FREE RESOLUTIONS OF (VARIANTS OF) BOREL FIXED IDEALS AND DISCRETE MORSE THEORY

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This report is a survey of a joint work with Dr. Ryota Okazaki of Osaka University and JST CREST. The detailed versions ([12, 13]) will be submitted for publication elsewhere.

1. Introduction

Let $S := \Bbbk[x_1, \dots, x_n]$ be a polynomial ring over a field $\Bbbk$. For a monomial ideal $I \subset S$, $G(I)$ denotes the set of minimal (monomial) generators of I. We say a monomial ideal $I \subset S$ is *Borel fixed* (or *strongly stable*), if $\mathsf{m} \in G(I)$, $x_i | \mathsf{m}$ and $j < i$ imply $(x_j/x_i) \cdot \mathsf{m} \in I$. Borel fixed ideals are important, since they appear as the *generic initial ideals* of homogeneous ideals (if $\mathrm{char}(\Bbbk) = 0$).

A squarefree monomial ideal I is said to be *squarefree strongly stable*, if $\mathsf{m} \in G(I)$, $x_i | \mathsf{m}$, $x_j \not| \mathsf{m}$ and $j < i$ imply $(x_j/x_i) \cdot \mathsf{m} \in I$. Any monomial $\mathsf{m} \in S$ with $\deg(\mathsf{m}) = e$ has a unique expression

$$\mathsf{m} = \prod_{i=1}^{e} x_{\alpha_i} \quad \text{with} \quad 1 \le \alpha_1 \le \alpha_2 \le \cdots \le \alpha_e \le n. \tag{1.1}$$

Now we can consider the squarefree monomial

$$\mathsf{m}^{\mathsf{sq}} = \prod_{i=1}^{e} x_{\alpha_i + i - 1}$$

in a larger polynomial ring $T = \Bbbk[x_1, \dots, x_N]$ with $N \gg 0$. If $I \subset S$ is Borel fixed, then

$$I^{\mathsf{sq}} := (\, \mathsf{m}^{\mathsf{sq}} \mid \mathsf{m} \in G(I) \,) \subset T \tag{1.2}$$

is squarefree strongly stable. This operation plays a role in the *shifting theory* for simplicial complexes (see [1]).

A minimal free resolution of a Borel fixed ideal I has been constructed by Eliahou and Kervaire [7]. While the minimal free resolution is unique up to isomorphism, its "description" depends on the choice of a free basis, and further analysis of the minimal free resolution is still an interesting problem. See, for example, [2, 6, 9, 10, 11]. In this paper, we will give a new approach which is applicable to both I and I^{sq}. Our main tool is the "alternative polarization" $\mathsf{b}\text{-}\mathsf{pol}(I)$ of I.

Let

$$\widetilde{S} := \Bbbk[\, x_{i,j} \mid 1 \le i \le n,\, 1 \le j \le d \,]$$

be the polynomial ring, and set

$$\Theta := \{ x_{i,1} - x_{i,j} \mid 1 \le i \le n,\, 2 \le j \le d \} \subset \widetilde{S}.$$

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Then there is an isomorphism $\widetilde{S}/(\Theta) \cong S$ induced by $\widetilde{S} \ni x_{i,j} \longmapsto x_i \in S$. Throughout this paper, $\widetilde{S}$ and Θ are used in this meaning.

Assume that $\mathsf{m} \in G(I)$ has the expression (1.1). If $\deg(\mathsf{m})\,(= e) \le d$, we set

$$\mathsf{b\text{-}pol}(\mathsf{m}) = \prod_{i=1}^{e} x_{\alpha_i, i} \in \widetilde{S}. \tag{1.3}$$

Note that $\mathsf{b\text{-}pol}(\mathsf{m})$ is a squarefree monomial. If there is no danger of confusion, $\mathsf{b\text{-}pol}(\mathsf{m})$ is denoted by $\widetilde{\mathsf{m}}$. If $\mathsf{m} = \prod_{i=1}^{n} x_i^{a_i}$, then we have

$$\widetilde{\mathsf{m}}\ (= \mathsf{b\text{-}pol}(\mathsf{m})) \ = \prod_{\substack{1 \le i \le n \\ b_{i-1}+1 \le j \le b_i}} x_{i,j} \in \widetilde{S}, \quad \text{where} \quad b_i := \sum_{l=1}^{i} a_l.$$

If $\deg(\mathsf{m}) \le d$ for all $\mathsf{m} \in G(I)$, we set

$$\mathsf{b\text{-}pol}(I) := (\mathsf{b\text{-}pol}(\mathsf{m}) \mid \mathsf{m} \in G(I)) \subset \widetilde{S}.$$

The second author ([14]) showed that if I is Borel fixed, then $\widetilde{I} := \mathsf{b\text{-}pol}(I)$ is a "polarization" of I, that is, Θ forms an $\widetilde{S}/\widetilde{I}$-regular sequence with the natural isomorphism

$$\widetilde{S}/(\widetilde{I} + (\Theta)) \cong S/I.$$

Note that $\mathsf{b\text{-}pol}(-)$ does not give a polarization for a general monomial ideal. We can obtain I^{sq} of a Borel fixed ideal I through $\mathsf{b\text{-}pol}(I)$, see Proposition 8 below.

In this paper, we will construct a minimal $\widetilde{S}$-free resolution $\widetilde{P}_\bullet$ of $\widetilde{S}/\widetilde{I}$, which is analogous to the Eliahou-Kervaire resolution of S/I. However, their description can *not* be lifted to $\widetilde{I}$, and we need modification. Clearly, $\widetilde{P}_\bullet \otimes_{\widetilde{S}} \widetilde{S}/(\Theta)$ gives the minimal free resolution of S/I. Similar construction also works for T/I^{sq} (Corollary 9). In some sense, our results are generalizations of those in [11], which concerns the case I is generated in one degree (i.e., all elements of $G(I)$ have the same degree).

In [2], Batzies and Welker tried to construct a minimal free resolutions of a monomial ideals J using Forman's *discrete Morse theory* ([8]). If J is *shellable* (also called *linear quotients* in literature), their method works, and we have a *Batzies-Welker type* minimal free resolution. However, it is very hard to compute their resolution explicitly.

