TENSOR STRUCTURES OF RIGHT BOUNDED DERIVED CATEGORIES OF COMMUTATIVE RINGS AND BALMER SPECTRA

RYO TAKAHASHI

Introduction

This article makes a report of the talk given by the author at the 62nd Algebra Symposium, which was held at Osaka University in September, 2017. The talk is based on joint work with Hiroki Matsui. The complete proofs of the results in this article that are due to Matsui and the author are all stated in [10], together with more detailed information and other related results.

Tensor triangular geometry is a theory established by Balmer [2] at the beginning of the current century. Let $\mathcal{T} = (\mathcal{T}, \otimes, \mathbb{1})$ be an (essentially small) *tensor triangulated category*, that is, a triangulated category $\mathcal{T}$ equipped with symmetric tensor product $\otimes$ and unit object $\mathbb{1}$. A (thick tensor) *ideal* of $\mathcal{T}$ is defined to be a thick subcategory of $\mathcal{T}$ which is closed under the action of $\mathcal{T}$ by $\otimes$. A proper ideal $\mathcal{P}$ of $\mathcal{T}$ is called *prime* if it satisfies:

$$X \otimes Y \in \mathcal{P} \implies X \in \mathcal{P} \text{ or } Y \in \mathcal{P}.$$

Prime ideals of tensor triangulated categories turn out to behave similarly to prime ideals of commutative rings; both share a lot of analogous properties. Among other things, the *Balmer spectrum* $\mathsf{Spc}\,\mathcal{T}$ of $\mathcal{T}$, which is defined as the set of prime ideals of $\mathcal{T}$, has the structure of a topological space, corresponding to the fact that the Zariski spectrum $\operatorname{Spec} R$ of a commutative ring R has a Zariski topology. Tensor triangular geometry studies Balmer spectra and develops commutative-algebraic and algebro-geometric observations. It is related to a lot of branches of mathematics, including commutative algebra, algebraic geometry, stable homotopy theory, modular representation theory, motivic theory, noncommutative topology and symplectic geometry. As Balmer [4] addressed an invited lecture at the International Congress of Mathematicians (ICM) in 2010, tensor triangular geometry has been attracting a great deal of attention.

Let R be a commutative noetherian ring. Let $\mathsf{D}^-(R)$ be the right bounded derived category of finitely generated R-modules. It is then a routine to verify that

$$(\mathsf{D}^-(R), \otimes_R^{\mathbf{L}}, R)$$

is a tensor triangulated category. The main topics of the talk at the symposium by the author concern the structure of the ideals of $\mathsf{D}^-(R)$ and the structure of the Balmer spectrum $\mathsf{Spc}\,\mathsf{D}^-(R)$ of $\mathsf{D}^-(R)$.

1. Tensor triangulated categories and Balmer spectra

In this section, we introduce some of Balmer's works on general tensor triangulated categories. All the materials in this section are taken from [2, 3, 4]. First of all, we recall the definition of a tensor triangulated category.

Definition 1.1. A *tensor triangulated category* $(\mathcal{T}, \otimes, \mathbb{1})$ is a triangulated category $\mathcal{T}$ equipped with symmetric tensor product $\otimes$ and unit object $\mathbb{1}$. To be more precise, $\mathcal{T}$ is both a triangulated category and a symmetric monoidal category such that the triangulated and symmetric monoidal structures are compatible.

Here are several examples of a tensor triangulated category. Note that all of them are essentially small.

Example 1.2.

(1) Let X be a (quasi-compact and quasi-separated) scheme. Denote by $\mathsf{D}^{\mathsf{perf}}(X)$ the derived category of perfect complexes of $\mathcal{O}_X$-modules. Then $(\mathsf{D}^{\mathsf{perf}}(X), \otimes^{\mathbf{L}}_{\mathcal{O}_X}, \mathcal{O}_X)$ is a tensor triangulated category.

(2) Let R be a commutative ring. Denote by $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\, R)$ the homotopy category of bounded complexes of finitely generated projective R-modules. Then $(\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\, R), \otimes_R, R)$ is a tensor triangulated category. This is nothing but the affine case of (1).

(3) Let k be a field of positive characteristic, and G a finite group (scheme over k). Denote by $\underline{\mathsf{mod}}\, kG$ the stable category of finitely generated kG-modules. Then $(\underline{\mathsf{mod}}\, kG, \otimes_k, k)$ is a tensor triangulated category.

(4) Let k, G be as in (3). Denote by $\mathsf{D}^{\mathsf{b}}(\mathsf{mod}\, kG)$ the derived category of bounded complexes of finitely generated kG-modules. Then $(\mathsf{D}^{\mathsf{b}}(\mathsf{mod}\, kG), \otimes_k, k)$ is a tensor triangulated category.

(5) Let R be a commutative noetherian ring. Denote by $\mathsf{D}^{-}(\mathsf{mod}\, R)$ the derived category of homologically right bounded complexes of finitely generated R-modules. Then $(\mathsf{D}^{-}(\mathsf{mod}\, R), \otimes^{\mathbf{L}}_R, R)$ is a tensor triangulated category. This tensor triangulated category plays a main role in this article.

Next, we give the definitions of a (thick tensor) ideal and a Balmer spectrum. We recall here that a *thick* subcategory of a triangulated category is by definition a nonempty full subcategory which is closed under direct summands, shifts and cones.

Definition 1.3. Let $\mathcal{T}$ be an essentially small tensor triangulated category.

(1) A thick subcategory $\mathcal{I}$ of $\mathcal{T}$ is a *(tensor) ideal* if it satisfies the following implication.

$$a \in \mathcal{T},\ x \in \mathcal{I} \implies a \otimes x \in \mathcal{I}.$$

This is an analogue of an ideal of a commutative ring.

(2) An ideal $\mathcal{I}$ of $\mathcal{T}$ is *radical* if $\mathcal{I} = \sqrt{\mathcal{I}}$, where

$$\sqrt{\mathcal{I}} := \{a \in \mathcal{T} \mid \underbrace{a \otimes \cdots \otimes a}_{n} \in \mathcal{I} \text{ for some } n > 0\}$$

is the *radical* of $\mathcal{I}$. These are analogues of a radical ideal and the radical of an ideal of a commutative ring, respectively.

(3) A proper ideal $\mathcal{P}$ of $\mathcal{T}$ is *prime* if it satisfies the following implication.

$$x \otimes y \in \mathcal{P} \implies x \in \mathcal{P} \text{ or } y \in \mathcal{P}.$$

This is an analogue of a prime ideal of a commutative ring.

(4) The *Balmer spectrum* of $\mathcal{T}$ is defined by:

$$\mathsf{Spc}\, \mathcal{T} = \{\text{Prime ideals of } \mathcal{T}\}.$$

This corresponds to the Zariski spectrum $\operatorname{Spec} R$ of a commutative ring R.

(5) The *Balmer support* of an object x of $\mathcal{T}$ is defined by:

$$\mathtt{Spp}(x) = \{\mathcal{P} \in \mathtt{Spc}\,\mathcal{T} \mid x \notin \mathcal{P}\}.$$

This corresponds to the subset $\mathrm{V}(f) = \{\mathfrak{p} \in \operatorname{Spec} R \mid f \in \mathfrak{p}\}$ of $\operatorname{Spec} R$ for an element f of R. Note that the containment is opposite.

(6) We put

$$\mathtt{U}(x) := \mathtt{Spp}(x)^{\complement} = \{\mathcal{P} \in \mathtt{Spc}\,\mathcal{T} \mid x \in \mathcal{P}\}.$$

This corresponds to the subset $\mathrm{D}(f) = \{\mathfrak{p} \in \operatorname{Spec} R \mid f \notin \mathfrak{p}\}$ of $\operatorname{Spec} R$.

Throughout the rest of this article, we assume that all tensor triangulated categories are essentially small, so that we can always define their Balmer spectra.

We make the definitions of a maximal ideal and a minimal prime of a tensor triangulated category.

Definition 1.4. Let $\mathcal{T}$ be a tensor triangulated category.

(1) An ideal of $\mathcal{T}$ is said to be a *maximal ideal* of $\mathcal{T}$ if it is a proper ideal of $\mathcal{T}$ which is maximal with respect to the inclusion relation. We denote by $\mathtt{Mx}\,\mathcal{T}$ the set of maximal ideals of $\mathcal{T}$.
(2) An ideal of $\mathcal{T}$ is said to be a *minimal prime* of $\mathcal{T}$ if it is a prime ideal of $\mathcal{T}$ which is minimal with respect to the inclusion relation. We denote by $\mathtt{Mn}\,\mathcal{T}$ the set of minimal primes of $\mathcal{T}$.

Each Balmer spectrum has the structure of a topological space such that the Balmer supports are closed subsets. We state this here together with several fundamental properties which will often be used later.

