行列因子化とミラー対称性

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Abstract. いろいろ考えた結果，代数学シンポジウムでの講演で用いたプロジェクター原稿をほぼそのままの形で報告集原稿とすることにしました．より詳しい内容は参考文献を御覧ください．

1. Motivation

Mirror Symmetry

$$\text{"Algebra"} \Longleftrightarrow \text{"Geometry"}$$

In many cases,

"Geometry" is difficult but "Algebra" is easy.

So, Mirror Symmetry tells us:

Use "Algebra" to study difficult "Geometry"

Our aim:

to study **geometry of vanishing cycles**
in the Milnor fiber of isolated singularities
(quite difficult)
by the **representation theory**
of finite dimensional algebras
(not easy but not too difficult)

Date: 2007 年 8 月 9 日（木）.

2. Application

Can give a correspondence

"Graded Cohen-Macauley modules"

$\Longleftrightarrow$

"Representations of finite dimensional algebras"

3. Preparations

Definition 3.1. Fix $a, b, c, h \in \mathbb{Z}_{>0}$ such that $\gcd(a, b, c, h) = 1$. $W := (a, b, c; h)$ is a **regular weight system** if

$$\chi(W, T) := \frac{(1 - T^{h-a})(1 - T^{h-b})(1 - T^{h-c})}{(1 - T^a)(1 - T^b)(1 - T^c)}$$

is a polynomial in T.

Theorem 3.2 (Kyoji Saito). *The followings are equivalent*:

(i) *$W = (a, b, c; h)$ is a regular weight system.*

(ii) *A generic Element $f \in \mathbb{C}[x, y, z]$ satisfying*

$$E_W f := [a \cdot x\frac{\partial}{\partial x} + b \cdot y\frac{\partial}{\partial y} + c \cdot z\frac{\partial}{\partial z}]f = hf,$$

(a polynomial of degree h) has an isolated singularity only at the origin.

$\mathbb{C}[x, y, z]$ is a graded ring with respect to E_W:

$$\mathbb{C}[x, y, z] = \bigoplus_{d \in \mathbb{Z}_{\geq 0}} \mathbb{C}[x, y, z]_d,$$

$$\mathbb{C}[x, y, z]_d := \{g \in \mathbb{C}[x, y, z] \mid E_W g = dg\}.$$

Remark 3.3. Fix $f_W \in \mathbb{C}[x, y, z]_h$ for a regular weight system W. Then,

$$J(f_W) := \mathbb{C}[x, y, z] \Big/ \left(\frac{\partial f_W}{\partial x}, \frac{\partial f_W}{\partial y}, \frac{\partial f_W}{\partial z}\right)$$

is a finite dimensional $\mathbb{C}$-algebra. In particular, $\chi(W, T)$ is a **Poincaré polynomial** of $J(f_W)$.

Example 3.4. (Regular weight sytem of type A_l)
$W = (1, b, l + 1 - b; l + 1)$.

$$\chi(W, T) = \frac{1 - T^l}{1 - T}.$$

$$f_W(x, y, z) = x^{l+1} + yz$$

$$J(f_W) \simeq \mathbb{C}[x]/(x^l)$$

Definition 3.5 (**Milnor number**).

$$\begin{aligned}\mu_W :=& \dim_{\mathbb{C}} J(f_W)\\ =&\chi(W,1)\\ =&\frac{(h-a)(h-b)(h-c)}{abc}.\end{aligned}$$

Definition 3.6. The integer

$$\epsilon_W := a+b+c-h$$

is called the **minimal exponent** or **Gorenstein parameter** of W.

Remark 3.7. The quotient ring $R_W := \mathbb{C}[x,y,z]/(f_W)$ is a Gorenstein ring such that

$$K_{R_W} \simeq R_W(-\epsilon_W).$$

$(1) \in \mathrm{Aut}(\mathrm{gr}\text{-}R_W)$: the grading shift by 1

Remark 3.8 (Classification). (i) If $\epsilon_W > 0$, then $\epsilon_W = 1$, in particular,

W	f_W	Type
$(1,b,l+1-b;l+1)$	$x^{l+1}+yz$	A_l
$(2,l-2,l-1;2l-2)$	$x^{l-1}+xy^2+z^2$	D_l
$(3,4,6;12)$	$x^4+y^3+z^2$	E_6
$(6,4,9;18)$	$x^3+xy^3+z^2$	E_7
$(6,10,15;30)$	$x^5+y^3+z^2$	E_8.

(ii) If $\epsilon_W = 0$, then W corresponds to a simple elliptic singularity:

W	f_W	type
$(1,1,1;3)$	$x^3+y^3+z^3+axyz$	$\widetilde{E}_6$
$(1,1,2;4)$	$x^4+y^4+z^2+axyz$	$\widetilde{E}_7$
$(1,2,3;6)$	$x^6+y^3+z^2+axyz$	$\widetilde{E}_8$.

(iii) If $\epsilon_W < 0$, then the number of regular weight systems is finite for each fixed ϵ_W.