A Borel fixed ideal I and its polarization $\widetilde{I} = \mathsf{b\text{-}pol}(I)$ is shellable. We will show that our resolution $\widetilde{P}_\bullet$ of $\widetilde{S}/\widetilde{I}$ and the induced resolutions of S/I and T/I^{sq} are Batzies-Welker type. In particular, these resolutions are cellular. As far as the author knows, an *explicit* description of a Batzies-Welker type resolution of a general Borel fixed ideal has never been obtained before. Finally, we show that the CW complex supporting $\widetilde{P}_\bullet$ is *regular*.

2. The Eliahou-Kervaire type resolution of $\widetilde{S}/\,\mathsf{b\text{-}pol}(I)$

Throughout the rest of the paper, I is a Borel fixed monomial ideal with $\deg \mathsf{m} \le d$ for all $\mathsf{m} \in G(I)$. For the definitions of the alternative polarization $\mathsf{b\text{-}pol}(I)$ of I and related concepts, consult the previous section. For a monomial $\mathsf{m} = \prod_{i=1}^{n} x_i^{a_i} \in S$, set $\mu(\mathsf{m}) := \min\{\, i \mid a_i > 0 \,\}$ and $\nu(\mathsf{m}) := \max\{\, i \mid a_i > 0 \,\}$. In [7], it is shown that any monomial $\mathsf{m} \in I$ has a unique expression $\mathsf{m} = \mathsf{m}_1 \cdot \mathsf{m}_2$ with $\nu(\mathsf{m}_1) \le \mu(\mathsf{m}_2)$ and $\mathsf{m}_1 \in G(I)$. Following [7], we set $g(\mathsf{m}) := \mathsf{m}_1$. For i with $i < \nu(\mathsf{m})$, let

$$\mathfrak{b}_i(\mathsf{m}) = (x_i/x_k) \cdot \mathsf{m}, \text{ where } k := \min\{\, j \mid a_j > 0,\ j > i\}.$$

Since I is Borel fixed, $\mathsf{m} \in I$ implies $\mathfrak{b}_i(\mathsf{m}) \in I$.

Definition 1 ([12, Definition 2.1])**.** For a finite subset $\widetilde{F} = \{\,(i_1, j_1), (i_2, j_2), \dots, (i_q, j_q)\,\}$ of $\mathbb{N} \times \mathbb{N}$ and a monomial $\mathsf{m} = \prod_{i=1}^{e} x_{\alpha_i} = \prod_{i=1}^{n} x_i^{a_i} \in G(I)$ with $1 \le \alpha_1 \le \alpha_2 \le \cdots \le \alpha_e \le n$, we say the pair $(\widetilde{F}, \widetilde{\mathsf{m}})$ is *admissible* (for $\mathsf{b\text{-}pol}(I)$), if the following are satisfied:

(a) $1 \le i_1 < i_2 < \cdots < i_q < \nu(\mathsf{m})$,

(b) $j_r = \max\{\, l \mid \alpha_l \le i_r\,\} + 1$ (equivalently, $j_r = 1 + \sum_{l=1}^{i_r} a_l$) for all r.

For $\mathsf{m} \in G(I)$, the pair $(\emptyset, \widetilde{\mathsf{m}})$ is also admissible.

Lemma 2. *Let $(\widetilde{F}, \widetilde{\mathsf{m}})$ be an admissible pair with $\widetilde{F} = \{\,(i_1, j_1), \dots, (i_q, j_q)\,\}$ and $\mathsf{m} = \prod x_i^{a_i} \in G(I)$. Then we have the following.*

(i) $j_1 \le j_2 \le \cdots \le j_q$.

(ii) $x_{k,j_r} \cdot \mathsf{b\text{-}pol}(\mathfrak{b}_{i_r}(\mathsf{m})) = x_{i_r,j_r} \cdot \mathsf{b\text{-}pol}(\mathsf{m})$*, where* $k = \min\{\, l \mid l > i_r, a_l > 0\,\}$.

For $\mathsf{m} \in G(I)$ and an integer i with $1 \le i < \nu(\mathsf{m})$, set $\mathsf{m}_{\langle i\rangle} := g(\mathfrak{b}_i(\mathsf{m}))$ and $\widetilde{\mathsf{m}}_{\langle i\rangle} := \mathsf{b\text{-}pol}(\mathsf{m}_{\langle i\rangle})$. If $i \ge \nu(\mathsf{m})$, we set $\mathsf{m}_{\langle i\rangle} := \mathsf{m}$ for the convenience. In the situation of Lemma 2, $\widetilde{\mathsf{m}}_{\langle i_r\rangle}$ divides $x_{i_r,j_r} \cdot \widetilde{\mathsf{m}}$ for all $1 \le r \le q$.

For $\widetilde{F} = \{\,(i_1, j_1), \dots, (i_q, j_q)\,\}$ and r with $1 \le r \le q$, set $\widetilde{F}_r := \widetilde{F} \setminus \{\,(i_r, j_r)\,\}$, and for an admissible pair $(\widetilde{F}, \widetilde{\mathsf{m}})$ for $\mathsf{b\text{-}pol}(I)$,

$$B(\widetilde{F}, \widetilde{\mathsf{m}}) := \{\, r \mid (\widetilde{F}_r, \widetilde{\mathsf{m}}_{\langle i_r\rangle}) \text{ is admissible}\,\}.$$

Lemma 3. *Let $(\widetilde{F}, \widetilde{\mathsf{m}})$ be as in Lemma 2.*

(i) *For all r with $1 \le r \le q$, $(\widetilde{F}_r, \widetilde{\mathsf{m}})$ is admissible.*

(ii) *We always have $q \in B(\widetilde{F}, \widetilde{\mathsf{m}})$.*

(iii) *Assume that $(\widetilde{F}_r, \widetilde{\mathsf{m}}_{\langle i_r\rangle})$ satisfies the condition (a) of Definition 1. Then $r \in B(\widetilde{F}, \widetilde{\mathsf{m}})$ if and only if either $j_r < j_{r+1}$ or $r = q$.*

(iv) *For r, s with $1 \le r < s \le q$ and $j_r < j_s$, we have $\mathfrak{b}_{i_r}(\mathfrak{b}_{i_s}(\mathsf{m})) = \mathfrak{b}_{i_s}(\mathfrak{b}_{i_r}(\mathsf{m}))$ and hence $(\widetilde{\mathsf{m}}_{\langle i_r\rangle})_{\langle i_s\rangle} = (\widetilde{\mathsf{m}}_{\langle i_s\rangle})_{\langle i_r\rangle}$.*

