String Theory and Moonshine Phenomenon

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About five years ago together with my collaborators I have found some curious phenomenon in string theory, i.e. appearance of exotic discrete symmetry in the theory [1]. This phenomenon is now called as Mathieu moonshine and is under intensive study. Today I would like to give you a brief introduction to moonshine phenomena which possibly play interesting role in string theory in the future.

Before going into the moonshine phenomenon in string theory let me briefly recall the story of monstrous moonshine which is very well-known. Modular J function has a q-series expansion

$$\begin{aligned} J(q) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + 864299970q^3 \\ +20245856256q^4 + 333202640600q^5 + \cdots \end{aligned}$$

$$q = e^{2\pi i\tau}, \;\; Im(\tau) > 0, \;\; J(\tau) = J(\frac{a\tau + b}{c\tau + d}), \;\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, Z)$$

It turns out q-expansion coefficients of J-function are decomposed into a sum of dimensions of irreducible representations of the monster group M as

$$\begin{aligned} &196884 = 1 + 196883, \quad 21493760 = 1 + 196883 + 21296876, \\ &864299970 = 2 \times 1 + 2 \times 196883 + 21296876 + 842609326, \\ &20245856256 = 1 \times 1 + 3 \times 196883 + 2 \times 21296876 \\ &\quad +842609326 + 19360062527, \cdots \end{aligned}$$

Dimensions of some irreducible representations of monster are in fact given by

$$\{1, 196883, 21296876, 842609326, \\ 18538750076, 19360062527 \cdots\}$$

Monster group is the largest sporadic discrete group, of order $\approx 10^{53}$ and the strange relationship between modular form and the largest discrete group was first noted by McKay.

To be precise we may write as

$$J_1(\tau) = J(q) - 744 = \sum_{n=-1} c(n)q^n, \qquad\qquad c(0) = 0$$
$$= \sum_{n=-1} Tr_{V(n)}\, 1 \times q^n, \qquad\qquad dimV(n) = c(n)$$

McKay-Thompson series is given by

$$J_g(\tau) = \sum_{n=-1} Tr_{V(n)}\, g \times q^n, \quad g \in M$$

where $Tr_{V(n)}\, g$ denotes the character of a group element g in the representation $V(n)$. This depends on the conjugacy class g of M. If McKay-Thompson series is known for all conjugacy classes, decomposition of $V(n)$ into irreducible representations become uniquely determined. Series J_g are modular forms with respect to subgroups of $SL(2,Z)$ and possess similar properties like the modular J-function such as the genus=0 (Hauptmodul) property.

Phenomenon of monstrous moonshine has been understood mathematically in early 1990's using the technology of vertex operator algebra. However, we still do not have a 'simple' physical explanation of this phenomenon.

1 Mathieu moonshine

K_3 surface :

We consider string theory compactified on K_3 surface. K_3 surface is a complex 2-dimensional hyperKähler manifold and ubiquitous in string theory. It possesses $SU(2)$ holonomy and a holomorphic 2-form. Thus the string theory on K_3 has an $\mathcal{N}$=4 superconformal symmetry with the central charge $c = 6$ which contains $SU(2)_{k=1}$ affine symmetry.

Now instead of modular J-function we consider the elliptic genus of K_3 surface. Elliptic genus describes the topological invariants of the target manifold and counts the number of BPS states in the theory. Using world-sheet variables it is written as

$$Z_{elliptic}(z;\tau) = Tr_{\mathcal{H}_L \times \mathcal{H}_R}(-1)^{F_L+F_R} e^{4\pi i z J^3_{L,0}} q^{L_0-\frac{c}{24}} \bar{q}^{\bar{L}_0-\frac{c}{24}}$$

Here L_0 denotes the zero mode of the Virasoro operators and F_L and F_R are left and right moving fermion numbers. J^3_0 denotes the Cartan generator of affine $SU(2)_1$. In elliptic genus the right moving sector is frozen to the supersymmetric ground states (BPS states) while in the left moving sector all the states in the left-moving Hilbert space $\mathcal{H}_L$ contribute.

In general it is difficult to compute elliptic genera, however, we were able to evaluate it by making use of Gepner models [2]. Elliptic genus is given by

$$Z_{K3}(\tau,z) = 8\left[\left(\frac{\theta_2(\tau,z)}{\theta_2(\tau,0)}\right)^2 + \left(\frac{\theta_3(\tau,z)}{\theta_3(\tau,0)}\right)^2 + \left(\frac{\theta_4(\tau,z)}{\theta_4(\tau,0)}\right)^2\right]$$

Here $\theta_i(\tau,z)$ are Jacobi theta functions.

