PURITY OF RECIPROCITY SHEAVES AND MOTIVES WITH MODULUS

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Abstract. This is a survey on recent development of theory of *motives with modulus*, which generalizes Voevodsky's theory of motives to a non-homotopy invariant framework.

Contents

1. Conjectural theory of motives

In 1980's Beilinson [1] and Deligne [8] independently formulated conjectures on theory of (mixed) motives:

Conjecture 1.1. *Fix a base field k. There exists an abelian tensor category $\mathcal{MM}_k$ of* motives over k *enjoying the following properties.*

(1) *$\mathcal{MM}_k$ contains Grothendieck's category $\mathcal{M}_k$ of* pure motives over k *as the full subcategory of semi-simple objects.*
(2) *Let* **Sch** *be the category of schemes separated of finite type over k. Then there exist a functor*[1] [2],

$$\mathbf{Sch} \to D(\mathcal{MM}_k)\ ;\ X \to M(X),$$

where $D(\mathcal{MM}_k)$ is the derived category of (unbounded[3]*) complexes in $\mathcal{MM}_k$ and natural isomorphisms for $X \in$* **Sch** *and*

[1]This is a covariant variant of the original formulation of Beilinson who used a contravariant functor.

[2]$M(X)$ is called the motive of $X \in \mathbf{Sch}$.

[3]In Beilinson's original version, bounded complexes are used.

$n \in \mathbb{Z}_{\geq 0}$

$$\mathrm{Hom}_{D(\mathcal{MM}_k)}(M(X), \mathbb{Z}(n)[i]) \otimes \mathbb{Q} \simeq K_{2n-i}(X)^{(n)}, \tag{1.1}$$

where $\mathbb{Z}(n)$ is a distinguished object of $D(\mathcal{MM}_k)$ called the Tate object [4] *and $K_j(X)^{(n)}$ is the weight n eigenspace for Adams operations on the algebraic K-group $K_j(X)$ of X.*

The abelian groups

$$H^i_{\mathcal{M}}(X, \mathbb{Z}(n)) := \mathrm{Hom}_{D(\mathcal{MM}_k)}(M(X), \mathbb{Z}(n)[i]) \ \text{ for } \ X \in \mathbf{Sch}$$

should form the universal cohomology theory on **Sch**, and it is called *motivic cohomology.*

Bloch [2] gave a cycle-theoretic description of motivic cohomology (at least for smooth k-schemes). He introduced higher Chow groups $CH^n(X, j)$ for $X \in \mathbf{Sch}$ and $n, j \in \mathbb{Z}_{\geq 0}$ as a generalization of Chow groups[5], and proved that the Chern class map induces an isomorphism

$$K_j(X)^{(n)} \simeq CH^n(X, j) \ \text{ for } \ X \in \mathbf{Sm}.$$

This leads us to the following.

Conjecture 1.2. *For X smooth over k, there is a natural isomorphism*

$$H^i_{\mathcal{M}}(X, \mathbb{Z}(n)) \simeq CH^n(X, 2n - i).$$

The category $\mathcal{MM}_k$ has not yet been constructed while Voevodsky [25] in 1990's brought about a big progress in theory of motives by constructing a triangulated category $\mathbf{DM}^{\mathrm{eff}}$ which have the properties expected for $D(\mathcal{MM}_k)$ at least restricted to smooth schemes over k [6].

2. Voevodsky's theory of motives

We recall Voevodsky's construction of triangulated categories $\mathbf{DM}^{\mathrm{eff}}$ of motives over k in [25].

Let **Sm** be the category of smooth separated k-schemes and **Cor** be the category which has the same objects as **Sm** and whose morphisms are *finite correspondences*: For $X, Y \in \mathbf{Sm}$, we define $\mathbf{Cor}(X, Y)$ to be the free abelian group on the set of integral closed subscheme $Z \subset X \times Y$ which are finite and surjective on a connected component of X.

Let **PST** be the category of additive contravariant functors from **Cor** to the category **Ab** of abelian groups. An object $F \in \mathbf{PST}$ is called

[4] See Remark 2.5 for the definition.

[5] $CH^n(X, 0)$ coincides with the Chow group $CH^n(X)$ of algebraic cycles of codimension n on X modulo rational equivalence

[6] Levine and Hanamura made independent constructions and all constructions are now known to be equivalent.

a *preshseaf with transfers*. Note that **PST** is an abelian category. For $X \in \mathbf{Sm}$ let $\mathbb{Z}_{\mathrm{tr}}(X) \in \mathbf{PST}$ be represented by X by Yoneda: For $Y \in \mathbf{Sm}$, we have $\mathbb{Z}_{\mathrm{tr}}(X)(Y) = \mathbf{Cor}(Y, X)$.

Recall that for $X \in \mathbf{Sm}$ the small Nisnevich site X_{Nis} is the category of étale morphisms $Y \to X$ equipped with the class of *Nisnevich coverings*: An étale covering $\{p_i : U_i \to X\}_{i\in I}$ is a Nisnevich covering if for all $x \in X$, there exists $i \in I$ and $y \in U_i$ such that $p_i(y) = x$ and $k(x) \simeq k(y)$.