(v) *For r, s with $1 \le r < s \le q$ and $j_r = j_s$, we have $\mathfrak{b}_{i_r}(\mathsf{m}) = \mathfrak{b}_{i_r}(\mathfrak{b}_{i_s}(\mathsf{m}))$ and hence $\widetilde{\mathsf{m}}_{\langle i_r\rangle} = (\widetilde{\mathsf{m}}_{\langle i_s\rangle})_{\langle i_r\rangle}$.*

Example 4. Let $I \subset S = \Bbbk[x_1, x_2, x_3, x_4]$ be the smallest Borel fixed ideal containing $\mathsf{m} = x_1^2 x_3 x_4$. In this case, $\mathsf{m}'_{\langle i\rangle} = \mathfrak{b}_i(\mathsf{m}')$ for all $\mathsf{m}' \in G(I)$. Hence, we have $\mathsf{m}_{\langle 1\rangle} = x_1^3 x_4$, $\mathsf{m}_{\langle 2\rangle} = x_1^2 x_2 x_4$ and $\mathsf{m}_{\langle 3\rangle} = x_1^2 x_3^2$. The following 3 pairs are all admissible.

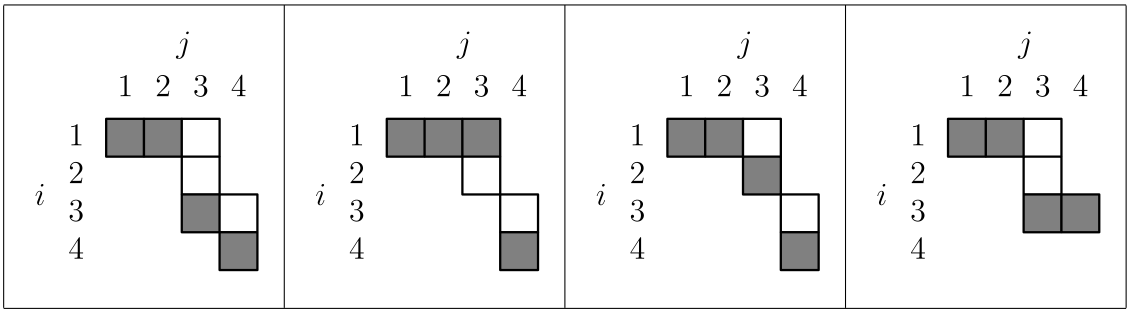
- $(\widetilde{F}, \widetilde{\mathsf{m}}) = (\{\,(1,3), (2,3), (3,4)\,\}, x_{1,1}\, x_{1,2}\, x_{3,3}\, x_{4,4})$
- $(\widetilde{F}_2, \widetilde{\mathsf{m}}_{\langle 2\rangle}) = (\{\,(1,3), (3,4)\,\}, x_{1,1}\, x_{1,2}\, x_{2,3}\, x_{4,4})$
- $(\widetilde{F}_3, \widetilde{\mathsf{m}}_{\langle 3\rangle}) = (\{\,(1,3), (2,3)\,\}, x_{1,1}\, x_{1,2}\, x_{3,3}\, x_{3,4})$

(For this $\widetilde{F}$, $i_r = r$ holds and the reader should be careful). However, $(\widetilde{F}_1, \widetilde{\mathsf{m}}_{\langle 1\rangle}) = (\{\,(2,3), (3,4)\,\}, x_{1,1}\, x_{1,2}\, x_{1,3}\, x_{4,4})$ does not satisfy the condition (b) of Definition 1. Hence $B(\widetilde{F}, \widetilde{\mathsf{m}}) = \{2, 3\}$.

The diagrams of (admissible) pairs are very useful for better understanding. To draw a diagram of $(\widetilde{F},\widetilde{\mathsf{m}})$, we put a white square in the (i,j)-th position if $(i,j)\in\widetilde{F}$ and the black square there if $x_{i,j}$ divides $\widetilde{\mathsf{m}}$. If $\widetilde{F}$ is maximal among $\widetilde{F}'$ such that $(\widetilde{F}',\widetilde{\mathsf{m}})$ is admissible, then the diagram of $(\widetilde{F},\widetilde{\mathsf{m}})$ forms a "right side down stairs" (see the leftmost and rightmost diagrams of the table below). If $(\widetilde{F},\widetilde{\mathsf{m}})$ is admissible but $\widetilde{F}$ is not maximal, then some white squares are removed from the diagram for the maximal case. If the pair is admissible, there is a unique black square in each column and this is the "lowest" of the squares in the column.

If $(\widetilde{F},\widetilde{\mathsf{m}})$ is admissible and $r\in B(\widetilde{F},\widetilde{\mathsf{m}})$, then we can get the diagram of $(\widetilde{F}_r,\widetilde{\mathsf{m}}_{\langle i_r\rangle})$ from that of $(\widetilde{F},\widetilde{\mathsf{m}})$ by the following procedure.

(i) Remove the (sole) black square in the j_r-th column.
(ii) Replace the white square in the (i_r,j_r)-th position by a black one.
(iii) If $\mathsf{m}_{\langle i_r\rangle}\neq\mathfrak{b}_{i_r}(\mathsf{m})$, erase some squares from the lower-right of the diagram.



$(\widetilde{F},\widetilde{\mathsf{m}})$	$(\widetilde{F}_1,\widetilde{\mathsf{m}}_{\langle 1\rangle})$	$(\widetilde{F}_2,\widetilde{\mathsf{m}}_{\langle 2\rangle})$	$(\widetilde{F}_3,\widetilde{\mathsf{m}}_{\langle 3\rangle})$
admissible	not admissible	admissible	admissible

Next let I' be the smallest Borel fixed ideal containing $\mathsf{m}=x_1^2x_3x_4$ and $x_1^2x_2$. For $\widetilde{F}=\{(1,3),(2,3),(3,4)\}$, $(\widetilde{F},\widetilde{\mathsf{m}})$ is admissible again. However $\widetilde{\mathsf{m}}_{\langle 2\rangle}=x_1^2x_2$ in this time, and $(\widetilde{F}_2,\widetilde{\mathsf{m}}_{\langle 2\rangle})=(\{(1,3),(3,4)\},x_{1,1}\,x_{1,2}\,x_{2,3})$ is no longer admissible. In fact, it does not satisfy (a) of Definition 1. Hence $B(\widetilde{F},\widetilde{\mathsf{m}})=\{3\}$ for $\mathsf{b\text{-}pol}(I')$.

