Symplectic varieties and Poisson deformations

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A symplectic variety X is a normal algebraic variety (defined over $\mathbf{C}$) which admits an everywhere non-degenerate d-closed 2-form ω on the regular locus X_{reg} of X such that, for any resolution $f : \tilde{X} \to X$ with $f^{-1}(X_{reg}) \cong X_{reg}$, the 2-form ω extends to a regular closed 2-form on $\tilde{X}$. There is a natural Poisson structure $\{\ ,\ \}$ on X determined by ω. Then we can introduce the notion of a Poisson deformation of $(X, \{\ ,\ \})$. A Poisson deformation is a deformation of the pair of X itself and the Poisson structure on it. When X is not a compact variety, the usual deformation theory does not work in general because the tangent object $\mathbf{T}^1_X$ may possibly have infinite dimension, and moreover, infinitesimal or formal deformations do not capture actual deformations of non-compact varieties. On the other hand, Poisson deformations work very well in many important cases where X is not a complete variety. Denote by $\mathrm{P}D_X$ the Poisson deformation functor of a symplectic variety. In this lecture, we shall study the Poisson deformation of an affine symplectic variety. The main result is:

Theorem 1. *Let X be an affine symplectic variety. Then the Poisson deformation functor PD_X is unobstructed.*

A Poisson deformation of X is controlled by the Poisson cohomology $\mathrm{HP}^2(X)$. When X has only terminal singularities, we have $\mathrm{HP}^2(X) \cong H^2((X_{reg})^{an}, \mathbf{C})$, where $(X_{reg})^{an}$ is the associated complex space with X_{reg}. This description enables us to prove that PD_X is unobstructed. But, in general, there is not such a direct, topological description of $\mathrm{HP}^2(X)$. Let us explain our strategy to describe $\mathrm{HP}^2(X)$. As remarked, $\mathrm{HP}^2(X)$ is identified with $\mathrm{PD}_X(\mathbf{C}[\epsilon])$ where $\mathbf{C}[\epsilon]$ is the ring of dual numbers over $\mathbf{C}$. First, note that there is an open locus U of X where X is smooth, or is locally a trivial deformation of a (surface) rational double point at each $p \in U$. Let Σ be the singular locus of U. Note that $X \setminus U$ has codimension ≥ 4 in X. Moreover, we have $\mathrm{PD}_X(\mathbf{C}[\epsilon]) \cong \mathrm{PD}_U(\mathbf{C}[\epsilon])$. Put $T^1_{U^{an}} := \underline{\mathrm{Ext}}^1(\Omega^1_{U^{an}}, \mathcal{O}_{U^{an}})$. As is well-known, a (local) section of $T^1_{U^{an}}$ corresponds to a 1-st order deformation of U^{an}. Let $\mathcal{H}$ be a locally constant $\mathbf{C}$-modules on Σ defined as the subsheaf of $T^1_{U^{an}}$ which consists of the sections coming from Poisson deformations of U^{an}. Now we have an exact sequence:

$$0 \to H^2(U^{an}, \mathbf{C}) \to \mathrm{PD}_U(\mathbf{C}[\epsilon]) \to H^0(\Sigma, \mathcal{H}).$$

Here the first term $H^2(U^{an}, \mathbf{C})$ is the space of locally trivial[1] Poisson deformations of U. By the definition of U, there exists a minimal resolution $\pi : \tilde{U} \to U$. Let m be the number of irreducible components of the exceptional divisor of π. A key result is:

Proposition 2. *The following equality holds:*

$$\dim H^0(\Sigma, \mathcal{H}) = m.$$

In order to prove Proposition 2, we need to know the monodromy action of $\pi_1(\Sigma)$ on $\mathcal{H}$. The idea is to compare two sheaves $R^2\pi^{an}_*\mathbf{C}$ and $\mathcal{H}$. Note that, for each point $p \in \Sigma$, the germ (U, p) is isomorphic to the product of an ADE surface singularity S and $(\mathbf{C}^{2n-2}, 0)$. Let $\tilde{S}$ be the minimal resolution of S. Then, $(R^2\pi^{an}_*\mathbf{C})_p$ is isomorphic to $H^2(\tilde{S}, \mathbf{C})$. A monodromy of $R^2\pi^{an}_*\mathbf{C}$ comes from a graph automorphism of the Dynkin diagram determined by the exceptional (-2)-curves on $\tilde{S}$. As is well known, S is described in terms of a simple Lie algebra $\mathfrak{g}$, and $H^2(\tilde{S}, \mathbf{C})$ is identified with the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$; therefore, one may regard $R^2\pi^{an}_*\mathbf{C}$ as a local system of the $\mathbf{C}$-module $\mathfrak{h}$ (on Σ), whose monodromy action coincides with the natural action of a graph automorphism on $\mathfrak{h}$. On the other hand, $\mathcal{H}$ is a local system of $\mathfrak{h}/W$, where $\mathfrak{h}/W$ is the linear space obtained as the quotient of $\mathfrak{h}$ by the Weyl group W of $\mathfrak{g}$. The action of a graph automorphism on $\mathfrak{h}$ descends to an action on $\mathfrak{h}/W$, which gives a monodromy action for $\mathcal{H}$. This description of the monodromy enables us to compute $\dim H^0(\Sigma, \mathcal{H})$.

Proposition 2 together with the exact sequence above gives an upper-bound of $\dim \mathrm{PD}_U(\mathbf{C}[\epsilon])$ in terms of some topological data of X (or U). We shall prove Theorem 1 by using this upper-bound. The rough idea is the following. There is a natural map of functors $\mathrm{PD}_{\tilde{U}} \to \mathrm{PD}_U$ induced by the resolution map $\tilde{U} \to U$. The tangent space $\mathrm{PD}_{\tilde{U}}(\mathbf{C}[\epsilon])$ to $\mathrm{PD}_{\tilde{U}}$ is identified with $H^2(\tilde{U}^{an}, \mathbf{C})$. We have an exact sequence

$$0 \to H^2(U^{an}, \mathbf{C}) \to H^2(\tilde{U}^{an}, \mathbf{C}) \to H^0(U^{an}, R^2\pi^{an}_*\mathbf{C}) \to 0,$$

[1]More exactly, this means that the Poisson deformations are locally trivial as usual flat deformations of U^{an}

and $\dim H^0(U^{an}, R^2\pi^{an}_*\mathbf{C}) = m$. In particular, we have $\dim H^2(\tilde{U}^{an}, \mathbf{C}) = \dim H^2(U^{an}, \mathbf{C}) + m$. But, this implies that $\dim \mathrm{PD}_{\tilde{U}}(\mathbf{C}[\epsilon]) \geq \dim \mathrm{PD}_U(\mathbf{C}[\epsilon])$. On the other hand, the map $\mathrm{PD}_{\tilde{U}} \to \mathrm{PD}_U$ has a finite closed fiber; or more exactly, the corresponding map $\mathrm{Spec} R_{\tilde{U}} \to \mathrm{Spec} R_U$ of pro-representable hulls, has a finite closed fiber. Since $\mathrm{PD}_{\tilde{U}}$ is unobstructed, this implies that PD_U is unobstructed and $\dim \mathrm{PD}_{\tilde{U}}(\mathbf{C}[\epsilon]) = \dim \mathrm{PD}_U(\mathbf{C}[\epsilon])$. Finally, we obtain the unobstructedness of PD_X from that of PD_U.

Theorem 1 is only concerned with the formal deformations of X; but, if we impose the following condition (*), then the formal universal Poisson deformation of X has an algebraization.

(*): X has a $\mathbf{C}^*$-action with positive weights with a unique fixed point $0 \in X$. Moreover, ω is positively weighted for the action.