For $F \in \mathbf{PST}$ and $X \in \mathbf{Sm}$ let F_X be the presheaf on X_{Nis} induced by F. Let **NST** be the full subcategory of **PST** consisting of $F \in \mathbf{PST}$ such that F_X is a sheaf on X_{Nis}. One easily sees $\mathbb{Z}_{\mathrm{tr}}(X) \in \mathbf{NST}$ for $X \in \mathbf{Sm}$.

Theorem 2.1. *(Voevodsky)*

(1) *The natural inclusion* $\mathbf{NST} \to \mathbf{PST}$ *admits an exact left adjoint* $a_{\mathrm{Nis}} : \mathbf{PST} \to \mathbf{NST}$ *such that for all* $X \in \mathbf{Sm}$, $(a_{\mathrm{Nis}}F)_X$ *is the Nisnevich sheafication of* F_X.
(2) *The category* **NST** *is Grothendieck abelian*[7].
(3) *Let* $D(\mathbf{NST})$ *be the derived category of unbounded complexes in* **NST**. *For* $F \in D(\mathbf{NST})$ *and* $X \in \mathbf{Sm}$, *there is a canonical isomorphism*

$$H^i(X_{\mathrm{Nis}}, F_X) \simeq \mathrm{Hom}_{D(\mathbf{NST})}(\mathbb{Z}_{\mathrm{tr}}(X), F[i]).$$

In what follows we write $F_{\mathrm{Nis}} = a_{\mathrm{Nis}}F$ *for* $F \in \mathbf{MPST}$.

Definition 2.2. Define $\mathbf{DM}^{\mathrm{eff}}$ as the localization of $D(\mathbf{NST})$ by the localising subcategory generated by the complexes

$$\mathbb{Z}_{\mathrm{tr}}(X \times \mathbf{A}^1) \to \mathbb{Z}_{\mathrm{tr}}(X) \ \text{ for } \ X \in \mathbf{Sm},$$

where the maps are induced by the projections $X \times \mathbf{A}^1 \to X$.

By the definition we have a functor

$$M : \mathbf{Sm} \to \mathbf{DM}^{\mathrm{eff}}, \tag{2.1}$$

which maps $X \in \mathbf{Sm}$ to $\mathbb{Z}_{\mathrm{tr}}(X) \in \mathbf{NST}$ considered as a complex by putting it in degree 0.[8] By a general result of Neeman [18], the localization functor $\pi : D(\mathbf{NST}) \to \mathbf{DM}^{\mathrm{eff}}$ admits a fully faithful right adjoint $j : \mathbf{DM}^{\mathrm{eff}} \to D(\mathbf{NST})$.

[7] see [24] for a definition of Grothendieck abelian categories. An important property is that it admits enough injectives.

[8] This should corresponds to the functor in Conjecture 1.1(2).

In order to state a fundamental theorem in Voevodsky's theory, we need introduce a construction due to Suslin[9].

Definition 2.3. For $X \in \mathbf{Sm}$ define $\widetilde{C}_n(X) \in \mathbf{NST}$ by

$$\widetilde{C}_n(X)(Y) = \mathbf{Cor}(Y \times (\mathbf{A}^1)^n, X).$$

We then put

$$C_n(X) = \mathrm{Coker}\,\big(\oplus\,(p_i^n)^* : \bigoplus_{1\le i\le n} \widetilde{C}_{n-1}(X) \to \widetilde{C}_n(X)\big),$$

where $(p_i^n)^*$ is induced by the projection $p_i^n : (\mathbf{A}^1)^n \to (\mathbf{A}^1)^{n-1}$ removing the i-th factor. For $n \geq 1$, $i \in \{1, \dots, n\}$ and $\varepsilon \in \{0, 1\}$, define $\delta_{i,\varepsilon}^n : (\mathbf{A}^1)^{n-1} \to (\mathbf{A}^1)^n$ to be the map inserting ε at the i-th component. It induce maps in **NST**

$$\delta_{i,\varepsilon}^{n,*} : C_n(X) \to C_{n-1}(X)$$

and define coboundary maps by

$$d_n := \sum_{i=1}^{n} (-1)^{i-1}(\delta_{i,0*}^n - \delta_{i,\infty*}^n) : C_n(X) \to C_{n-1}(X).$$

Thus we obtain a chain complex $C_*(X)$ in **NST**.

The following result is a key to Voevodsky's theory of motives.

Theorem 2.4. *(Voevodsky) Assume k is perfect. For $X, Y \in \mathbf{Sm}$, there is a canonical isomorphism for $j \in \mathbb{Z}$:*

$$\mathrm{Hom}_{\mathbf{DM}^{\mathrm{eff}}}(M(Y), M(X)[j]) \simeq \mathbb{H}^j(Y_{\mathrm{Nis}}, C_*(X)_Y). \tag{2.2}$$

where $C_*(X)_Y$ is the complex of sheaves on Y_{Nis} induced by $C_*(X)$.