4. Geometry of Regular Weight Systems

Fix a polynomial f_W for W.

$$f_W : \mathbb{C}^3 \backslash f_W^{-1}(0) \to \mathbb{C}\backslash\{0\}$$

is a topologically locally trivial fiber bundle.

$$X_{W,1} := f_W^{-1}(1) \quad \textbf{Milnor fiber}$$

is a complex manifold of dimension 2, therefore, there exists an **intersection form**

$$I : H_2(X_{W,1}, \mathbb{Z}) \times H_2(X_{W,1}, \mathbb{Z}) \to \mathbb{Z}.$$

Milnor's theorem implies that

$$H_2(X_{W,1}, \mathbb{Z}) \simeq \mathbb{Z}^{\mu_W}.$$

It is generated by **vaniching cycles**.

$$\rho : \pi_1(\mathbb{C}\backslash\{0\}, *)(\simeq \mathbb{Z}) \to \mathrm{Aut}(H_2(X_{W,1}, \mathbb{Z}), -I)$$

$$c_W := \rho(1) \quad \textbf{Milnor Monodromy} .$$

$$\mathcal{R} := \{[L] \in H_2(X_{W,1}, \mathbb{Z}) \mid L : \text{vanishing cycle}\}$$

Claim 4.1. *The data* $(H_2(X_{W,1}, \mathbb{Z}), -I, \mathcal{R}, c_W)$ *satifies axioms of the* **generalized root system** *introduced by K. Saito (a generalization of classical root systems).*

$(H_2(X_{W,1}, \mathbb{Z}), -I)$	**root lattice**
$\mathcal{R}$	**set of roots**
c_W	**Coxeter transformation**

In particular, if W *gives a singularity of type ADE, then it is the classical root system of the corresponding type.*

Remark 4.2. Generalized root systems will play important roles in the study of **Frobenius structures** (K.Saito's flat structures) on the base space of the universal unfolding. Indeed, it is $\mathfrak{h}/\mathcal{W}$ for an ADE singularity where $\mathfrak{h}$ is the **Cartan subalgebra** and $\mathcal{W}$ is the **Weyl group** of the corresponding type.

Problem 4.3 (K.Saito, in transl. AMS)**.** Construct directly from $W = (a, b, c; h)$, **algebraically, arithmetically or combinatorically**, without the geometry of the Milnor fiber, the generalized root system isomorphic to $(H_2(X_{W,1}, \mathbb{Z}), -I, \mathcal{R}, c_W)$.

Remark 4.4. Beyond the classical root system, there is no **canonical** choice of a **simple basis** (or a **distinguished basis** of vanishing cycles). As a result, **Dynkin diagram** given by the intersection matrix of a distinguished basis is not unique. Indeed, the group $B_{\mu_W} \ltimes (\mathbb{Z}/2\mathbb{Z})^{\mu_W}$ acts on the set of Dynkin diagrams. (B_{μ_W} is the braid group on μ_W-strings.)

Problem 4.5. Define a notion of a "good simple basis" for the generalized root system $(H_2(X_{W,1}, \mathbb{Z}), -I, \mathcal{R}, c_W)$.

5. An Approach to Problems

Claim 5.1 (T, math.AG/0506347)**.** *Consider*

- **Categorification of Generalized Root Systems**.
- *Use the idea of* **Homological Mirror Symmetry**.
- *Construct algebraically* **Triangulated Category** *mirror dual to the singularity associated to* W*.*

Root systems	Categorification
$H_2(X_{W,1},\mathbb{Z}) = K_0(\mathcal{T})$ **Grothendieck group**	$\mathcal{T}$ **triangulated category**
$L = [\mathcal{E}]$ vanishing cycle	$\mathcal{E}$ **indecomposable object**
$L_1 \cap L_2$ intersection	$\mathcal{E}_1 \to \mathcal{E}_2$ **morphism**
$c_W = [\tau_{AR}]$ Milnor monodromy	$\tau_{AR} := \mathcal{S} \circ T^{-1}$ **Coxeter functor**
$(L_1, \ldots, L_{\mu_W})$ distinguished basis of vanishing cycles	$(\mathcal{E}_1, \ldots, \mathcal{E}_{\mu_W})$ full strongly exceptional collection

T: the translation functor on $\mathcal{T}$.

S: the Serre functor on $\mathcal{T}$.

τ_{AR}: **Auslander–Reiten translation**.

6. "Nice" Triangulated Categories

A triangulated category is

- an additive category $\mathcal{T}$ with
- $T \in \mathrm{Auteq}(\mathcal{T})$ called a **translation**
- which has a class of **exact triangles**:

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$$

  satisfying certain axioms.

Consider $\mathcal{T}$ with the following properties:

- $\mathcal{T}$ is $\mathbb{C}$**-linear**, i.e., $\mathrm{Hom}_{\mathcal{T}}(E,F)$ is a $\mathbb{C}$-vector space for all $E, F \in \mathcal{T}$.
- $\mathcal{T}$ is **locally finite**, i.e.,

$$\sum_i \dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{T}}(E, T^i F) < \infty, \quad {}^\forall E, F \in \mathcal{T}.$$

- $\mathcal{T}$ is **Krull-Scmidt**, i.e., any object $E \in \mathcal{T}$ is a finite direct sum of indecomposable objects.

