ON THE ENERGY OF QUASICONFORMAL MAPPINGS AND PSEUDOHOLOMORPHIC CURVES IN COMPLEX PROJECTIVE SPACES

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Abstract. We prove that the energy density of uniformly continuous, quasiconformal mappings, omitting two points in $\mathbb{CP}^1$, is equal to zero. We also prove the sharpness of this result, constructing a family of uniformly continuous, quasiconformal mappings, whose areas grow asymptotically quadratically. Finally, we prove that the energy density of pseudoholomorphic Brody curves, omitting three "complex lines" in general position in $\mathbb{CP}^2$, is equal to zero.

According to the Picard Theorem, a holomorphic function f defined on the complex plane $\mathbb{C}$ is constant as soon as $f(\mathbb{C})$ omits at least three values in $\mathbb{CP}^1$. This result has different generalizations in at least two directions.

S.Rickman [11] proved that for every $n \geq 2$ and for every $K > 1$, a nonconstant entire K-quasiregular mapping in $\mathbb{R}^n$ omits at most $m = m(n, K)$ values.

M.Green [6] proved that a holomorphic map from $\mathbb{C}$ to the complex projective space $\mathbb{CP}^n$, omitting $(2n+1)$ hyperplanes in general position, is constant. An almost complex version of that result was proved by J.Duval [5] for entire pseudoholomorphic curves in the complement of five J-lines, in general position in $\mathbb{CP}^2$ endowed with an almost complex structure J tamed by the Fubini Study metric ω_{FS}.

Let f be a mapping defined on $\mathbb{C}$ with values in $\mathbb{CP}^n$, $f \in W^{1,2}_{loc}(\mathbb{C})$. We recall that if $D \subset\subset \mathbb{C}$, then $Area(f(D)) := \int_D f^*\omega_{FS}$ is the area of $f(D)$, counted with multiplicity. Then, the energy density $E(f)$ defined by

$$E(f) = \limsup_{R\to\infty} \frac{1}{\pi R^2} Area(f(D_R)) = \limsup_{R\to\infty} \frac{1}{\pi R^2} \int_{D_R} f^*\omega_{FS}$$

gives the infinity behaviour of the area of the map f. Here D_R denotes the open disk centered at the origin in $\mathbb{C}$, with radius R.

For instance, in case $n = 1$ we obtain:

$$E(f) = \limsup_{R\to\infty} \frac{1}{\pi R^2} \int_{D_R} f^*\left(\frac{dz \wedge d\overline{z}}{(1+|z|^2)^2}\right) = \limsup_{R\to\infty} \frac{1}{\pi R^2} \int_{D_R} (\rho(f)(z))^2 dz \wedge d\overline{z},$$

where $\rho(f)$ denotes the spherical derivative of f.

It is natural, for a non constant function f, to understand how the curve $f(\mathbb{C})$ spreads over the complex projective space, namely to study the value of the energy density $E(f)$ of f. That question was studied by the second named author [12, 13] in different complex manifolds. He proved for instance that $E(f) = 0$ for every holomorphic map f such that $f(\mathbb{C})$ omits $(n+1)$ hyperplanes in general position in $\mathbb{CP}^n$.

On another side, J.Clunie and W.Hayman [3] considered entire holomorphic functions f for which the spherical derivative $\rho(f)$ satisfies $\sup_{|z|=R} \rho(f(z)) < KR^\sigma$, where $K > 0$, $\sigma > -1$ and R

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is sufficiently large. For the definition of the spherical derivative, see Definition 1.2. They proved that given $K > 0$, every such entire holomorphic curve satisfies the estimate

$$T(R, f) = O(R^{\sigma+1}) \quad \text{as } R \to \infty.$$

Here $T(R, f)$ denotes the Shimizu-Ahlfors characteristic function of f, defined by

$$T(R, f) = \int_1^R \left(\int_{D_t} f^*\omega_{FS} \right) \frac{dt}{t}.$$

In particular, for every $1 < \alpha \leq 2$, every entire Brody curve $f : \mathbb{C} \to \mathbb{CP}^1$ (case where $\sigma = 0$) satisfies:

$$\lim_{R\to\infty} \frac{1}{R^\alpha} T(R, f) = 0. \tag{0.1}$$

The paper deals with the behaviour of the energy density of quasiconformal mappings in the complex projective space $\mathbb{CP}^1$ and of pseudoholomorphic entire curves in the complex projective space $\mathbb{CP}^2$.

