QUANTUM RULED SURFACES

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Abstract. One of the main projects in noncommutative algebraic geometry is to classify all noncommutative projective surfaces. A conjecture due to Artin says that every noncommutative projective surface is birationally equivalent to either (1) a quantum projective plane, (2) a quantum ruled surface, or (3) a surface finite over its center. Quantum projective planes have been studied intensively by many people, in particular, they have been classified by Artin, Tate and Van den Bergh. On the other hand, there are many open questions on quantum ruled surfaces. In this note, we will introduce quantum ruled surfaces and explain some of the recent results on them.

1. Motivation

One of the fundamental motivations of noncommutative algebraic geometry is to study noncommutative algebras using ideas and techniques of algebraic geometry. Since classification of low dimensional projective schemes has been active and successful in algebraic geometry for many years, one of the major projects in noncommutative algebraic geometry is to classify low dimensional noncommutative projective schemes defined by Artin and Zhang. Although classification of noncommutative projective curves were completed by Artin and Stafford, classification of noncommutative projective surfaces is still nowhere in sight. The purpose of this note is to introduce a quantum ruled surface, which is an important class of noncommutative projective surfaces, and explain some of the recent results. Since intersection theory plays an essential role in the classification of commutative schemes, it should be extended to noncommutative settings. In this note, we will particularly see that the intersection theory defined by Smith and the author of this note works well over a quantum ruled surface.

2. Quasi-schemes

Throughout, let k be a fixed field. The following theorem motivates the definition of a quasi-scheme below.

Theorem 2.1 (Gabriel [7], Rosenberg [16]). *Every scheme can be reconstructed from the category of quasi-coherent sheaves on it.*

This is an expository paper.

Definition 2.2 (Rosenberg [15], Van den Bergh [22]). *A quasi-scheme X is a Grothendieck category* $\operatorname{Mod} X$*, that is,* $\operatorname{Mod} X$ *is an abelian category having a generator and the exact direct limits.*

An object in $\operatorname{Mod} X$ is called an X-module. As we see below, the notion of quasi-scheme includes both commutative and noncommutative schemes.

Example 2.3. *A quasi-compact, quasi-separated scheme X is a quasi-scheme where* $\operatorname{Mod} X$ *is the category of quasi-coherent sheaves on X* [20].

Let R be a ring, and $\operatorname{Mod} R$ the category of right R-modules. If R is commutative and $X = \operatorname{Spec} R$, then it is well-known that $\operatorname{Mod} X \cong \operatorname{Mod} R$, which motivates the following definition.

Definition 2.4 (Artin-Zhang [5]). *For a ring R not necessarily commutative, the noncommutative affine scheme associated to R is a quasi-scheme $X = \operatorname{Spec} R$ where* $\operatorname{Mod} X = \operatorname{Mod} R$*.*

Let A be a graded algebra, and $\operatorname{GrMod} A$ the category of graded right A-modules. We say that $M \in \operatorname{GrMod} A$ is torsion if $M_n = 0$ for all $n \gg 0$. Let $\operatorname{Tors} A \subset \operatorname{GrMod} A$ be the full subcategory consisting of direct limits of torsion modules, and $\operatorname{Tails} A = \operatorname{GrMod} A / \operatorname{Tors} A$ the quotient category. The following well-known theorem motivates the definition of a noncommutative projective scheme below.

Theorem 2.5 (Serre [17])**.** *If A is a commutative graded algebra finitely generated in degree 1 over k and $X = \operatorname{Proj} A$, then* $\operatorname{Mod} X \cong \operatorname{Tails} A$*.*

Definition 2.6 (Artin-Zhang [5]). *For a graded ring A not necessarily commutative, the noncommutative projective scheme associated to A is a quasi-scheme $X = \operatorname{Proj} A$ where* $\operatorname{Mod} X = \operatorname{Tails} A$*.*

If A is a noetherian graded domain, then we define the function field of X by

$$k(X) := \{a/b \in Q(A) \mid a, b \in A \text{ homogeneous of the same degree}\}.$$

If A is a graded domain finitely generated in degree 1 over k of $\operatorname{GKdim} A = d+1$, then it is reasonable to call $\operatorname{Proj} A$ a noncommutative projective variety of dimension d.

