Z-STRUCTURES IN LOG MIXED HODGE THEORY

Sampei Usui (Osaka University)

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Contents

§0. Introduction

This is a brief guide of log mixed Hodge theory.

As a joint work of Kazuya Kato, Chikara Nakayama, and Usui, the notion of log mixed Hodge was introduced and studied, and various partial compactifications of a classifying space D of mixed Hodge structures were constructed, and their relation was described as a fundamental diagram.

In this note, we focus on the **Z**-structure of a log mixed Hodge structure.

§1. Ringed space $(X^{\log}, \mathcal{O}_X^{\log})$

We review a ringed space $(X^{\log}, \mathcal{O}_X^{\log})$ which is the best place to live for logarithms. The references of this section are [KN99], [KU09, Ch. 2].

Let X be an analytic space (more generally, an object of the category $\mathcal{B}$) with an fs log structure. Here a log structure on X is a morphism $\alpha : M_X \to \mathcal{O}_X$ of monoid sheaves such that $\alpha^{-1}(\mathcal{O}_X^*) \simeq \mathcal{O}_X^*$, and fs is a good property for it. Define a set

$$X^{\log} := \{(x, h) \,|\, x \in X, h : M_{X,x}^{\mathrm{gp}} \to \mathbf{S}^1 \text{ homomorphism s.t. } h(u) = u/|u| \text{ for } u \in \mathcal{O}_{X,x}^*\}.$$

Introduce on $X^{\log}$ the weakest topology such that the following two maps are continuous:

(1) $\tau : X^{\log} \to X, \quad (x, h) \mapsto x.$

(2) For any open $U \subset X$ and any $f \in \Gamma(U, M_X^{\mathrm{gp}}), \tau^{-1}(U) \to \mathbf{S}^1, (x, h) \mapsto h(f).$

Then, $\tau : X^{\log} \to X$ is proper and surjective, and fibers are $\tau^{-1}(x) \simeq (\mathbf{S}^1)^r$ with $r = \mathrm{rank}(M_X^{\mathrm{gp}}/\mathcal{O}_X^\times)_x$.

Define the *sheaf of logarithms* $\mathcal{L} = \mathcal{L}_X$ *of* M_X^{gp} *on* $X^{\log}$ as the fiber product of $\mathrm{Cont}(\ \ , i\mathbf{R}) \xrightarrow{\exp} \mathrm{Cont}(\ \ , \mathbf{S}^1) \leftarrow \tau^{-1}(M_X^{\mathrm{gp}})$, where the second map is (f at $(x, f)) \mapsto h(f)$. The homomorphisms $\tau^{-1}(\mathcal{O}_X) \xrightarrow{\exp} \tau^{-1}(\mathcal{O}_X^*) \subset \tau^{-1}(M_X^{\mathrm{gp}})$ and $\tau^{-1}(\mathcal{O}_X) \to \mathrm{Cont}(\ \ , i\mathbf{R})$, $f \mapsto (f - \bar{f})/2$, induce a homomorphism $\iota : \tau^{-1}(\mathcal{O}_X) \to \mathcal{L}$. Define a sheaf of rings $\mathcal{O}_X^{\mathrm{gp}}$ on $X^{\log}$ by

$$\mathcal{O}_X^{\log} := \frac{\tau^{-1}(\mathcal{O}_X) \otimes \mathrm{Sym}_{\mathbf{Z}}(\mathcal{L})}{(f \otimes 1 - 1 \otimes \iota(f) \mid f \in \tau^{-1}(\mathcal{O}_x))}.$$

Then, $\tau : (X^{\log}, \mathcal{O}_X^{\log}) \to (X, \mathcal{O}_X)$ is a morphism of ringed spaces over $\mathbf{C}$.

§2. Local systems on $X^{\log}$

Let X be an analytic space (more generally, an object of the category $\mathcal{B}$) endowed with an fs log structure.

We review here the following result about local systems on $X^{\log}$ in [KU09, §2.3]: If L is a locally constant sheaf on $X^{\log}$ of free $\mathbf{Z}$-modules of finite rank with "unipotent local monodromy" (see 2.1 below), then locally on X, L is embedded in

$$\mathcal{O}_X^{\log} \otimes L_0$$

in a special way, where L_0 is a stalk of L regarded as a constant sheaf.

2.1. Let L be a locally constant sheaf on $X^{\log}$. For $x \in X$ and $y \in X^{\log}$ lying over x, we call the action of $\pi_1(x^{\log}) = \pi_1(\tau^{-1}(x))$ on L_y the *local monodromy* of L at y.

Assume L is a locally constant sheaf of abelian groups on $X^{\log}$. We say the local monodromy of L is *unipotent* if the local monodromy of L at y is unipotent for any $y \in X^{\log}$.