Remark 2.5. Let $\mathbb{Z}(1) = \widetilde{M}(\mathbf{P}^1)[-2] \in \mathbf{DM}^{\mathrm{eff}}$, where $\widetilde{M}(\mathbf{P}^1)$ is the kernel of the splitting epimorphism $M(\mathbf{P}^1) \to M(\mathrm{Spec}(k))$ induced by the projection $\mathbf{P}^1 \to \mathrm{Spec}(k)$. Define the Tate object $\mathbb{Z}(n) = \mathbb{Z}(1)^{\otimes n}$. Voevodsky defines *motivic cohomology* as

$$H^i_{\mathcal{M}}(X, \mathbb{Z}(n)) := \mathrm{Hom}_{\mathbf{DM}^{\mathrm{eff}}}(M(X), \mathbb{Z}(n)[i]) \ \text{ for } \ X \in \mathbf{Sm}. \tag{2.3}$$

Using moving lemmas of algebraic cycles from [9], [23] and [2], Voevodsky deduced from Theorem 2.4 a natural isomorphism

$$H^i_{\mathcal{M}}(X, \mathbb{Z}(n)) \simeq \mathrm{CH}^n(X, 2n-i) \tag{2.4}$$

and thus proved Conjecture 1.2.

[9]This is a cubical variant of the original definition of Suslin who used a simplicial version.

Remark 2.6. Assuming $\mathrm{ch}(k) = 0$, the funcotor (2.1) is extended to a functor [10]

$$M : \mathbf{Sch} \to \mathbf{DM}^{\mathrm{eff}}.$$

Hence we can define motivic cohomology for $X \in \mathbf{Sch}$ by the formula (2.3). However this is not expected to serve as motivic cohomology envisioned in Conjecture 1.1 (See §3).

The proof of Theorem 2.4 is quite involved and long. It is based on Voevosdky's theory of *homotopy invariant presheaves with transfers*.

Definition 2.7. An object $F \in \mathbf{PST}$ is called homotopy invariant if for any $X \in \mathbf{Sm}$, the projection $X \times \mathbf{A}^1 \to X$ induces an isomorphism $F(X) \simeq F(X \times \mathbf{A}^1)$. Let $\mathbf{HI} \subset \mathbf{PST}$ be the full subcategory of homotopy invariant presheaves with transfers.

It turns out that $\mathbf{HI} \subset \mathbf{PST}$ is an abelian subcategory.

Now we state Voevodsky's fundamental results on $\mathbf{HI}$ [11]. Theorem 2.4 is deduced from them by general arguments in homological algebra.

Theorem 2.8. *Let $F \in \mathbf{HI}$. Then*

(1) $F_{\mathrm{Nis}} \in \mathbf{HI}$.
(2) *For a dense open immersion $U \hookrightarrow X$ in $\mathbf{Sm}$, the restriction $F_{\mathrm{Nis}}(X) \to F_{\mathrm{Nis}}(U)$ is injective.*

To state the second main result on $\mathbf{HI}$, we introduce some notations. For $X \in \mathbf{Sm}$ and $n \in \mathbb{Z}_{\geq 0}$, let $X^{(n)}$ be the set of points $x \in X$ such that the closure of x in X is of dimension n. For $F \in \mathbf{PST}$ and $n \in \mathbb{Z}_{>0}$ and $S \in \mathbf{Sm}$, define

$$F_{-n}(S) = \mathrm{Coker}\,\Big(\bigoplus_{1\leq i\leq n} F((\mathbf{G}_m)^{i-1}\times\mathbf{A}^1\times(\mathbf{G}_m)^{n-i}\times S) \to F((\mathbf{G}_m)^n\times S)\Big),$$

where $\mathbf{A}^1$ is the affine line over k and $\mathbf{G}_m = \mathbf{A}^1 - 0$. This gives an endofunctor $\mathbf{PST} \to \mathbf{PST}; F \to F_{-n}$.

Theorem 2.9. *Assume k is perfect. For $F \in \mathbf{HI} \cap \mathbf{NST}$, $X \in \mathbf{Sm}$ and $x \in X^{(n)}$ with $n \in \mathbb{Z}_{>0}$, we have*

$$H^i_x(X_{\mathrm{Nis}}, F_X) = 0 \ \ \text{for} \ \ i \neq n, \tag{2.5}$$

and there exists a natural isomorphism:

$$\theta_x : F_{-n}(x) \simeq H^n_x(X_{\mathrm{Nis}}, F_X). \tag{2.6}$$

[10] Shane Kelly proved that such an extension exists also in case $\mathrm{ch}(k) > 0$ if one inverts the exponential characteristic of k for $\mathbf{DM}^{\mathrm{eff}}$.