For $F=\{i_1,\dots,i_q\}\subset\mathbb{N}$ with $i_1<\cdots<i_q$ and $\mathsf{m}\in G(I)$, Eliahou and Kervaire ([7]) called the pair (F,m) admissible for I, if $i_q<\nu(\mathsf{m})$. In this case, there is a unique sequence $j_1,\dots,j_q$ such that $(\widetilde{F},\widetilde{\mathsf{m}})$ is admissible for $\widetilde{I}$, where $\widetilde{F}=\{(i_1,j_1),\dots,(i_q,j_q)\}$. In this way, there is a one-to-one correspondence between the admissible pairs for I and those of $\widetilde{I}$. As the free summands of the Eliahou-Kervaire resolution of I are indexed by the admissible pairs for I, our resolution of $\widetilde{I}$ are indexed by the admissible pairs for $\widetilde{I}$.

We will define a $\mathbb{Z}^{n\times d}$-graded chain complex $\widetilde{P}_\bullet$ of free $\widetilde{S}$-modules as follows. First, set $\widetilde{P}_0:=\widetilde{S}$. For each $q\geq 1$, we set

$$A_q:=\text{the set of admissible pairs }(\widetilde{F},\widetilde{\mathsf{m}})\text{ for }\mathsf{b\text{-}pol}(I)\text{ with }\#\widetilde{F}=q,$$

and

$$\widetilde{P}_q:=\bigoplus_{(\widetilde{F},\widetilde{\mathsf{m}})\in A_{q-1}}\widetilde{S}\,e(\widetilde{F},\widetilde{\mathsf{m}}),$$

where $e(\widetilde{F}, \widetilde{\mathsf{m}})$ is a basis element with

$$\deg\left(e(\widetilde{F}, \widetilde{\mathsf{m}})\right) = \deg\left(\widetilde{\mathsf{m}} \times \prod_{(i_r,j_r)\in\widetilde{F}} x_{i_r,j_r}\right) \in \mathbb{Z}^{n\times d}.$$

We define the $\widetilde{S}$-homomorphism $\partial : \widetilde{P}_q \to \widetilde{P}_{q-1}$ for $q \geq 2$ so that $e(\widetilde{F}, \widetilde{\mathsf{m}})$ with $\widetilde{F} = \{(i_1, j_1), \ldots, (i_q, j_q)\}$ is sent to

$$\sum_{1\leq r\leq q} (-1)^r \cdot x_{i_r,j_r} \cdot e(\widetilde{F}_r, \widetilde{\mathsf{m}}) - \sum_{r\in B(\widetilde{F},\widetilde{\mathsf{m}})} (-1)^r \cdot \frac{x_{i_r,j_r} \cdot \widetilde{\mathsf{m}}}{\widetilde{\mathsf{m}}_{\langle i_r\rangle}} \cdot e(\widetilde{F}_r, \widetilde{\mathsf{m}}_{\langle i_r\rangle}),$$

and $\partial : \widetilde{P}_1 \to \widetilde{P}_0$ by $e(\emptyset, \widetilde{\mathsf{m}}) \longmapsto \widetilde{\mathsf{m}} \in \widetilde{S} = \widetilde{P}_0$. Clearly, ∂ is a $\mathbb{Z}^{n\times d}$-graded homomorphism.

Set

$$\widetilde{P}_\bullet : \cdots \xrightarrow{\partial} \widetilde{P}_i \xrightarrow{\partial} \cdots \xrightarrow{\partial} \widetilde{P}_1 \xrightarrow{\partial} \widetilde{P}_0 \longrightarrow 0.$$

Theorem 5 ([12, Theorem 2.6])**.** *The complex $\widetilde{P}_\bullet$ is a $\mathbb{Z}^{n\times d}$-graded minimal $\widetilde{S}$-free resolution for $\widetilde{S}/\,\mathsf{b\text{-}pol}(I)$.*

Sketch of Proof. Calculation using Lemma 3 shows that $\partial \circ \partial(e(\widetilde{F}, \widetilde{\mathsf{m}})) = 0$ for each admissible pair $(\widetilde{F}, \widetilde{\mathsf{m}})$. That is, $\widetilde{P}_\bullet$ is a chain complex.

Let $I = (\mathsf{m}_1, \ldots, \mathsf{m}_t)$ with $\mathsf{m}_1 \succ \cdots \succ \mathsf{m}_t$, and set $I_r := (\mathsf{m}_1, \ldots, \mathsf{m}_r)$. Here $\succ$ is the lexicographic order with $x_1 \succ x_2 \succ \cdots \succ x_n$. Then I_r are also Borel fixed. The acyclicity of the complex $\widetilde{P}$ can be shown inductively by means of mapping cones. □

3. Applications and Remarks

Let $I \subset S$ be a Borel fixed ideal, and $\Theta \subset \widetilde{S}$ the sequence defined in Introduction. As remarked before, there is a one-to-one correspondence between the admissible pairs for $\widetilde{I}$ and those for I, and if $(\widetilde{F}, \widetilde{\mathsf{m}})$ corresponds to (F, m) then $\#\widetilde{F} = \#F$. Hence we have

$$\operatorname{Tor}_i^{\widetilde{S}}(\Bbbk, \widetilde{I}) \cong \operatorname{Tor}_i^S(\Bbbk, I) \tag{3.1}$$

as $\mathbb{Z}$-graded $\Bbbk$-vector spaces for all i, where S and $\widetilde{S}$ are considered to be $\mathbb{Z}$-graded. Of course, this is clear, if one knows the fact that $\widetilde{I}$ is a polarization of I ([14, Theorem 3.4]). Conversely, we can show that $\widetilde{I}$ is a polarization by the equation (3.1) and [11, Lemma 6.9].

The next result also follows from [11, Lemma 6.9].