From the definition of the Shimizu-Ahlfors characteristic function we have:

$$\forall R > 1,\ T(R, f) \geq \frac{1}{R} \int_{R/2}^R \left(\int_{D_t} f^*\omega_{FS} \right) dt \geq \frac{1}{2} \int_{D_{R/2}} f^*\omega_{FS}.$$

Hence, it follows from (0.1) that every entire Brody curve $f : \mathbb{C} \to \mathbb{CP}^1$ satisfies:

$$\forall \alpha > 1,\ \lim_{R\to\infty} \frac{1}{R^\alpha} \int_{D_R} f^*\omega_{FS} = 0. \tag{0.2}$$

The following Theorems 0.1 and 0.3 give generalizations of (0.2) for uniformly continuous quasiconformal mappings from $\mathbb{C}$ to $\mathbb{CP}^1$, omitting two points. We first prove in Theorem 0.1 that in case $\alpha = 2$ then the estimate (0.2) is still valid, namely that the energy of such uniformly continuous quasiconformal mappings is equal to zero. In Theorem 0.3 we prove that (0.2) is not valid anymore for $\alpha < 2$. Hence, the value $\alpha = 2$ is optimal for uniformly continuous quasiconformal mappings and that result presents a striking difference between holomorphic entire Brody curves and uniformly continuous quasiconformal mappings in $\mathbb{CP}^1$.

Theorem 0.1. *Let $f : \mathbb{C} \to \mathbb{CP}^1$ be a uniformly continuous, quasiconformal mapping. If $f(\mathbb{C})$ omits two points then:*

$$\lim_{R\to+\infty} \frac{1}{\pi R^2} \int_{D_R} f^*\omega_{FS} = 0.$$

We apply the method of the proof of Theorem 0.1 to the study of pseudoholomorphic entire curves. According to [12], the energy density of holomorphic Brody curves omitting three complex lines in general position in $\mathbb{CP}^2$ is equal to zero. We extend this result to pseudoholomorphic Brody curves omitting three almost complex lines in general position in $\mathbb{CP}^2$. We recall that a Brody curve is a pseudoholomorphic map from $(\mathbb{C}, J_{st})$ to $(\mathbb{CP}^2, J)$ with bounded derivative (see Section 1 for the precise definitions).

Theorem 0.2. *Let J be a smooth ($\mathcal{C}^\infty$) almost complex structure on $\mathbb{CP}^2$, tamed by ω_{FS}. Let $f : \mathbb{C} \to \mathbb{CP}^2$ be a smooth ($\mathcal{C}^\infty$) J-holomorphic Brody curve. If $f(\mathbb{C})$ omits three J-lines in general position, then:*

$$\lim_{R\to+\infty} \frac{1}{\pi R^2} \int_{D_R} f^*\omega_{FS} = 0.$$

As we already mentioned, the quadratic growth of the area of uniformly continuous quasiconformal mappings, given by Theorem 0.1, is optimal:

Theorem 0.3. *For every* $0 < \alpha < 2$, *there exists a uniformly continuous, quasiconformal mapping* $f : \mathbb{C} \to \mathbb{CP}^1$ *such that* $f(\mathbb{C})$ *omits two points and:*

$$\lim_{R \to +\infty} \frac{1}{R^\alpha} \int_{D_R} f^*\omega_{FS} = +\infty.$$

The proof of Theorem 0.3 relies on the construction of a family of quasiconformal curves for which the Hausdorff dimension tends asymptotically to two.

It is classically known (see [3, 1]) that if a Brody curve $f : \mathbb{C} \to \mathbb{CP}^n$ omits n hyperplanes in general position then

$$\int_{D_R} f^*\omega_{FS} = O(R) \quad \text{as } R \to \infty.$$

There exist some discrepancies between this classical result and the above Theorems 0.1 and 0.2. This suggests the following open problem.

Problem 0.4.