Moreover, we need

- $\mathcal{T}$ is an **enhanced** triangulated category, i.e, there exists an A_∞**-category** (or a differential graded category) whose derived category is triangulated equivalent to $\mathcal{T}$.
- $\mathcal{T}$ has a **strongly exceptional collection** $(E_1, \dots, E_n)$, i.e.,
  (i) $\mathrm{Hom}_{\mathcal{T}}(E_i, E_i) = \mathbb{C}$ for all $i = 1, \dots, n$,
  (ii) $\mathrm{Hom}_{\mathcal{T}}(E_i, T^k E_j) \neq 0$ only if $k = 0$ and $i < j$,
  which is **full**, i.e., the smallest full triangulated subcategory containing the objects $\{E_1, \dots, E_n\}$ is equivalent to $\mathcal{T}$.

Proposition 6.1. *$\mathcal{T}$ has the* **Serre functor** *$\mathcal{S}$, i.e., $\mathcal{S} \in \mathrm{Auteq}(\mathcal{T})$ which induces bifunctorial isomorphisms*

$$\mathrm{Hom}_{\mathcal{T}}(E, F) \simeq \mathrm{Hom}_{\mathcal{T}}(F, \mathcal{S}E)^*, \quad {}^\forall E, F \in \mathcal{T}.$$

$A := \mathrm{End}_{\mathcal{T}}(\bigoplus_{i=1}^n E_i)$ is a basic (i.e., $A/\mathrm{rad}A \simeq \mathbb{C} \times \cdots \times \mathbb{C}$) finite dimensional algebra over $\mathbb{C}$.

Proposition 6.2 (Gabriel)**.** *Let A be a basic finite dimensional algebra. Then, there exists a unique* **quiver** *(an oriented graph) $\vec{\Delta}$ such that $A \simeq \mathbb{C}\vec{\Delta}/I$ for some ideal $I \subset \mathbb{C}\vec{\Delta}$. ($\mathbb{C}\vec{\Delta}$ is the* **path algebra** *of the quiver $\vec{\Delta}$.)*

Proposition 6.3. *$\mathcal{T} \simeq D^b(\text{mod-}\mathbb{C}\vec{\Delta}/I)$.*

7. Triangulated Category $\mathcal{T}_W$

W: a regular weight system of **dual type** (i.e., W has a dual W^*, explained later). Fix f_W and set $R_W := \mathbb{C}[x, y, z]/(f_W)$.

Consider the triangulated category

$$D^{gr}_{Sg}(R_W) := D^b(\text{gr-}R_W)/K^b(\text{grproj-}R_W),$$

and set

$$\mathcal{T}_W := D^{gr}_{Sg}(R_{W^*}).$$

Remark 7.1. If $\mathrm{gl.dim}(R) < \infty$, then

$$K^b(\text{grproj-}R) \simeq D^b(\text{gr-}R).$$

8. Properties of $\mathcal{T}_W$

Definition 8.1. $M \in \text{gr-}R_W$ is a **Cohen-Macauley** module if

$$\mathrm{Ext}^i_{R_W}(R_W/\mathfrak{m}, M) = 0, \quad i < \dim R_W = 2.$$

Definition 8.2.

$$\mathrm{CM}^{gr}(R_W) \subset \text{gr-}R_W$$

an exact category of CM-modules.

Lemma 8.3 (Auslander). *$\mathrm{CM}^{gr}(R_W)$ is a* **Frobenius category***.*

A Frobenius category is an exact category with enough injectives and projectives and its class of injectives coincides with that of projectives.

Definition 8.4. Define a category $\underline{\mathrm{CM}}^{gr}(R)$ as follows:

$$Ob(\underline{\mathrm{CM}}^{gr}(R_W)) = Ob(\mathrm{CM}^{gr}(R_W)).$$

$$\underline{\mathrm{Hom}}_{R_W}(M,N) := \mathrm{Hom}_{\text{gr-}R_W}(M,N)/\mathcal{P}(M,N)$$

($g \in \mathcal{P}(M,N)$ iff there exist a projective object P and homomorphisms $g' : M \to P$ and $g'' : P \to N$ such that $g = g'' \circ g'$.)

$\underline{\mathrm{CM}}^{gr}(R_W)$: **stable category** of $\mathrm{CM}^{gr}(R_W)$.

Proposition 8.5 (Happel). *$\underline{\mathrm{CM}}^{gr}(R_W)$ is a triangulated category.*

$S := \mathbb{C}[x,y,z]$.

For $M \in \mathrm{CM}^{gr}(R_W)$,

$^\exists$ graded free resolution of M in gr-S

$$0 \to \tau^{-h} F_1 \xrightarrow{f_1} F_0 \to M \to 0, \quad F_0, F_1.$$

$^\exists f_0 : F_0 \to F_1$ of degree 0 such that

$$f_1 f_0 = f_W \cdot \mathrm{id}_{F_0}, \quad f_0 f_1 = f_W \cdot \mathrm{id}_{F_1}.$$

Definition 8.6 (Eisenbud).

$$\overline{F} := \left(F_0 \underset{f_1}{\overset{f_0}{\rightleftarrows}} F_1 \right),$$

is called a graded **matrix factorization** of f_W.