Corollary 6. *$\widetilde{P}_\bullet \otimes_{\widetilde{S}} \widetilde{S}/(\Theta)$ is a minimal S-free resolution of S/I.*

Remark 7. The correspondence between the admissible pairs for I and those for $\widetilde{I}$, does *not* give a chain map between the Eliahou-Kervaire resolution and our $\widetilde{P}_\bullet \otimes_{\widetilde{S}} \widetilde{S}/(\Theta)$. In this sense, two resolutions are not the same. See Example 19 below.

Let $T = \Bbbk[x_1, \ldots, x_{n+d-1}]$ be a polynomial ring, and

$$\Theta' := \{x_{i,j} - x_{i+1,j-1} \mid 1 \leq i < n,\ 1 < j \leq d\}$$

a subset of $\widetilde{S}$. Then the ring homomorphism $\widetilde{S} \to T$ with $x_{i,j} \mapsto x_{i+j-1}$ induces the isomorphism $\widetilde{S}/(\Theta') \cong T$.

Proposition 8 ([14, Proposition 4,1]). *With the above notation, Θ' forms an $\widetilde{S}/\widetilde{I}$-regular sequence, and we have $(\widetilde{S}/(\Theta') \otimes_{\widetilde{S}} (\widetilde{S}/\widetilde{I}) \cong T/I^{\mathsf{sq}}$, where I^{sq} is the one defined in* (1.2).

Applying Proposition 8 and [5, Proposition 1.1.5], we have the following.

Corollary 9. *The complex $\widetilde{P}_\bullet \otimes_{\widetilde{S}} \widetilde{S}/(\Theta')$ is a minimal T-free resolution of T/I^{sq}.*

For a Borel fixed ideal I generated in one degree, Nagel and Reiner [11] constructed a CW complex, which supports a minimal free resolution of $\widetilde{I}$ (or I, I^{sq}). Note that if I is generated in one degree then $\mathsf{m}_{\langle i \rangle} = \mathfrak{b}_i(\mathsf{m})$ for all $\mathsf{m} \in G(I)$, and $\widetilde{P}_\bullet$ is simpler.

Proposition 10 ([12, Proposition 4.9]). *Let I be a Borel fixed ideal generated in one degree. Then Nagel-Reiner description of a free resolution of $\widetilde{I}$ coincides with our $\widetilde{P}_\bullet$.*

4. Relation to Batzies-Welker theory

In [2], Batzies and Welker connected the theory of *cellular resolutions* of monomial ideals with Forman's discrete Morse theory ([8]).

Definition 11. A monomial ideal J is called *shellable* if there is a total order $\sqsubset$ on $G(J)$ satisfying the following condition.

(∗) For any $\mathsf{m}, \mathsf{m}' \in G(J)$ with $\mathsf{m} \sqsupset \mathsf{m}'$, there is an $\mathsf{m}'' \in G(J)$ such that $\mathsf{m} \sqsupseteq \mathsf{m}''$, $\deg\left(\frac{\mathrm{lcm}(\mathsf{m},\mathsf{m}'')}{\mathsf{m}}\right) = 1$ and $\mathrm{lcm}(\mathsf{m}, \mathsf{m}'')$ divides $\mathrm{lcm}(\mathsf{m}, \mathsf{m}')$.

For a Borel fixed ideal I, let $\sqsubset$ be the total order on $G(\widetilde{I}) = \{\, \widetilde{\mathsf{m}} \mid \mathsf{m} \in G(I) \,\}$ such that $\widetilde{\mathsf{m}}' \sqsubset \widetilde{\mathsf{m}}$ if and only if $\mathsf{m}' \succ \mathsf{m}$ in the lexicographic order on S with $x_1 \succ x_2 \succ \cdots \succ x_n$. In the rest of this section, $\sqsubset$ means this order.

Lemma 12. *The order $\sqsubset$ makes $\widetilde{I}$ shellable.*

The following construction is taken from [2, Theorems 3.2 and 4.3]. For the background of their theory, the reader is recommended to consult the original paper.

For a non-empty subset $\sigma \subset G(\widetilde{I})$, let $\widetilde{\mathsf{m}}_\sigma$ denote the largest element of σ with respect to the order $\sqsubset$, and set $\mathrm{lcm}(\sigma) := \mathrm{lcm}\{\, \widetilde{\mathsf{m}} \mid \widetilde{\mathsf{m}} \in \sigma \,\}$.

Definition 13. We define a total order $\prec_\sigma$ on $G(\widetilde{I})$ as follows. Set

$$N_\sigma := \{\, (\widetilde{\mathsf{m}}_\sigma)_{\langle i \rangle} \mid 1 \le i < \nu(\mathsf{m}_\sigma),\ (\widetilde{\mathsf{m}}_\sigma)_{\langle i \rangle} \text{ divides } \mathrm{lcm}(\sigma) \,\},$$

where $(\widetilde{\mathsf{m}}_\sigma)_{\langle i \rangle}$ denotes $\mathsf{b}\text{-}\mathsf{pol}((\mathsf{m}_\sigma)_{\langle i \rangle})$. For all $\widetilde{\mathsf{m}} \in N_\sigma$ and $\widetilde{\mathsf{m}}' \in G(\widetilde{I}) \setminus N_\sigma$, define $\widetilde{\mathsf{m}} \prec_\sigma \widetilde{\mathsf{m}}'$. The restriction of $\prec_\sigma$ to N_σ is set to be $\sqsubset$, and the same is true for the restriction to $G(\widetilde{I}) \setminus N_\sigma$.

Let X be the $(\#G(\widetilde{I}) - 1)$-simplex associated with $2^{G(\widetilde{I})}$ (more precisely, $2^{G(\widetilde{I})} \setminus \{\emptyset\}$). Hence we freely identify $\sigma \subset G(\widetilde{I})$ with the corresponding cell of the simplex X. Let G_X be the directed graph defined as follows. The vertex set of G_X is $2^{G(\widetilde{I})} \setminus \{\emptyset\}$. For $\emptyset \neq \sigma, \sigma' \subset G(\widetilde{I})$, there is an arrow $\sigma \to \sigma'$ if and only if $\sigma \supset \sigma'$ and $\#\sigma = \#\sigma' + 1$. For $\sigma = \{\, \widetilde{\mathsf{m}}_1, \widetilde{\mathsf{m}}_2, \dots, \widetilde{\mathsf{m}}_k \,\}$ with $\widetilde{\mathsf{m}}_1 \prec_\sigma \widetilde{\mathsf{m}}_2 \prec_\sigma \cdots \prec_\sigma \widetilde{\mathsf{m}}_k \ (= \widetilde{\mathsf{m}}_\sigma)$ and $l \in \mathbb{N}$ with $1 \le l < k$, set $\sigma_l := \{\, \widetilde{\mathsf{m}}_{k-l}, \widetilde{\mathsf{m}}_{k-l+1}, \dots, \widetilde{\mathsf{m}}_k \,\}$ and

$$u(\sigma) := \sup\{\, l \mid \exists \widetilde{\mathsf{m}} \in G(\widetilde{I}) \text{ s.t. } \widetilde{\mathsf{m}} \prec_\sigma \widetilde{\mathsf{m}}_{k-l} \text{ and } \widetilde{\mathsf{m}} | \,\mathrm{lcm}(\sigma_l) \,\}.$$