We want to see how the Hilbert space $\mathcal{H}_L$ in elliptic genus decompose into irreducible representations of $\mathcal{N}$=4 superconformal algebra (SCA).

Highest weight states of $\mathcal{N}$=4 SCA are parametrized by the eigenvalues of L_0 and J_0^3.

$$L_0|h,\ell\rangle = h|h,\ell\rangle, \qquad\qquad J_0^3|h,\ell\rangle = \ell|h,\ell\rangle$$

There are two different types of representations in $c=6$ SCA.
In the *Ramond* sector

BPS (massless) rep.	$h=\frac{1}{4}$;	$\ell=0,\ \frac{1}{2}$
non-BPS (massive) rep.	$h>\frac{1}{4}$;	$\ell=\frac{1}{2}$

Character of a representation is defined as

$$Tr_{\mathcal{R}}(-1)^F q^{L_0} e^{4\pi i z J_0^3}$$

where $\mathcal{R}$ denotes the representation space.
Index is given by the value of the character at $z=0$,

$$\text{Index}(\mathcal{R}) = Tr_{\mathcal{R}}(-1)^F q^{L_0}$$

BPS representations have a non-vanishing index

$$\begin{aligned}\text{index (BPS, }\ell=0) &= 1\\ \text{index (BPS, }\ell=\frac{1}{2}) &= -2\end{aligned}$$

while non-BPS reps. have vanishing indices

$$\text{index (non-BPS, }\ell=\frac{1}{2}) = 0.$$

Characters are given explicitly as [3]

$$ch^{BPS}_{h=\frac{1}{4},\ell=0}(\tau,z) = \frac{\theta_1(z;\tau)^2}{\eta(\tau)^3}\mu(z;\tau)$$

where

$$\mu(z;\tau)=\frac{-ie^{\pi iz}}{\theta_1(z;\tau)}\sum_n(-1)^n\frac{q^{\frac{1}{2}n(n+1)}e^{2\pi inz}}{1-q^ne^{2\pi iz}}$$

while non-BPS characters are given by

$$ch^{non-BPS}_{h,\ell=\frac{1}{2}}=q^{h-\frac{3}{8}}\ \frac{\theta_1(z;\tau)^2}{\eta(\tau)^3},\qquad h>\frac{1}{4}$$

Function $\mu(\tau,z)$ is a typical example of Mock theta function (Lerch sum or

Appel function). Mock theta functions look like theta functions but they have anomalous modular transformation laws and are difficult to handle. Recently there were developments in understanding the nature of Mock theta functions due to Zwegers[4].

He has introduced a method of regularization which is similar to the ones used in physics. It improves the modular property of Mock theta functions so that they transform as analytic Jacobi forms.

Now let us make a decomposition of elliptic genus into a sum of characters of $\mathcal{N}$=4 representaions

$$Z_{K3}(\tau,z)=24ch^{BPS}_{h=\frac{1}{4},\ell=0}(\tau,z)+2\sum_{n\geq 0}A(n)ch^{non-BPS}_{h=\frac{1}{4}+n,\ell=\frac{1}{2}}(\tau,z)$$

At smaller values of n, expansion coefficients $A(n)$ may be found by direct series expansion of Z_{K3}. We find, $A(0)=-1$ and

n	1	2	3	4	5	6	7	8	$\cdots$
$A(n)$	45	231	770	2277	5796	13915	30843	65550	$\ldots$

(1)

Dimensions of some irreducible reps. of Mathieu group M_{24} appear in (1)

$$\text{dimensions}:\{\ 45\quad 231\quad 770\quad 990\quad 1771\quad 2024\quad 2277\quad 3312\quad 3520\quad 5313\quad 5544\quad 5796\quad 10395\quad\cdots\}\qquad (2)$$

$$A(6)=13915=3520+10395,$$
$$A(7)=30843=10395+5796+5544+5313+2024+1771$$

[1]

M_{24} is a subgroup of S_{24} (permutation group of 24 objects) and its order is given by $\approx 10^9$.

M_{24} is known for its many interesting arithmetic properties and in particular intimately tied to the Golay code of efficient error corrections.