Remark 8.7.

$$Q := \begin{pmatrix} 0 & f_1 \\ f_0 & 0 \end{pmatrix}, \quad Q^2 = f_W \cdot \mathrm{Id}.$$

Example 8.8.

$$Q := \begin{pmatrix} 0 & x^2 \\ x & 0 \end{pmatrix}, \quad Q^2 = x^3.$$

Example 8.9.

$$Q_0 := \begin{pmatrix} 0 & f \\ 1 & 0 \end{pmatrix}, \quad Q_f := \begin{pmatrix} 0 & 1 \\ f & 0 \end{pmatrix}, \quad Q_f^2 = f.$$

Example 8.10.

$$Q := \begin{pmatrix} 0 & 0 & y & x \\ 0 & 0 & x & -y \\ y & x & 0 & 0 \\ x & -y & 0 & 0 \end{pmatrix}, \quad Q^2 = x^2 + y^2.$$

Lemma 8.11. *The category* $\mathrm{MF}_S^{gr}(f_W)$ *of graded matrix factorizations of* f_W *is a Frobenius category. Therefore, its stable category*

$$\mathrm{HMF}_S^{gr}(f_W) := \underline{\mathrm{MF}}_S^{gr}(f_W)$$

is triangulated.

Remark 8.12.

$$\overline{F} = \left(F_0 \underset{f_1}{\overset{f_0}{\rightleftarrows}} F_1 \right) \mapsto \mathrm{Coker}(f_1) \in \mathrm{CM}^{gr}(R_W).$$

Proposition 8.13 (Buchweitz '85, Orlov '05)**.**

$$\begin{aligned} D^b(\text{gr-}R_W)/D^b(\text{grproj-}R_W) \simeq \quad & \underline{\mathrm{CM}}^{gr}(R_W) \\ \simeq \quad & \mathrm{HMF}_S^{gr}(f_W). \end{aligned}$$

Proposition 8.14. $\underline{\mathrm{CM}}^{gr}(R_W)$ *is locally finite and Krull-Schmidt.*

Proposition 8.15 (Auslander-Reiten)**.**

$$\mathcal{S} = T \circ (-\epsilon_W)$$

is the Serre functor on $\underline{\mathrm{CM}}^{gr}(R_W)$*.*

Proposition 8.16. $T^2 \simeq (h)$*. Therefore*

$$\mathcal{S}^h \simeq T^{h-2\epsilon_W}, \quad \tau_{AR}^h \simeq T^{-2\epsilon_W}.$$

Proposition 8.17 (T)**.** $\mathrm{HMF}_S^{gr}(f_W)$ *is an enhanced triangulated category.*

9. Conjecture

Let W be a regular weight system of **dual type** (explained later). Fix f_{W^*} and set $R_W := \mathbb{C}[x,y,z]/(f_{W^*})$.

$$\mathcal{T}_W := D^{gr}_{Sg}(R_{W^*}) \simeq \underline{\mathrm{CM}}^{gr}(R_{W^*}) \simeq \mathrm{HMF}^{gr}_S(f_{W^*}).$$

Conjecture 9.1 (T)**.** *Let W be a regular weight system of dual type.*

(i) *$\mathcal{T}_W$ has a full strongly exceptional collection $(E_1,\dots,E_{\mu_W})$.*

(ii) *$(K_0(\mathcal{T}_W), \chi + {}^t\chi) \simeq (H_2(X_{W,1},\mathbb{Z}), -I)$, where*

$$\chi(E,F) := \sum_{i\in\mathbb{Z}} (-1)^i \dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{T}_W}(E, T^i F).$$

10. Theorems

Theorem 10.1. *The conjecture is true if $\epsilon_W = 1$, i.e., if W corresponds to ADE singularities which is self-dual ($W = W^*$). In particular, $D^{gr}_{Sg}(R_W) \simeq D^b(\text{mod-}\mathbb{C}\vec{\Delta})$, where $\vec{\Delta}$ is a* **Dynkin quiver** *(Dynkin diagram with an orientation) of the corresponding type.*

[T] for A_l-singularities, [H.Kajiura-K.Saito-T, math.AG/0511155], for general cases.