Since classification of low dimensional projective varieties have been successful in algebraic geometry, one of the major projects in noncommutative algebraic geometry is to classify low dimensional noncommutative projective varieties.

Since noncommutative projective curves were classified by Artin and Stafford [2] (1995), the next project is to classify noncommutative projective surfaces. This project is still nowhere in sight. We only have a conjecture below.

Conjecture 2.7 (Artin [1]). *Every noncommutative projective surface is birationally equivalent to one of the following:*

(1) *a quantum projective plane.*
(2) *a quantum ruled surface.*
(3) *a surface finite over its center.*

Since quantum projective planes were classified by Artin, Tate and Van den Bergh [4] (1990), the next project is to classify quantum ruled surfaces, which is still wide open. The purpose of this note is to introduce quantum ruled surfaces and explain some of the recent results on them.

3. Bimodules

Blowing up plays an essential role in the classification of commutative schemes, so it should be extended to the noncommutative setting. Van den Bergh introduced a notion of bimodule below in order to define blowing up of a point on a noncommutative surface.

Definition 3.1 (Van den Bergh [22]). *Let X, Y be quasi-schemes. An X-Y bimodule M is an adjoint pair of functors*

$$\begin{aligned} - \otimes_X M \quad &: \operatorname{Mod} X \to \operatorname{Mod} Y \\ \mathcal{H}om_Y(M, -) \quad &: \operatorname{Mod} Y \to \operatorname{Mod} X, \end{aligned}$$

that is,

$$\operatorname{Hom}_Y(- \otimes_X M, -) \cong \operatorname{Hom}_X(-, \mathcal{H}om_Y(M, -)).$$

Of course, this definition was motivated by the following fact.

Example 3.2. *If R, S are rings and $X = \operatorname{Spec} R, Y = \operatorname{Spec} S$, then M is an X-Y bimodule if and only if M is an R-S bimodule.*

In the commutative case, modules are naturally bimodules.

Example 3.3. *A coherent $\mathcal{O}_X$-module $\mathcal{M}$ on a scheme X can be viewed as an X-bimodule by*

$$\begin{aligned} - \otimes_{\mathcal{O}_X} \mathcal{M} \quad &: \operatorname{Mod} X \to \operatorname{Mod} X \\ \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, -) \quad &: \operatorname{Mod} X \to \operatorname{Mod} X. \end{aligned}$$

There is a special bimodule over any quasi-scheme X corresponding to the "structure sheaf" on X.

Example 3.4. *Let X be a quasi-scheme. The identity functor, denoted by*

$$o_X : \operatorname{Mod} X \to \operatorname{Mod} X,$$

can be viewed as an X-bimodule.

Let X, Y, Z be quasi-schemes. If M is an X-Y bimodule, and N is a Y-Z bimodule, then $M \otimes_Y N$ is an X-Z bimodule defined by the composition of functors, that is,

$$\begin{aligned} - \otimes_X (M \otimes_Y N) \quad &:= (- \otimes_X M) \otimes_Y N \\ \mathcal{H}om_Z(M \otimes_Y N, -) \quad &:= \mathcal{H}om_Y(M, \mathcal{H}om_Z(N, -)). \end{aligned}$$

Since a Grothendieck group has enough injectives, Ext-groups can be defined using injective resolution. We will define Tor-groups as follows: let M be an X-Y bimodule. We define $\mathcal{T}\mathrm{or}_i^X(-, M) : \operatorname{Mod} X \to \operatorname{Mod} Y$ by the formula

$$\operatorname{Hom}_Y(\mathcal{T}\mathrm{or}_i^X(-, M), I) \cong \operatorname{Ext}^i_X(-, \mathcal{H}om_Y(M, I))$$

for all injective objects $I \in \operatorname{Mod} Y$.