[11] Here we state the results for Nisnevich sheaves. Voevodsky proves also the similar results for Zariski sheaves. In the application to theory of motives, only the results for Nisnevich sheaves are used

To state the last result on **HI**, we note the following fact: For $F \in$ **NST** and $i \in \mathbb{Z}_{\geq 0}$, the presheaf $X \to H^i(X_{\mathrm{Nis}}, F_X)$ on **Sm** is considered as an object H^i_F of **PST**.[12]

Theorem 2.10. *Assume k is perfect. For $F \in$ **HI** $\cap$ **NST**, we have $H^i_F \in$ **HI**.*

3. Non-homotopy invariant theory of motives

As we have seen, Voevodsky's theory of motives is based on homotopy invariance (see Definition 2.7). This implies that the invariants for $X \in$ **Sch** arising from his category $\mathbf{DM}^{\mathrm{eff}}$ such as motivic cohomology defined as (2.3) should have this property (see Remark 2.6). The homotopy invariance implies also the nil-invariance, which means the invariance when X is replaced by its reduced part X_{red}. However algebraic K-theory does not have these properties: For $X \in$ **Sch**, $K_n(X) \to K_n(X \times \mathbf{A}^1)$ is not an isomorphism in general unless $X \in$ **Sm**, and $K_n(X)$ is not nil-invariant in general, either.

On the other hand there are phenomena which motivates us to extend theory of motives to non-homotopy invariant (and non-nil-invariant) framework. One of them is the works on Grothendieck's variational Hodge conjecture by Bloch-Esnault-Kerz [6] and Morrow [16]. Another is the work of Kerz-Saito [15] on wildly ramified higher dimensional class field theory. Here we give a brief explanation on the latter.

For $X \in$ **Sm** choose a dense open immersion $X \hookrightarrow \overline{X}$ such that $\overline{X}$ is integral and proper over k and that $\overline{X} - X$ is the support of some $D \in \mathrm{Div}(\overline{X})^+$, where $\mathrm{Div}(\overline{X})^+$ denotes the monoid of effective Cartier divisors on $\overline{X}$. In [15], for $D \in \mathrm{Div}(\overline{X})^+$, the *Chow group* $\mathrm{CH}_0(\overline{X}|D)$ *of zero-cycles with modulus* is introduced as a generalization of the Chow group $\mathrm{CH}_0(X)$ of zero-cycles on X. It is defined as a quotient of the group of zero-cycles on X by an equivalence relation given by rational functions on curves on X which satisfies a certain modulus condition with respect to D. Then, putting

$$C(X) := \varprojlim_{D \in \mathrm{Div}(\overline{X})^+} \mathrm{CH}_0(\overline{X}|D)$$

where D ranges over all elements of $\mathrm{Div}(\overline{X})^+$ such that $|D| = \overline{X} - X$, one can show that $C(X)$ is independent of the choice of $\overline{X}$ and hence is an invariant of $X \in$ **Sm**.

[12] This is a consequence of Theorem 2.1(3).

Theorem 3.1. *([15]) Assume k is finite and $\mathrm{ch}(k) \neq 2$. Then there exists a canonical isomorphism*

$$C(X) \simeq W^{ab}(X),$$

where the right hand side denote the abelian Weil group of X, namely the subgroup of the abelian fundamental group $\pi_1^{ab}(X)$ consisting of elements whose images in the absolute Galois group of k are integral powers of the Frobenius substitution.

Recall that $W^{ab}(X)$ carries information on wild ramification of abelian coverings of X along $\overline{X} - X$ which is known to be not homotopy invariant. Hence there is no hope to recover $C(X)$ (and also $\mathrm{CH}_0(\overline{X}|D)$) from Voevodsky's category $\mathbf{DM}^{\mathrm{eff}}$.

4. Reciprocity sheaves

In order to extend Voevodsky's paradigm to a non-homotopy invariant framework, we use a new full abelian subcategory $\mathbf{RSC} \subset \mathbf{PST}$ of *reiprocity presheaves*, which was introduced by Kahn-Saito-Yamazaki in [10] and [14].[13] It contains $\mathbf{HI}$ and many objects of $\mathbf{PST}$ which are not in $\mathbf{HI}$, such as the sheaf of the additive group $\mathbf{G}_a$ and the sheaf Ω^i of Kähler differential forms.

First we recall the following (see [17, Lem. 2.16]).

Lemma 4.1. *A given $F \in \mathbf{PST}$ is in $\mathbf{HI}$ if and only if for any $X \in \mathbf{Sm}$ and $a \in F(X)$, the map $a : \mathbb{Z}_{\mathrm{tr}}(X) \to F$ in $\mathbf{PST}$ associated to a by the Yoneda functor, factors through the map $\mathbb{Z}_{\mathrm{tr}}(X) \to h_0(X)$. Here $h_0(X)$ is a quotient of $\mathbb{Z}_{\mathrm{tr}}(X)$ in $\mathbf{PST}$ defined by*

(4.1)

$$\begin{aligned} h_0(X)(Y) &= \mathrm{Coker}\left(\mathbb{Z}_{\mathrm{tr}}(X)(Y \times \mathbf{A}^1) \xrightarrow{i_0^*-i_1^*} \mathbb{Z}_{\mathrm{tr}}(X)(Y)\right) \quad (Y \in \mathbf{Sm}) \\ &= \mathrm{Coker}\left(\mathbf{Cor}(Y \times \mathbf{A}^1, X) \xrightarrow{i_0^*-i_1^*} \mathbf{Cor}(Y, X)\right), \end{aligned}$$

where i_ε^* for $\varepsilon = 0, 1$ is the pullback by the section $i_\varepsilon : \mathrm{Spec}(k) \to \mathbf{A}^1$.