Theorem 10.2 (KST, arXiv:0708.0210)**.** *The conjecture is true if $\epsilon_W = -1$, i.e., if W corresponds to one of Arnold's 14 exceptional singularities. In particular, $D^{gr}_{Sg}(R_W) \simeq D^b(\text{mod-}\mathbb{C}\vec{\Delta}_{A_W}/I)$, where A_W is the* **Dolgachev number** *of W (=***Gabrielov numver** *of W^*) and $\vec{\Delta}_{A_W}$ with I is a quiver with relations as follows:*

W	f_W	A_W	W^*
$(6,14,21;42)$	$x^7+y^3+z^2$	(2,3,7)	$(6,14,21;42)$
$(6,8,15;30)$	$x^5+xy^3+z^2$	(2,3,8)	$(4,10,15;30)$
$(4,10,15;30)$	$x^5y+y^3+z^2$	(2,4,5)	$(6,8,15;30)$
$(6,8,9;24)$	$x^4+y^3+xz^2$	(2,3,9)	$(3,8,12;24)$
$(3,8,12;24)$	$zx^4+y^3+z^2$	(3,3,4)	$(6,8,9;24)$
$(4,6,11;22)$	$yx^4+xy^3+z^2$	(2,4,6)	$(4,6,11;22)$
$(4,5,10;20)$	$x^5+y^2z+z^2$	(2,5,5)	$(4,5,10;20)$
$(3,5,9;18)$	$zx^3+xy^3+z^2$	(3,3,5)	$(4,6,7;18)$
$(4,6,7;18)$	$x^3y+y^3+xz^2$	(2,4,7)	$(3,5,9;18)$
$(3,4,8;16)$	$x^4y+y^2z+z^2$	(3,4,4)	$(4,5,6;16)$
$(4,5,6;16)$	$x^4+zy^2+z^2$	(2,5,6)	$(3,4,8;16)$
$(3,5,6;15)$	$zx^3+y^3+xz^2$	(3,3,6)	$(3,5,6;15)$
$(3,4,5;13)$	$x^3y+y^2z+z^2x$	(3,4,5)	$(3,4,5;13)$
$(3,4,4;12)$	$x^4+y^2z+yz^2$	(4,4,4)	$(3,4,4;12)$

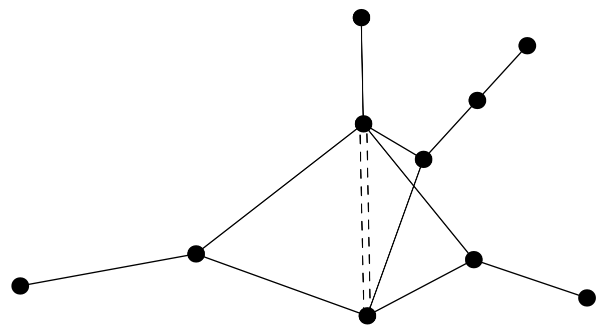


FIGURE 1. Diagram for $A_W = (3,3,4)$. I: two relations along the double dotted line.

11. Sketch of Proof

(i) **Find enough "good" matrix factorizations**.
(ii) Show that these matrix factorizations form a strongly exceptional collection. (Use Serre duality.)
(iii) Use the following to prove the above strongly exceptional collection is full.

Lemma 11.1 (Category Generating Lemma)**.** *Let* $\mathcal{T}' := \langle E_0, \dots, E_n \rangle$ *be a full triangulated subcategory of* $D^{gr}_{Sg}(R_W)$ *generated by an exceptional collection* $(E_0, \dots, E_n)$ *satisfying the following*:

(i) $(1) \in \mathrm{Auteq}(\mathcal{T}')$,
(ii) $\mathcal{T}'$ *has an object* E *isomorphic to* $R_W/\mathfrak{m}$

Then $\mathcal{T}' \simeq D^{gr}_{Sg}(R)$.

This follows from the well-known facts:

Lemma 11.2 ($\mathcal{T}'$ is **right admissible**)**.** *For any* $X \in \mathcal{T}$ *there is an exact triangle*

$$N \to X \to M \to TN$$

where $N \in \mathcal{T}'$ *and* $\mathrm{Hom}_{\mathcal{T}}(N, M) = 0$.

Lemma 11.3. *$M \in \mathrm{CM}^{gr}(R_W)$ is (graded) free if and only if $\mathrm{Ext}^i_{R_W}(R_W/\mathfrak{m}, M) = 0$ for $i \neq d$.*

Remark 11.4. $M \simeq 0$ in $\underline{\mathrm{CM}^{gr}}(R_W)$ if and only if M is free.

Example 11.5.

$$Q := \begin{pmatrix} 0 & f_1 \\ f_0 & 0 \end{pmatrix}, \quad Q^2 = f_W = x^4 z + y^3 + z^2,$$