Let X be a quasi-scheme. If $\operatorname{Mod} X$ is k-linear, that is, Hom_X-set has a k-vector space structure compatible with compositions, then we say that X is a quasi-scheme over k. We say that X is noetherian if $\operatorname{Mod} X$ is locally noetherian, that is, $\operatorname{Mod} X$ has a set of noetherian generators. If X is a noetherian quasi-scheme, then we use lower letter case $\operatorname{mod} X \subset \operatorname{Mod} X$ to denote the full subcategory consisting of noetherian objects. For example, if X is a noetherian scheme, then $\operatorname{mod} X$ is the category of coherent sheaves on X, and if R is a right noetherian ring, then $\operatorname{mod} R$ is the category of finitely generated right R-modules.

4. Quantum Projective Space Bundle

In this section, let X be a noetherian quasi-scheme. We will define a noncommutative projective scheme over X, extending the idea of Artin and Zhang.

Definition 4.1 (Van den Bergh [22]). *A graded X-algebra is a direct sum*

$$A = \bigoplus_{i \in \mathbf{Z}} A_i$$

of X-bimodules equipped with natural transformations

$$\begin{aligned} o_X \quad &\to A_0 \quad \text{(unit)} \\ A_i \otimes_X A_j \quad &\to A_{i+j} \quad \text{(multiplication)} \end{aligned}$$

satisfying the usual axioms of a ring (commutative diagrams below):

$$\begin{array}{ccc} A_i \otimes_X o_X & \to & A_i \otimes_X A_0 \\ & \searrow & \downarrow \\ & & A_i \end{array} \qquad \begin{array}{ccc} o_X \otimes_X A_j & \to & A_0 \otimes_X A_j \\ & \searrow & \downarrow \\ & & A_j \end{array}$$

$$\begin{array}{ccc} A_i \otimes_X A_j \otimes_X A_k & \to & A_{i+j} \otimes_X A_k \\ \downarrow & & \downarrow \\ A_i \otimes_X A_{j+k} & \to & A_{i+j+k}. \end{array}$$

A graded right A-module is a direct sum

$$M = \bigoplus_{i \in \mathbf{Z}} M_i$$

of X-modules equipped with functors

$$M_i \otimes_X A_j \to M_{i+j} \text{ (action)}$$

satisfying the usual axioms of a module (commutative diagrams below):

$$\begin{array}{ccc} M_i \otimes_X o_X & \to & M_i \otimes_X A_0 \\ & \searrow & \downarrow \\ & & M_i \end{array}$$

$$\begin{array}{ccc} M_i \otimes_X A_j \otimes_X A_k & \to & M_{i+j} \otimes_X A_k \\ \downarrow & & \downarrow \\ M_i \otimes_X A_{j+k} & \to & M_{i+j+k}. \end{array}$$

We would like to point out that a graded X-algebra A itself has no canonical structure of a graded right A-module.

Let A be a graded X-algebra, and $\operatorname{GrMod} A$ the category of graded right A-modules. We say that $M \in \operatorname{GrMod} A$ is torsion if $M_n = 0$ for all $n \gg 0$. Let $\operatorname{Tors} A \subset \operatorname{GrMod} A$ be the full subcategory consisting of direct limits of torsion modules, and $\operatorname{Tails} A = \operatorname{GrMod} A / \operatorname{Tors} A$ the quotient category as before.