The key idea to define $\mathbf{RSC}$ is to introduce bigger quotients $h_0(\mathcal{X})$ of $\mathbb{Z}_{\mathrm{tr}}(X)$ associated to $\mathcal{X} \in \mathbf{MSm}(X)$, where $\mathbf{MSm}(X)$ is the set of pairs $\mathcal{X} = (\overline{X}, X_\infty)$ of locally integral proper schemes $\overline{X}$ over k and effective Cartier divisors X_∞ on $\overline{X}$ such that $X = \overline{X} - |X_\infty| \in \mathbf{Sm}$. It is defined by

(4.2)

$$h_0(\mathcal{X})(Y) = \mathrm{Coker}\left(\mathbf{MCor}(Y \times \overline{\square}, \mathcal{X}) \xrightarrow{i_0^*-i_1^*} \mathbf{Cor}(Y, X)\right) \quad (Y \in \mathbf{Sm}),$$

[13]The category $\mathbf{RSC}$ is denoted by $\mathbf{SCRec}$ in [14]. It is slightly smaller than the category $\mathbf{Rec}$ studied in [10].

where $\mathbf{MCor}(Y\times\overline{\square},\mathcal{X})$ is the subgroup of $\mathbf{Cor}(Y\times\mathbf{A}^1,X)$ generated by integral closed subschemes $Z\subset Y\times\mathbf{A}^1\times X$ which are finite and surjective over a component of $Y\times\mathbf{A}^1$ and satisfies the following additional condition (called modulus condition): Let $\overline{Z}\subset Y\times\mathbf{P}^1\times\overline{X}$ be the closure of Z and $\overline{Z}^N$ be its normalization with the projections $p:\overline{Z}^N\to\overline{X}$ and $q:\overline{Z}^N\to\mathbf{P}^1$. Then we have the following inequality of Caritier divisors on $\overline{Z}^N$

$$q^*\infty\geq p^*X_\infty.$$

Definition 4.2. Let $F\in\mathbf{PST}$ and $X\in\mathbf{Sm}$. We say F *has reciprocity* if for any $X\in\mathbf{Sm}$ and $a\in F(X)$, there exists $\mathcal{X}\in\mathbf{MSm}(X)$ such that the map $a:\mathbb{Z}_{\mathrm{tr}}(X)\to F$ associated to $a\in F(X)$ factors through $h_0(\mathcal{X})$. We write $\mathbf{RSC}\subset\mathbf{PST}$ for the full subcategory of reciprocity presheaves.

By definition $h_0(X)$ is a quotient of $h_0(\mathcal{X})$ for any $\mathcal{X}\in\mathbf{MSm}(X)$ so that $\mathbf{HI}\subset\mathbf{RSC}$. It turns out that $\mathbf{RSC}$ is an abelian category closed under subobjects and quotients in $\mathbf{PST}$.

We now state our main results for reciprocity sheaves. The first result generalizes Theorem 2.8.

Theorem 4.3. *([22]) Let $F\in\mathbf{RSC}$. Then*

(1) $F_{\mathrm{Nis}}\in\mathbf{RSC}$.
(2) *For a dense open immersion $U\hookrightarrow X$ in $\mathbf{Sm}$, the restriction $F_{\mathrm{Nis}}(X)\to F_{\mathrm{Nis}}(U)$ is injective.*

Take $X\in\mathbf{Sm}$ and $x\in X$. Using the perfectness of k, one can show that there is an isomorphism

$$\varepsilon:X^h_{|x}\simeq\operatorname{Spec}K\{t_1,\dots,t_n\}.$$

where $X^h_{|x}$ be the henselization of X at x and $K=k(x)$ and $(t_1,\dots,t_n)$ is a system of regular parameter of X at x, and $K\{x_1,\dots,x_d\}$ is the henselization of $K[x_1,\dots,x_d]$ at $(t_1,\dots,t_n)$. The second result generalizes Theorem 2.9.

Theorem 4.4. *([22]) Let $F\in\mathbf{RSC}\cap\mathbf{NST}$. For $X\in\mathbf{Sm}$ and $x\in X^{(n)}$ with $n\in\mathbb{Z}_{>0}$, we have*

$$H^i_x(X_{\mathrm{Nis}},F_X)=0\quad\text{for}\quad i\neq n,\tag{4.3}$$

and there exists an isomorphism depending on ε:

$$\theta_\varepsilon:F_{-n}(x)\simeq H^n_x(X_{\mathrm{Nis}},F_X).\tag{4.4}$$

The last result is a variant of Theorem 2.10 for $\mathbf{RSC}$.

Theorem 4.5. *([22]) Assume* $\mathrm{ch}(k) = 0$ *or the following condition:*

(RS) *For any pair (X, D) of a locally integral scheme X and an effective Cartier divisor D on X such that $X - |D| \in \mathbf{Sm}$ and is dense in X, there exists a proper birational map $\pi : X' \to X$ such that $X' \in \mathbf{Sm}$ and $\pi^{-1}(D)_{red}$ is a simple normal crossing divisor and that π is an isomorphism over $X - |D|$.*

Then, for $F \in \mathbf{RSC} \cap \mathbf{NST}$, we have $H^i_F \in \mathbf{RSC}$.