Theorem 2.2. *Let X be an object of $\mathcal{B}(\log)$, and let L be a locally constant sheaf on $X^{\log}$ of free abelian groups of finite rank. Fix a point $x \in X$ and a point $y \in X^{\log}$ lying over x, and assume that the local monodromy of L at y is unipotent. Let $(q_j)_{1\le j\le n}$ be a finite family of elements of $M_{X,x}^{\mathrm{gp}}$ whose image in $(M_X^{\mathrm{gp}}/\mathcal{O}_X^\times)_x$ is a $\mathbf{Z}$-basis, and let $(\gamma_j)_{1\le j\le n}$ be the dual $\mathbf{Z}$-basis of $\pi_1(x^{\log})$ in the duality (cf.* [KU09, 2.2.9]*). Then if we replace X by some open neighborhood of x, we have an isomorphism of $\mathcal{O}_X^{\log}$-modules*

$$\nu : \mathcal{O}_X^{\log} \otimes L \xrightarrow{\sim} \mathcal{O}_X^{\log} \otimes L_0, \quad L_0 := \textit{the stalk } L_y,$$

where L_0 is regarded as a constant sheaf, satisfying the following (1). *Let*

$$N_j = \log(\gamma_j) : L_{0,\mathbf{Q}} \to L_{0,\mathbf{Q}},$$

lift q_j *in* $\Gamma(X, M_X^{\mathrm{gp}})$ *(by replacing* X *by an open neighborhood of* x*), and let*

$$\xi = \exp(\textstyle\sum_{j=1}^{n}(2\pi i)^{-1}\log(q_j)\otimes N_j) : \mathcal{O}_X^{\log}\otimes_A L_0 \xrightarrow{\sim} \mathcal{O}_X^{\log}\otimes_A L_0$$

(nilpotent rewinding along $(\gamma_j)_{1\le j\le n}$*).*

Note that the operator $\xi = \exp(\sum_{j=1}^{n}(2\pi i)^{-1}\log(q_j)\otimes N_j)$ *depends on the local choices of the branches of* $\log(q_j)$*, but that the subsheaf* $\xi^{-1}(1\otimes L_0)$ *of* $\mathcal{O}_X^{\log}\otimes L_0$*, which we consider in* (1) *below, is independent of the choices and hence is defined globally on* $X^{\log}$*.*

(1) *The restriction of* ν *to* $L = 1\otimes L$ *induces an isomorphism of locally constant sheaves*

$$\nu : L \xrightarrow{\sim} \xi^{-1}(1\otimes L_0).$$

If we fix branches of the germs $\log(q_j)_y$ *at* y $(1\le j\le n)$*, we can take an isomorphism* ν *satisfying* (1) *as above which satisfies furthermore the following* (2)*.*

(2) *The branch of* ξ_y *defined by the fixed branches of* $\log(q_j)_y$ *satisfies*

$$\nu_y(1\otimes v) = \xi_y^{-1}(1\otimes v) \quad \textit{for any } v\in L_0.$$

(ξ_y^{-1} *is nilpotent twist along* $(\gamma_j)_{1\le j\le n}$*.)*

Proof. Let L' be the locally constant subsheaf $\xi^{-1}(1\otimes L_0)$ of $\mathcal{O}_X^{\log}\otimes L_0$. Fix a branch of $\log(q_j)_y$ at y for $1\le j\le n$, and let $\nu : L_y \to (L')_y$ be the isomorphism of $\mathbf{Z}$-modules $v\mapsto \xi_y^{-1}(1\otimes v)$ where ξ_y is defined by the fixed branches of $\log(q_j)_y$. Note here that $\pi_1(x^{\log})$ acts on the first v and ξ_y^{-1} but $\pi_1(x^{\log})$ acts trivially on the second v.

Then ν preserves the local monodromy actions of $\pi_1(x^{\log})$ on these stalks of the locally constant sheaves L and L'. In fact, for $v\in L_0 = L_y$ and for $1\le k\le n$,

$$\begin{aligned}
&\gamma_k(\xi_y^{-1}(1\otimes v) \text{ in } L'_y) = \gamma_k(\xi_y)^{-1}\cdot(1\otimes v)\\
&= \exp(-(\textstyle\sum_{j=1}^{n}((2\pi i)^{-1}\log(q_j)_y - \delta_{jk})\otimes N_j)\cdot(1\otimes v)\\
&= \xi_y^{-1}\exp(1\otimes N_k)(1\otimes v) = \xi_y^{-1}(1\otimes\gamma_k(v \text{ in } L_y))
\end{aligned}$$

(for the signature "$-$" before the Kronecker symbol δ_{jk}, see [KU09, Appendix A1]). Hence there is a unique isomorphism $\nu : L|_{x^{\log}} \to L'|_{x^{\log}}$ between the pullbacks of L and L' to $x^{\log}$ which induces the above isomorphism ν on the stalks at y.