Definition 4.2 (Van den Bergh [22]). *For a graded X-algebra A,* $\operatorname{Proj} A$ *is the quasi-scheme where* $\operatorname{Mod}(\operatorname{Proj} A) = \operatorname{Tails} A$*.*

We denote the quotient functor by

$$\pi : \operatorname{GrMod} A \to \operatorname{Tails} A,$$

which is an exact functor, and the section functor by

$$\omega : \operatorname{Tails} A \to \operatorname{GrMod} A,$$

which is the right adjoint to π. The structure map

$$f : \operatorname{Proj} A \to X$$

is the adjoint pair of functors defined by

$$f_* : \mathrm{Tails}\, A \xrightarrow{\omega} \mathrm{GrMod}\, A \xrightarrow{(-)_0} \mathrm{Mod}\, X$$

$$f^* : \mathrm{Mod}\, X \xrightarrow{-\otimes_X A} \mathrm{GrMod}\, A \xrightarrow{\pi} \mathrm{Tails}\, A.$$

A quantum projective space bundle over X should be $\mathrm{Proj}\, A$ where A is a graded X-algebra analogous to a polynomial algebra over X. It is natural to impose the following conditions to A. We say that:

1. A is connected if $A_i = 0$ for $i < 0$ and $A_0 \cong o_X$.
2. A is regular if $\mathcal{T}\mathrm{or}_i^A(-, o_X) = 0$ for all $i \gg 0$.
3. A is flat if $- \otimes_X A : \mathrm{Mod}\, X \to \mathrm{GrMod}\, A$ is an exact functor.
4. A is noetherian if the functor $- \otimes_X A : \mathrm{Mod}\, X \to \mathrm{GrMod}\, A$ sends neotherian objects to noetherian objects, that is, $- \otimes_X A : \mathrm{mod}\, X \to \mathrm{grmod}\, A$.

Definition 4.3 (Mori-Smith [12]). *A quantum projective space bundle over X is a quasi-scheme* Proj A *where A is a noetherian, flat, regular, connected graded X-algebra.*

As in the commutative case, it is important to calculate the Grothendieck group of a quantum projective space bundle to perform intersection theory (see section 6). The Grothendieck group of X is defined by

$$K_0(X) := K_0(\mathrm{mod}\, X).$$

Let A be a graded X-algebra and $M \in \mathrm{grmod}\, A$. The Hilbert series of M is defined by

$$H_M(t) := \sum_{i \in \mathbf{Z}} [M_i] t^i \in K_0(X)[[t]][t^{-1}].$$

Theorem 4.4 (Mori-Smith [12])**.** *If* Proj A *is a quantum projective space bundle over a noetherian smooth projective scheme X, then*

$$K_0(\mathrm{Proj}\, A) \cong K_0(X)[t, t^{-1}]/(\sum_{i \in \mathbf{N}} (-1)^i H_{\mathcal{T}\mathrm{or}_i^A(\mathcal{O}_X, o_X)}(t)).$$

Applying the above theorem to $X = \mathrm{Mod}\, k$, we have:

Corollary 4.5 (Mori-Smith [11])**.** *If A is a right noetherian regular connected graded algebra over k, then*

$$K_0(\mathrm{Proj}\, A) \cong \mathbf{Z}[t]/(H_A(t)^{-1}).$$

Example 4.6. *If $A = k[x_1, \ldots, x_n]$ is a commutative polynomial algebra over k so that* Proj $A = \mathbf{P}^{n-1}$*, then A is a noetherian regular connected graded algebra over k with $H_A(t) = (1-t)^{-n}$, so*

$$K_0(\mathbf{P}^{n-1}) \cong \mathbf{Z}[t]/((1-t)^n),$$

which is well-known.

5. Quantum Ruled Surfaces

Let X be a smooth projective curve over k. We will define a quantum ruled surface over X. First, we recall a commutative ruled surface over X. One of the characterizations of a ruled surface over X is a scheme defined by $\mathbf{P}(\mathcal{E}) := \operatorname{Proj} S(\mathcal{E})$ where

- $\mathcal{E}$ is a locally free $\mathcal{O}_X$-module of rank 2, and
- $S(\mathcal{E})$ is the symmetric algebra of $\mathcal{E}$ over X.