The above theorems give an affirmative answer to [10, Conjecture 1] except the part on the coincidence of Nisnevich and Zariski cohomology.

5. Theory of motives with modulus

In this section we explain the construction of Kahn-Saito-Yamazaki [14] of a new triangulated category of *motives with modulus*, which extends Voevodsky's construction of his category of motives to a non-homotopy invariant setting. For this we first need generalize the theory of preshseaves with transfers to *preshseaves with transfers with modulus*.

Definition 5.1. (see [14, Definitions 1.1 and 1.8]) The category $\underline{\mathbf{M}}\mathbf{Cor}$ of *modulus pairs* has objects $\mathcal{X} = (\overline{X}, X_\infty)$, where $\overline{X} \in \mathbf{Sch}$ is locally integral and X_∞ is an effective Cartier divisor on $\overline{X}$ such that $\overline{X} - |X_\infty| \in \mathbf{Sm}$ and is dense in $\overline{X}$ (The case $|X_\infty| = \emptyset$ is allowed). For $\mathcal{X} = (\overline{X}, X_\infty), \mathcal{X}' = (\overline{X}', X'_\infty) \in \underline{\mathbf{M}}\mathbf{Cor}$ with $X = \overline{X} - |X_\infty|$ and $X' = \overline{X}' - |X'_\infty|$, the morphism group $\underline{\mathbf{M}}\mathbf{Cor}(\mathcal{X}', \mathcal{X})$ is the subgroup of $\mathbf{Cor}(X', X)$ freely generated by integral closed subschemes $Z \subset X' \times X$ finite and surjective over a connected component of X' satisfying the following additional condition: Let $\bar{Z}^N$ be the normalization of the closure $\overline{Z}$ of Z in $\overline{X}' \times \overline{X}$ with $p : \bar{Z}^N \to \overline{X}$ and $q : \bar{Z}^N \to \overline{X}'$ the projections. Then $\overline{Z}$ is proper over $\overline{X}'$ and we have the inequality $q^* X'_\infty \geq p^* X_\infty$ of Cartier divisors on $\overline{Z}^N$.

We call $\mathcal{X}$ *proper* if $\overline{X}$ is proper over k and let $\mathbf{MCor}$ denote the full subcategory of $\underline{\mathbf{M}}\mathbf{Cor}$ whose objects are proper modulus pairs.

Definition 5.2. Let $\underline{\mathbf{M}}\mathbf{PST}$ (resp. $\mathbf{MPST}$) be the abelian category of contravariant additive functors $\underline{\mathbf{M}}\mathbf{Cor} \to \mathbf{Ab}$ (resp. $\mathbf{MCor} \to \mathbf{Ab}$). For $\mathcal{X} \in \underline{\mathbf{M}}\mathbf{Cor}$ (resp. $\mathcal{X} \in \mathbf{MCor}$) let $\mathbb{Z}_{\mathrm{tr}}(\mathcal{X}) \in \underline{\mathbf{M}}\mathbf{PST}$ (resp. $\mathbb{Z}_{\mathrm{tr}}(\mathcal{X}) \in \underline{\mathbf{M}}\mathbf{PST}$) be the object represented by $\mathcal{X}$.

We have a functor

$$\omega : \mathbf{MCor} \to \mathbf{Cor} \; ; \; (\overline{X}, X_\infty) \to \overline{X} - |X_\infty|,$$

and a pair of adjunction

$$\mathbf{MPST} \underset{\tau^*}{\overset{\tau_!}{\rightleftarrows}} \underline{\mathbf{M}}\mathbf{PST}, \quad \mathbf{MPST} \underset{\omega^*}{\overset{\omega_!}{\rightleftarrows}} \mathbf{PST}, \tag{5.1}$$

where τ^* is induced by the natural inclusion $\tau : \mathbf{MCor} \to \underline{\mathbf{M}}\mathbf{Cor}$ and $\tau_!$ is its left Kan extension, and ω^* is induced by ω and $\omega_!$ is its left Kan extension.

Definition 5.3. For $F \in \underline{\mathbf{M}}\mathbf{PST}$ and $\mathcal{X} = (\overline{X}, X_\infty) \in \underline{\mathbf{M}}\mathbf{Cor}$ write $F_{\mathcal{X}}$ for the presheaf on the Nisnevich site $\overline{X}_{\mathrm{Nis}}$ over $\overline{X}$ given by $U \to F(\mathcal{X}_U)$ for $U \to \overline{X}$ étale, where $\mathcal{X}_U = (U, X_\infty \times_{\overline{X}} U) \in \underline{\mathbf{M}}\mathbf{Cor}$. Let $\underline{\mathbf{M}}\mathbf{NST} \subset \underline{\mathbf{M}}\mathbf{PST}$ be the full subcategory of such $F \in \underline{\mathbf{M}}\mathbf{PST}$ that $F_{\mathcal{X}}$ are Nisnevich sheaves for all $\mathcal{X} \in \underline{\mathbf{M}}\mathbf{Cor}$. Let $\mathbf{MNST} \subset \mathbf{MPST}$ be the full subcategory of such $F \in \mathbf{MPST}$ that $\tau_! F \in \underline{\mathbf{M}}\mathbf{NST}$.