By the proper base change theorem ([KU09, Appendix A2]) applied to the proper map $\tau : X^{\log}\to X$ and to the sheaf $\mathcal{F}$ of isomorphisms from L to L' on $X^{\log}$, the isomorphism ν extends to an isomorphism $\nu : L \xrightarrow{\sim} L'$ if we replace X by some open neighborhood of x in X. This isomorphism ν induces an isomorphism of $\mathcal{O}_X^{\log}$-modules

$$\nu : \mathcal{O}_X^{\log}\otimes L \xrightarrow{\sim} \mathcal{O}_X^{\log}\otimes L' = \mathcal{O}_X^{\log}\otimes L_0. \quad \square$$

2.3. *Example.* Let $\Delta = \{q \in \mathbf{C} \mid |q| < 1\}$ be the unit disc. We have a standard family of degenerating elliptic curve

$$f : E \to \Delta,$$

which is a morphism of analytic manifolds, having the following property.

(1) For $q \in \Delta$ with $q \neq 0$, $f^{-1}(q) = \mathbf{C}^\times/q^{\mathbf{Z}}$. This is an elliptic curve. In fact, taking $\tau \in \mathbf{C}$ with $q = \exp(2\pi i \tau)$, we have $\tau \in \mathfrak{h}$, and

$$\mathbf{C}/(\mathbf{Z}\tau + \mathbf{Z}) \xrightarrow{\sim} \mathbf{C}^\times/q^{\mathbf{Z}}, \quad (t \bmod (\mathbf{Z}\tau + \mathbf{Z})) \mapsto (\exp(2\pi i t) \bmod q^{\mathbf{Z}}).$$

(2) $f^{-1}(0) = \mathbf{P}^1(\mathbf{C})/(0 \sim \infty)$.

Explicitly, this E is defined as $X/\sim$, where $X = \{(t_1, t_2) \in \mathbf{C}^2 \mid |t_1 t_2| < 1\}$ and $\sim$ is the following equivalence relation. Let $g : X \to \Delta$, $(t_1, t_2) \mapsto t_1 t_2$. For $a, b \in X$, if $a \sim b$, then $g(a) = g(b)$. The restriction of $\sim$ to $g^{-1}(q)$ for each $q \in \Delta$ is defined as follows. Assume first $q \neq 0$. Consider the map $g^{-1}(q) \simeq \mathbf{C}^\times \to \mathbf{C}^\times/q^{\mathbf{Z}}$ where the first isomorphism is $(t_1, t_2) \mapsto t_1$. For $a, b \in g^{-1}(q)$, $a \sim b$ if and only if the images of a, b in $\mathbf{C}^\times/q^{\mathbf{Z}}$ coincide, i.e., $(b_1, b_2) = (q^n a_1, q^{-n} a_2)$ for some $n \in \mathbf{Z}$. Next assume $q = 0$. Consider $g^{-1}(0) = \{(t_1, t_2) \mid t_1 t_2 = 0\} \to \mathbf{P}^1(\mathbf{C})/(0 \sim \infty)$, where the arrow sends $(t_1, 0)$ to t_1 and $(0, t_2)$ to t_2^{-1}. Then, for $a, b \in g^{-1}(0)$, $a \sim b$ if and only if the images of a, b in $\mathbf{P}^1(\mathbf{C})/(0 \sim \infty)$ coincide, i.e., $a = b$ or $\{a, b\} = \{(s, 0), (0, s^{-1})\}$ for some $s \in \mathbf{C}^\times$.

The projection $X \to E$ is a local homeomorphism. The analytic structure and the log structure of E are the unique ones for which this projection is locally an isomorphism of analytic spaces with log structures.

Let $f : E \to \Delta$ be as above and consider the locally constant sheaf $L = R^1 f_*^{\log}\mathbf{Z}$ on $\Delta^{\log}$. In Theorem 2.2, take $X = \Delta$, $x = 0 \in \Delta$, and take the coordinate function q of Δ as q_1 ($n = 1$ in this situation). Then the element γ_1, which we denote here by γ, is the positive generator of $\pi_1(\Delta^{\log})$ (represented by a circle in Δ^* in the counterclockwise direction). L has a $\mathbf{Z}$-basis (e_1, e_2) locally on $\Delta^{\log}$ (e_1 is defined globally but e_2 is determined by a local choice of the branch of $\log q$). Fix a branch of $\log q$ at y and take the corresponding $e_{2,y}$. We have

$$\begin{aligned} &\gamma(e_1) = e_1,\ \gamma(e_{2,y}) = e_1 + e_{2,y}, \\ &N(e_1) = 0,\ N(e_{2,y}) = e_1, \ \text{ where } N = \log \gamma. \end{aligned}$$

Let $\omega := (2\pi i)^{-1} \log q \otimes e_1 + 1 \otimes e_{2,y}$. Then $\mathcal{O}_X^{\log}$-module $\mathcal{O}_X^{\log} \otimes L$ has a global base $(1 \otimes e_1, \omega)$. We have an isomorphism of $\mathcal{O}_X^{\log}$-modules

$$\nu : \mathcal{O}_X^{\log} \otimes L \xrightarrow{\sim} \mathcal{O}_X^{\log} \otimes L_0, \quad 1 \otimes e_1 \to 1 \otimes e_1, \ \ \omega \mapsto 1 \otimes e_{2,y}.$$

This ν has the property stated in Theorem 2.2 globally on Δ. In fact, ν sends $1 \otimes e_1$ to $1 \otimes e_1 = \xi^{-1}(1 \otimes e_1)$ and $1 \otimes e_{2,y}$ to $-(2\pi i)^{-1} \log q \otimes e_1 + 1 \otimes e_{2,y} = \xi^{-1}(1 \otimes e_{2,y})$.