where $f_0 = f_1$ is given by

$$\begin{pmatrix}
z & -y^2 & 0 & 0 & 0 & 0 & -x^2y & 0 & -x^4 & 0 & 0 & 0 & 0 & 0 \\
-xy & -z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x^4 & 0 & x^2y & 0 \\
0 & -xy & z & 0 & 0 & 0 & -x^3 & 0 & xy^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & z & y^2 & 0 & 0 & 0 & 0 & xy^2 & 0 & x^4 & 0 & -x^3y \\
0 & 0 & 0 & 0 & -z & 0 & 0 & 0 & 0 & 0 & -xy^2 & 0 & -x^4 & 0 \\
0 & 0 & 0 & 0 & 0 & z & y^2 & 0 & 0 & -x^2y & 0 & xy^2 & 0 & x^4 \\
0 & 0 & -x^2 & 0 & 0 & 0 & -z & 0 & 0 & 0 & x^2y & 0 & -xy^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -xy & z & 0 & x^3 & 0 & -x^2y & 0 & xy^2 \\
-x & 0 & y & 0 & 0 & 0 & 0 & 0 & -z & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -y & y & 0 & 0 & 0 & x^2 & 0 & -z & y^2 & 0 & 0 & 0 \\
0 & -x & 0 & 0 & -y & 0 & 0 & 0 & xy & 0 & z & 0 & 0 & 0 \\
0 & 0 & 0 & x & 0 & y & 0 & 0 & 0 & 0 & 0 & -z & y^2 & 0 \\
0 & 0 & 0 & 0 & -x & 0 & -y & 0 & 0 & 0 & 0 & 0 & z & 0 \\
0 & 0 & 0 & 0 & 0 & x & 0 & y & 0 & 0 & 0 & 0 & 0 & -z
\end{pmatrix}$$

This matrix factorization (with suitable gradings) gives the "bottom vertex".

Remark 11.6. The size of matrix factorizations Q corresponding to the "bottom vertex" are very large!! Generally, $Q \in M(S, n)$, $\quad \mathbf{n \geq 20}$.

12. Duality of Regular Weight Systems

Remark 12.1. For W with $\epsilon_W = -1$, W^* is the **Arnold's strange dual** partner of W.

A natural generalization of strange duality $\Longrightarrow$ (topological) **mirror symmetry**.

Definition 12.2 (Vafa's formula)**.** $G \subset GL(3,k)$: finite, diagonal.

$$\begin{aligned}&\chi(W,G)(y,\bar{y})\\ :=& \frac{(-1)^3}{|G|}\sum_{\alpha\in G}\sum_{\beta\in G}\prod_{\omega_i\alpha_i\notin\mathbb{Z}}(y\bar{y})^{\frac{1-2\omega_i}{2}}\left(\frac{y}{\bar{y}}\right)^{-\omega_i\alpha_i+[\omega_i\alpha_i]+\frac{1}{2}}\\ &\times\prod_{\omega_i\alpha_i\in\mathbb{Z}}\mathbf{e}\left[\omega_i\beta_i+\frac{1}{2}\right]\frac{1-\mathbf{e}\left[(1-\omega_i\beta_i)\right](y\bar{y})^{1-\omega_i}}{1-\mathbf{e}\left[\omega_i\beta_i\right](y\bar{y})^{\omega_i}},\end{aligned}$$

where $\omega_i := a_i/h$.

$\chi(W,G)(y,\bar{y})$: **orbifoldized Poincare polynomial**.

Remark 12.3. $\chi_W(T) = \chi(W,\{1\})(T^{\frac{1}{h}},1)$.

Definition 12.4 (Topological Mirror Symmetry)**.** (W^*,G^*) is **topological mirror dual** to (W,G) if

$$\chi(W^*,G^*)(y,\bar{y}) = (-1)^3\bar{y}^{\hat{c}_W}\chi(W,G)(y,\bar{y}^{-1}),$$

where $\hat{c}_W := 1-2\frac{\epsilon_W}{h}$ ($\hat{c}_W = \hat{c}_{W^*}$).

Remark 12.5. Serre functor on $\mathcal{T}_W$ satisfies

$$\mathcal{S}^h \simeq T^{h\cdot\hat{c}_W}.$$

$\mathcal{T}_W$ is **fractional Calabi–Yau** of dimension $\hat{c}_W$.

Definition 12.6. $W=(a,b,c;h)$ is dual to $W^*=(a^*,b^*,c^*;h^*)$ if the pair $(W,\{1\})$ is topological mirror symmetric to the pair $(W^*,\mathbb{Z}/h^*\mathbb{Z})$.

Theorem 12.7 (T '98)**.** *W has the dual W^* if and only if W (f_W) is one of the following 5 types:*

Type I: ($W=W^*$).

$$\mathbf{f_W(x,y,z) := x^{p_1}+y^{p_2}+z^{p_3}},$$

where $(p_i,p_j)=1, i=1,2,3$.

Type II:

$$\mathbf{f_W(x,y,z) := x^{p_1}+y^{p_2}+yz^{\frac{p_3}{p_2}}},$$

$$\mathbf{f_{W^*}(x_*,y_*,z_*) = x_*^{p_1}+y_*^{\frac{p_3}{p_2}}+y_*z_*^{p_2}},$$

where $p_2\neq p_3$, $p_2|p_3$, $(p_1,p_3)=1$, $(p_2-1,p_3)=1$ and $(p_3/p_2-1,p_3)=1$.

Type III: ($W=W^*$).

$$\mathbf{f_W(x,y,z) := x^{p_1}+y^{q_3+1}z+yz^{q_2+1}},$$

where $(p_1, p_2) = 1$, $p_2 + 1 = (q_2 + 1)(q_3 + 1)$ and $(q_2, q_3) = 1$.