The following variant of Theorem 2.1 is proved in [14] [14].

Theorem 5.4. (1) *The natural inclusion* $\mathbf{MNST} \to \mathbf{MPST}$ *admits an exact left adjoint* $a_{\mathrm{Nis}} : \mathbf{MPST} \to \mathbf{MNST}$ *such that for all* $\mathcal{X} \in \mathbf{MCor}$, $(a_{\mathrm{Nis}}F)_{\mathcal{X}}$ *is the Nisnevich sheafication of* $F_{\mathcal{X}}$.
(2) *The category* $\mathbf{MNST}$ *is Grothendieck abelian.*
(3) *Let* $D(\mathbf{MNST})$ *be the derived category of (unbounded) complexes in* $\mathbf{MNST}$*. For* $F \in D(\mathbf{MNST})$ *and* $\mathcal{X} = (\overline{X}, X_\infty) \in \mathbf{MCor}$*, there is a canonical isomorphism*

$$\mathbb{H}^i(\overline{X}_{\mathrm{Nis}}, F_{\mathcal{X}}) \simeq \mathrm{Hom}_{D(\mathbf{MNST})}(\mathbb{Z}_{\mathrm{tr}}(\mathcal{X}), F[i]).$$

In what follows we write $F_{\mathrm{Nis}} = a_{\mathrm{Nis}}F$ *for* $F \in \mathbf{MPST}$*.*

Definition 5.5. An object $F \in \mathbf{MPST}$ is called $\overline{\square}$-invariant if $F(\mathcal{X}) \simeq F(\mathcal{X} \times \overline{\square})$ for all $\mathcal{X} \in \mathbf{MCor}$, where $\overline{\square} = (\mathbf{P}^1, \infty)$ and $\mathcal{X} \times \overline{\square} = (\overline{X} \times \mathbf{P}^1, \overline{X} \times \infty + X_\infty \times \mathbf{P}^1) \in \mathbf{MCor}$ for $\mathcal{X} = (\overline{X}, X_\infty)$.

Definition 5.6. Define $\mathbf{MDM}$ as the localization of $D(\mathbf{MNST})$ by the localising subcategory generated by the complexes

$$\mathbb{Z}_{\mathrm{tr}}(\mathcal{X} \times \overline{\square}) \to \mathbb{Z}_{\mathrm{tr}}(\mathcal{X}) \ \text{ for } \ \mathcal{X} \in \mathbf{MCor},$$

where the maps are induced by the projections $\mathcal{X} \times \overline{\square} \to \mathcal{X}$.

By the definition we have a functor $M : \mathbf{MCor} \to \mathbf{MDM}$ which maps $\mathcal{X} \in \mathbf{MCor}$ to $\mathbb{Z}_{\mathrm{tr}}(\mathcal{X}) \in \mathbf{MNST}$ considered as a complex by

[14] The similar result holds for $\underline{\mathbf{M}}\mathbf{NST}$ instead of $\mathbf{MNST}$.

putting it in degree 0. We have a commutative diagram

$$\begin{array}{ccc} \mathbf{MCor} & \xrightarrow{M} & \mathbf{MDM}^{\mathrm{eff}} \\ {\scriptstyle\omega}\downarrow & & {\scriptstyle\omega_{\mathrm{eff}}}\downarrow \\ \mathbf{Cor} & \xrightarrow{M} & \mathbf{DM}^{\mathrm{eff}}, \end{array} \tag{5.2}$$

where ω_{eff} is induced by $\omega_!$ from (5.1). In [14] is shown the following.

Theorem 5.7. *The functor ω_{eff} is a localization and admits a fully faithful adjoint ω^{eff}.*

Definition 5.8. For $\mathcal{X} \in \mathbf{MCor}$ define $\widetilde{C}_n(\mathcal{X}) \in \mathbf{MNST}$ by

$$\widetilde{C}_n(\mathcal{X})(\mathcal{Y}) = \mathbf{MCor}(\mathcal{Y} \times \overline{\square}^n, \mathcal{X}).$$

Then the same construction as Definition 2.3 produces a chain complex $C_*(\mathcal{X})$ in $\mathbf{MNST}$.

We have the following variant of Theorem 2.4. Its proof uses the results in §4[15]

Theorem 5.9. *([22]) Assume* $\mathrm{ch}(k) = 0$ *or* (RS) *from Theorem 4.5. For $\mathcal{X}, \mathcal{Y} \in \mathbf{MCor}$ and $j \in \mathbb{Z}$, there is a canonical isomorphism*

$$\mathrm{Hom}_{\mathbf{MDM}}(M(\mathcal{Y}), M(\mathcal{X})[i]) \simeq H^i(\overline{Y}_{\mathrm{Nis}}, C_*(\mathcal{X})_{\mathcal{Y}}) \tag{5.3}$$

where $C_*(\mathcal{X})_{\mathcal{Y}}$ is the complex of sheaves on $\overline{Y}_{\mathrm{Nis}}$ induced by $C_*(\mathcal{X})$ (it depends on $\mathcal{Y}$, not only $\overline{Y}$. See Definition 5.3).