§3. Extensions of Z-structure of higher direct image

We review some results in [U01, 6] concerning extensions of $\mathbf{Z}$-structure of a higher direct image of a semi-stable degeneration.

3.1. Let $f : X \to \Delta$ be a semi-stable degeneration of complex manifolds over a disc with coordinate q, i.e., a proper, surjective, flat, holomorphic map from a complex manifold onto a unit disc which is smooth over the punctured disc Δ^* and the central fiber X_0 is reduced and normal crossing. Assume moreover that $X \subset \mathbf{P}^n \times \Delta$, for some n, and f is the restriction of the second projection.

3.2. In the situation 3.1, we endow X (resp. Δ) the log structure associated to the divisor X_0 (resp. $\{0\}$). Let $\omega^\bullet_{X/\Delta} := \Omega^\bullet_{X/\Delta}(\log(X_0))$, $\omega^1_\Delta := \Omega^1_\Delta(\log(\{0\}))$. Let $\omega^{1,\log}_\Delta := \mathcal{O}^{\log}_\Delta \otimes_{\tau^*\mathcal{O}_\Delta} \omega^1_\Delta$. Let w be an integer, and let $\mathcal{V} := R^w f_* \omega^\bullet_{X/\Delta}$ and $L_\mathbf{C} := R^w f^{\log}_* \mathbf{C}$. Then we have

$$(1) \qquad L_\mathbf{C} \simeq \operatorname{Ker}\left(\nabla : \tau^*_\Delta \mathcal{V} \to \omega^{1,\log}_\Delta \otimes_{\mathcal{O}^{\log}_\Delta} \tau^*_\Delta \mathcal{V}\right) \quad \text{on } \Delta^{\log},$$

or, equivalently,

$$(2) \qquad \mathcal{V} \simeq (\tau_\Delta)_*\left(\mathcal{O}^{\log}_\Delta \otimes_\mathbf{C} L_\mathbf{C}\right) \quad \text{on } \Delta.$$

3.3. We use the notation in 3.1 and 3.2. Choose now a multi-valued, flat frame

$$(1) \qquad \{e_1, \dots, e_r\}$$

of $\mathcal{V}|\Delta^*$ from the image of

$$(R^w f^{\log}_* \mathbf{Z})|(\tau_\Delta)^{-1}(\Delta^*) \to (\tau^*_\Delta \mathcal{V})|\tau^{-1}_\Delta(\Delta^*) \ = \mathcal{V}|\Delta^*.$$

We regard (1) also as a multi-valued, flat frame of $\tau^*_\Delta \mathcal{V}$, by abuse of notation. Putting

$$(2) \qquad \tilde{e}_j := \exp(tN) \cdot e_j, \ \ t := (2\pi i)^{-1} \log q, \ N : \log \gamma \text{ for the monodromy},$$

we have an invariant frame

$$(3) \qquad \{\tilde{e}_1, \dots, \tilde{e}_r\}$$

of $\mathcal{V}|\Delta^*$ (in fact, $\gamma \tilde{e}_j = \exp((t-1)N) \cdot \gamma e_j = \tilde{e}_j$, cf. [KU09, Appendix A1]), which extends over Δ and induces a frame of the canonical extension $\mathcal{V}$ of $\mathcal{V}|\Delta^*$. We use the same letters for the induced frame of $\mathcal{V}$, by abuse of notation.

Theorem 3.4. *In the above notation, we have two types of integral structure on* $\mathcal{V} = R^w f_* \omega^\bullet_{X/\Delta}\langle Y\rangle$:

(i) *The integral structure determined by the multi-valued, flat frame* 2.3 (1) *of*

$$\tau_\Delta^* \mathcal{V} \simeq \mathcal{O}_\Delta^{\log} \otimes_{\mathbf{Z}} R^w f_*^{\log}\mathbf{Z} \quad \textit{on } \Delta^{\log}.$$

Here the local monodromy is induced by the $\mathbf{S}^1$*-action on* $\Delta^{\log}$*.*

(ii) *The integral structure determined by the invariant frame* 3.3 (3) *of*

$$\mathcal{V} \simeq \mathcal{O}_\Delta \otimes_{\mathbf{Z}} (\tau_\Delta)_* R^w f_*^{\log} (f^{\log})^{-1}\mathbf{Z}[t] \quad \textit{on } \Delta.$$