Note that

$$S(\mathcal{E}) = T(\mathcal{E})/(\mathcal{Q})$$

where

- $T(\mathcal{E})$ is the tensor algebra of $\mathcal{E}$ over X, and
- $\mathcal{Q} \subset \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{E}$ is an invertible $\mathcal{O}_X$-subbimodule locally generated by the sections of the form $xy - yx$.

We will extend this construction.

Recall that if R is a commutative ring, then R-R bimodules can be identified with $R \otimes R$-modules. If $X = \operatorname{Spec} R$, then $\operatorname{Spec}(R \otimes R) = X \times X$, so X-X bimodules can be identified with $X \times X$-modules.

Definition 5.1 (Artin-Van den Bergh [3]). *Let X be a smooth projective variety over k. A coherent $\mathcal{O}_X$-bimodule is a coherent sheaf $\mathcal{M}$ on $X \times X$ such that*

$$pr_i : \operatorname{Supp} \mathcal{M} \subset X \times X \to X$$

are finite for $i = 1, 2$ where $pr_i(x_1, x_2) = x_i$ are projection maps.

If $\mathcal{M}$ is a coherent $\mathcal{O}_X$-bimodule, then

$$\begin{aligned} - \otimes_X \mathcal{M} &:= pr_{2*}(pr_1^*(-) \otimes_{\mathcal{O}_{X\times X}} \mathcal{M}) \\ \mathcal{H}om_X(\mathcal{M}, -) &:= pr_{1*}(\mathcal{H}om_{\mathcal{O}_{X\times X}}(\mathcal{M}, pr_2^!(-))) \end{aligned}$$

are adjoint pair of functors, so $\mathcal{M}$ can be viewed as an X-bimodule in the earlier sense.

We say that a coherent $\mathcal{O}_X$-bimodule $\mathcal{E}$ is locally free of rank r if $pr_{i*}\mathcal{E}$ are locally free of rank r on X for $i = 1, 2$. Every coherent locally free $\mathcal{O}_X$-bimodule $\mathcal{E}$ of rank r has a right adjoint $\mathcal{E}^*$ and a left adjoint ${}^*\mathcal{E}$ which are also locally free $\mathcal{O}_X$ bimodules of rank r, that is,

$$\operatorname{Hom}_X(- \otimes_{\mathcal{O}_X} \mathcal{E}, -) \cong \operatorname{Hom}_X(-, - \otimes_{\mathcal{O}_X} \mathcal{E}^*),$$
$$\operatorname{Hom}_X(- \otimes_{\mathcal{O}_X} {}^*\mathcal{E}, -) \cong \operatorname{Hom}_X(-, - \otimes_{\mathcal{O}_X} \mathcal{E}).$$

We say that an invertible $\mathcal{O}_X$-subbimodule $\mathcal{Q} \subset \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{E}$ is non-degenerate if the compositions

$$\mathcal{E}^* \otimes_{\mathcal{O}_X} \mathcal{Q} \to \mathcal{E}^* \otimes_{\mathcal{O}_X} \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{E} \to o_X \otimes_{\mathcal{O}_X} \mathcal{E} \to \mathcal{E},$$
$$\mathcal{Q} \otimes_{\mathcal{O}_X} {}^*\mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{E} \otimes_{\mathcal{O}_X} {}^*\mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} o_X \to \mathcal{E}$$

are isomorphisms.

For the rest of this section, let X be a smooth projective curve over k.

Definition 5.2 (Van den Bergh [21], Patrick [14]). *A quantum ruled surface over X is a quasi-scheme $\mathbf{P}(\mathcal{E}) := \operatorname{Proj} \mathcal{A}$ where*

- *$\mathcal{E}$ is a locally free $\mathcal{O}_X$-bimodule of rank 2,*
- *$\mathcal{Q} \subset \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{E}$ is a non-degenerate invertible $\mathcal{O}_X$-subbimodule, and*
- *$\mathcal{A} = T(\mathcal{E})/(\mathcal{Q})$ is the graded X-algebra.*

It is known that $\mathbf{P}(\mathcal{E})$ is independent of the choice of a non-degenerate $\mathcal{Q}$. In fact, $\mathcal{Q}$ is not even needed to define $\mathbf{P}(\mathcal{E})$ [23].