We shall briefly explain how this condition (*) is used in the algebraization. Let $R_X := \lim R_X/(m_X)^{n+1}$ be the pro-representable hull of PD_X. Then the formal universal deformation $\{X_n\}$ of X defines an m_X-adic ring $A := \lim \Gamma(X_n, \mathcal{O}_{X_n})$ and let $\hat{A}$ be the completion of A along the maximal ideal of A. The rings R_X and $\hat{A}$ both have the natural $\mathbf{C}^*$-actions induced from the $\mathbf{C}^*$-action on X, and there is a $\mathbf{C}^*$-equivariant map $R_X \to \hat{A}$. By taking the $\mathbf{C}^*$-subalgebras of R_X and $\hat{A}$ generated by eigen-vectors, we get a map

$$\mathbf{C}[x_1, ..., x_d] \to S$$

from a polynomial ring to a $\mathbf{C}$-algebra of finite type. We also have a Poisson structure on S over $\mathbf{C}[x_1, ..., x_d]$ by the second condition of (*). As a consequence, there is an affine space $\mathbf{A}^d$ whose completion at the origin coincides with $\mathrm{Spec}(R_X)$ in such a way that the formal universal Poisson deformation over $\mathrm{Spec}(R_X)$ is algebraized to a $\mathbf{C}^*$-equivariant map

$$\mathcal{X} \to \mathbf{A}^d.$$

According to a result of Birkar-Cascini-Hacon-McKernan, we can take a crepant partial resolution $\pi : Y \to X$ in such a way that Y has only $\mathbf{Q}$-factorial terminal singularities. This Y is called a $\mathbf{Q}$-*factorial terminalization* of X. In our case, Y is a symplectic variety and the $\mathbf{C}^*$-action on X uniquely extends to that on Y. Since Y has only terminal singularities, it is relatively easy to show that the Poisson deformation functor PD_Y is unobstructed. Moreover, the formal universal Poisson deformation of Y has an algebraization over an affine space $\mathbf{A}^d$:

$$\mathcal{Y} \to \mathbf{A}^d.$$

There is a $\mathbf{C}^*$-equivariant commutative diagram

$$\begin{array}{ccc} \mathcal{Y} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathbf{A}^d & \xrightarrow{\psi} & \mathbf{A}^d \end{array} \qquad (1)$$

We have the following.

Theorem 3 *(a) ψ is a finite Galois covering.*

(b) $\mathcal{Y} \to \mathbf{A}^d$ is a locally trivial deformation of Y.

(c) The induced map $\mathcal{Y}_t \to \mathcal{X}_{\psi(t)}$ is an isomorphism for a general point $t \in \mathbf{A}^d$.

The Galois group of ψ is described as follows. Let Σ be the singular locus of X. There is a closed subset $\Sigma_0 \subset \Sigma$ such that X is locally isomorphic to $(S, 0) \times (\mathbf{C}^{2n-2}, 0)$ at every point $p \in \Sigma - \Sigma_0$ where S is an ADE surface singularity. We have $\mathrm{Codim}_X \Sigma_0 \geq 4$. Let $\mathcal{B}$ be the set of connected components of $\Sigma - \Sigma_0$. Let $B \in \mathcal{B}$. Pick a point $b \in B$ and take a transversal slice $S_B \subset Y$ of B passing through b. In other words, X is locally isomorphic to $S_B \times (B, b)$ around b. S_B is a surface with an ADE singularity. Put $\tilde{S}_B := \pi^{-1}(S_B)$. Then $\tilde{S}_B$ is a minimal resolution of S_B. Put $T_B := S_B \times (B, b)$ and $\tilde{T}_B := \pi^{-1}(T_B)$. Note that $\tilde{T}_B = \tilde{S}_B \times (B, b)$. Let C_i $(1 \leq i \leq r)$ be the (-2)-curves contained in $\tilde{S}_B$ and let $[C_i] \in H^2(\tilde{S}_B, \mathbf{R})$ be their classes in the 2-nd cohomology group. Then