Here $t = (2\pi i)^{-1}\log q$ *as before, and the monodromy logarithm is given by* $2\pi i \operatorname{Res}_0(\nabla)$*.*

We explain (ii). We may assume that the basis of $\mathcal{V}(0)$ induced by $\{\tilde{e}_1, \dots, \tilde{e}_r\}$ respects the monodromy weight filtration M. Then, by using the frame $\{\tilde{e}_1, \dots, \tilde{e}_r\}$, we extend M over $\mathcal{V}$. For $\tilde{e}_j \in M_k$, we have

$$\begin{aligned}\nabla \tilde{e}_j &= dt \otimes N \cdot \exp(tN)\cdot e_j = dt \otimes \exp(tN)\cdot Ne_j \\ &= dt \otimes \exp(tN)\cdot \big(\textstyle\sum_\ell a_\ell e_\ell\big) = (2\pi i)^{-1} d\log q \otimes \big(\textstyle\sum_\ell a_\ell \tilde{e}_\ell\big),\end{aligned}$$

where $\sum_i a_\ell e_\ell := Ne_j \in M_{k-2}$, $a_\ell \in \mathbf{C}$. Hence

$$\nabla(h(q)\tilde{e}_j) = dh(q)\otimes \tilde{e}_j + (2\pi i)^{-1} h(q) d\log q \otimes \big(\textstyle\sum_\ell a_\ell \tilde{e}_\ell\big) \in \omega^1_\Delta \otimes M_k.$$

So we have

$$N = 2\pi i \operatorname{Res}_0(\nabla)$$

under the identification

$$\varpi^*(\mathcal{V}|\Delta^*)(u) \xrightarrow{\sim} \mathcal{V}(0), \quad \tilde{e}_j(u) \mapsto \tilde{e}_j(0),$$

where $\varpi : \mathfrak{h} \to \Delta^*$ is the universal covering and $u \in \mathfrak{h}$. In fact,

$$\begin{aligned}N(\tilde{e}_j(u)) &= N(\exp(tN)\cdot e_j(u)) = \exp(tN)N(\tilde{e}_j(u)) \\ &= \exp(tN)(\textstyle\sum_\ell a_\ell e_\ell(u)) = \textstyle\sum_\ell a_\ell \tilde{e}_\ell(u).\end{aligned}$$

3.5. Note that the integral structures (i) and (ii) of Theorem in 3.4 are independent of the choice of a multi-valued, flat frame 3.3 (1). However, the integral structure (ii) depends on the choice of a coordinate q on Δ. Note also that the integral structure (i) of Theorem in 3.4 is the one in the limiting mixed Hodge structure of Schmid [Sc73], whereas the integral structure (ii) is the one in the limiting mixed Hodge structure of Steenbrink [St76].

3.6. The result of [KU01] is widely generalized as “relative rounding” by Nakayama and Ogus [NO10].

§4. Examples of spaces of nilpotent orbits

4.1. *Example* ([KU09, §0]). Let $H_0 = \mathbf{Z}^2 = \mathbf{Z}e_1 \oplus \mathbf{Z}e_2$, weight is 1, and $\langle e_2, e_1 \rangle = 1$. Let N be the nilpotent endomorphism of H_0 defined by $H(e_2) = e_1$, $N(e_1) = 0$. Then $W := W(N)[-1]$ is

$$W_{-1} = 0 \subset W_0 = W_1 = \mathbf{R}e_1 \subset W_2 = H_{0,\mathbf{R}}.$$

We identify $\mathrm{gr}^W = H_{0,\mathbf{R}}$ by this basis e_1, e_2.

For $\tau \in \mathfrak{h}$, $F = F(\tau e_1 + e_2)$ is

$$F^2 = 0 \subset F^1 = \mathbf{C} \cdot (\tau e_1 + e_2) \subset F^0 = H_{0,\mathbf{C}}.$$

Then

$$F^1(\mathrm{gr}_2^W) = \mathbf{C}e_2,\ F^0(\mathrm{gr}_0^W) = \mathbf{C}e_1,\ F^p(\mathrm{gr}_w^W) = 0 \text{ for the other } (p, w).$$

Let

$$\Sigma := \{\mathbf{R}_{\geq 0}N \mid N \text{ is a nilpotent element of } \mathfrak{g}_{\mathbf{Q}}\}.$$

Then Σ is a fan in $\mathfrak{g}_{\mathbf{Q}}$. We see that the set of nilpotent orbits is $D_\Sigma = D \cup \mathbf{P}^1(\mathbf{Q})$. This is explained as follows. For $a \in \mathbf{P}^1(\mathbf{Q})$, let V_a be the one-dimensional $\mathbf{R}$-vector subspace of $H_{0,\mathbf{R}}$ corresponding to a, that is, $V_a = \mathbf{R}(ae_1 + e_2)$ if $a \in \mathbf{Q}$, and $V_\infty = \mathbf{R}e_1$. For $a \in \mathbf{P}^1(\mathbf{Q})$, define a sharp rational nilpotent cone σ_a by

$$\sigma_a = \{N \in \mathfrak{g}_{\mathbf{R}} \mid N(H_{0,\mathbf{R}}) \subset V_a,\ \ N(V_a) = \{0\},\ \ \langle x, N(x) \rangle_0 \geq 0 \text{ for any } x \in H_{0,\mathbf{R}}\}.$$