(1) Let $f : \mathbb{C} \to \mathbb{CP}^1$ be a uniformly continuous quasiconformal mapping. Suppose f omits one point. Can we prove that the energy growth satisfies the following?

$$\lim_{R \to \infty} \frac{1}{R^2} \int_{D_R} f^*\omega_{FS} = 0.$$

(2) Let J be a smooth almost complex structure on $\mathbb{CP}^2$, tamed by ω_{FS}. Let $f : \mathbb{C} \to \mathbb{CP}^2$ be a smooth J-holomorphic Brody curve. Suppose $f(\mathbb{C})$ omits two J-lines in general position. Can we prove that the energy growth satisfies the following?

$$\int_{D_R} f^*\omega_{FS} = O(R) \quad \text{as } R \to \infty.$$

In Section 2 we give the definitions. In Section 3 we prove Theorem 0.1 and Theorem 0.2. In Section 4 we prove Theorem 0.3.

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1. Definitions and preliminaries

The complex projective space $\mathbb{CP}^n$ will be endowed with the Fubini Study form denoted by ω_{FS} (not mentioning the complex dimension). The notion of uniform continuity for a map from $\mathbb{C}$ to $\mathbb{CP}^n$ will be defined with respect to the Euclidean distance on $\mathbb{C}$ and the Fubini Study distance on $\mathbb{CP}^n$. For $n = 1$, the Fubini Study distance is the spherical distance defined by

$$\begin{cases} d_{spher}(x, y) & = & \dfrac{2|x - y|}{\sqrt{1 + x^2}\sqrt{1 + y^2}} & \text{if } x, y \in \mathbb{C} \\ \\ d_{spher}(x, \infty) & = & \dfrac{2}{\sqrt{1 + x^2}} & \text{if } x \in \mathbb{C}. \end{cases}$$

Let f be a continuous map from an open set $G \subset \mathbb{C}$ to $\mathbb{CP}^1$ such that $f \in W^{1,2}_{loc}(G)$. We denote by ∂f (resp. $\overline{\partial} f$) the $\mathbb{C}$-linear derivative (resp. $\mathbb{C}$-anti linear derivative) of f.

Definition 1.1. If there is $K \geq 1$ such that for almost every $z \in G$:

$$|\overline{\partial} f|(z) \leq \frac{K - 1}{K + 1} |\partial f|(z)$$

then f is a K-quasiconformal mapping. Here $|\cdot|$ denotes the norm with respect to the Fubini Study metric on $\mathbb{CP}^1$.

We recall that given a real smooth manifold M of real dimension $2n$, a section J of the bundle $End(TM)$ is an almost complex structure if it satisfies $J^2 = -id_{TM}$. We denote by J_{st} the standard complex structure on $\mathbb{C}$.

Definition 1.2. Let J be an almost complex structure on $\mathbb{CP}^n$.

(i) J is tamed by ω_{FS} if $\omega_{FS}(v, J(x)v) > 0$, for every $x \in \mathbb{CP}^n$, $v \in T_x\mathbb{CP}^n \backslash \{0\}$.

(ii) A map $f : \mathbb{C} \to \mathbb{CP}^n$, of class $\mathcal{C}^1$, is a J-holomorphic entire curve if it satisfies
$$df \circ J_{st} = J(f) \circ df.$$

(iii) A J-holomorphic entire curve f is a *Brody curve* if there exists $C > 0$ such that $\|df\| \leq C$, where $\|df\| = \sup_{z\in\mathbb{C}} |df|(z)$ and $f^*\omega_{FS} = |df|^2 dx \wedge dy$. In case $n = 1$, $|df| = \rho(f)$, the *spherical derivative* of f.

(iv) A J-line in $\mathbb{CP}^2$ is a J-holomorphic entire curve embedded in $\mathbb{CP}^2$, diffeomorphic to $\mathbb{CP}^1$, with degree 1 in homology.

(v) Three J-lines in $\mathbb{CP}^2$ are in general position if they do not have triple intersection point.

Throughout the paper, we use the following notations. For a subset $\Omega \subset \mathbb{C}$ and a positive number r we set $r\Omega = \{rz/ \ \ z \in \Omega\}$.

Finally, for $r > 0$, we set $D_r := \{z \in \mathbb{C}/ \ \ |z| < r\}$.