Proposition 1.5 ([2]). *Let $\mathcal{T}$ be a tensor triangulated category.*

(1) $\mathtt{Spc}\,\mathcal{T}$ *is a topological space with an open basis* $\{\mathtt{U}(x)\}_{x\in\mathcal{T}}$.
(2) *Every proper ideal of $\mathcal{T}$ is contained in a maximal ideal.*
(3) *Maximal ideals of $\mathcal{T}$ are prime.*
(4) *Every prime ideal of $\mathcal{T}$ contains a minimal prime.*
(5) *For each $\mathcal{P} \in \mathtt{Spc}\,\mathcal{T}$ the closure $\overline{\{\mathcal{P}\}}$ of $\{\mathcal{P}\}$ is irreducible, and described as follows.*

$$\overline{\{\mathcal{P}\}} = \{\mathcal{Q} \in \mathtt{Spc}\,\mathcal{T} \mid \mathcal{Q} \subseteq \mathcal{P}\}. \tag{1.5.1}$$

Conversely, any nonempty irreducible closed subset of $\mathtt{Spc}\,\mathcal{T}$ *has this form.*

(6) *The open subset* $\mathtt{U}(x)$ *of* $\mathtt{Spc}\,\mathcal{T}$ *is quasi-compact for each $x \in \mathcal{T}$. Conversely, any nonempty quasi-compact open subset of* $\mathtt{Spc}\,\mathcal{T}$ *has this form.*
(7) *For an ideal $\mathcal{I}$ of $\mathcal{T}$ one has*

$$\sqrt{\mathcal{I}} = \bigcap_{\mathcal{I}\subseteq\mathcal{P}\in\mathtt{Spc}\,\mathcal{T}} \mathcal{P}.$$

The equality (1.5.1) corresponds to the equality

$$\overline{\{\mathfrak{p}\}} = \{\mathfrak{q} \in \operatorname{Spec} R \mid \mathfrak{q} \supseteq \mathfrak{p}\}$$

of subsets of $\operatorname{Spec} R$ for a commutative ring R and a prime ideal $\mathfrak{p}$ of R. Again, the containment is opposite.

Thus, ideals of tensor triangulated categories have a lot of similar properties to ideals of commutative rings.

For a full subcategory $\mathcal{X}$ of $\mathcal{T}$ and a subset S of $\mathtt{Spc}\,\mathcal{T}$, set

$$\mathtt{Spp}\,\mathcal{X} = \bigcup_{x\in\mathcal{X}} \mathtt{Spp}(x),$$
$$\mathtt{Spp}^{-1}\,S = \{x \in \mathcal{T} \mid \mathtt{Spp}(x) \in S\}.$$

The following theorem is a celebrated result due to Balmer [2, Theorem 4.10].

Theorem 1.6 (Balmer (2005))**.** *Let $\mathcal{T}$ be a tensor triangulated category. Then there is a one-to-one correspondence*

$$\{\textit{Radical ideals of } \mathcal{T}\} \underset{\mathtt{Spp}^{-1}}{\overset{\mathtt{Spp}}{\underset{1-1}{\rightleftarrows}}} \{\textit{Thomason subsets of } \mathtt{Spc}\,\mathcal{T}\}.$$

Here, a subset A of a topological space X is said to be *Thomason* if one can write

$$A = \bigcup_{i\in I} B_i$$

for some family $\{B_i\}_{i\in I}$ of subsets of X such that $B_i^{\complement} = X \setminus B_i$ is a quasi-compact open subset. A subset C of X is said to be *specialization-closed* if it satisfies the implication

$$x \in C \implies \overline{\{x\}} \subseteq C.$$

We notice that this condition is equivalent to saying that C is a (possibly infinite) union of closed subsets. Therefore, a Thomason subset is always specialization-closed. The name of a Thomason subset comes from the fact that for a quasi-compact quasi-separated scheme X, Thomason [13] gives a complete classification of the ideals of $\mathsf{D}^{\mathsf{perf}}(X)$ in terms of the Thomason subsets of the underlying topological space of X.

Theorem 1.7 says that for a given tensor triangulated category $\mathcal{T}$ the understanding of the structure of the Balmer spectrum of $\mathcal{T}$ provides a complete classification of the radical ideals of $\mathcal{T}$. Since each ideal of $\mathcal{T}$ is the kernel of some tensor triangulated functor from $\mathcal{T}$ and vice versa, classifying ideals of $\mathcal{T}$ leads us to the understanding of the structure of tensor triangulated functors from $\mathcal{T}$. In this sense, the above theorem is quite meaningful.

For each tensor triangulated category $\mathcal{T}$ one can define the structure sheaf $\mathcal{O}_{\mathcal{T}}$ on $\mathcal{T}$, and then the Balmer spectrum $\mathtt{Spc}\,\mathcal{T}$ has the structure of a locally ringed space [4, Constructions 24 and 29]. More precisely, for each quasi-compact open subset U of $\mathtt{Spc}\,\mathcal{T}$ we define

$$\mathcal{T}(U) := (\mathcal{T}/\,\mathtt{Spp}^{-1}(U^{\complement}))^{\natural},$$

where $(-)^{\natural}$ stands for the idempotent completion. Then it holds that

$$\mathtt{Spc}\,\mathcal{T}(U) \cong U.$$

The assignment $U \mapsto \mathrm{End}_{\mathcal{T}(U)}(\mathbb{1})$ induces a presheaf of commutative rings, and we define the structure sheaf $\mathcal{O}_{\mathcal{T}}$ on $\mathcal{T}$ as its sheafification. Thus we obtain a locally ringed space

$$\mathrm{Spec}\,\mathcal{T} := (\mathtt{Spc}\,\mathcal{T}, \mathcal{O}_{\mathcal{T}}).$$

The following theorem due to Balmer is also well-known. We refer the reader to [2, Theorem 6.3] and [4, Theorem 57]; see also [3, Proposition 6.11].

Theorem 1.7 (Balmer (2005, 2010))**.**

(1) *Let X be a quasi-compact quasi-separated scheme. Then there is an isomorphism*

$$\operatorname{Spec}\mathsf{D}^{\mathsf{perf}}(X)\cong X$$

of locally ringed spaces.

(2) *Let k be a field of positive characteristic, and G a finite group (scheme over k). Then there are isomorphisms*

$$\operatorname{Spec}\mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,kG)\cong \operatorname{Spec}^{\mathrm{h}}\mathsf{H}^{\bullet}(G,k),$$
$$\operatorname{Spec}(\underline{\mathsf{mod}}\,kG)\cong \operatorname{Proj}\mathsf{H}^{\bullet}(G,k)$$

of locally ringed spaces.

Here $\mathsf{H}^{\bullet}(G,k)$ stands for the group cohomology ring. For a graded-commutative ring A, we denote by $\operatorname{Spec}^{\mathrm{h}} A$ the set of homogeneous prime ideals of A. For a commutative nonnegatively graded ring R we denote by $\operatorname{Proj} R$ the set of homogeneous prime ideals of R that do not contain $R_+ = \bigoplus_{i>0} R_i$. Note that $\operatorname{Proj}\mathsf{H}^{\bullet}(G,k)$ is nothing but the (projective) support variety $\mathcal{V}_G(k)$.

The isomorphism in Theorem 1.7(1) says that a scheme X is reconstructed from its derived category $\mathsf{D}^{\mathsf{perf}}(X)$; see also [1]. This is actually because of the tensor structure of $\mathsf{D}^{\mathsf{perf}}(X)$. Indeed, only from the triangulated structure of $\mathsf{D}^{\mathsf{perf}}(X)$ the original scheme X cannot be reconstructed, since there are a lot of derived equivalences of nonsingular algebraic varieties (e.g. the Fourier–Mukai transformation).

The second isomorphism in Theorem 1.7(2) is obtained by restricting the first one. Key roles in the proof of Theorem 1.7 are played by the classification theorems of ideals due to Hopkins [9], Neeman [11], Thomason [13], Benson–Carlson–Rickard [5] and Friedlander–Pevtsova [8]; see also the works of Benson–Iyengar–Krause [6] and Benson–Iyengar–Krause–Pevtsova [7]. The Balmer spectra are described for some other tensor triangulated categories by several authors; details can be found in [4].

Let $(\mathcal{T},\otimes,\mathbb{1})$ be a tensor triangulated category. Balmer [3] constructs a continuous map

$$\rho^{\bullet}_{\mathcal{T}}:\mathsf{Spc}\,\mathcal{T}\to\operatorname{Spec}^{\mathrm{h}}\mathrm{R}^{\bullet}_{\mathcal{T}},$$

which is given by

$$\rho^{\bullet}_{\mathcal{T}}(\mathcal{P}):=(f\in\mathrm{R}^{\bullet}_{\mathcal{T}}\mid \operatorname{cone}(f)\notin\mathcal{P}).$$

Here,

$$\mathrm{R}^{\bullet}_{\mathcal{T}}=\operatorname{Hom}_{\mathcal{T}}(\mathbb{1},\Sigma^{\bullet}\mathbb{1})$$

is a graded-commutative ring.

It is seen that for $\mathcal{T}=\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,R)$ with R being a commutative ring we have $\mathrm{R}^{\bullet}_{\mathcal{T}}=R$, and it is also observed that for $\mathcal{T}=\mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,kG)$ with k being a field k of positive characteristic and G being a finite group (scheme over k) we have $\mathrm{R}^{\bullet}_{\mathcal{T}}=\mathsf{H}^{\bullet}(G,k)$. It is shown by Balmer [3, Propositions 8.1 and 8.5] that the isomorphism in Theorem 1.7(1) in the affine case, and the first isomorphism in Theorem 1.7(2) are given by the map $\rho^{\bullet}_{\mathcal{T}}$ given above. Thus the following conjecture has been presented by Balmer [4, Conjecture 72] in his invited lecture at the International Congress of Mathematicians (ICM), which was held in 2010 at Hyderabad.

Conjecture 1.8 (Balmer, ICM 2010)**.** The map $\rho^{\bullet}_{\mathcal{T}}$ is (locally) injective if $\mathcal{T}$ is algebraic as a triangulated category.

Let $f : X \to Y$ be a continuous map of topological spaces. We say that f is *locally injective* at a point $x \in X$ if there exists a neighborhood N of x such that the restriction $f|_N$ of f on N is injective. The map f is called *locally injective* if for all points $x \in X$ it is locally injective at x. Also, recall that a triangulated category is called *algebraic* if it is described as the stable category of a Frobenius exact category.

It is known that the conjecture does not hold for a non-algebraic triangulated category; indeed, if $\mathcal{T}$ is the Spanier–Whitehead stable homotopy category $\mathrm{SH}^{\mathrm{fin}}$ of finite pointed CW-complexes, then $\rho^{\bullet}_{\mathcal{T}}$ is not injective; see [4, Theorem 51]. On the other hand, as we have seen above, the conjecture does hold for $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\, R)$ and $\mathsf{D}^{\mathsf{b}}(\mathsf{mod}\, kG)$.