We identify $a \in \mathbf{P}^1(\mathbf{Q})$ with the nilpotent orbit $(\sigma_a, Z_a) \in D_\Sigma$ where $Z_a = \{F \in \check{D} = \mathbf{P}^1(\mathbf{C}) \mid F^1 \neq V_{a,\mathbf{C}}\}$. For example,

$$\sigma_\infty = \begin{pmatrix} 0 & \mathbf{R}_{\geq 0} \\ 0 & 0 \end{pmatrix}, \quad Z_\infty = \mathbf{C} \subset \mathbf{P}^1(\mathbf{C}) = \check{D}.$$

For

$$\Gamma = \begin{pmatrix} 1 & \mathbf{Z} \\ 0 & 1 \end{pmatrix} \subset \mathrm{SL}(2, \mathbf{Z}), \quad \sigma = \sigma_\infty = \begin{pmatrix} 0 & \mathbf{R}_{\geq 0} \\ 0 & 0 \end{pmatrix},$$

we have a commutative diagram of analytic spaces

$$\begin{array}{ccc} \Delta^* & \simeq & \Gamma\backslash D \\ \cap & & \cap \\ \Delta & \simeq & \Gamma\backslash D_\sigma. \end{array}$$

Here the upper isomorphism sends $e^{2\pi i\tau} \in \Delta^*$ ($\tau \in \mathfrak{h} = D$) to ($\tau \bmod \Gamma$), the lower isomorphism extends the upper isomorphism by sending $0 \in \Delta$ to the class of the nilpotent orbit $((\sigma_\infty, \mathbf{C}) \bmod \Gamma)$.

4.2. *Example* $0 \to \mathbf{Z}(1) \to H \to \mathbf{Z} \to 0$ ([KNU.p, 7.1.1]).

Description 1. Let $H_0 = \mathbf{Z}^2 = \mathbf{Z}e_1 \oplus \mathbf{Z}e_2$, and let

$$W_{-3} = 0 \subset W_{-2} = W_{-1} = \mathbf{R}e_1 \subset W_0 = H_{0,\mathbf{R}}.$$

Let $e'_1 := e_1 \in \mathrm{gr}^W_{-2} = W_{-2}$ and let e'_2 be the image of e_2 in gr^W_0. Let $\langle\ ,\ \rangle_{-2}$ (resp. $\langle\ ,\ \rangle_0$) be the bilinear form on gr^W_{-2} (resp. gr^W_0) defined by $\langle e'_1, e'_1\rangle_{-2} = 1$ (resp. $\langle e'_2, e'_2\rangle_0 = 1$). Let $h^{0,0} = h^{-1,-1} = 1$ and $h^{p,q} = 0$ for other (p, q). Then the classifying space D of mixed Hodge structures with polarized grades quotients becomes $\mathbf{C} \simeq D$, $t \mapsto F = F(te_1 + e_2)$, defined by

$$F^1 = 0 \subset F^0 = \mathbf{C} \cdot (te_1 + e_2) \subset F^{-1} = H_{0,\mathbf{C}}.$$

Let $G_{\mathbf{Z}} := \mathrm{Aut}(H_0, W, (\langle\ ,\ \rangle_w)_w)$, and let $\Gamma := G_{\mathbf{Z},u} = \begin{pmatrix} 1 & \mathbf{Z} \\ 0 & 1 \end{pmatrix}$. Let $\Sigma := \{\sigma, -\sigma, \{0\}\}$ with $\pm\sigma = \begin{pmatrix} 0 & \pm\mathbf{R}_{\geq 0} \\ 0 & 0 \end{pmatrix} \subset \mathfrak{g}_{\mathbf{R}} := \mathrm{Lie}\, G_{\mathbf{R}}$. Then

$$\begin{array}{ccc} D = \mathbf{C} \to \mathbf{C}^\times = \Gamma\backslash D & \subset & \mathbf{P}^1(\mathbf{C}) = \Gamma\backslash D_\Sigma \\ t \mapsto q = e^{2\pi i t}, & & 0 \mapsto (\sigma, D), \\ & & \infty \mapsto (-\sigma, D). \end{array}$$

Endow $\Gamma\backslash D_\Sigma = \mathbf{P}^1(\mathbf{C})$ with the log structure corresponding to the divisor $\{0, \infty\}$. Then $(\Gamma\backslash D_\Sigma)^{\log} = [0, \infty] \times \mathbf{S}^1$.