Type IV:

$$\mathbf{f_W}(\mathbf{x}, \mathbf{y}, \mathbf{z}) := \mathbf{x}^{\mathbf{p_1}} + \mathbf{x}\mathbf{y}^{\frac{\mathbf{p_2}}{\mathbf{p_1}}} + \mathbf{y}\mathbf{z}^{\frac{\mathbf{p_3}}{\mathbf{p_2}}},$$

$$\mathbf{f_{W^*}}(\mathbf{x}_*, \mathbf{y}_*, \mathbf{z}_*) = \mathbf{x}_*^{\frac{\mathbf{p_3}}{\mathbf{p_2}}} + \mathbf{x}_*\mathbf{y}_*^{\frac{\mathbf{p_2}}{\mathbf{p_1}}} + \mathbf{y}_*\mathbf{z}_*^{\mathbf{p_1}},$$

where $p_1 \neq p_2 \neq p_3$, $p_1|p_3$, $p_2|p_3$, $(p_1-1, p_2) = 1$, $(p_2-p_1+1, p_3) = 1$, $(p_3/p_2-1, p_3/p_1) = 1$ and $(p_3/p_1 - p_3/p_2 + 1, p_3) = 1$.

Type V:

$$\mathbf{f_W}(\mathbf{x}, \mathbf{y}, \mathbf{z}) := \mathbf{z}\mathbf{x}^{\mathbf{k}} + \mathbf{x}\mathbf{y}^{\mathbf{m}} + \mathbf{y}\mathbf{z}^{\mathbf{l}},$$

$$\mathbf{f_{W^*}}(\mathbf{x}_*, \mathbf{y}_*, \mathbf{z}_*) = \mathbf{z}_*\mathbf{x}_*^{\mathbf{k}} + \mathbf{x}_*\mathbf{y}_*^{\mathbf{l}} + \mathbf{y}_*\mathbf{z}_*^{\mathbf{m}},$$

where $(lm-m+1, klm+1) = 1$, $(mk-k+1, klm+1) = 1$, and $(kl-l+1, klm+1) = 1$.

Remark 12.8. If W^* is dual to W, then W^* is dual to W in the sense of K. Saito (duality defined by Coxeter transformations).

Remark 12.9. Regular weight systems with $\epsilon_W = 0$ (simple elliptic singularities) are not of dual type.

13. Homological Mirror Symmetry

Conjecture 13.1. *There should exists an A_∞-category*

$$\mathrm{Fuk}^{\rightarrow}(X_{W,1}),$$

(*objects are finite number of vaniching cycles in the Milnor fiber and homomorphism spaces are given by* **Floer homology**) *satisfying certain properties, such that*

$$D^b\mathrm{Fuk}^{\rightarrow}(X_{W,1}) \simeq \mathrm{HMF}^{gr}_S(f_{W^*}).$$

Remark 13.2. $\mathrm{Fuk}^{\rightarrow}(X_{W,1})$ can be considered as a geometrical categorification of a distinguished basis of vanishing cycles.

14. Beyond ADE and 14 Exceptionals

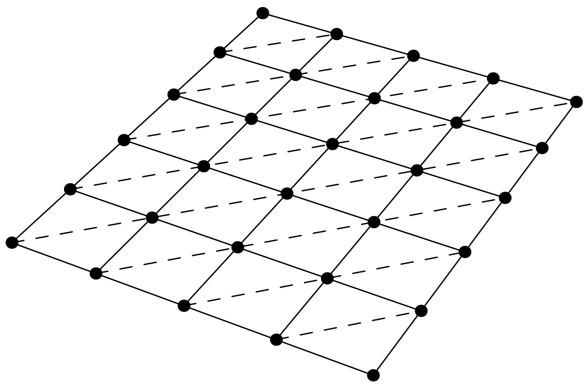
Theorem 14.1 (T, in progress)**.** (i) *For any regular weight system W of Type I and II, $\mathcal{T}_W$ has a full strongly exceptional collection $(E_1, \ldots, E_{\mu_W})$ and $(K_0(\mathcal{T}_W), \chi + {}^t\chi) \simeq (H_2(X_{W,1}, \mathbb{Z}), -I)$.*

(ii) *For any W of Type III, $\mathcal{T}_W$ has a full strongly exceptional collection $(E_1, \ldots, E_{\mu_W})$.*

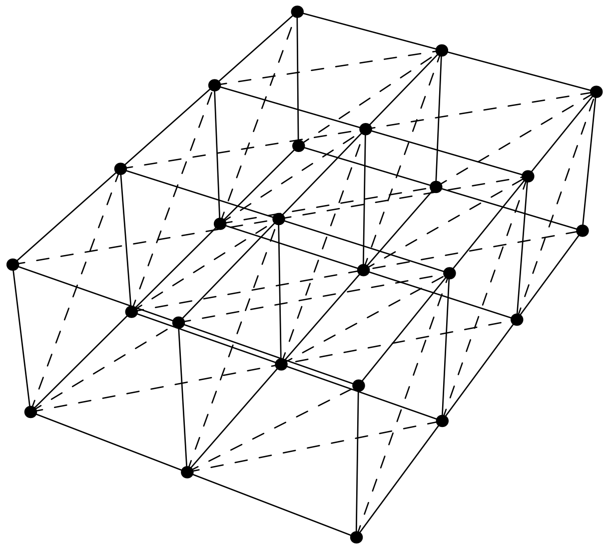
Proofs for these are similar to previous theorems: calculations of matrix factorizations, calculations of homomorphisms based on Serre duality and the "Category Generating Lemma".