If $u := u(\sigma) \neq -\infty$, we can define $\widetilde{\mathsf{n}}_\sigma := \min_{\prec_\sigma}\{\, \widetilde{\mathsf{m}} \mid \widetilde{\mathsf{m}} \text{ divides } \mathrm{lcm}(\sigma_u)\,\}$. Let E_X be the set of edges of G_X. We define a subset A of E_X by

$$A := \{\, \sigma \cup \{\widetilde{\mathsf{n}}_\sigma\} \to \sigma \mid u(\sigma) \neq -\infty, \widetilde{\mathsf{n}}_\sigma \notin \sigma \,\}.$$

It is easy to see that A is a *matching*, that is, every σ occurs in at most one edges of A. We say $\emptyset \neq \sigma \subset G(\widetilde{I})$ is *critical*, if it does not occur in any edge of A.

We have the directed graph G_X^A with the vertex set $2^{G(\widetilde{I})} \setminus \{\emptyset\}$ (i.e., same as G_X) and the set of edges $(E_X \setminus A) \cup \{\, \sigma \to \tau \mid (\tau \to \sigma) \in A \,\}$. By the proof of [2, Theorem 3.2], we see that the matching A is *acyclic*, that is, G_X^A has no directed cycle. A directed path in G_X^A is called a *gradient path*.

The discrete Morse theory ([8]) gives a CW complex X_A with the following conditions.

- There is a one-to-one correspondence between the i-cells of X_A and the *critical* i-cells of X (equivalently, the critical subsets of $G(\widetilde{I})$ consisting of $i+1$ elements).
- X_A is contractible, that is, homotopy equivalent to X.

The cell of X_A corresponding to a critical cell σ of X is denoted by σ_A. By [2, Proposition 7.3], the closure of σ_A contains τ_A if and only if there is a gradient path from σ to τ. See also Proposition 16 below and the argument before it.

Assume that $\emptyset \neq \sigma \subset G(\widetilde{I})$ is critical. Recall that $\widetilde{\mathsf{m}}_\sigma$ denotes the largest element of σ with respect to $\sqsubset$. Take $\mathsf{m}_\sigma = \prod_{l=1}^n x_l^{a_l} \in G(I)$ with $\widetilde{\mathsf{m}}_\sigma = \mathsf{b\text{-}pol}(\mathsf{m}_\sigma)$, and set $q := \#\sigma - 1$. Then there are integers $i_1, \ldots, i_q$ with $1 \leq i_1 < \ldots < i_q < \nu(\mathsf{m}_\sigma)$ and

$$\sigma = \{\, (\widetilde{\mathsf{m}}_\sigma)_{\langle i_r \rangle} \mid 1 \leq r \leq q \,\} \cup \{\widetilde{\mathsf{m}}_\sigma\} \tag{4.1}$$

(see the proof of [2, Proposition 4.3]). Equivalently, we have $\sigma = N_\sigma \cup \{\widetilde{\mathsf{m}}_\sigma\}$. Set $j_r := 1 + \sum_{l=1}^{i_r} a_l$ for each $1 \leq r \leq q$, and $\widetilde{F}_\sigma := \{\, (i_1, j_1), \ldots, (i_q, j_q) \,\}$. Then $(\widetilde{F}_\sigma, \widetilde{\mathsf{m}}_\sigma)$ is an admissible pair for $\widetilde{I}$. Conversely, any admissible pair comes from a critical cell $\sigma \subset G(\widetilde{I})$ in this way. Hence there is a one-to-one correspondence between critical cells and admissible pairs.

Let X_A^i denote the set of all the critical subset $\sigma \subset G(\widetilde{I})$ with $\#\sigma = i+1$, and for (not necessarily critical) subsets σ, τ of $G(\widetilde{I})$, let $\mathcal{P}_{\sigma,\tau}$ denote the set of all the gradient paths from σ to τ. For $\sigma \in X_A^q$ of the form (4.1), $e(\sigma)$ denotes a basis element with degree $\deg(\mathrm{lcm}(\sigma)) \in \mathbb{Z}^{n \times d}$. Set

$$\widetilde{Q}_q = \bigoplus_{\sigma \in X_A^q} \widetilde{S}\, e(\sigma) \qquad (q \geq 0).$$

The differential map $\widetilde{Q}_q \to \widetilde{Q}_{q-1}$ sends $e(\sigma)$ to

$$\sum_{r=1}^{q} (-1)^r x_{i_r, j_r} \cdot e(\sigma \setminus \{(\widetilde{\mathsf{m}}_\sigma)_{\langle i_r \rangle}\}) - (-1)^q \sum_{\substack{\tau \in X_A^{q-1} \\ \mathcal{P} \in \mathcal{P}_{\sigma \setminus \{\widetilde{\mathsf{m}}_\sigma\}, \tau}}} m(\mathcal{P}) \cdot \frac{\mathrm{lcm}(\sigma)}{\mathrm{lcm}(\tau)} \cdot e(\tau), \tag{4.2}$$

where $m(\mathcal{P}) = \pm 1$ is the one defined in [2, p.166].

The following is a direct consequence of [2, Theorem 4.3] (and [2, Remark 4.4]).

Proposition 14 (Batzies-Welker, [2]). *$\widetilde{Q}_\bullet$ is a minimal free resolution of $\widetilde{I}$, and has a cellular structure supported by X_A.*

Theorem 15 ([12, Theorem 5.11]). *Our description of $\widetilde{P}_\bullet$ (more precisely, the truncation $\widetilde{P}_{\geq 1}$) coincides with the Batzies-Welker resolution $\widetilde{Q}_\bullet$. That is, $\widetilde{P}_\bullet$ is a cellular resolution supported by a CW complex X_A, which is obtained by the discrete Morse theory.*

First, note that the following hold.