Now we introduce some notation, which will be used throughout the rest of this article.

Notation 1.9.

(1) Let R be a commutative noetherian ring.
(2) We denote by $\operatorname{Spec} R$ the *Zariski spectrum* of R, namely, the set of prime ideals of R equipped with the Zariski topology.
(3) For an ideal I of R we define $\mathrm{V}(I)$ the set of prime ideals of R containing I, and put $\mathrm{D}(I) = \mathrm{V}(I)^{\complement} = \operatorname{Spec} R \setminus \mathrm{V}(I)$.
(4) The set of maximal ideals (respectively, minimal primes) of R is denoted by $\operatorname{Max} R$ (respectively, $\operatorname{Min} R$).
(5) We denote by $\mathsf{mod}\, R$ the category of finitely generated R-modules, and by $\mathsf{proj}\, R$ the full subcategory of $\mathsf{mod}\, R$ consisting of finitely generated projective R-modules.
(6) We denote by $\mathsf{D}^*(R)$ the derived category $\mathsf{D}^*(\mathsf{mod}\, R)$ of the abelian category $\mathsf{mod}\, R$, and by $\mathsf{K}^*(R)$ the homotopy category $\mathsf{K}^*(\mathsf{proj}\, R)$ of the additive category $\mathsf{proj}\, R$, where $* \in \{-, \mathrm{b}\}$. There are obvious inclusions
$$\mathsf{K}^{\mathsf{b}}(R) \subseteq \mathsf{D}^{\mathsf{b}}(R) \subseteq \mathsf{D}^{-}(R).$$
Taking projective resolutions induces an equivalence
$$\mathsf{D}^{-}(R) \cong \mathsf{K}^{-}(R)$$
of tensor triangulated categories. We will often identify $\mathsf{D}^{-}(R)$ with $\mathsf{K}^{-}(R)$ via this equivalence.

From the next section on, we will investigate the structure of $\mathsf{D}^{-}(R)$ as a tensor triangulated category. We close this section by giving comments about how hard it is.

Difficulities for $\mathsf{D}^{-}(R)$. The tensor triangulated category $\mathsf{D}^{-}(R)$ possesses a lot of defects on its structure, compared with the other well-established tensor triangulated categories:

(1) $\mathsf{D}^{-}(R)$ does not have arbitrary products or coproducts. (However, it does have some specific infinite coproducts, which will somehow play a crucial role in the proofs of our results.)
(2) $\mathsf{D}^{-}(R)$ is not closed under duals. For example, in the case where R is an algebra over a field k, $\mathsf{D}^{-}(R)$ is not closed under k-duals.
(3) In particular, $\mathsf{D}^{-}(R)$ is never rigid. Recall that a triangulated category $\mathcal{T}$ is called *rigid* if there exist an exact functor $D : \mathcal{T}^{\mathrm{op}} \to \mathcal{T}$ and a functorial isomorphism
$$\operatorname{Hom}_{\mathcal{T}}(a \otimes b, c) \cong \operatorname{Hom}_{\mathcal{T}}(a, D(b) \otimes c)$$
for $a, b, c \in \mathcal{T}$. In fact, $\mathsf{D}^{-}(R)$ is even never closed as a symmetric monoidal category. There are a lot of results on rigid tensor triangulated categories, but we cannot use them for $\mathsf{D}^{-}(R)$.

(4) One has

$$\mathsf{thick}_{\mathsf{D}^-(R)} R \neq \mathsf{D}^-(R).$$

Indeed, the left hand side coincides with $\mathsf{K}^{\mathsf{b}}(R)$. There are several results on tensor triangulated categories $(\mathcal{T}, \otimes, \mathbb{1})$ satisfying $\mathsf{thick}_{\mathcal{T}} \mathbb{1} = \mathcal{T}$, but they are not available for $\mathsf{D}^-(R)$.

Thus, results in the literature are quite limited on tensor triangulated categories that can be applied to our tensor triangulated category $\mathsf{D}^-(R)$.

2. Compactly and cocompactly generated thick tensor ideals of $\mathsf{D}^-(R)$

In this section, we classify compactly or cocompactly generated ideals of the tensor triangulated category $\mathsf{D}^-(R)$. We begin with recalling the definitions of compact and cocompact objects.

Definition 2.1. Let $\mathcal{T}$ be a triangulated category.

(1) An object $M \in \mathcal{T}$ is called *compact* (respectively, *cocompact*) if the natural morphism

$$\bigoplus_{\lambda\in\Lambda} \mathrm{Hom}_{\mathcal{T}}(M, N_\lambda) \to \mathrm{Hom}_{\mathcal{T}}(M, \bigoplus_{\lambda\in\Lambda} N_\lambda)$$
$$\left(\text{respectively, } \bigoplus_{\lambda\in\Lambda} \mathrm{Hom}_{\mathcal{T}}(N_\lambda, M) \to \mathrm{Hom}_{\mathcal{T}}(\prod_{\lambda\in\Lambda} N_\lambda, M)\right)$$

is an isomorphism for all families $\{N_\lambda\}_{\lambda\in\Lambda}$ of objects of $\mathcal{T}$ such that the coproduct $\bigoplus_{\lambda\in\Lambda} N_\lambda$ (respectively, the product $\prod_{\lambda\in\Lambda} N_\lambda$) exists in $\mathcal{T}$.

(2) We denote by $\mathcal{T}^{\mathsf{c}}$ (respectively, $\mathcal{T}^{\mathsf{cc}}$) the full subcategory of $\mathcal{T}$ consisting of compact (respectively, cocompact) objects of $\mathcal{T}$.

(3) An ideal of $\mathcal{T}$ is said to be *compactly generated* (respectively, *cocompactly generated*) if it is generated by some compact (respectively, cocompact) objects of $\mathcal{T}$ as an ideal.

The following equalities hold for compactly and cocompactly generated ideals of $\mathsf{D}^-(R)$.

Fact 2.2. There are equalities

$$\mathsf{D}^-(R)^{\mathsf{c}} = \mathsf{K}^{\mathsf{b}}(R),$$
$$\mathsf{D}^-(R)^{\mathsf{cc}} = \mathsf{D}^{\mathsf{b}}(R).$$

The second equality in the above fact is due to Oppermann–Stovicek [12, Theorem 18]. The first equality is well-known, and actually proved along the same lines as in the proof of the fact that the compact objects of the unbounded derived category of all R-modules are the perfect complexes over R.

Next, let us recall the definition of the (usual) support of a chain complex. Note that this notion is different from that of a Balmer support introduced in the previous section.

Definition 2.3.

(1) Let $X \in \mathsf{D}^-(R)$ be a complex. The *support* of X is defined to be the union of the supports (as R-modules) of homologies of X. One has equalities

$$\begin{aligned}\operatorname{Supp} X &= \bigcup_{i\in\mathbb{Z}} \operatorname{Supp} \mathsf{H}^i(X) \\ &= \{\mathfrak{p} \in \operatorname{Spec} R \mid X_\mathfrak{p} \neq 0\} \\ &= \{\mathfrak{p} \in \operatorname{Spec} R \mid \kappa(\mathfrak{p}) \otimes_R^{\mathbf{L}} X \neq 0\}\end{aligned} \tag{2.3.1}$$

of subsets of $\operatorname{Spec} R$, where $\kappa(\mathfrak{p})$ denotes the residue field $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ of the local ring $R_\mathfrak{p}$.

(2) For a full subcategory $\mathcal{X}$ of $\mathsf{D}^-(R)$, set

$$\operatorname{Supp} \mathcal{X} = \bigcup_{X\in\mathcal{X}} \operatorname{Supp} X.$$

It is easy to see that the following hold.
- $\operatorname{Supp} \mathcal{X}$ is a specialization-closed subset of $\operatorname{Spec} R$.
- There is an equality $\operatorname{Supp} \mathcal{X} = \operatorname{Supp}(\mathsf{thick}^\otimes \mathcal{X})$.

Here, $\mathsf{thick}^\otimes \mathcal{X}$ stands for the ideal generated by $\mathcal{X}$, that is, the smallest ideal of $\mathsf{D}^-(R)$ containing $\mathcal{X}$.

(3) For a subset S of $\operatorname{Spec} R$, set

$$\langle S\rangle = \mathsf{thick}^\otimes_{\mathsf{D}^-(R)}\{R/\mathfrak{p} \mid \mathfrak{p} \in S\}.$$

The second equality in (2.3.1) holds even for unbounded complexes of non-finitely generated R-modules, while the third equality only holds for complexes in $\mathsf{D}^-(R)$.

The following theorem is the first main result of this article.

Theorem 2.4 ([10, Theorem 2.12]). *There is a one-to-one correspondence*

$$\left\{\begin{matrix}\textit{Cocompactly generated}\\ \textit{ideals of } \mathsf{D}^-(R)\end{matrix}\right\} \underset{\langle\rangle}{\overset{\operatorname{Supp}}{\underset{1-1}{\rightleftarrows}}} \left\{\begin{matrix}\textit{Specialization-closed}\\ \textit{subsets of } \operatorname{Spec} R\end{matrix}\right\}.$$

Thus the cocompactly generated ideals of $\mathsf{D}^-(R)$ are completely classified.

