elements of* $\widehat{\mathrm{Pic}}(X) \otimes \mathbb{Q}$. *Then there is a constant* C *such that*

$$|\widehat{\mathrm{vol}}(\overline{L} + \epsilon_1 \overline{A}_1 + \cdots + \epsilon_r \overline{A}_r) - \widehat{\mathrm{vol}}(\overline{L})| \leq C(|\epsilon_1| + \cdots + |\epsilon_r|)$$

for all rational numbers $\epsilon_1, \ldots, \epsilon_r$ *with* $|\epsilon_1| \leq 1, \ldots, |\epsilon_r| \leq 1$. *In particular, we have*

$$\lim_{\substack{\epsilon_1,\ldots,\epsilon_r \in \mathbb{Q} \\ \epsilon_1 \to 0,\ldots,\epsilon_r \to 0}} \widehat{\mathrm{vol}}(\overline{L} + \epsilon_1 \overline{A}_1 + \cdots + \epsilon_r \overline{A}_r) = \widehat{\mathrm{vol}}(\overline{L}).$$

6. Normed $\mathbb{Z}$-module

Let us recall a normed $\mathbb{Z}$-module and their properties. Let M be a finitely generated $\mathbb{Z}$-module and $\|\cdot\|$ a norm of $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$. A pair $\overline{M} = (M, \|\cdot\|)$ is called a *normed* $\mathbb{Z}$*-module*. Define

$$\hat{h}^0(\overline{M}) = \log \#\{m \in M \mid \|m\| \leq 1\}.$$

For $\phi \in M_{\mathbb{R}}^\vee = \mathrm{Hom}_{\mathbb{R}}(M_{\mathbb{R}}, \mathbb{R})$, we set

$$\|\phi\|^\vee = \sup\left\{ \frac{|\phi(x)|}{\|x\|} \;\middle|\; x \in M_{\mathbb{R}} \setminus \{0\} \right\}.$$

Then $(M^\vee, \|\cdot\|^\vee)$ is a normed $\mathbb{Z}$-module. Define

$$\hat{h}^1(\overline{M}) = \hat{h}^0\left(M^\vee, \|\cdot\|^\vee\right).$$

Define

$$\hat{\chi}(\overline{M}) = \frac{\mathrm{vol}(\{x \in M_{\mathbb{R}} \mid \|x\| \leq 1\})}{\mathrm{vol}(M_{\mathbb{R}}/(M/M_{tor}))},$$

where vol is a Haar measure of $M_{\mathbb{R}}$. Note that $\hat{\chi}(\overline{M})$ does not depend on the choice of the Haar measure vol. Then the following inequalities are known as Riemann-Roch theorem for normed $\mathbb{Z}$-modules due to Gillet-Soulé:

$$
\begin{aligned}
&-\log(3/2)\operatorname{rk}(M) - 2\log((\operatorname{rk}(M))!) \\
&\qquad\qquad \leq \hat{\chi}(\overline{M}) - (\hat{h}^0(\overline{M}) - \hat{h}^1(\overline{M})) \leq \log(6)\operatorname{rk}(M).
\end{aligned}
$$

7. Proof of Generalized Hodge index theorem

Let us give a sketch of the proof of "Generalized Hodge index theorem" using "Continuity of arithmetic volumes".

Let $\overline{A}$ be a C^∞-hermitian invertible sheaf on X such that A is ample with respect to $X \to \operatorname{Spec}(\mathbb{Z})$ and $c_1(\overline{A})$ is positive.

$$
\begin{cases}
\text{(Gillet-Soulé Riemann-Roch for arithmetic varieties)} \\
\text{(Aymptotic estimate of analytic torsions due to Bismut-Vasserot)}
\end{cases}
$$

or

(Arithmetic Hilbert-Samuel formula due to Abbes-Bouche)

$$
\Longrightarrow \quad \hat{\chi}\left(H^0(nA), \|\cdot\|_{\sup}\right) = \frac{\widehat{\deg}(\widehat{c}_1(\overline{A})^d)}{d!} n^d + o(n^d)
$$

Thus, using Gillet-Soulé Riemann-Roch for a normed $\mathbb{Z}$-modules, we have

$$
\hat{h}^0\left(H^0(nA), \|\cdot\|_{\sup}\right) \geq \frac{\widehat{\deg}(\widehat{c}_1(\overline{A})^d)}{d!} n^d + o(n^d).
$$