(1) If σ is critical, so is $\sigma \setminus \{ (\widetilde{\mathsf{m}}_\sigma)_{\langle i_r \rangle} \}$ for $1 \leq r \leq q$.
(2) Let σ and τ be (not necessarily critical) cells with $\mathcal{P}_{\sigma,\tau} \neq \varnothing$. Then $\mathrm{lcm}(\tau)$ divides $\mathrm{lcm}(\sigma)$.
(3) Let $\sigma \in X_A^q$, $\tau \in X_A^{q-1}$ and assume that there is a gradient path $\sigma \to \sigma \setminus \{\widetilde{\mathsf{m}}\} = \sigma_0 \to \sigma_1 \to \cdots \to \sigma_l = \tau$. Then $\#\sigma_{l-1} = \#\tau + 1 = q+1$, $\#\sigma_i = q$ or $q+1$ for each i, and σ_i is not critical for all $0 \leq i < l$. Hence, if $l \geq 1$, then $\widetilde{\mathsf{m}}$ must be $\widetilde{\mathsf{m}}_\sigma$.

Next, we will show the following.

Proposition 16. *Let σ, τ be critical cells with $\#\sigma = \#\tau + 1$, and $(\widetilde{F}_\sigma, \widetilde{\mathsf{m}}_\sigma)$ and $(\widetilde{F}_\tau, \widetilde{\mathsf{m}}_\tau)$ the admissible pairs corresponding to σ and τ respectively. Set $\widetilde{F}_\sigma = \{ (i_1, j_1), \dots, (i_q, j_q) \}$ with $i_1 < \cdots < i_q$. Then $\mathcal{P}_{\sigma \setminus \{\widetilde{\mathsf{m}}_\sigma\}, \tau} \neq \varnothing$ if and only if there is some $r \in B(\widetilde{F}_\sigma, \widetilde{\mathsf{m}}_\sigma)$ with $(\widetilde{F}_\tau, \widetilde{\mathsf{m}}_\tau) = ((\widetilde{F}_\sigma)_r, (\widetilde{\mathsf{m}}_\sigma)_{\langle i_r \rangle})$. If this is the case, we have $\#\mathcal{P}_{\sigma \setminus \{\widetilde{\mathsf{m}}_\sigma\}, \tau} = 1$.*

Sketch of Proof. Only if part follows from the above remark. Note that the second index j of each $x_{i,j} \in \widetilde{S}$ restricts the choice of paths and it makes the proof easier.

Next, assuming $\widetilde{F}_\tau = (\widetilde{F}_\sigma)_r$ and $\widetilde{\mathsf{m}}_\tau = (\widetilde{\mathsf{m}}_\sigma)_{\langle i_r \rangle}$ for some $r \in B(\widetilde{F}_\sigma, \widetilde{\mathsf{m}}_\sigma)$, we will construct a gradient path from $\sigma \setminus \{\widetilde{\mathsf{m}}_\sigma\}$ to τ. For short notation, set $\widetilde{\mathsf{m}}_{[s]} := (\widetilde{\mathsf{m}}_\sigma)_{\langle i_s \rangle}$ and $\widetilde{\mathsf{m}}_{[s,t]} := ((\widetilde{\mathsf{m}}_\sigma)_{\langle i_s \rangle})_{\langle i_t \rangle}$. By (4.1), we have $\sigma_0 := (\sigma \setminus \{\widetilde{\mathsf{m}}_\sigma\}) = \{ \widetilde{\mathsf{m}}_{[s]} \mid 1 \leq s \leq q \}$ and $\tau = \{ \widetilde{\mathsf{m}}_{[r,s]} \mid 1 \leq s \leq q, s \neq r \} \cup \{\widetilde{\mathsf{m}}_{[r]}\}$. We can inductively construct a gradient path $\sigma_0 \to \sigma_1 \to \cdots \to \sigma_t \to \cdots \sigma_{2(q-r+1)r-2}$ as follows. Write $t = 2pr + \lambda$ with $t \neq 0$, $0 \leq p \leq q - r$, and $0 \leq \lambda < 2r$. For $0 < t \leq 2(q-r)$, we set

$$\sigma_t = \begin{cases} \sigma_{t-1} \cup \{ \widetilde{\mathsf{m}}_{[q-p,s]} \} & \text{if } \lambda = 2s-1 \text{ for some } 1 \leq s \leq r; \\ \sigma_{t-1} \setminus \{ \widetilde{\mathsf{m}}_{[q-p+1,s]} \} & \text{if } \lambda = 2s \text{ for some } 0 < s < r; \\ \sigma_t \setminus \{ \widetilde{\mathsf{m}}_{[q-p+1]} \} & \text{if } \lambda = 0, \end{cases}$$

where we set $\widetilde{\mathsf{m}}_{[q+1,s]} = \widetilde{\mathsf{m}}_{[s]}$ for all s. In the case $\widetilde{\mathsf{m}}_{[s,t]} = \widetilde{\mathsf{m}}_{[s+1,t]}$, it seems to cause a problem, but skipping the corresponding part of path, we can avoid the problem. Since $r \in B(\widetilde{F}_\sigma, \widetilde{\mathsf{m}}_\sigma)$, we have $\widetilde{\mathsf{m}}_{[s,r]} = \widetilde{\mathsf{m}}_{[r,s]}$ for all $s > r$ by Lemma 3 (iv). Hence

$$\sigma_{2(q-r)} = \{ \widetilde{\mathsf{m}}_{[r+1,s]} \mid 1 \leq s < r \} \cup \{ \widetilde{\mathsf{m}}_{[r]} \} \cup \{ \widetilde{\mathsf{m}}_{[r,s]} \mid r < s \leq q \}.$$

Now for s with $0 < s \leq r-1$, set σ_t with $2(q-r)r < t \leq 2(q-r+1)r - 2$ to be $\sigma_{t-1} \cup \{ \widetilde{\mathsf{m}}_{[r,s]} \}$ if s is odd and otherwise $\sigma_{t-1} \setminus \{ \widetilde{\mathsf{m}}_{[r+1,s]} \}$. Then we have $\sigma_{2(q-r+1)r-2} = \tau$, and the gradient path $\sigma \rightsquigarrow \tau$.