2. Quadratic growth of the energy density of quasiconformal mappings and of pseudoholomorphic entire curves

The aim of this Section is to prove Theorem 0.1 and Theorem 0.2. Theorem 0.2 will be a consequence of Theorem 0.1, by considering some quasiconformal projections of the given Brody curve and following the same approach as the one considered in Theorem 0.1.

2.1. **Proof of Theorem 0.1.** Let $f : \mathbb{C} \to \mathbb{CP}^1$ be a uniformly continuous K-quasiconformal mapping omitting 0 and ∞. Here $K \geq 1$.

Lemma 2.1. *There is a holomorphic polynomial P, $deg(P) = d \geq 1$, and a K-quasiconformal homeomorphism h of $\mathbb{C}$ such that $f = \exp(P \circ h)$.*

Proof. This is very close to [5, Lemme analytique page 2364]. By [8, Chapter VI], we can write $f = \exp(P \circ h)$ where P is an entire function and h is a K-quasiconformal homeomorphism of the plane. The problem is how to show that P is a polynomial. Set $q(z) = \exp(P(z))$. If P is a transcendental entire function then q is of infinite order. So it is enough to show that q has finite order.

By [8, Chapter I, Theorem 8.1], h can be extended to a K-quasiconformal homeomorphism of the Riemann sphere $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ by setting $h(\infty) = \infty$. Then h and h^{-1} become $(1/K)$-Hölder homeomorphisms of the sphere by a theorem of Mori [10]. Considering a neighborhood around ∞, we can find $R_0 > 1$ and $C_0 > 1$ such that
$$C_0^{-1}\left|\frac{1}{z}\right|^K \leq \left|\frac{1}{h^{-1}(z)}\right| \leq C_0\left|\frac{1}{z}\right|^{1/K}, \quad (|z| \geq R_0)$$
$$\left|\frac{1}{h^{-1}(z)} - \frac{1}{h^{-1}(z')}\right| \leq C_0\left|\frac{1}{z} - \frac{1}{z'}\right|^{1/K}, \quad (|z|, |z'| \geq R_0).$$
It follows from these inequalities and from the condition $R_0 > 1$ that
$$|h^{-1}(z) - h^{-1}(z')| \leq C_0^3 \cdot |zz'|^K \cdot |z - z'|^{1/K}, \quad (|z|, |z'| \geq R_0). \tag{2.1}$$

Since f is uniformly continuous, we can find $0 < \delta < 1$ such that if $|z_1 - z_2| \leq \delta$ then the spherical distance $d_{spher}(f(z_1), f(z_2)) \leq 1$. (Note that $d_{spher}(0, \infty) = 2$ in our convention.)

Suppose the following condition on two points z and z' on $\mathbb{C}$:

$$|z| \geq R_0 + 1, \quad |z - z'| \leq 2^{-K^2} C_0^{-3K} \delta^K |z|^{-2K^2}. \tag{2.2}$$

The right-hand side of the second inequality is smaller than 1. Hence $R_0 < |z'| < |z| + 1 < 2|z|$. Then, from (2.1), we have $|h^{-1}(z) - h^{-1}(z')| \leq \delta$ and hence $d_{spher}\left(f\left(h^{-1}(z)\right), f\left(h^{-1}(z')\right)\right) \leq 1$. Recall $q = \exp \circ P = f \circ h^{-1}$. Therefore we conclude that (2.2) implies $d_{spher}(q(z), q(z')) \leq 1$.

Now we claim that the spherical derivative satisfies

$$|dq|(z) \leq \text{const} \cdot |z|^{2K^2}, \quad (|z| \geq R_0 + 1). \tag{2.3}$$

Indeed, fix $z \in \mathbb{C}$ with $|z| \geq R_0 + 1$ and set $\varepsilon = 2^{-K^2} C_0^{-3K} \delta^K |z|^{-2K^2}$. For making the calculation simpler, we consider a rotation of the Riemann sphere $S : \mathbb{C}P^1 \to \mathbb{C}P^1$ satisfying $S(q(z)) = 0$. Set $w = S \circ q$. Then $w(z) = 0$ and $|dw| = |dq|$. (Recall that the point z was fixed.)