Description 2. Let S be an object in the category $\mathcal{B}(\log)$ ([KU09, 3.2.4], [KNU.p, 1.1.4]). Let $Q = (H_{(w)})_w$ with $H_{-2} = \mathbf{Z}(1)$, $H_{(0)} = \mathbf{Z}$, and $H_{(w)} = 0$ for other w. For Σ in the above Description 1, we have

$$\begin{array}{ccc} \mathrm{LMH}_Q & = \mathcal{E}xt^1_{\mathrm{LMH}}(\mathbf{Z}, \mathbf{Z}(1)) = & M^{\mathrm{gp}} \\ \cup & & \cup \\ \mathrm{LMH}_Q^{(\Sigma)} & = \mathcal{M}or_{\mathcal{B}(\log)}(-, \mathbf{P}^1(\mathbf{C})) = & M \cup M^{-1}. \end{array}$$

In fact, for $T \in \mathcal{B}(\log)$, $a \in \Gamma(T, M_T^{\mathrm{gp}})$ corresponds to the following $H \in \mathrm{LMH}_Q(T)$:

Local system $H_{\mathbf{Z}}$ on $T^{\log}$ is given by

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{Z}(1) & \longrightarrow & H_{\mathbf{Z}} & \longrightarrow & \mathbf{Z} & \longrightarrow & 0 \\ & & \| & & {\scriptstyle c}\downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbf{Z}(1) & \longrightarrow & \mathcal{L}_T & \longrightarrow & \tau^{-1}(M_T^{\mathrm{gp}}) & \longrightarrow & 0, \end{array}$$

where the right vertical arrow is defined by $1 \mapsto a^{-1}$, and the right square is defined to be cartesian.

Weight filtration is the same as in Description 1.

Hodge filtration is given by

$$F^1 = 0 \subset F^0 = \operatorname{Ker}(\tilde{c} : \mathcal{O}_T^{\log} \otimes H_{\mathbf{Z}} \to \mathcal{O}_T^{\log}) \subset F^{-1} = \mathcal{O}_T^{\log} \otimes H_{\mathbf{Z}},$$

where $\tilde{c}$ is the $\mathcal{O}_T^{\log}$-linear map induced by $H_{\mathbf{Z}} \xrightarrow{c} \mathcal{L}_T \subset \mathcal{O}_T^{\log}$.

4.3. *Example* $h^{p,q} = 1$ $(p+q=3,\ p,q \geq 0)$ ([KU09, 12.3]).

Let $H_0 = \mathbf{Z}^4 = \mathbf{Z}e_1 \oplus \mathbf{Z}e_2 \oplus \mathbf{Z}e_3 \oplus \mathbf{Z}e_4$, weight is 3, and $\langle e_3, e_1\rangle = \langle e_4, e_2\rangle = 1$. Define $M(2, \mathbf{C}) \times \mathbf{C} \hookrightarrow \check{D}$, $(\tau = (\tau_{jk}), v) \mapsto F = F(\tau, v)$ by

$$\begin{aligned}
&F^4 = 0 \subset F^3 = \mathbf{C} \cdot (v(\tau_{11}e_1 + \tau_{12}e_2 + e_3) + (\tau_{12}e_1 + \tau_{22}e_2 + e_4)) \\
&\subset F^2 = \mathbf{C} \cdot (\tau_{11}e_1 + \tau_{12}e_2 + e_3) + \mathbf{C} \cdot (\tau_{12}e_1 + \tau_{22}e_2 + e_4) \\
&\subset F^1 = (F^3)^{\perp} = F^2 + \mathbf{C} \cdot (e_1 - ve_2 +) \subset F^0 = H_{0,\mathbf{C}}.
\end{aligned}$$

We see that $F(\tau, v) \in D$ if and only if

$$\begin{aligned}
&v\bar{v}\operatorname{Im}(\tau_{11}) + (v + \bar{v})\operatorname{Im}(\tau_{12}) + \operatorname{Im}(\tau_{22}) < 0 \quad \text{and} \\
&\det(\operatorname{Im}(\tau)) < 0.
\end{aligned}$$

$G_{\mathbf{C}} := \operatorname{Sp}(2, \mathbf{C}) \ni \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ acts on these $F(\tau, v) \in \check{D}$ by

$$\begin{aligned}
&F(\tau, v) \mapsto F((A\tau + B)(C\tau + D)^{-1}, v'), \quad \text{where} \\
&\qquad v' = (pv + q)(rv + s)^{-1} \quad \text{if } C\tau + D = \begin{pmatrix} p & q \\ r & s \end{pmatrix}
\end{aligned}$$

Let N_α, N_β, and N_m with $m \in S :=$ (square free positive integer) be nilpotent elements in $\mathfrak{g}_{\mathbf{Q}} := \operatorname{Lie} G_{\mathbf{Q}}$ defined by

$$\begin{aligned}
&N_\alpha(e_3) = e_1, N_a(e_j) = 0 \ \text{ for } \ j \neq 3, \\
&N_\beta(e_4) = e_3, N_\beta(e_3) = -e_1, N_\beta(e_1) = -e_2, N_\beta(e_2) = 0. \\
&N_m(e_1) = e_3, N_m(e_3) = 0, N_m(e_4) = -me_2, N_m(e_2) = 0.
\end{aligned}$$