Since a quantum ruled surface over X is a quantum projective space bundle over X, we can calculate its Grothendieck group.

Theorem 5.3 (Mori-Smith [12]). *If $\mathbf{P}(\mathcal{E})$ is a quantum ruled surface over X, then*

$$K_0(\mathbf{P}(\mathcal{E})) \cong K_0(X)[t]/([\mathcal{O}_X] - [pr_{2*}\mathcal{E}]t + [pr_{2*}\mathcal{Q}]t^2).$$

We define the structure sheaf on $\mathbf{P}(\mathcal{E})$ by

$$\mathcal{O}_{\mathbf{P}(\mathcal{E})} := f^*\mathcal{O}_X \in \operatorname{mod} \mathbf{P}(\mathcal{E}),$$

and the canonical sheaf on $\mathbf{P}(\mathcal{E})$ by

$$\omega_{\mathbf{P}(\mathcal{E})} := f^*(\omega_X \otimes_{\mathcal{O}_X} \mathcal{Q})(-2) \in \operatorname{mod} \mathbf{P}(\mathcal{E}),$$

where ω_X is the canonical sheaf on X. The following result says that a quantum ruled surface satisfies classical Serre duality.

Theorem 5.4 (Mori [9]). *If $\mathbf{P}(\mathcal{E})$ is a quantum ruled surface over X, then*

$$\operatorname{Ext}^i_{\mathbf{P}(\mathcal{E})}(\mathcal{M}, \omega_{\mathbf{P}(\mathcal{E})}) \cong \operatorname{Ext}^{2-i}_{\mathbf{P}(\mathcal{E})}(\mathcal{O}_{\mathbf{P}(\mathcal{E})}, \mathcal{M})'$$

for all $\mathcal{M} \in \operatorname{mod} \mathbf{P}(\mathcal{E})$ where $(-)'$ is the functor taking the k-vector space dual.

6. Intersection Theory

Intersection theory plays an essential role in the classification of commutative schemes, so it should be extended to the noncommutative setting. Let X be a noetherian quasi-scheme over k. We say that X is Ext-finite if

$$\dim_k \operatorname{Ext}^i_X(\mathcal{M}, \mathcal{N}) < \infty$$

for all $i \in \mathbf{N}$, and all $\mathcal{M}, \mathcal{N} \in \operatorname{mod} X$. The homological dimension of X is defined by

$$\operatorname{hd}(X) := \sup\{i \mid \operatorname{Ext}^i_X(\mathcal{M}, \mathcal{N}) \neq 0 \text{ for some } \mathcal{M}, \mathcal{N} \in \operatorname{mod} X\}.$$

If X is a noetherian Ext-finite quasi-scheme over k of finite homological dimension, then we can extend the Euler form

$$(\mathcal{M},\mathcal{N}) := \sum_{i\in\mathbf{N}} (-1)^i \dim_k \operatorname{Ext}^i_X(\mathcal{M},\mathcal{N})$$

where $\mathcal{M},\mathcal{N} \in \operatorname{mod} X$ to the Grothendieck group

$$(-,-) : K_0(X) \times K_0(X) \to \mathbf{Z}.$$

Following [11], we define the intersection multiplicity of $\mathcal{M},\mathcal{N} \in \operatorname{mod} X$ by

$$\mathcal{M} \cdot \mathcal{N} := (-1)^{\operatorname{codim} \mathcal{M}} (\mathcal{M},\mathcal{N}),$$

which we also extend to the Grothendieck group. This new intersection theory agrees with the commutative one by the following theorem.