If $|z - z'| \leq \varepsilon$ then

$$d_{spher}\left(0, w(z')\right) = d_{spher}\left(w(z), w(z')\right) = d_{spher}\left(q(z), q(z')\right) \leq 1.$$

Hence we have $|w(z')| \leq 1$ for $|z - z'| \leq \varepsilon$. By the Cauchy formula for derivative

$$w'(z) = \frac{1}{2\pi i} \int_{|z'-z|=\varepsilon} \frac{w(z')}{(z'-z)^2} dz',$$

we obtain $|w'(z)| \leq \varepsilon^{-1}$. Hence, the spherical derivative satisfies

$$|dq|(z) = |dw|(z) = \frac{|w'(z)|}{1 + |w(z)|^2} = |w'(z)| \leq \varepsilon^{-1} = \text{const} \cdot |z|^{2K^2}.$$

This proves the claim (2.3).

Then we can estimate the characteristic function $T(R, q)$ by

$$T(R, q) = \int_1^R \left(\int_{D_r} |dq|^2 \, dxdy \right) \frac{dr}{r} = O(R^{4K^2+2}).$$

Thus q has finite order. As we pointed out in the beginning, this implies that P is a polynomial. □

We denote by $g := P \circ h$. By applying a translation and a scale change we may assume that P is a unitary polynomial (i.e. the top coefficient is one) and $h(0) = 0$. Then for sufficiently large $|z|$ we get:

$$\frac{1}{2}|z|^d \leq |P(z)| \leq 2|z|^d. \tag{2.4}$$

In this subsection we use the notation $A \lesssim B$, or equivalently $B \gtrsim A$, if there exists a positive constant C depending only on h and P satisfying $A \leq CB$.

To prove Theorem 0.1 by a contradiction, we suppose that there exists a sequence $1 < R_1 < R_2 < R_3 < \ldots$, with $\lim_{n\to\infty} R_n = \infty$, satisfying

$$\int_{D_{R_n}} f^* \omega_{FS} \gtrsim R_n^2. \tag{2.5}$$

Lemma 2.2. *There exists a positive number c such that for all $n \geq 1$ and $z \in \partial D_{R_n}$ we have $|h(z)| \geq 2cR_n^{2/d}$. In particular $h(D_{R_n})$ contains $2cD_{R_n^{2/d}}$. Here a slightly tricky factor $2c$ is introduced for a later convenience of the exposition.*

Proof. Suppose the statement is false. By taking a subsequence if necessary, there exist $z_n \in \partial D_{R_n}$ with $|h(z_n)| < (1/n)R_n^{2/d}$. Since h is a K-quasiconformal homeomorphism of $\mathbb{C}$, it follows from a result of Mori [10]

$$\sup_{\partial D_{R_n}} |h(z)| \lesssim \inf_{\partial D_{R_n}} |h(z)| < \frac{R_n^{2/d}}{n}.$$

Since P is a polynomial of degree d, we get

$$\sup_{\partial D_{R_n}} |g(z)| \lesssim \frac{R_n^2}{n^d}.$$

Then $g(D_{R_n})$ is contained in $CD_{R_n^2/n^d}$ for some positive constant C. But this implies

$$\int_{D_{R_n}} f^*\omega_{FS} \lesssim \int_{CD_{R_n^2/n^d}} \exp^* \omega_{FS} \lesssim \frac{R_n^2}{n^d}.$$

This contradicts (2.5). □

Define $S := \{z \in \mathbb{C}/ \ -1 \le Re(z) \le 1\}$. The spherical derivative of the exponential map is bounded from below over this strip:

$$|d\exp|(x + y\sqrt{-1}) = \frac{e^x}{1+e^{2x}} \gtrsim 1, \quad (-1 \le x \le 1).$$

Combining this with the uniform continuity of $f = \exp(g)$, we conclude that g is also uniformly continuous on $g^{-1}(S)$. In particular we can find $\delta > 0$ such that if $z, w \in g^{-1}(S)$ satisfy $|z - w| < \delta$ then we have $|g(z) - g(w)| < 1/2$.

Set $\Omega_n := P^{-1}(S) \cap c\overline{D_{R_n^{2/d}}}$, where c is the positive constant introduced in Lemma 2.2. From this lemma, we have $h^{-1}(\Omega_n) \subset D_{R_n}$.