In fact, this one-to-one correspondence is not just a bijection of sets. For ideals $\mathcal{X},\mathcal{Y}$ of $\mathsf{D}^-(R)$, define $\mathcal{X}\wedge\mathcal{Y}$ and $\mathcal{X}\vee\mathcal{Y}$ by:

$$\begin{cases}\mathcal{X}\wedge\mathcal{Y} = \mathsf{thick}^\otimes\{X \otimes_R^{\mathbf{L}} Y \mid X\in\mathcal{X},\ Y\in\mathcal{Y}\},\\ \mathcal{X}\vee\mathcal{Y} = \mathsf{thick}^\otimes(\mathcal{X}\cup\mathcal{Y}).\end{cases}$$

It is then seen that for specialization-closed subsets A, B of $\operatorname{Spec} R$ there are equalities

$$\begin{cases}\langle A\rangle \wedge \langle B\rangle = \langle A\cap B\rangle,\\ \langle A\rangle \vee \langle B\rangle = \langle A\cup B\rangle.\end{cases}$$

Using these equalities, one can show that the set of cocompactly generated ideals of $\mathsf{D}^-(R)$ forms a lattice with join $\vee$ and meet $\wedge$, and that the bijections in the theorem are lattice isomorphisms; see [10, Proposition 2.18].

On the other hand, using the above theorem, we observe that the assignments $\mathcal{X} \mapsto \mathcal{X}\cap\mathsf{K}^\mathsf{b}(R)$ and $\mathsf{thick}^\otimes \mathcal{Y} \leftarrow\!\shortmid \mathcal{Y}$ make a one-to-one correspondence

$$\{\text{Cocompactly generated ideals of } \mathsf{D}^-(R)\} \rightleftarrows \{\text{Thick subcategories of } \mathsf{K}^\mathsf{b}(R)\};$$

see [10, Corollary 2.14].

To prove the theorem, we need to extend the Hopkins–Neeman smash nilpotence theorem as follows; see [10, Theorem 2.7].

Lemma 2.5 (Generalized smash nilpotence)**.** *Let $f : X \to Y$ be a morphism in $\mathsf{K}^{-}(R)$ such that $Y \in \mathsf{K}^{\mathsf{b}}(R)$. If $f \otimes_R \kappa(\mathfrak{p}) = 0$ for all prime ideals $\mathfrak{p}$ of R, then $f^{\otimes t} = 0$ for some integer $t > 0$.*

We do not state the proof of this lemma, but give several comments on the proof.

Remark 2.6.

(1) If we assume further that $X \in \mathsf{K}^{\mathsf{b}}(R)$, then the assertion of the lemma is nothing but the original smash nilpotence due to Hopkins [9, Theorem 10] and Neeman [11, Theorem 1.1]. In the proof of the original smash nilpotence, one can reduce to the case where $X = R$ by replacing the morphism $f : X \to Y$ with a morphism $f' : R \to \mathbf{R}\mathrm{Hom}_R(X, Y)$ via the isomorphim

$$\mathrm{Hom}_{\mathsf{K}^{\mathsf{b}}(R)}(R, \mathbf{R}\mathrm{Hom}_R(X, Y)) \cong \mathrm{Hom}_{\mathsf{K}^{\mathsf{b}}(R)}(X, Y).$$

Thanks to this reduction, one can identify the morphism $f \in \mathrm{Hom}_{\mathsf{K}^{\mathsf{b}}(R)}(R, Y)$ with the element $f(1) \in \mathsf{H}^0 Y$, which plays a key role in the proof of the original smash nilpotence.

(2) We show and use the following statements; see [10, Lemmas 2.5 and 2.6].
   (a) Let $\mathcal{T}$ be a tensor triangulated category. Let f, g be a morphism in $\mathcal{T}$, and let $\mathcal{X}, \mathcal{Y}$ be full subcategories of $\mathcal{T}$. If $f \otimes \mathcal{X} = 0$ and $g \otimes \mathcal{Y} = 0$, then $(f \otimes g) \otimes (\mathcal{X} * \mathcal{Y}) = 0$.
   (b) Let $\boldsymbol{x} = x_1, \ldots, x_n$ be a sequence of elements of R. Let f be a morphism in $\mathsf{K}^{-}(R)$. If $f \otimes_R R/(\boldsymbol{x}) = 0$, then $f^{\otimes 2^n} \otimes_R \mathrm{K}(\boldsymbol{x}) = 0$.

   Here, $\mathcal{X} * \mathcal{Y}$ stands for the full subcategory of $\mathsf{K}^{-}(R)$ consisting of objects E such that there exists an exact triangle

$$X \to E \to Y \rightsquigarrow$$

   in $\mathcal{T}$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$, and $\mathrm{K}(\boldsymbol{x})$ stands for the Koszul complex of R with respect to $\boldsymbol{x}$. The statement (b) is deduced by using (a).

(3) We need the assumption that $Y \in \mathsf{K}^{\mathsf{b}}(R)$ to have the equality

$$\mathrm{ann}_{R_\mathfrak{p}}(f_\mathfrak{p}) = \mathrm{ann}_R(f)_\mathfrak{p}$$

   for all prime ideals $\mathfrak{p}$ of R. Here, the *annihilator* of a morphism $f : X \to Y$ in $\mathsf{D}^{-}(R)$ is defined by

$$\mathrm{ann}_R(f) := \{a \in R \mid af = 0 \text{ in } \mathsf{D}^{-}(R)\},$$

   which is nothing but the kernel of the morphism $R \to \mathrm{Hom}_{\mathsf{D}^{-}(R)}(X, Y)$ given by $a \mapsto af$.

By virtue of the generalized smash nilpotence, we can prove the following key proposition. For an object X of $\mathsf{D}^{-}(R)$ we define the *annihilator* $\mathrm{ann}\, X$ of X as the annihilator of the identity morphism of X.

Proposition 2.7 ([10, Proposition 2.9])**.** *Let $X, Y \in \mathsf{D}^{-}(R)$ be complexes. Then the following implication holds true.*

$$\mathrm{V}(\mathrm{ann}\, X) \subseteq \mathrm{Supp}\, Y \implies X \in \mathsf{thick}^{\otimes} Y.$$

Again, we do not state the proof of this proposition but give some comments on it.

Remark 2.8.

(1) For every $X \in \mathsf{D}^-(R)$ one has

$$\mathrm{V}(\operatorname{ann} X) \supseteq \operatorname{Supp} X.$$

The equality holds if $X \in \mathsf{D}^{\mathsf{b}}(R)$.

(2) The original statement that is due to Hopkins and Neeman and corresponds to the proposition asserts that for perfect complexes X, Y over R the implication

$$\operatorname{Supp} X \subseteq \operatorname{Supp} Y \implies X \in \mathsf{thick}^{\otimes} Y$$

holds true; see [11, Lemma 1.2].

(3) Proposition 2.7 does not hold if $\mathrm{V}(\operatorname{ann} X)$ is replaced with $\operatorname{Supp} X$ or if $\operatorname{Supp} Y$ is replaced with $\mathrm{V}(\operatorname{ann} Y)$; we will see this in Remark 3.14.

(4) In the proof of the proposition, we first take a truncation $Y' \in \mathsf{K}^{\mathsf{b}}(R)$ of Y such that $\mathrm{V}(\operatorname{ann} X)$ is contained in $\operatorname{Supp} Y'$. Then we consider the morphism $R \to \operatorname{Hom}_R(Y', Y)$ sending $1 \in R$ to the inclusion morphism $Y' \to Y$. The stream of the proof is similar to [11, Lemma 1.2], but we need to make various modifications.

(5) In the proposition, we can replace the object Y of $\mathsf{D}^-(R)$ with any full subcategory $\mathcal{Y}$ of $\mathsf{D}^-(R)$. Indeed, we find an object $Y \in \mathcal{Y}$ such that $\operatorname{Supp} Y$ contains all the prime ideals (minimally) containing $\operatorname{ann} X$. Then $\mathrm{V}(\operatorname{ann} X)$ is contained in $\operatorname{Supp} Y$, and we can reduce to the case where the subcategory $\mathcal{Y}$ consists only of Y.

As a corollary of Proposition 2.7 we have the following result. This result will be used in the proof of Theorem 2.4, and several other places.

Corollary 2.9 ([10, Corollary 2.11 and Proposition 4.11])**.**

(1) *Let X be a complex in $\mathsf{D}^-(R)$. Then it holds that*

$$\operatorname{Supp} X = \operatorname{Spec} R \iff \mathsf{thick}^{\otimes} X = \mathsf{D}^-(R).$$

(2) *Let I be an ideal of R, and let $\mathcal{X}$ be an ideal of $\mathsf{D}^-(R)$. Take a system of generators $\boldsymbol{x} = x_1, \dots, x_n$ of I. Then it holds that*

$$\mathrm{V}(I) \subseteq \operatorname{Supp} \mathcal{X} \iff R/I \in \mathcal{X} \iff \mathrm{K}(\boldsymbol{x}) \in \mathcal{X}.$$

Proof. (1) The implication ($\Leftarrow$) follows from the equalities

$$\operatorname{Supp} X = \operatorname{Supp}(\mathsf{thick}^{\otimes} X) = \operatorname{Supp} \mathsf{D}^-(R) = \operatorname{Spec} R.$$

As for the implication ($\Rightarrow$), for all objects $M \in \mathsf{D}^-(R)$ one has that $\mathrm{V}(\operatorname{ann} M)$ is contained in $\operatorname{Supp} X$. Hence M belongs to $\mathsf{thick}^{\otimes} X$ by Proposition 2.7.

(2) We have

$$\operatorname{Supp} R/I = \mathrm{V}(\operatorname{ann} R/I) = \mathrm{V}(I) = \mathrm{V}(\operatorname{ann} \mathrm{K}(\boldsymbol{x})) = \operatorname{Supp} \mathrm{K}(\boldsymbol{x}).$$

Using Proposition 2.7 completes the proof of the assertion. □

Now we can obtain the proof of the main result of this section.