Monster $\supset$ Conway $\supset$ Mathieu $\supset\cdots$

2 Mathieu moonshine conjecture

Expansion coefficients of K_3 elliptic genus into $\mathcal{N}$=4 characters are given by the sum of dimensions of representations of Mathieu group M_{24}

We were able to derive analogues of McKay-Thompson series [5, 6]. And then the multiplicities $C_R(n)$ of the decomposition of $A(n)$ into representations R

$$A(n) = \sum_R C_R(n)\, dimR$$

were unambiguously determined. It tuned out that $C_R(n)$ are all positive integers up to $n \approx 1000$ and this gives a very strong evidence of Mathieu moonshine conjecture.

The conjecture is now proved mathematically using the method of mathematical induction. [7]

Unfortunately the proof so far did not provide much insight into the nature of Mathieu moonshine. The situation looks a bit like the case of monstrous moonshine. 24 of M_{24} will certainly be the Euler number of K_3 and M_{24} permutes homology classes. There are, however, various indications that string theory on K_3 can not have such a high symmetry as M_{24}. Instead of the total Hilbert space the BRS subsector of the theory may possibly possess an enhanced symmetry. It will be interesting to look into the algebraic structures of BPS states to explain Mathieu moonshine.

3 More Moonshine Phenomena

Mathieu moonshine exists at the intersection of string theory, K_3 surface (geometry), (Mock) modular forms and sporadic discrete symmetry and appears to possess interesting mixture of physics and mathematics. Recently there have been intense interests in exploring new types of moonshine phenomena other than Mathieu moonshine. Already several types of new moonshine phenomena have been discovered.

Umbral moonshine [8]

fermions on 24 dim. lattice

spin 7 manifold

Due to time limitation we discuss only about Umbral moonshine. Umbral moonshine has a mysterious relationship to the Niemeier lattice. It is known there are 23 (24, if we include Leech lattice) types of self-dual lattices in

24 dimensions. It is given by the combination of root lattices of A-D-E type together with appropriate weight vectors so that the lattice becomes self-dual. The simplest examples are

$$\begin{array}{ll} (A_1)^{24} & (k=1) \\ (A_2)^{12} & (k=2) \\ (A_3)^{8} & (k=3) \\ \cdots & \cdots \end{array}$$

etc. If one divides the automorphism groups of Niemeier lattice by the automorphism group of A-D-E lattice, one obtain isolated discrete groups

$$G_k = \frac{[\text{automorphism group of lattice}]_k}{[\text{Weyl group of root lattice}]_k}.$$

It turns out that G_k's become the symmetry groups of the Umbral moonshine. In fact the first one agrees with the Mathieu group $G_1 = M_{24}$ and reproduces the Mathieu moonshine. The second one G_2 agrees with the Mathieu group M_{12} and is assumed to be related to 4-dimensional hyperKähler manifold with $c = 12$ $(k = 2)$.
Analogue of K_3 elliptic genus is given by

$$Z(k=2) = 4\left[\left(\frac{\theta_2(z)\theta_3(z)}{\theta_2(0)\theta_3(0)}\right)^2 + \left(\frac{\theta_2(z)\theta_4(z)}{\theta_2(0)\theta_4(0)}\right)^2 + \left(\frac{\theta_3(z)\theta_4(z)}{\theta_3(0)\theta_4(0)}\right)^2\right]$$

By expanding $Z(k = 2)$ in terms of characters of representations of $c = 12$, $\mathcal{N} = 4$ algebra one finds the expansion coefficients decompose into the symmetry group M_{12}.

Here, however, there is something awkward: $Z(k = 2)$ does not contain the contribution of vacuum operator ($h = 0$ in NS sector) thus the theory appears to describe the geometry of a (singular) non-compact four-fold. The rest of Umbral moonshine series has the same property (absence of identity operator) and their geometrical interpretation is somewhat obscure.

Recently, we have used $\mathcal{N} = 4$ Liouville theory [9] which is known to possess some special duality property [10]. It is possible to embed Umbral series into $\mathcal{N} = 4$ Liouville theory and by using duality we can map Umbral theory at $c = 6k$ to its dual theory at $c = 6$. Thus a Umbral moonshine at $c = 6k$ can be mapped to a dual moonshine at $c = 2$. We hope this is going to help geometrical interpretation of Umbral moonshine.

Moonshine symmetries recently discovered in string theory are still very mysterious and we may encounter many more surprises in the near future.

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