Lemma 2.3. *$Area(h^{-1}(\Omega_n)) \gtrsim R_n^2$ for sufficiently large n.*

Proof. According to (2.4), if $P(z) \in [-1,1] \times \left[-\frac{c^d R_n^2}{2}, \frac{c^d R_n^2}{2}\right]$ then $|z|^d \le c^d R_n^2$ for every sufficiently large n. This implies

$$P^{-1}\left([-1,1] \times \left[-\frac{c^d R_n^2}{2}, \frac{c^d R_n^2}{2}\right]\right) \subset cD_{R_n^{2/d}}$$

for sufficiently large n. Set $m = \lfloor c^d R_n^2/2 \rfloor$ and consider m points $k\sqrt{-1} \in S$ $(k = 0, 1, 2, \dots, m-1)$. Note that the discs $D_{1/2}(k\sqrt{-1})$ of radius $1/2$ centered at these points are disjoint with each other. We have $P^{-1}(D_{1/2}(k\sqrt{-1})) \subset cD_{R_n^{2/d}}$. Then this is contained in Ω_n and hence $g^{-1}(D_{1/2}(k\sqrt{-1})) = h^{-1}\left(P^{-1}(D_{1/2}(k\sqrt{-1}))\right)$ is contained in $h^{-1}(\Omega_n) \subset D_{R_n}$.

Take $z_k \in D_{R_n}$ with $g(z_k) = k\sqrt{-1}$. We claim $D_\delta(z_k) \subset g^{-1}(D_{1/2}(k\sqrt{-1}))$. If not, then there exists $w \in D_\delta(z_k)$ with $|g(w) - k\sqrt{-1}| \ge 1/2$. If $g(w) \in S$, then this directly contradicts the definition of δ. If $g(w) \notin S$, then there exists another point w' on the line segment between z_k and w satisfying $g(w') \in \partial S$. But then $|g(w') - k\sqrt{-1}| \ge 1$, which also contradicts the definition of δ.

Thus we can conclude that the m discs $D_\delta(z_k)$ $(k = 0, 1, 2, \dots, m-1)$ are disjoint and contained in $h^{-1}(\Omega_n)$. This implies the statement. (Recall $m = \lfloor c^d R_n^2/2 \rfloor$.) □

Define $P_n(z) := R_n^{-2} P(R_n^{2/d} z)$ and $h_n(z) := R_n^{-2/d} h(R_n z)$. Since P is unitary and of degree d, the polynomials P_n converge to the monomial z^d as $n \to \infty$, uniformly on compact subsets of $\mathbb{C}$. Define

$$X := \{z \in \overline{D_c}/ \ Re(z^d) = 0\}.$$

Note

$$R_n^{-2/d}\Omega_n = P_n^{-1}\left(\left[-\frac{1}{R_n^2}, \frac{1}{R_n^2}\right] \times \mathbb{R}\right) \cap \overline{D_c}.$$

These sets converge to X in the sense of the Hausdorff convergence.

The maps $h_n^{-1}|_{2D_c}$ are all K-quasiconformal mappings (with *fixed* K) from $2D_c$ to D_1 with $h_n^{-1}(0) = 0$. So they become a normal family, according to [9], Theorem 4 p.40. In particular we may extract a subsequence, still denoted $h_n^{-1}|_{2D_c}$, that converges uniformly on compact subsets of $2D_c$ to some K-quasiconformal mapping φ from $2D_c$ to D_1. Then the sets $h_n^{-1}(R_n^{-2/d}\Omega_n)$ converge to $\varphi(X)$ in the Hausdorff topology. As is classically known, the image of a plane curve under a quasiconformal map cannot have a positive area. (Indeed its Hausdorff dimension is strictly smaller than 2.) In particular $\varphi(X)$ has zero area. Hence

$$\lim_{n\to\infty} Area\left(h_n^{-1}(R_n^{-2/d}\Omega_n)\right) = 0.$$

On the other hand, we have

$$h_n^{-1}(R_n^{-2/d}\Omega_n) = R_n^{-1}h^{-1}(\Omega_n).$$

From Lemma 2.3

$$Area\left(h_n^{-1}(R_n^{-2/d}\Omega_n)\right) = Area\left(R_n^{-1}h^{-1}(\Omega_n)\right) \gtrsim 1.$$