(5.3) implies an isomorphism for $\mathcal{X} = (\overline{X}, X_\infty) \in \mathbf{MCor}$

$$\mathrm{CH}_0(\overline{X}|X_\infty) \simeq \mathrm{Hom}_{\mathbf{MDM}}(M(\mathrm{Spec}(k), \emptyset), M(\mathcal{X})),$$

where the left hand side is the Chow group of zero-cycles with modulus which appeared in §3.

6. Motivic cohomology with modulus and open questions

In view of Voevodsky's definition of motivic cohomology (cf. (2.3)), we may define *motivic cohomology with modulus* as[16]

$$H^i_{\mathcal{M}}(\mathcal{X}, \mathbb{Z}(n)) := \mathrm{Hom}_{\mathbf{MDM}}(M(\mathcal{X}), \mathbb{Z}(n)[i]) \quad \text{for} \quad \mathcal{X} \in \mathbf{MCor} \tag{6.1}$$

Natural questions are the following.

[15] Indeed we need refine the results of §4 in the new categorical framework explained above.

[16] It is not yet completely clear to the author what is the right definition of the Tate object $\mathbb{Z}(n)$ in $\mathbf{MDM}$. One option is the image of $\mathbb{Z}(n)$ in $\mathbf{DM}^{eff}$ under ω^{eff} from Theorem 5.7 .

Question 6.1. Can one establish an analogue of the isomorphism (2.4) for (6.1)?

Question 6.2. Can one establish an analogue of the isomorphism (1.1) for (6.1) by replacing algebraic K-theory with relative algebraic K-theory for the pair $\mathcal{X}$?

In order to answer Question 6.1, one should generalize Bloch's higher Chow groups. Several attempts have been already made in this direction. The first attempt was due to S. Bloch and H. Esnault ([4], [5]) who introduced *additive higher Chow groups* of a field k. It is conceived as a variant of Bloch's higher Chow group for the modulus pair $(\mathbf{A}^1_k, 2\cdot 0)$. They showed that a part of these groups coincides with the absolute differential forms of k. Rülling [21] generalized it to the case $(\mathbf{A}^1_k, m\cdot 0)$ for $m \in \mathbb{Z}_{\geq 1}$ and proved that these groups give a cycle theoretic description of the big deRham-Witt complex of Hesselholt-Madsen. Park [19] extended the definition of Bloch-Esnault to introduce additive higher Chow groups $\mathrm{TCH}^r(X, n; m)$ for a k-scheme X. The groups studied by Bloch-Esnault and Rülling correspond to the case $X = \operatorname{Spec} k$ and $r = n$. Motivated by a work [15] of Kerz and the author, Park's definition is extended in [7] to higher Chow groups $\mathrm{CH}^r(X|D, n)$ for a modulus pair (X, D). We have

$$\mathrm{CH}^r(X|D, n) = \mathrm{TCH}^r(Y, n{+}1; m) \ \text{ for } \ (X, D) = (Y\times\mathbf{A}^1_k, m{\cdot}(Y\times\{0\})).$$

The definition of $\mathrm{CH}^r(X|D, n)$ is given by

$$\mathrm{CH}^r(X|D, n) = H_n(z^r(X|D, \bullet)),$$

where $z^r(X|D, \bullet)$ is the *cycle complex with modulus*, which is a subcomplex of the cubical version of Bloch's cycle complex $z^r(X, \bullet)$ introduced in [2], consisting of those cycles satisfying a certain modulus condition.

As in the case of Bloch's cycle complex, $z^r(X|D, \bullet)$ gives rise to a complex $z^r(-|D, \bullet)$ of sheaves on the small étale site $X_{\text{ét}}$. We then consider the complex $\mathbb{Z}(r)_{X|D} := z^r(-|D, 2r - \bullet)$ and put

$$\mathbb{H}^i(X_{\mathrm{Nis}}, \mathbb{Z}(r)_{X|D}). \tag{6.2}$$

There is a natural map

$$\mathrm{CH}^r(X|D, n) \to \mathbb{H}^{2r-n}(X_{\mathrm{Nis}}, \mathbb{Z}(r)_{X|D}).$$

A fundamtal fact due to Bloch [3] is that this map is an isomorphism in case $D = \emptyset$. However this is not true any more in general (see the remark above [20, Th.3]). This implies that one can not expect to use $\mathrm{CH}^r(X|D, n)$ to answer Question 6.1 for $\mathcal{X} = (X, D)$. A naive hope is then to use (6.2) for the aim. There have been further developments in this direction: Kai [11] proved a modulus analogue of the so-called "easy" moving lemma in [2] to establish basic functoriality for (6.2). Kai-Miyazaki [12] proved a modulus analogue of Suslin's moving lemma proved in [23]. [17]

Finally we mention a work [13] of Iwasa and Kai who defined Chern class maps from relative K-theory of the pair (X, D) to (6.2). This is expected to give the first step toward Question 6.2.

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