Proof of Theorem 2.4. Let $\mathcal{X}$ be a cocompactly generated ideal of $\mathsf{D}^-(R)$. Then one can write $\mathcal{X} = \mathsf{thick}^{\otimes} \mathcal{C}$ for some full subcategory $\mathcal{C}$ of $\mathsf{D}^{\mathsf{b}}(R)$. What we want to show is the equality $\mathcal{X} = \langle \operatorname{Supp} \mathcal{X} \rangle$. As to the inclusion ($\supseteq$), Corollay 2.9(2) implies that $R/\mathfrak{p}$ is in $\mathcal{X}$ for all $\mathfrak{p} \in \operatorname{Supp} \mathcal{X}$. As for the inclusion ($\subseteq$), it suffices to show that $\mathcal{C}$ is contained in $\langle \operatorname{Supp} \mathcal{X} \rangle = \langle \operatorname{Supp} \mathcal{C} \rangle$. Pick an object $M \in \mathcal{C}$. Then M is a bounded complex of finitely generated R-modules, whence it is in $\mathsf{thick}\{R/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp} M\}$. Now we are done. □

As a corollary of Theorem 2.4 we have the following result.

Corollary 2.10 ([10, Corollary 2.16])**.** *The following are equivalent for an ideal $\mathcal{X}$ of $\mathsf{D}^-(R)$.*
(1) *$\mathcal{X}$ is compactly generated.*
(2) *$\mathcal{X}$ is cocompactly generated.*
When this is the case, we simply say that $\mathcal{X}$ is compact.

Proof. Since $\mathsf{K}^{\mathsf{b}}(R)$ is contained in $\mathsf{D}^{\mathsf{b}}(R)$, compact generation implies cocompact generation. Therefore (1) implies (2). Let us show that (2) implies (1). Let W be a specialization-closed subset of $\operatorname{Spec} R$. Put

$$\mathcal{A} := \mathsf{thick}^{\otimes}\{R/\mathfrak{p} \mid \mathfrak{p} \in W\},$$
$$\mathcal{B} := \mathsf{thick}^{\otimes}\{\mathrm{K}(\boldsymbol{x}) \mid \boldsymbol{x}R \in W\}.$$

Then $\mathcal{A}$ is cocompactly generated, while $\mathcal{B}$ is compactly generated. We see that $\operatorname{Supp}\mathcal{A} = \operatorname{Supp}\mathcal{B} = W$. Using Theorem 2.4, we obtain $\mathcal{A} = \mathcal{B}$. ☐

As another corollary of Proposition 2.7, we get the following result.

Corollary 2.11 ([10, Corollary 2.20])**.** *If R is artinian, then all ideals of $\mathsf{D}^-(R)$ are compact. Therefore one has a one-to-one correspondence*

$$\{\textit{Ideals of } \mathsf{D}^-(R)\} \rightleftarrows \{\textit{Subsets of } \operatorname{Spec} R\}.$$

3. The Balmer spectrum of $\mathsf{D}^-(R)$ and classifications of thick tensor ideals

In this section, we consider the structure of the Balmer spectrum of $\mathsf{D}^-(R)$, and make correspondences among some classes of ideals of $\mathsf{D}^-(R)$ and subsets of $\operatorname{Spec} R$ and $\mathsf{Spc}\,\mathsf{D}^-(R)$. The section consists of three subsections.

3.1. The structure of $\mathsf{Spc}\,\mathsf{D}^-(R)$.

We investigate the structure of the Balmer spectrum of $\mathsf{D}^-(R)$ as a topological space, comparing it with the Zariski spectrum of R. We start by defining a tame ideal of $\mathsf{D}^-(R)$.

Definition 3.1.
(1) For a subset S of $\operatorname{Spec} R$, we define the full subcategory $\operatorname{Supp}^{-1} S$ of $\mathsf{D}^-(R)$ by

$$\operatorname{Supp}^{-1} S = \{X \in \mathsf{D}^-(R) \mid \operatorname{Supp} X \subseteq S\}.$$

One easily sees that $\operatorname{Supp}^{-1} S$ is an ideal of $\mathsf{D}^-(R)$, and furthermore, the following equalities hold.
- $\operatorname{Supp}^{-1} S = \operatorname{Supp}^{-1} S_{\mathsf{spcl}}$.
- $\operatorname{Supp}(\operatorname{Supp}^{-1} S) = S_{\mathsf{spcl}}$.

Here, S_{spcl} stands for the largest specialization-closed subset of $\operatorname{Spec} R$ contained in S. (This is the spcl-*interior* of S in $\operatorname{Spec} R$ if we use the terminology in the next Subsection 3.2.)

(2) An ideal $\mathcal{X}$ of $\mathsf{D}^-(R)$ is called *tame* if $\mathcal{X} = \operatorname{Supp}^{-1} S$ for some subset S of $\operatorname{Spec} R$. We set

$$^{\mathsf{t}}\mathsf{Spc}\,\mathsf{D}^-(R) = \{\text{tame prime ideals of } \mathsf{D}^-(R)\}.$$

One can construct the following correspondence between $\operatorname{Spec} R$ and $\mathsf{Spc}\,\mathsf{D}^-(R)$; see [10, Propositions 3.4 and 3.7].

Proposition 3.2.

(1) *For* $\mathfrak{p} \in \operatorname{Spec} R$*, the full subcategory*

$$\mathcal{S}(\mathfrak{p}) := \{X \in \mathsf{D}^-(R) \mid X_\mathfrak{p} = 0\}$$

of $\mathsf{D}^-(R)$ *is a prime ideal of* $\mathsf{D}^-(R)$

(2) *For* $\mathcal{P} \in \mathsf{Spc}\,\mathsf{D}^-(R)$*, the set*

$$\{I \subseteq R \mid R/I \notin \mathcal{P}\}$$

of ideals of R *has a unique maximal element* $\mathfrak{s}(\mathcal{P})$ *with respect to the inclusion relation, which is a prime ideal of* R

Concerning the correspondence constructed in the above proposition, the following statements hold.

Theorem 3.3 ([10, Theorems 3.9, 4.5, 4.7, 4.12 and 4.14])**.**

(1) *One has the order-reversing maps*

$$\mathcal{S} : \operatorname{Spec} R \rightleftarrows \mathsf{Spc}\,\mathsf{D}^-(R) : \mathfrak{s}$$

such that

$$\begin{cases} \mathfrak{s} \cdot \mathcal{S} = 1, \\ \mathcal{S} \cdot \mathfrak{s} = \operatorname{Supp}^{-1} \operatorname{Supp}. \end{cases}$$

In particular, the inequality

$$\dim(\mathsf{Spc}\,\mathsf{D}^-(R)) \geq \dim R$$

between the Krull dimensions holds.

(2) *The subset* ${}^{\mathsf{t}}\mathsf{Spc}\,\mathsf{D}^-(R)$ *of* $\mathsf{Spc}\,\mathsf{D}^-(R)$ *is dense. There is a commutative diagram*

$$\begin{array}{ccccc}
\operatorname{Spec} R & \xrightarrow{\ \mathcal{S}\ } & \mathsf{Spc}\,\mathsf{D}^-(R) & \xrightarrow{\ \mathfrak{s}\ } & \operatorname{Spec} R \\
& {\scriptstyle \mathcal{S}'} \searrow & \uparrow {\scriptstyle \mathsf{inc}} & \nearrow {\scriptstyle \mathfrak{s}'} & \\
& & {}^{\mathsf{t}}\mathsf{Spc}\,\mathsf{D}^-(R) & &
\end{array}$$

such that $\mathcal{S}'$ *is an open bijection,* $\mathfrak{s}'$ *is a continuous bijection and* $\mathfrak{s}$ *is a continuous map. In particular, the image of* $\mathcal{S}$ *coincides with* ${}^{\mathsf{t}}\mathsf{Spc}\,\mathsf{D}^-(R)$*.*

(3) *There is a commutative diagram*

$$\begin{array}{ccc}
\operatorname{Min} R & \xrightarrow{\ \mathcal{S}_{\min}\ } & \mathtt{Mx}\,\mathsf{D}^-(R) \\
{\scriptstyle \mathsf{inc}} \downarrow & & \downarrow {\scriptstyle \mathsf{inc}} \\
\operatorname{Spec} R & \xrightarrow{\ \mathcal{S}\ } & \mathsf{Spc}\,\mathsf{D}^-(R) \\
{\scriptstyle \mathsf{inc}} \uparrow & & \uparrow {\scriptstyle \mathsf{inc}} \\
\operatorname{Max} R & \xhookrightarrow{\ \mathcal{S}_{\max}\ } & \mathtt{Mn}\,\mathsf{D}^-(R)
\end{array}$$

such that $\mathcal{S}_{\min}$ *is a homeomorphism, and the injective map* $\mathcal{S}_{\max}$ *is also a homeomorphism if* R *is a semilocal ring.*

(4) *The following are equivalent.*

(a) *The map* $\mathcal{S}$ *is continuous.*
(b) *The map* $\mathcal{S}'$ *is homeomorphic.*
(c) *The map* $\mathfrak{s}'$ *is homeomorphic.*
(d) *The set* $\operatorname{Spec} R$ *is finite.*

Here are several comments on this theorem.

Remark 3.4.