This is a contradiction. □

2.2. **Proof of Theorem 0.2.** Let $f : \mathbb{C} \to \mathbb{CP}^2$ be a J-holomorphic entire curve where J is a smooth almost complex structure on $\mathbb{CP}^2$, tamed by the Fubini-Study form ω_{FS}. We assume that f is a Brody curve, meaning that $\sup_{z\in\mathbb{C}} \rho(f)(z) \leq 1$.

We assume that $f(\mathbb{C})$ omits three J-lines in general position in $\mathbb{CP}^2$. We denote by L_1, L_2 and L_3 these lines.

We set $p_1 = L_2 \cap L_3$, $p_2 = L_1 \cap L_3$ and $p_3 = L_1 \cap L_2$. According to [5, Proposition page 2362], for $j = 1, 2, 3$, there is a projection $\pi_j : \mathbb{CP}^2\backslash\{p_j\} \to \mathbb{CP}^1$ and there exists $k > 0$ such that $f_j := \pi_j \circ f$ is a κ-quasiconformal mapping from $\mathbb{C}$ to $\mathbb{CP}^1$ omitting 0 and ∞.

The map f being a Brody curve, it satisfies:

$$\forall R > 0, \ \int_{D_R} f^*\omega_{FS} \leq \pi R^2.$$

According to [5], there exists $C > 0$ such that we have for every $j = 1, 2, 3$ and for every $R > 0$:

$$\int_{D_R} f_j^*\omega_{FS} \leq CR^2. \tag{2.6}$$

For $j = 1, 2, 3$ and $t > 0$, let $G_{j,t} := \{z \in \mathbb{C}/\ d_{spher}(f(z), p_j) > t\}$.

Lemma 2.4. *There exists $t_0 > 0$ and there exists $\alpha > 0$ such that*

$$\forall R > 0, \ \int_{D_R} f^*\omega_{FS} \leq \alpha \sum_{j=1}^{3} \int_{G_{j,t_0}\cap D_R} f_j^*\omega_{FS}.$$

Proof of Lemma 2.4. We choose t_0 such that $(G_{1,t_0} \cap G_{2,t_0}) \cup (G_{1,t_0} \cap G_{3,t_0}) \cup (G_{2,t_0} \cap G_{3,t_0}) = \mathbb{C}$. There exists $\alpha_{1,2} > 0$ such that for all $z \in G_{1,t_0} \cap G_{2,t_0}$

$$|df|(z) \leq \alpha_{1,2}\left(|df_1|(z) + |df_2|(z)\right).$$

We can argue similarly for $G_{1,t_0} \cap G_{3,t_0}$ and $G_{2,t_0} \cap G_{3,t_0}$. □

Hence, in order to prove Theorem 0.2, it is sufficient to prove that for every $j = 1, 2, 3$:

$$\lim_{R\to\infty} \frac{1}{R^2} \int_{G_{j,t_0}\cap D_R} f_j^*\omega_{FS} = 0.$$

Assume, to get a contradiction, that there exists $c_0 > 0$ and an increasing sequence $(R_k)_k$ of positive numbers, with $\lim_{k\to\infty} R_k = \infty$, such that:

$$\forall k \in \mathbb{N}, \ \int_{G_{1,t_0}\cap D_{R_k}} f_1^*\omega_{FS} \geq c_0 R_k^2. \tag{2.7}$$

It follows from the "Lemme analytique" in [5] that there exists a holomorphic polynomial P and a κ-quasiconformal homeomorphism h such that $f_1 = \exp(P \circ h)$. We denote $g := P \circ h$. We can assume that $h(0) = 0$ and P is a unitary polynomial with degree equal to d.