$$\Phi := \{\Sigma a_i [C_i]; a_i \in \mathbf{Z},\ (\Sigma a_i [C_i])^2 = -2\}$$

is a root system of the same type as that of the ADE-singularity S_B. Let W be the Weyl group of Φ. Let $\{E_i(B)\}_{1 \leq i \leq \bar{r}}$ be the set of irreducible exceptional divisors of π lying over B, and let $e_i(B) \in H^2(X, \mathbf{Z})$ be their classes. Clearly, $\bar{r} \leq r$. If $\bar{r} = r$, then we define $W_B := W$. If $\bar{r} < r$, the Dynkin diagram of Φ has a non-trivial graph automorphism. When Φ is of type A_r with $r > 1$, $\bar{r} = [r + 1/2]$ and the Dynkin diagram has a graph automorphism τ of order 2. When Φ is of type D_r with $r \geq 5$, $\bar{r} = r - 1$ and the Dynkin diagram has a graph automorphism τ of order 2. When Φ is of type D_4, the Dynkin diagram has two different graph automorphisms of order 2 and 3. There are two possibilities of $\bar{r}$; $\bar{r} = 2$ or $\bar{r} = 3$. In the first case, let τ be the graph automorphism of order 3. In the latter case, let τ be the graph automorphism of order 2. Finally, when Φ is of type E_6, $\bar{r} = 4$ and the Dynkin diagram has a graph automorphism τ of order 2. In all these cases, we define

$$W_B := \{w \in W; \tau w \tau^{-1} = w\}.$$

The Galois group of ψ coincides with W_B.

As an application of Theorem 3, we have

Corollary 4: *Let (X, ω) be an affine symplectic variety with the property (*). Then the following are equivalent.*

(1) *X has a crepant projective resolution.*

(2) *X has a smoothing by a Poisson deformation.*

Example 5 (i) Let $O \subset \mathfrak{g}$ be a nilpotent orbit of a complex simple Lie algebra. Let $\tilde{O}$ be the normalization of the closure $\bar{O}$ of O in $\mathfrak{g}$. Then $\tilde{O}$ is an affine symplectic variety with the Kostant-Kirillov 2-form ω on O. Let G be a complex algebraic group with $Lie(G) = \mathfrak{g}$. By [Fu], $\tilde{O}$ has a crepant projective resolution if and only if O is a Richardson orbit (cf. [C-M]) and there is a parabolic subgroup P of G such that its Springer map $T^*(G/P) \to \tilde{O}$ is birational. In this case, every crepant resolution of $\tilde{O}$ is actually obtained as a Springer map for some P. If $\tilde{O}$ has a crepant resolution, $\tilde{O}$ has a smoothing by a Poisson deformation. The smoothing of $\tilde{O}$ is isomorphic to the affine variety G/L, where L is the Levi subgroup of P. Conversely, if $\tilde{O}$ has a smoothing by a Poisson deformation, then the smoothing always has this form.

(ii) In general, $\tilde{O}$ has no crepant resolutions. But, by [Na 4], at least when $\mathfrak{g}$ is a classical simple Lie algebra, every $\mathbf{Q}$-factorial terminalization of $\tilde{O}$ is given by a generalized Springer map. More explicitly, there is a parabolic subalgebra $\mathfrak{p}$ with Levi decomposition $\mathfrak{p} = \mathfrak{n} \oplus \mathfrak{l}$ and a nilpotent orbit O' in $\mathfrak{l}$ so that the generalized Springer map $G \times^P (\mathfrak{n} + \bar{O}') \to \tilde{O}$ is a crepant, birational map, and the normalization of $G \times^P (\mathfrak{n} + \bar{O}')$ is a $\mathbf{Q}$-factorial terminalization of $\tilde{O}$. By a Poisson deformation, $\tilde{O}$ deforms to the normalization of $G \times^L \bar{O}'$. Here $G \times^L \bar{O}'$ is a fiber bundle over G/L with a typical fiber $\bar{O}'$, and its normalization can be written as $G \times^L \tilde{O}'$ with the normalization $\tilde{O}'$ of $\bar{O}'$.

We can apply Theorem 3 to the Poisson deformations of an affine symplectic variety related to a nilpotent orbit in a complex simple Lie algebra.