Theorem 6.1 (Chan [6]). *If X is a smooth variety over k, and C, D are subschemes of X such that $\dim C + \dim D \le \dim X$, then*

$$C \cdot D = \mathcal{O}_C \cdot \mathcal{O}_D.$$

Here the left-hand-side $C \cdot D$ is the intersection multiplicity of C and D defined in terms of Tor due to Serre [18], and the right-hand-side $\mathcal{O}_C \cdot \mathcal{O}_D$ is the intersection multiplicity of $\mathcal{O}_C$ and $\mathcal{O}_D$ defined in terms of Ext as above. Note that the condition $\dim C + \dim D \le \dim X$ is necessary in order for $C \cdot D$ to be well-defined (see [18]), however, if X is a smooth projective variety over k, then X is a noetherian Ext-finite quasi-scheme over k of finite homological dimension, so $\mathcal{O}_C \cdot \mathcal{O}_D$ is always well-defined.

For the rest of this section, let X be a smooth projective curve over k. The following two theorems guarantee that we can apply the above new intersection theory to a quantum ruled surface.

Theorem 6.2 (Nyman [13]). *A quantum ruled surface $\mathbf{P}(\mathcal{E})$ over X is* Ext*-finite.*

Theorem 6.3 (Mori-Smith [12]). *A quantum ruled surface $\mathbf{P}(\mathcal{E})$ over X has finite homological dimension.*

We define the following "divisors" on $\mathbf{P}(\mathcal{E})$ as elements of the Grothendieck group:

(1) The section H of $f : \mathbf{P}(\mathcal{E}) \to X$ is defined by

$$\mathcal{O}_H := [\mathcal{O}_{\mathbf{P}(\mathcal{E})}] - [\mathcal{O}_{\mathbf{P}(\mathcal{E})}(-1)] \in K_0(\mathbf{P}(\mathcal{E})).$$

(2) The fiber $f^{-1}p$ of a closed point $p \in X$ is defined by

$$\mathcal{O}_{f^{-1}p} := [f^*\mathcal{O}_X] - [f^*\mathcal{O}_X(-p)] \in K_0(\mathbf{P}(\mathcal{E})).$$

(3) The quasi-canonical divisor K is defined by

$$\mathcal{O}_K := [\omega_{\mathbf{P}(\mathcal{E})}] - [\mathcal{O}_{\mathbf{P}(\mathcal{E})}] \in K_0(\mathbf{P}(\mathcal{E})).$$

If C and D are "divisors" as above, then we define

$$C \cdot D := \mathcal{O}_C \cdot \mathcal{O}_D = -(\mathcal{O}_C, \mathcal{O}_D).$$

We define the Picard group of $\mathbf{P}(\mathcal{E})$ by

$$\operatorname{Pic} \mathbf{P}(\mathcal{E}) = f^* \operatorname{Pic} X \oplus \mathbf{Z}.H.$$

We refer to [12] for the justification of this definition. The following theorem, which says that "fibers do not meet and a fiber and the section meet exactly once" as in the commutative case, completely determines the intersection theory on $\operatorname{Pic} \mathbf{P}(\mathcal{E})$.

Theorem 6.4 (Mori-Smith [12]). *If $\mathbf{P}(\mathcal{E})$ is a quantum ruled surface over X, and $p, q \in X$ are closed points of X, then*
(1) $f^{-1}p \cdot f^{-1}q = 0$.
(2) $f^{-1}p \cdot H = 1$.
(3) $H \cdot f^{-1}q = 1$.
(4) $H \cdot H = \deg(pr_{2*}\mathcal{E})$.

The following three results describe how the canonical divisor intersects with other divisors, extending the commutative results. First one says that an Adjunction Formula holds for $\operatorname{Pic} \mathbf{P}(\mathcal{E})$.

Theorem 6.5 (Mori [9]). *If $\mathbf{P}(\mathcal{E})$ is a quantum ruled surface over X, K is the quasi-canonical divisor on $\mathbf{P}(\mathcal{E})$, and $D = H$ or $D = f^{-1}p$, then*

$$2g - 2 = D \cdot D + D \cdot K,$$

where $g := 1 - (\mathcal{O}_X, \mathcal{O}_D)$ is the genus of D.