(1) Recall that for a topological space X the *Krull dimension* $\dim X$ of X is by definition the supremum of the lengths of chains of nonempty irreducible closed subsets of X. For a tensor triangulated category $\mathcal{T}$ we have

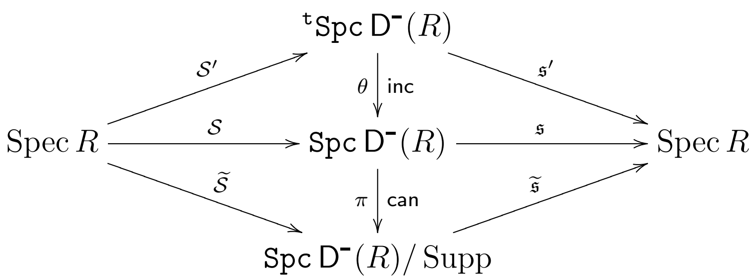
$$\begin{aligned}\dim(\mathtt{Spc}\,\mathcal{T}) &= \sup\{n \geq 0 \mid \exists\,\text{chain } \overline{\{\mathcal{P}_0\}} \subsetneq \cdots \subsetneq \overline{\{\mathcal{P}_n\}} \text{ of subsets of } \mathtt{Spc}\,\mathcal{T}\} \\ &= \sup\{n \geq 0 \mid \exists\,\text{chain } \mathcal{P}_0 \subsetneq \cdots \subsetneq \mathcal{P}_n \text{ of points of } \mathtt{Spc}\,\mathcal{T}\}.\end{aligned}$$

(2) Note that $\operatorname{Max} R$, $\operatorname{Min} R$, $\mathtt{Mx}\,\mathcal{T}$ and $\mathtt{Mn}\,\mathcal{T}$ are all T_1-spaces, and that, in general, any finite subset of a T_1-space is closed. Thus, to show Theorem 3.3(3), it is enough to check that the top and bottom horizontal maps are bijective and injective, respectively (after we verify that they are induced).

(3) The following are equivalent ([10, Lemma 4.6]).
- All specialization-closed subsets of $\operatorname{Spec} R$ are closed.
- There are only finitely many specialization-closed subsets of $\operatorname{Spec} R$.
- There are only finitely many closed subsets of $\operatorname{Spec} R$.
- There are only finitely many prime ideals of R.

Using this equivalences, we can deduce Theorem 3.3(4).

(4) More precisely than Theorem 3.3(1), we actually have a commutative diagram



such that $\mathfrak{s}\mathcal{S}$ is identity, $\mathcal{S}', \widetilde{\mathcal{S}}, \mathfrak{s}', \widetilde{\mathfrak{s}}, \pi\theta$ are bijections, $\mathcal{S}', \widetilde{\mathcal{S}}$ are open and closed, and $\mathfrak{s}, \mathfrak{s}', \widetilde{\mathfrak{s}}$ are continuous ([10, Theorem 4.5]). Here, $\mathtt{Spc}\,\mathsf{D}^{-}(R)/\operatorname{Supp}$ denotes the quotient topological space by the equivalence relation induced by taking $\operatorname{Supp}(-)$, and π the canonical surjection. (To be precise, we define a relation $\sim$ in $\mathtt{Spc}\,\mathsf{D}^{-}(R)$ by

$$\mathcal{P} \sim \mathcal{Q} \iff \operatorname{Supp}\mathcal{P} = \operatorname{Supp}\mathcal{Q}$$

for $\mathcal{P}, \mathcal{Q} \in \mathtt{Spc}\,\mathsf{D}^{-}(R)$. Then $\sim$ is an equivalence relation in $\mathtt{Spc}\,\mathsf{D}^{-}(R)$. We denote by $\mathtt{Spc}\,\mathsf{D}^{-}(R)/\sim$ the set of equivalence classes, and by $\pi : \mathtt{Spc}\,\mathsf{D}^{-}(R) \to \mathtt{Spc}\,\mathsf{D}^{-}(R)/\sim$ the map sending each $\mathcal{P} \in \mathtt{Spc}\,\mathsf{D}^{-}(R)$ to its equivalence class $[\mathcal{P}] \in \mathtt{Spc}\,\mathsf{D}^{-}(R)/\sim$. The set $\mathtt{Spc}\,\mathsf{D}^{-}(R)/\sim$ is a topological space, where a subset S of $\mathtt{Spc}\,\mathsf{D}^{-}(R)/\sim$ is open if and only if $\pi^{-1}(S)$ is an open subset of $\mathtt{Spc}\,\mathsf{D}^{-}(R)$.)

(5) More precisely than Theorem 3.3(4), the following assertion holds true ([10, Theorem 4.7]). Consider the three conditions

(a) The map $\widetilde{\mathcal{S}}$ is a homeomorphism,

(b) The map $\widetilde{\mathfrak{s}}$ is a homeomorphism,
(c) The map $\pi\theta$ is a homeomorphism.

Then (a) is equivalent to (b), and (b)$\wedge$(c) is equivalent to the four conditions in Theorem 3.3(4).

Suppose that R is artinian. Then R is semilocal, has only finitely many prime ideals and satisfies $\operatorname{Max} R = \operatorname{Spec} R = \operatorname{Min} R$. Hence, Theorem 3.3 yields the following corollary.

Corollary 3.5. *Let R be an artinian ring. Then the following statements hold true.*

(1) *The maps $\mathcal{S} : \operatorname{Spec} R \rightleftarrows \mathsf{Spc}\,\mathsf{D}^{-}(R) : \mathfrak{s}$ are mutually inverse homeomorphisms.*
(2) *One has $\dim \mathsf{Spc}\,\mathsf{D}^{-}(R) = \dim R = 0 < \infty$.*
(3) *All prime ideals of $\mathsf{D}^{-}(R)$ are tame.*

In fact, a more complete statement holds true; see Theorem 3.11.

3.2. **Classifications of ideals of $\mathsf{D}^{-}(R)$.**
In this subsection, we consider making correspondences among compact, radical and tame ideals of $\mathsf{D}^{-}(R)$, and specialization-closed subsets of $\operatorname{Spec} R$, $\mathsf{Spc}\,\mathsf{D}^{-}(R)$ and ${}^{\mathsf{t}}\mathsf{Spc}\,\mathsf{D}^{-}(R)$. First of all, we explore the relationships among these three properties of ideals of $\mathsf{D}^{-}(R)$.

Proposition 3.6 ([10, Lemma 5.8])**.** *Let $\mathcal{X}$ be an ideal of $\mathsf{D}^{-}(R)$.*

(1) *There are equalities of ideals of $\mathsf{D}^{-}(R)$:*

$$\mathcal{X}_{\mathsf{cpt}} = \langle \operatorname{Supp} \mathcal{X} \rangle,$$
$$\mathcal{X}^{\mathsf{rad}} = \sqrt{\mathcal{X}},$$
$$\mathcal{X}^{\mathsf{tame}} = \operatorname{Supp}^{-1} \operatorname{Supp} \mathcal{X}.$$

(2) *There are inclusions*

$$\mathcal{X}_{\mathsf{cpt}} \subseteq \mathcal{X} \subseteq \mathcal{X}^{\mathsf{rad}} \subseteq \mathcal{X}^{\mathsf{tame}}$$

of ideals of $\mathsf{D}^{-}(R)$, all of whose supports are equal. In particular, every tame ideal of $\mathsf{D}^{-}(R)$ is radical.

Here, $\mathcal{X}^{\mathbb{P}}$ (respectively, $\mathcal{X}_{\mathbb{P}}$) stands for the $\mathbb{P}$-*closure* (respectively, $\mathbb{P}$-*interior*) of $\mathcal{X}$, namely, the smallest (respectively, largest) $\mathbb{P}$-ideal containing (respectively, contained in) $\mathcal{X}$. Also, cpt and rad denote the compact and radical properties, respectively.

The assertion (1) of the above proposition is seen to hold just by checking the defintions. In relation to (2), the following statement holds: Let W be a specialization-closed subset of $\operatorname{Spec} R$. Then $\langle W \rangle$ (respectively, $\operatorname{Supp}^{-1} W$) is the smallest (respectively, largest) ideal of $\mathsf{D}^{-}(R)$ whose support coincides with W; see [10, Theorem 6.6(2)].

To state the main result of this section, we introduce notation.

Notation 3.7. We use the following sets in the rest of this subsection.

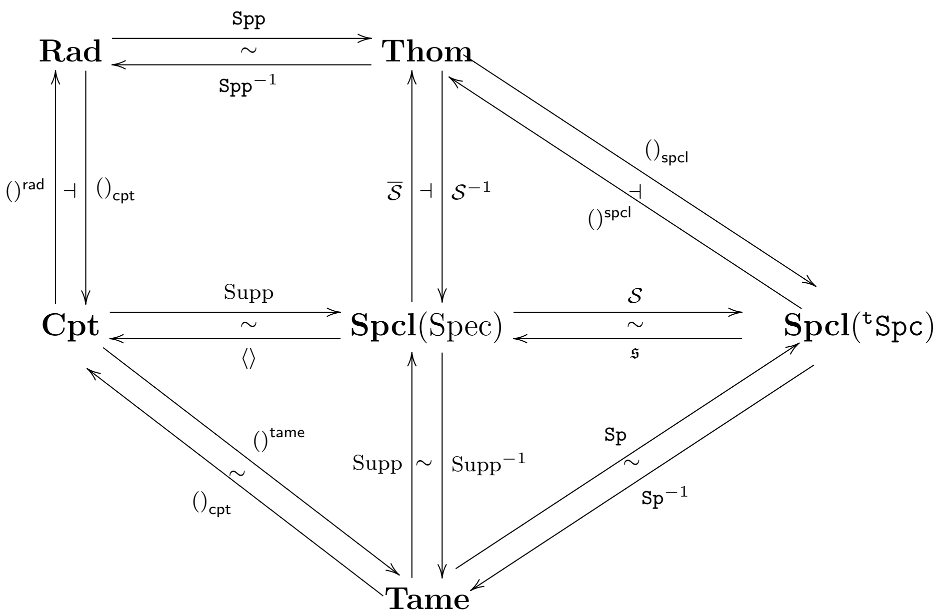
$$\mathbf{Rad} = \{\text{Radical ideals of } \mathsf{D}^{-}(R)\},$$
$$\mathbf{Tame} = \{\text{Tame ideals of } \mathsf{D}^{-}(R)\},$$
$$\mathbf{Cpt} = \{\text{Compact ideals of } \mathsf{D}^{-}(R)\},$$
$$\mathbf{Spcl}(\mathrm{Spec}) = \{\text{Specialization-closed subsets of } \operatorname{Spec} R\},$$
$$\mathbf{Spcl}({}^{\mathsf{t}}\mathsf{Spc}) = \{\text{Specialization-closed subsets of } {}^{\mathsf{t}}\mathsf{Spc}\,\mathsf{D}^{-}(R)\},$$
$$\mathbf{Thom} = \{\text{Thomason subsets of } \mathsf{Spc}\,\mathsf{D}^{-}(R)\}.$$

Proposition 3.6(2) implies the inclusion

$$\mathbf{Rad} \supseteq \mathbf{Tame}.$$

The main result of this section makes correspondences among the above six sets.