Quivers and relations for these types are given as follows：
(注：非常に込み入った図のため，頂点・辺の数は正確ではない．また矢印も省略)
Type I

$$f_W(x,y,z) := x^{p_1} + y^{p_2}$$

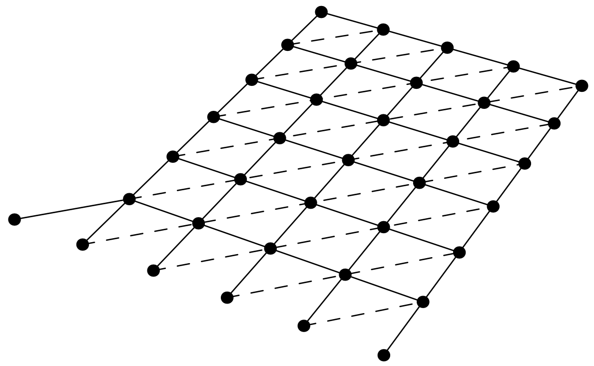


$$f_W(x,y,z) := x^{p_1} + y^{p_2} + z^{p_3}$$

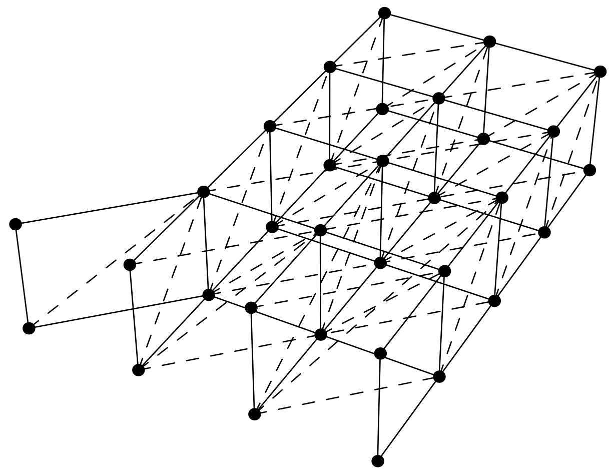


Type II

$$f_W(x,y,z) := y^{p_2} + yz^{\frac{p_3}{p_2}}$$



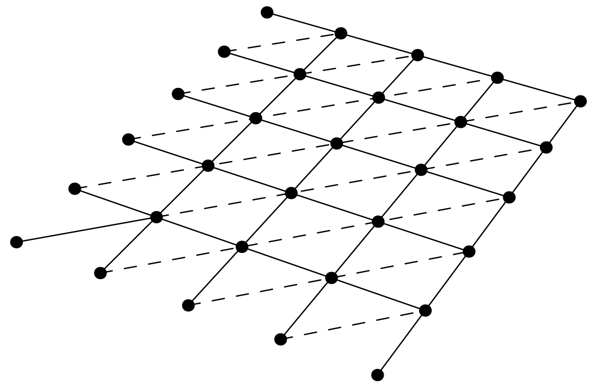
$$f_W(x,y,z) := x^{p_1} + y^{p_2} + yz^{\frac{p_3}{p_2}}$$

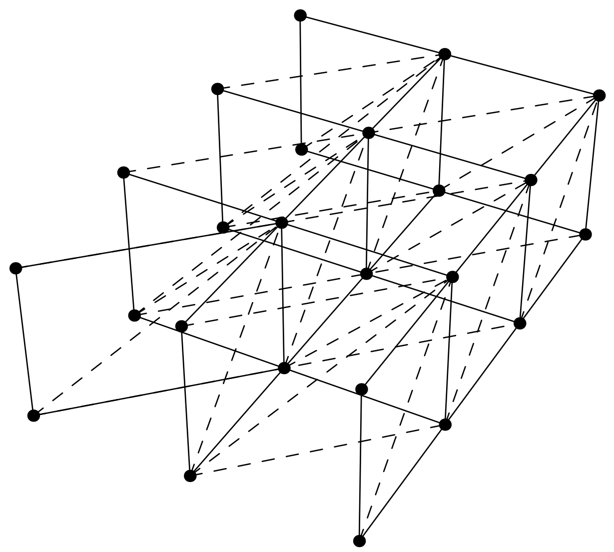


Type III

$$f_W(x,y,z) := y^{q_3+1}z + yz^{q_2+1}$$

$$f_W(x,y,z) := x^{p_1} + y^{q_3+1}z + yz^{q_2+1}$$





15. Future Dreams, Fancies and ...

Want to construct

- Lie algebras
- period maps
- automorphic forms

from "nice" triangulated categories.

"nice" triangulated categories

$\Downarrow$ Lie algebra, Weyl group, invariant theory, ...

Frobenius (K.Saito's flat) structures on Space of stability conditions (Bridgeland)

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