The quasi-canonical divisor is determined in $\operatorname{Pic} \mathbf{P}(\mathcal{E})$ up to numerically equivalent as in the commutative case.

Theorem 6.6 (Mori [9]). *If $\mathbf{P}(\mathcal{E})$ is a quantum ruled surface over X, then the quasi-canonical divisor K on $\mathbf{P}(\mathcal{E})$ is numerically equivalent to*

$$-2H + (2g - 2 - e)f^{-1}p$$

where $p \in X$ is a closed point, g is the genus of X, and $e := -H \cdot H$.

In the commutative case, the theorem below is an easy consequence of the theorem above, however, in the noncommutative case, a separate proof is needed because we do not know so far that the new intersection theory is commutative.

Theorem 6.7 (Mori [9]). *Let $\mathbf{P}(\mathcal{E})$ be a quantum ruled surface over X, and K the quasi-canonical divisor on $\mathbf{P}(\mathcal{E})$. If $\mathcal{E}$ commutes with shifts, then*

$$K \cdot K = 8(1-g)$$

where g is the genus of X.

7. Classification (In Progress)

In this section, let X be a smooth projective curve over k. Classification of quantum ruled surfaces is still wide open. This project can be divided into two subprojects.

Question 7.1.
(1) *Classify all locally free $\mathcal{O}_X$-bimodules of rank 2.*
(2) *If $\mathcal{E}, \mathcal{F}$ are locally free $\mathcal{O}_X$-bimodules of rank* 2, *when $\mathbf{P}(\mathcal{E}) \cong \mathbf{P}(\mathcal{F})$?*

The first question can be regarded as a question in commutative algebraic geometry. It seems to be difficult even over $X = \mathbf{P}^1$ (see [14]). For the second question, we have the following result as in the commutative case.

Theorem 7.2 (Mori [10]). *If $\mathcal{E}$ is a locally free $\mathcal{O}_X$-bimodule of rank 2, and $\mathcal{L}, \mathcal{M}$ are invertible $\mathcal{O}_X$-bimodules, then*

$$\mathbf{P}(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{M}) \cong \mathbf{P}(\mathcal{E}).$$

We say that a locally free $\mathcal{O}_X$-bimodule $\mathcal{E}$ is decomposable if $\mathcal{E} \cong \mathcal{L} \oplus \mathcal{M}$ for some locally free $\mathcal{O}_X$-bimodules $\mathcal{L}, \mathcal{M}$ of rank 1. Since every locally free $\mathcal{O}_X$-bimodule of rank 1 is invertible [14], we have the following corollary.

Corollary 7.3 (Mori [10]). *If $\mathcal{E}$ is a decomposable locally free $\mathcal{O}_X$-bimodule of rank 2, then*

$$\mathbf{P}(\mathcal{E}) \cong \mathbf{P}(\mathcal{O}_X \oplus \mathcal{L})$$

for some invertible $\mathcal{O}_X$-bimodule $\mathcal{L}$.

Every invertible $\mathcal{O}_X$-bimodule is uniquely determined by the pair of an invertible $\mathcal{O}_X$-module $\mathcal{L} \in \operatorname{Pic} X$ and an automorphism $\sigma \in \operatorname{Aut} X$ by

$$\mathcal{L}_\sigma := pr_2^* \mathcal{L} \otimes_{\mathcal{O}_{X\times X}} \mathcal{O}_\Gamma$$

where

$$\Gamma = \{(p, \sigma(p)) \mid p \in X\} \subset X \times X$$

is the graph of X under σ [3]. It follows that quantum ruled surfaces $\mathbf{P}(\mathcal{E})$ where $\mathcal{E}$ are decomposable can be parameterized by the pairs $(\mathcal{L}, \sigma)$.

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