Theorem 3.8. [10, Theorems 5.13 and 5.20] *One has the following diagram, which is naturally commutative. (More precisely, the diagram with sections and bijections and the diagram with retractions and bijections are commutative.)*



Here:

- $\begin{cases} f \sim g \iff gf = 1 \text{ and } fg = 1 \text{ (i.e. } (f,g) \text{ is a bijection pair)}, \\ f \dashv g \iff gf = 1 \text{ (i.e. } (f,g) \text{ is a section-retraction pair)}. \end{cases}$
- $\begin{cases} A_{\mathsf{spcl}} = \text{ the } \mathsf{spcl}\text{-interior of } A \text{ in } {}^{\mathsf{t}}\mathsf{Spc}\,\mathsf{D}^{-}(R), \\ B^{\mathsf{spcl}} = \text{ the } \mathsf{spcl}\text{-closure of } B \text{ in } \mathsf{Spc}\,\mathsf{D}^{-}(R). \end{cases}$
- $\begin{cases} \overline{\mathcal{S}}(W) = \bigcup_{\mathfrak{p}\in W} \overline{\{\mathcal{S}(\mathfrak{p})\}}, \\ \mathcal{S}^{-1}(A) = \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathcal{S}(\mathfrak{p}) \in A\}, \\ \mathsf{Sp}(-) = \mathsf{Spp}(-) \cap {}^{\mathsf{t}}\mathsf{Spc}\,\mathsf{D}^{-}(R), \\ \mathsf{Sp}^{-1}(B) = \{M \in \mathsf{D}^{-}(R) \mid \mathsf{Sp}\,M \subseteq B\}, \\ \mathcal{S}(W) = \{\mathcal{S}(\mathfrak{p}) \mid \mathfrak{p} \in W\}, \\ \mathfrak{s}(B) = \{\mathfrak{s}(\mathcal{P}) \mid \mathcal{P} \in B\}. \end{cases}$

Moreover, the following are equivalent.

(1) *The pair* $\mathcal{S} : \operatorname{Spec} R \rightleftarrows \mathsf{Spc}\,\mathsf{D}^{-}(R) : \mathfrak{s}$ *of maps is a one-to-one correspondence.*
(2) *The pair* $(()^{\mathsf{rad}}, ()_{\mathsf{cpt}})$ *of maps is a one-to-one correspondence.*
(3) *The pair* $(\overline{\mathcal{S}}, \mathcal{S}^{-1})$ *of maps is a one-to-one correspondence.*
(4) *The pair* $(()^{\mathsf{spcl}}, ()_{\mathsf{spcl}})$ *of maps is a one-to-one correspondence.*
(5) *The equality* $\mathbf{Rad} = \mathbf{Tame}$ *holds.*

The spcl-*interior* A_{spcl} is the largest specialization-closed subset of ${}^{\mathsf{t}}\mathsf{Spc}\,\mathsf{D}^{-}(R)$ contained in A, while spcl-*closure* B^{spcl} is the smallest specialization-closed subset of $\mathsf{Spc}\,\mathsf{D}^{-}(R)$ containing B.

Here are some comments on the above theorem.

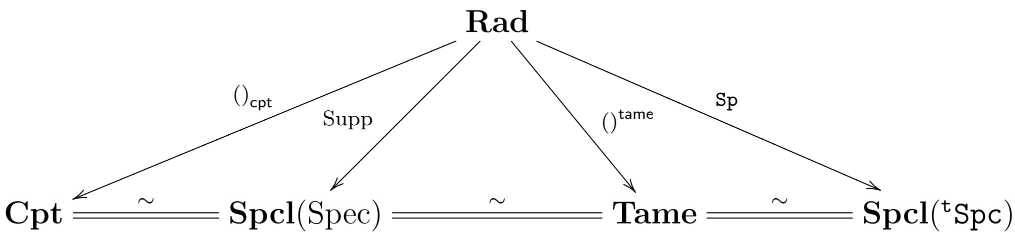
Remark 3.9.

(a) The one-to-one correspondence **Rad** $\cong$ **Thom** in the diagram of Theorem 3.8 is nothing but Theorem 1.6 due to Balmer, while the one-to-one correspondence **Cpt** $\cong$ **Spcl**(Spec) is nothing but Theorem 2.4. Thus this diagram connects Theorems 1.6 and 2.4, and gives rise to several related correspondences.

(b) The proof of Theorem 3.8 proceeds step by step; for example, we show and use the equalities

$$\begin{cases} A_{\mathsf{spcl}} = A \cap {}^{\mathsf{t}}\mathsf{Spc}\,\mathsf{D}^{-}(R), \\ B^{\mathsf{spcl}} = \{\mathcal{P} \in \mathsf{Spc}\,\mathsf{D}^{-}(R) \mid \mathcal{P}^{\mathsf{tame}} \in B\} = \bigcup_{\mathcal{P} \in B^{\mathsf{spcl}}} \mathsf{Spp}(R/\mathfrak{s}(\mathcal{P})). \end{cases}$$

(c) Theorem 3.8 yields a commutative diagram



where the bottom bijections are the ones in the diagram of Theorem 3.8. Furthermore, the conditions (1)–(5) in Theorem 3.8 are also equivalent to the following three conditions.

(6) The map Supp : **Rad** $\to$ **Spcl**(Spec) is a bijection.
(7) The map $()^{\mathsf{tame}}$: **Rad** $\to$ **Tame** is a bijection.
(8) The map Sp : **Rad** $\to$ **Spcl**(${}^{\mathsf{t}}\mathsf{Spc}$) is a bijection.

For the details, we refer the reader to [10, Corollary 5.21].

The corollary below is immediately obtained from the above theorem.

Corollary 3.10. *If every radical ideal of* $\mathsf{D}^{-}(R)$ *is compact, then* **Rad** $=$ **Tame**.

Proof. For each radical ideal $\mathcal{X}$ of $\mathsf{D}^{-}(R)$ one has $\mathcal{X} = \mathcal{X}_{\mathsf{cpt}} = (\mathcal{X}_{\mathsf{cpt}})^{\mathsf{rad}}$. Hence

$$()^{\mathsf{rad}} : \mathbf{Cpt} \rightleftarrows \mathbf{Rad} : ()_{\mathsf{cpt}}$$

is a one-to-one correspondence. Theorem 3.8 implies **Rad** $=$ **Tame**. □

We are interested in what rings R are characterized by the eight conditions (1)–(8) appearing in Theorem 3.8 and Remark 3.9.

Theorem 3.11 ([10, Theorem 6.5])**.** *The equivalent conditions* (1)–(8) *are also equivalent to the condition that*

(9) *the ring* R *is artinian.*

Furthermore, when this is the case, every ideal of $\mathsf{D}^{-}(R)$ *is compact, tame and radical.*

The most difficult part of the proof of this theorem is to show the necessity of the condition (9). Here, let us only check the last assertion of the theorem. Suppose that R is artinian, and pick any ideal $\mathcal{X}$ of $\mathsf{D}^-(R)$. Then Corollary 3.5(3) implies that $\mathcal{X}$ is compact, and that taking $\operatorname{Supp}(-)$ makes an injective map. Hence the equality

$$\operatorname{Supp}(\mathcal{X}) = \operatorname{Supp}(\operatorname{Supp}^{-1} \operatorname{Supp} \mathcal{X})$$

implies that $\mathcal{X}$ coincides with $\operatorname{Supp}^{-1} \operatorname{Supp} \mathcal{X}$, which shows that $\mathcal{X}$ is tame. In general, a tame ideal of $\mathsf{D}^-(R)$ is radical, and hence $\mathcal{X}$ is radical.

Using Theorem 3.11 and Corollary 3.10, we immediately obtain the following.

Corollary 3.12. *Suppose that R is not artinian. Then there exists a non-compact radical ideal of $\mathsf{D}^-(R)$.*

3.3. **On Balmer's conjecture for $\mathsf{D}^-(R)$.**

From now on, we consider Balmer's conjecture stated in Section 1 for our tensor triangulated category $\mathsf{D}^-(R)$. First of all, we investigate the difference between radical and tame ideals of $\mathsf{D}^-(R)$. We have already learned that the following holds.

$$\mathcal{X}^{\mathsf{rad}} \subseteq \mathcal{X}^{\mathsf{tame}}.$$

The following theorem says that if $\mathcal{X}$ is compact, then the equality does not hold under mild assumptions.

Theorem 3.13 ([10, Theorem 6.6])**.** *Let W be a nonempty proper specialization-closed subset of $\operatorname{Spec} R$, and put $\mathcal{X} = \langle W \rangle$. Assume that R is either a domain or a local ring. Then*

$$\mathcal{X}^{\mathsf{rad}} \subsetneq \mathcal{X}^{\mathsf{tame}}.$$

Proof. Since W is nonempty, it contains a prime ideal P. Take a system of generators $\boldsymbol{x} = x_1, \dots, x_r$ of P. It is essential to think of the following complex.

$$C := \bigoplus_{i>0} \mathrm{K}(\boldsymbol{x}^i)[i].$$

Thanks to the shifts, this infinite direct sum exists in our tensor triangulated category $\mathsf{D}^-(R)$. Since $\operatorname{Supp} C = \mathrm{V}(P)$ is contained in W, the complex C is in $\operatorname{Supp}^{-1} W = \mathcal{X}^{\mathsf{tame}}$ by Proposition 3.6(1).