The uniqueness of the path follows from elementally (but lengthy) argument. ☐

Sketch of Proof of Theorem 15. Recall that there is the one-to-one correspondence between the critical cells $\sigma \subset G(\widetilde{I})$ and the admissible pairs $(\widetilde{F}_\sigma, \widetilde{\mathsf{m}}_\sigma)$. Hence, for each q, we have the isomorphism $\widetilde{Q}_q \to \widetilde{P}_q$ induced by $e(\sigma) \longmapsto e(\widetilde{F}_\sigma, \widetilde{\mathsf{m}}_\sigma)$.

By Proposition 16, if we forget "coefficients" (more precisely, ± 1), the differential map of $\widetilde{Q}_\bullet$ and that of $\widetilde{P}_\bullet$ are compatible with the maps $e(\sigma) \longmapsto e(\widetilde{F}_\sigma, \widetilde{\mathsf{m}}_\sigma)$. So it is enough to check the equality of the coefficients. But it follows from direct computation. $\square$

Corollary 17 ([12, Corollary 5.12]). *The free resolution $\widetilde{P}_\bullet \otimes_{\widetilde{S}} \widetilde{S}/(\Theta)$ (resp. $\widetilde{P}_\bullet \otimes_{\widetilde{S}} \widetilde{S}/(\Theta')$) of S/I (resp. T/I^{sq}) is also a cellular resolution supported by X_A. In particular, these resolutions are Batzies-Welker type.*

We say a CW complex is *regular*, if for all i the closure $\overline{\sigma}$ of any i-cell σ is homeomorphic to an i-dimensional closed ball, and $\overline{\sigma} \setminus \sigma$ is the closure of the union of some $(i-1)$-cells. This is a natural condition especially in combinatorics.

Mermin [10] (see also Clark [6]) showed that the Eliahou-Kervaire resolution is cellular and supported by a regular CW complex. Hence it is a natural question whether the CW complex X_A supporting our $\widetilde{P}_\bullet$ is regular. (Since the discrete Morse theory is an "existence theorem" and X_A is not unique, the correct statement might be "*can be* regular".)

Theorem 18 ([13]). *The CW complex X_A of Theorem 15 is regular. In particular, our resolution $\widetilde{P}_\bullet$ is supported by a regular CW complex.*

Sketch of Proof. We define a finite poset P_A as follows:

(i) As the underlying set, $P_A = \{\text{the cells of } X_A\} \cup \{\hat{0}\}$. Here $\hat{0}$ is the least element.
(ii) For cells σ and τ of X_A, $\sigma \succeq \tau$ in P_A if and only if the closure of σ contains τ.

It suffices to show that P_A is a *CW poset* in the sense of [4], and we can use [4, Proposition 5.5]. We can easily check that P_A satisfies the following condition.

- For $\sigma, \tau \in P_A$ with $\sigma \succ \tau$ and $\mathrm{rank}(\sigma) = \mathrm{rank}(\tau) + 2$, there are exactly two elements between σ and τ.

Now it remains to show that the interval $[\,\hat{0},\, \sigma\,]$ is shellable for all σ, but we can imitate the argument of Clark [6]. In fact, $[\,\hat{0},\, \sigma\,]$ is *EL shellable* in the sense of [3]. $\square$

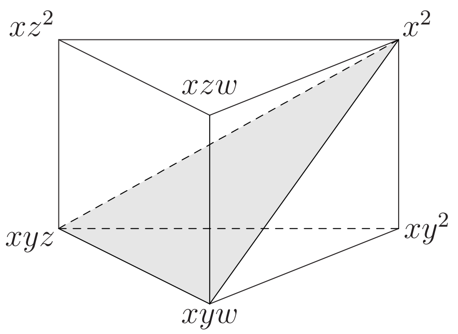


FIGURE 1

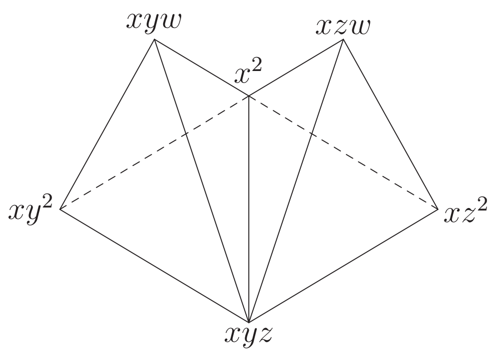


FIGURE 2

Example 19. Consider the Borel fixed ideal $I = (x^2, xy^2, xyz, xyw, xz^2, xzw)$. Then $\mathsf{b\text{-}pol}(I) = (x_1x_2, x_1y_2y_3, x_1y_2z_3, x_1y_2w_3, x_1z_2z_3, x_1z_3w_3)$, and easy computation shows that the CW complex X_A, which supports our resolutions $\widetilde{P}_\bullet$ of $\widetilde{S}/\widetilde{I}$ and $\widetilde{P}_\bullet \otimes_{\widetilde{S}} \widetilde{S}/(\Theta)$ of S/I, is the one illustrated in Figure 1. The complex consists of a square pyramid and a tetrahedron glued along trigonal faces of each. For a Borel fixed ideal generated in one degree, any face of the Nagel-Reiner CW complex is a product of several simplices. Hence a square pyramid can not appear in the case of Nagel and Reiner.

We remark that the Eliahou-Kervaire resolution of I is supported by the CW complex illustrated in Figure 2. This complex consists of two tetrahedrons glued along edges of each. These figures show visually that the description of the Eliahou-Kervaire resolution and that of ours are really different.

Anyway, the minimal free resolution of I is of the form $0 \to S^2 \to S^8 \to S^{11} \to S^6 \to 0$.

Theorem 20. *If S/I is Cohen-Macaulay, the underlying space of the regular CW complex X_A is homeomorphic to a closed ball of dimension* $\mathrm{codim}(I) - 1$.

The prove Theorem 20, we show and use the fact that the order complex of the poset P_A is *constructible* (if S/I is Cohen-Macaulay). We also remark that the converse of Theorem 20 does not hold. In fact, S/I is not Cohen-Macaulay in Example 19, while the underlying space of X_A is homeomorphic to a ball.

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