Let $\sigma_\mu := \mathbf{R}_{\geq 0}N_\mu$ for $\mu = \alpha, \beta, m$. Then we have

Proposition. *The fan* Σ *consisting of* $\{0\}$ *and all rational nilpotent cones of rank* 1 *in* $\mathfrak{g}_{\mathbf{R}}$ *is given as* $\{\operatorname{Ad}(g)\sigma \,|\, \sigma = \{0\}, \sigma_\alpha, \sigma_b, \sigma_m \ (m \in S),\ g \in G_{\mathbf{Q}}\}$.

Σ *is a complete fan, i.e., if there exists* $Z \subset \check{D}$ *such that* (σ, Z) *is a nilpotent orbit, then* $\sigma \in \Sigma$.

Let $D_\Sigma := \{(\sigma, Z) \text{ nilpotent orbit} \,|\, \sigma \in \Sigma, Z \subset \check{D}\}$. We consider a boundary point $(\sigma_\beta, Z) \in D_\Sigma$. The monodromy weight filtration $W := W(N)[-3]$ is

$$\begin{aligned}
&W_{-1} = 0 \subset W_0 = W_1 = \mathbf{R}e_2 \subset W_2 = W_3 = W_1 + \mathbf{R}e_1 \\
&\subset W_4 = W_5 = W_3 + \mathbf{R}e_3 \subset W_6 = H_{0,\mathbf{R}}.
\end{aligned}$$

We identify $\mathrm{gr}^W = H_{0,\mathbf{R}}$ by the basis e_2, e_1, e_3, e_4. Then, for $F \in Z$, we have

$$\begin{aligned}&F^3(\mathrm{gr}^W_6) = \mathbf{C}e_4,\ F^2(\mathrm{gr}^W_4) = \mathbf{C}e_3,\ F^1(\mathrm{gr}^W_2) = \mathbf{C}e_1,\ F^0(\mathrm{gr}^W_0) = \mathbf{C}e_2,\\ &F^p(\mathrm{gr}^W_w) = 0 \text{ for the other } (p, w).\end{aligned}$$

Remark. By using the above toroidal partial compactification $\Gamma\backslash D_\Sigma$, generaic global Torelli theorem was proved for mirror quintic threefolds by the author [U08], and for similar one-parameter mirror families to weighted hypersurfaces by Shirakawa [Sh09].

§5. Comments on general theory

5.1. Fundamental diagram [KNU.p].

Let D be a classifying space of mixed Hodge structures with polarized graded quotients ([U84]). It is a complex analytic manifold, and is a mixed Hodge theoretic generalization of a classifying space of polarized Hodge structures defined by Griffiths ([G68]).

In the series of papers [KNU08], [KNU09], [KNU11a], [KNU.p], we constructed the fundamental diagram consisting of various partial compactifications of D:

$$\begin{array}{ccccccc}
 & & & & D_{\mathrm{SL}(2),\mathrm{val}} & \hookrightarrow & D_{\mathrm{BS},\mathrm{val}} \\
 & & & & \downarrow & & \downarrow \\
D_{\Sigma,\mathrm{val}} & \leftarrow & D^\sharp_{\Sigma,\mathrm{val}} & \rightarrow & D_{\mathrm{SL}(2)} & & D_{\mathrm{BS}} \\
\downarrow & & \downarrow & & & & \\
D_\Sigma & \leftarrow & D^\sharp_\Sigma & & & &
\end{array}$$

Here D_Σ is the space of nilpotent orbits in the directions in Σ, $D_{\mathrm{SL}(2)}$ is the space of SL(2)-orbits, and D_{BS} is the space of Borel-Serre.

For the degeneration of polarized (pure) Hodge structures, this diagram was constructed in [KU09].

5.2. The following is known:

1 Local fan always exist ([KU09], [KNU.p]).
2 Global fan does not exist even for connected Néron model ([KNU.p]).
3 Weak fan always exist for Néron model with any gr^W ([KNU.p]).
4 Valuative formulation of period map $S_{\mathrm{val}} \to \Gamma\backslash D_{\mathrm{val},(\Gamma)}$ ([KNU11b]). Here $D_{\mathrm{val},(\Gamma)} \subset D_{\mathrm{val}}$ is the union of $D_{\sigma,\mathrm{val}}$ for $\sigma \in \mathcal{C}_\Gamma$ with $\mathcal{C}_\Gamma$ being the set of all sharp rational nilpoent cones in $\mathfrak{g}_{\mathbf{R}}$ generated by the logarithms of finite number of elements of a given subgroup Γ of $G_{\mathbf{Z}}$. This valuative formulation has an advantage that we do not need any fan or a weak fan Σ.

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