Hence $\widehat{\operatorname{vol}}(\overline{A}) \geq \widehat{\deg}(\widehat{c}_1(\overline{A})^d)$. For a positive integer n, $nL + A$ is ample with respect to $X \to \operatorname{Spec}(\mathbb{Z})$ and $c_1(n\overline{L} + \overline{A})$ is positive. Thus

$$
\widehat{\operatorname{vol}}(n\overline{L} + \overline{A}) \geq \widehat{\deg}((n\widehat{c}_1(\overline{L}) + \widehat{c}_1(\overline{A}))^d),
$$

which means that

$$
\widehat{\operatorname{vol}}(\overline{L} + (1/n)\overline{A}) \geq \widehat{\deg}((\widehat{c}_1(\overline{L}) + (1/n)\widehat{c}_1(\overline{A}))^d).
$$

Therefore, using the continuity of arithmetic volumes, we have

$$
\widehat{\operatorname{vol}}(\overline{L}) \geq \widehat{\deg}(\widehat{c}_1(\overline{L})^d).
$$

8. abc-ESTIMATE

We assume that the generic fiber of $X \to \operatorname{Spec}(\mathbb{Z})$ is smooth over $\mathbb{Q}$. The continuity of arithmetic volumes is a consequence of the following theorem.

Theorem 8.1 (abc-estimate)**.** *Let $\overline{L}$ and $\overline{A}$ be C^∞-hermitian invertible sheaves on X. Then there are positive constants a_0, C and D depending only on X, $\overline{L}$ and $\overline{A}$ such that*

$$\hat{h}^0(a\overline{L} + (b-c)\overline{A}) - \hat{h}^0(a\overline{L} - c\overline{A}) \leq Cba^{d-1} + Da^{d-1}\log(a)$$

for all integers a, b, c with $a \geq b \geq c \geq 0$ and $a \geq a_0$.

We can easily reduce to the case $\overline{L}$ and $\overline{A}$ with the following properties:

(1) A is very ample on X.
(2) The first Chern forms $c_1(\overline{A})$ and $c_1(\overline{L} + \overline{A})$ on $X(\mathbb{C})$ are positive.

In order to explain the technical aspects of the above theorem, let us consider it in the geometric case, namely, we assume that X is a smooth projective variety over $\mathbb{C}$, and we try to estimate

$$\Delta = h^0(X, aL + (b-c)A) - h^0(X, aL - cA).$$

The first way (elegant geometric way): Let us choose an infinite sequence $\{Y_i\}_{i=1}^\infty$ of distinct smooth members of $|A|$ such that

$$h^0(Y_i, nL + mA|_{Y_i}) = h^0(Y_j, nL + mA|_{Y_j})$$

for all i, j and all integers n, m. Then an exact sequence

$$0 \to H^0(aL - cA) \to H^0(aL + (b-c)A) \to \bigoplus_{i=1}^{b} H^0(aL + (b-c)A|_{Y_i})$$

gives rise to $\Delta \leq b \cdot h^0(a(L+A)|_{Y_1})$. This argument does not work in the arithmetic situation.

The second way (Yuan's way): For a fixed smooth member $Y \in |A|$, exact sequences

$$0 \to aL + (k-1-c)A \to aL + (k-c)A \to aL + (k-c)A|_Y \to 0$$

for $1 \leq k \leq b$ yield

$$\Delta \leq \sum_{k=1}^{b} h^0(aL + (k-c)A|_Y) \leq b \cdot h^0(a(L+A)|_Y).$$

This second way works if we consider $\hat{\chi}$ instead of $\hat{h}^0$. In this way, Yuan obtained an arithmetic analogue of a theorem of Siu. Unfortunately it does not work well for our purpose.

The third way (our way): An exact sequence

$$0 \to aL - cA \to aL + (b-c)A \to aL + (b-c)A|_{bY} \to 0$$

gives rise to

$$\Delta \le h^0(aL + (b-c)A|_{bY}).$$

On the other hand, look at exact sequences

$$0 \to aL + (b-c-k)A|_Y \to aL + (b-c)A|_{(k+1)Y} \to aL + (b-c)A|_{kY} \to 0.$$

These imply

$$h^0(aL + (b-c)A|_{bY}) \le \sum_{k=0}^{b-1} h^0(aL + (b-c-k)A|_Y) \le b \cdot h^0(a(L+A)|_Y).$$

In the arithmetic context, the behavior of the error terms by this way is better than the second way, so that we could get the desired estimate. Of course, this way is very complicated because it involves non-reduced schemes.

References

[1] A. Moriwaki, Continuity of volumes on arithmetic varieties, preprint.