Let $\mathfrak{g}$ be a complex simple Lie algebra and let G be the adjoint group. For a parabolic subgroup P of G, denote by $T^*(G/P)$ the cotangent bundle of G/P. The image of the Springer map $s : T^*(G/P) \to \mathfrak{g}$ is the closure $\bar{O}$ of a nilpotent (adjoint) orbit O in $\mathfrak{g}$. Then the normalization $\tilde{O}$ of $\bar{O}$ is an affine symplectic variety with the Kostant-Kirillov 2-form. If s is birational onto its image, then the Stein factorization $T^*(G/P) \to \tilde{O} \to \bar{O}$ of s gives a crepant resolution of $\tilde{O}$. In this situation, we have the following commutative diagram

$$\begin{array}{ccc} G \times^P r(\mathfrak{p}) & \longrightarrow & \widetilde{G \cdot r(\mathfrak{p})} \\ \downarrow & & \downarrow \\ \mathfrak{k}(\mathfrak{p}) & \longrightarrow & \mathfrak{k}(\mathfrak{p})/W' \end{array} \tag{2}$$

where $r(\mathfrak{p})$ is the solvable radical of $\mathfrak{p}$, $\widetilde{G \cdot r(\mathfrak{p})}$ is the normalization of the adjoint G-orbit of $r(\mathfrak{p})$ and $\mathfrak{k}(\mathfrak{p})$ is the centralizer of the Levi part $\mathfrak{l}$ of $\mathfrak{p}$. Moreover, $W' := N_W(L)/W(L)$, where L is the Levi subgroup of P and $W(L)$ is the Weyl group of L.

Theorem 6. *The diagram above coincides with the* $\mathbf{C}^*$*-equivariant commutative diagram of the universal Poisson deformations of* $T^*(G/P)$ *and* $\tilde{O}$.

Note that W' has been extensively studied by Howlett and others. Another important example is a transversal slice of $\mathfrak{g}$. In the commutative diagram above, put $\mathfrak{p} = \mathfrak{b}$ the Borel subalgebra. Then we have:

$$\begin{array}{ccc} G \times^B \mathfrak{b} & \xrightarrow{\pi_B} & \mathfrak{g} \\ \downarrow & & \varphi \downarrow \\ \mathfrak{h} & \longrightarrow & \mathfrak{h}/W. \end{array} \tag{3}$$

Let $x \in \mathfrak{g}$ be a nilpotent element of $\mathfrak{g}$ and let O be the adjoint orbit containing x. Let $\mathcal{V} \subset \mathfrak{g}$ be a transversal slice for O passing through x. Put $\mathcal{V}_B := \pi_B^{-1}(\mathcal{V})$. Denote by V (resp. $\tilde{V}_B$) the central fiber of $\mathcal{V} \to \mathfrak{h}/W$ (resp. $G \times^B \mathfrak{b} \to \mathfrak{h}$). Note that $\tilde{V}_B$ is somorphic to the cotangent bundle $T^*(G/B)$ of G/B, and $\tilde{V}_B \to V$ is a crepant resolution.

Theorem 7 *The commutative diagram*

$$\begin{array}{ccc} \tilde{\mathcal{V}}_B & \longrightarrow & \mathcal{V} \\ \Big\downarrow & & \varphi_{\mathcal{V}} \Big\downarrow \\ \mathfrak{h} & \longrightarrow & \mathfrak{h}/W \end{array} \tag{4}$$

is the $\mathbf{C}^*$-equivariant commutative diagram of the universal Poisson deformations of $\tilde{V}_B$ and V if $\mathfrak{g}$ is simply laced.

When $\mathfrak{g}$ is not simply-laced, Theorem 7 is no more true. In fact, Slodowy pointed out that the transversal slice $\mathcal{V}$ for a subregular nilpotent orbit of non-simply-laced $\mathfrak{g}$ does not give the universal deformation. However, we have a criterion of the universality. Let

$$\rho : A(O) \to GL(H^2(\pi_{B,0}^{-1}(x), \mathbf{Q}))$$

be the monodromy representation of the component group $A(O)$ of O.

Theorem 8. *Let $\mathfrak{g}$ be a comple simple Lie algebra which is not necessarilly simply-laced. Then the above commutative diagram is universal if and only if ρ is trivial.*

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