Suppose $\mathcal{X}^{\mathsf{rad}} = \mathcal{X}^{\mathsf{tame}}$. Then C belongs to $\mathcal{X}^{\mathsf{rad}} = \sqrt{\mathcal{X}}$. Hence there is an integer $n > 0$ such that

$$C' := \underbrace{C \otimes^{\mathbf{L}}_R \cdots \otimes^{\mathbf{L}}_R C}_{n}$$

belongs to $\mathcal{X}$. Note that C' contains

$$D := \bigoplus_{i>0} \mathrm{K}(\boldsymbol{x}^i)[ni]$$

as a direct summand. Therefore D is in $\mathcal{X} = \langle W \rangle = \mathsf{thick}^{\otimes}\{R/\mathfrak{p} \mid \mathfrak{p} \in W\}$, and we find a finite number of prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ in W such that

$$\operatorname{ann} D \supseteq (\operatorname{ann} R/\mathfrak{p}_1) \cdots (\operatorname{ann} R/\mathfrak{p}_m) = \mathfrak{p}_1 \cdots \mathfrak{p}_m.$$

Krull's intersection theorem implies

$$\operatorname{ann} D = \bigcap_{i>0} \boldsymbol{x}^i R = 0,$$

and we have $\mathfrak{p}_1 \cdots \mathfrak{p}_m = 0$. Thus for every prime ideal $\mathfrak{p}$ of R there exists an integer $1 \le t \le m$ such that $\mathfrak{p}$ contains $\mathfrak{p}_t$. Since W is specialization-closed and contains $\mathfrak{p}_t$, the prime ideal $\mathfrak{p}$ belongs to W. This shows that $W = \operatorname{Spec} R$, contrary to the assumption on W. □

Using the above proof, we have an observation related to Proposition 2.7.

Remark 3.14. We use the same notation as in the proof of Theorem 3.13.

(1) It holds that $\operatorname{Supp} C$ is contained in $\operatorname{Supp} R/P$, but C does not belong to $\mathsf{thick}^{\otimes} R/P$. Indeed, we have $\operatorname{Supp} C = \mathrm{V}(P) = \operatorname{Supp} R/P$. Assume that C is in $\mathsf{thick}^{\otimes} R/P$. Then

$$0 = \operatorname{ann} C \supseteq (\operatorname{ann} R/P)^u = P^u$$

for some integer $u > 0$. Hence the equality $\operatorname{Spec} R = \mathrm{V}(P)$ holds, which is contained in W since W is specialization-closed. Therefore W coincides with $\operatorname{Spec} R$, which is a contradiction.

(2) It holds that $\mathrm{V}(\operatorname{ann} R)$ is contained in $\mathrm{V}(\operatorname{ann} C)$, but R does not belong to $\mathsf{thick}^{\otimes} C$. In fact, we have $\mathrm{V}(\operatorname{ann} R) = \mathrm{V}(0) = \mathrm{V}(\operatorname{ann} C)$. As $\operatorname{Supp} C = \mathrm{V}(P)$ is a proper subset of $\operatorname{Spec} R$, it is observed from Corollary 2.9(1) that R is not in $\mathsf{thick}^{\otimes} C$.

Now, we consider Balmer's conjecture (Conjecture 1.8) for our tensor triangulated category $\mathsf{D}^-(R)$. First of all, let us check that the triangulated category $\mathsf{D}^-(R)$ is algebraic. The category $\mathsf{C}^-(R)$ of right bounded complexes of finitely generated R-modules is a Frobenius exact category with respect to the split short exact sequences of complexes in $\mathsf{C}^-(R)$, and $\mathsf{K}^-(R)$ is the stable category of $\mathsf{C}^-(R)$. Thus $\mathsf{K}^-(R)$ is an algebraic triangulated category.

Recall that Conjecture 1.8 concerns the continuous map

$$\rho^{\bullet}_{\mathsf{D}^-(R)} : \mathsf{Spc}\,\mathsf{D}^-(R) \to \operatorname{Spec}^{\mathrm{h}} \mathrm{R}^{\bullet}_{\mathsf{D}^-(R)}.$$

One can actually observe that

(a) $\mathrm{R}^{\bullet}_{\mathsf{D}^-(R)} = \mathrm{R}^{0}_{\mathsf{D}^-(R)} = R$,
(b) $\operatorname{Spec}^{\mathrm{h}} \mathrm{R}^{\bullet}_{\mathsf{D}^-(R)} = \operatorname{Spec} R$, and
(c) $\rho^{\bullet}_{\mathsf{D}^-(R)} = \mathfrak{s}$.

Thus, Conjecture 1.8 for $\mathsf{D}^-(R)$ just claims the local injectivity of the map $\mathfrak{s}$.

We can show that under quite mild assumptions the algebraic tensor triangulated category $\mathsf{D}^-(R)$ does not satisfy Balmer's conjecture.

Corollary 3.15 ([10, Corollary 6.10])**.** *Assume that* $\dim R > 0$*, and that* R *is either a domain or a local ring. Then* $\mathfrak{s}$ *is not locally injective. Hence, Balmer's Conjecture 1.8 does not hold true for* $\mathsf{D}^-(R)$.

Proof. By assumption we find a nonunit $x \in R$ such that the principal ideal xR of R has positive height. We apply Theorem 3.13 to $\mathcal{X} = \langle \mathrm{V}(x) \rangle$ to get

$$\bigcap_{\mathcal{X} \subseteq \mathcal{P} \in \mathsf{Spc}\,\mathsf{D}^-(R)} \mathcal{P} = \mathcal{X}^{\mathsf{rad}} \subsetneq \mathcal{X}^{\mathsf{tame}} = \bigcap_{\mathcal{X} \subseteq \mathcal{P} \in {}^{\mathsf{t}}\mathsf{Spc}\,\mathsf{D}^-(R)} \mathcal{P}.$$

Hence we can choose a prime ideal $\mathcal{P}$ of $\mathsf{D}^{-}(R)$ such that $\mathcal{X} \subseteq \mathcal{P} \subsetneq \mathcal{P}^{\mathsf{tame}}$.

Assume that $\mathfrak{s}$ is locally injective at the point $\mathcal{P}$. Then there exists an object $M \in \mathsf{D}^{-}(R)$ with $\mathcal{P} \in \mathtt{U}(M)$ such that $\mathfrak{s}|_{\mathtt{U}(M)}$ is injective. Then $\mathtt{U}(M)$ contains two distinct points $\mathcal{P}$ and $\mathcal{P}^{\mathsf{tame}}$, which are sent by $\mathfrak{s}$ to the same point in $\operatorname{Spec} R$. This contradicts the injectivity of the map $\mathfrak{s}|_{\mathtt{U}(M)}$. □

We end this section by stating a bit about the case where R is a discrete valuation ring. Since everything is clarified when R is artinian, the case of discrete valuation rings should be the first nontrivial case, but in fact, it turns out that even in this case the structure of $\mathsf{D}^{-}(R)$ is highly complicated. For the details, we refer the reader to [10, Section 7].

References

[1] P. Balmer, Presheaves of triangulated categories and reconstruction of schemes, *Math. Ann.* **324** (2002), no. 3, 557–580.

[2] P. Balmer, The spectrum of prime ideals in tensor triangulated categories, *J. Reine Angew. Math.* **588** (2005), 149–168.

[3] P. Balmer, Spectra, spectra, spectra–tensor triangular spectra versus Zariski spectra of endomorphism rings, *Algebr. Geom. Topol.* **10** (2010), no. 3, 1521–1563.

[4] P. Balmer, Tensor triangular geometry, *Proceedings of the International Congress of Mathematicians, Volume II*, 85–112, *Hindustan Book Agency, New Delhi*, 2010.

[5] D. J. Benson; J. F. Carlson; J. Rickard, Thick subcategories of the stable module category, *Fund. Math.* **153** (1997), no. 1, 59–80.

[6] D. J. Benson; S. B. Iyengar; H. Krause, Stratifying modular representations of finite groups, *Ann. of Math. (2)* **174** (2011), no. 3, 1643–1684.

[7] D. Benson; S. B. Iyengar; H. Krause; J. Pevtsova, Stratification for module categories of finite group schemes, *J. Amer. Math. Soc.* **31** (2018), no. 1, 265–302.

[8] E. M. Friedlander; J. Pevtsova, Π-supports for modules for finite group schemes, *Duke Math. J.* **139** (2007), no. 2, 317–368.

[9] M. J. Hopkins, Global methods in homotopy theory, *Homotopy theory (Durham, 1985)*, 73–96, London Math. Soc. Lecture Note Ser., 117, *Cambridge Univ. Press, Cambridge*, 1987.

[10] H. Matsui; R. Takahashi, Thick tensor ideals of right bounded derived categories, *Algebra Number Theory* **11** (2017), no. 7, 1677–1738.

[11] A. Neeman, The chromatic tower for $D(R)$, With an appendix by Marcel Bökstedt, *Topology* **31** (1992), no. 3, 519–532.

[12] S. Oppermann; J. Šťovíček, Duality for bounded derived categories of complete intersections, *Bull. Lond. Math. Soc.* **46** (2014), no. 2, 245–257.

[13] R. W. Thomason, The classification of triangulated subcategories, *Compos. Math.* **105** (1997), no. 1, 1–27.

Graduate School of Mathematics, Nagoya University, Furocho, Chikusaku, Nagoya, Aichi 464-8602, Japan

E-mail address: takahashi@math.nagoya-u.ac.jp

URL: http://www.math.nagoya-u.ac.jp/~takahashi/