According to (2.6), we may assume that

$$\forall R > 1, \ g(D_R) \subset \mathrm{const} \cdot D_{R^2}, \tag{2.8}$$

where const is a positive constant independent of R. Indeed, if this does not hold, then there exists a sequence $1 < r_1 < r_2 < r_3 < \ldots$ going to infinity and satisfying

$$g(D_{r_n}) \not\subset nD_{r_n^2}.$$

Then, by the cross-ratio distortion theorem of Mori [10], there exists a positive constant const satisfying

$$\mathrm{const} \cdot nD_{r_n^2} \subset g(D_{r_n}).$$

But this implies

$$\int_{D_{r_n}} f_1^* \omega_{FS} \gtrsim n\, r_n^2,$$

which contradicts (2.6). Hence (2.8) holds. After renormalization if necessary, we may assume that $g(D_R) \subset D_{R^2}$ for every $R > 1$.

It follows from (2.7) and from the same argument as in Lemma 2.2 that there exists $c_1 > 0$ satisfying

$$\forall k \geq 0, \ 3D_{R_k^2} \subset g(c_1 D_{R_k}).$$

We recall that d is the degree of the polynomial P. Let $a > 0$ such that for all sufficiently large R:

$$\int_{D_{R^2} \setminus \{z \in \mathbb{C}/ \ |Re(z)| \leq a\}} \exp^* \omega_{FS} < \frac{c_0 R^2}{2d}.$$

Since it follows from (2.7) that for every $k \geq 0$:

$$\int_{g(D_{R_k} \cap G_{1,t_0})} \exp^* \omega_{FS} \geq \frac{c_0 R_k^2}{d},$$

then, for every $k \geq 0$:

$$\int_{g(D_{R_k} \cap G_{1,t_0}) \cap \{z \in \mathbb{C}/ \ |Re(z)| \leq a\}} \exp^* \omega_{FS} > \frac{c_0 R_k^2}{2d}.$$

Notice that the derivative of f_1 is bounded over G_{1,t_0} and that $|d(\exp)|$ is bounded from below by a positive constant on $\{z \in \mathbb{C}/ \ |Re(z)| \leq a\}$. Then g has a bounded derivative on the set $G_{1,t_0} \cap g^{-1}(\{z \in \mathbb{C}/ \ |Re(z)| \leq a\})$. This implies that for every $k \geq 0$:

$$Area\left(D_{R_k} \cap G_{1,t_0} \cap g^{-1}(\{z \in \mathbb{C}/ \ |Re(z)| \leq a\})\right) \gtrsim R_k^2.$$

Set $S = \{z \in \mathbb{C}/ \ |Re(z)| \leq a\}$. Then in particular we get

$$Area\left(g^{-1}\left(D_{R_k^2} \cap S\right)\right) \gtrsim R_k^2.$$

The end of the proof of Theorem 0.2 follows the scheme of proof of Theorem 0.1. In the rest of the argument we always assume that k is sufficiently large. We set $P_k(z) = R_k^{-2} P(R_k^{2/d} z)$ and $h_k(z) = R_k^{-2/d} h(R_k z)$. We define $\varphi_k : 2^{1/d} D_1 \to c_1 D_1$ by $\varphi_k(z) = h_k^{-1}(z)$. The polynomials P_k converge to a monomial $P_\infty(z) = z^d$ uniformly over every compact subset of $\mathbb{C}$.

It follows from what preceeds:

$$Area\left(\varphi_k\left(P_k^{-1}\left(\{z \in D_1/ \ |Re(z)| \leq 1/R_k^2\}\right)\right)\right) \gtrsim 1. \tag{2.9}$$

For every $k \geq 0$, the function φ_k is a κ-quasiconformal mapping and $\varphi_k(2^{1/d}D_1) \subset c_1 D_1$. Hence, extracting a subsequence if necessary, we may assume that $\varphi_k \to_{k\to+\infty} \varphi$ uniformly on compact subsets of $2^{1/d}D_1$, where $\varphi : 2^{1/d}D_1 \to c_1 D_1$ is a κ-quasiconformal function. It follows from (2.9) that φ satisfies

$$Area(\varphi(P_\infty^{-1}\{z \in D_1/\ Re(z) = 0\})) \gtrsim 1.$$

This is a contradiction since $Area(P_\infty^{-1}\{z \in D_1/\ Re(z) = 0\}) = 0$ and the image of a plane curve under a quasiconformal mapping cannot have positive area. □

3. Proof of Theorem 0.3.

The proof of Theorem 0.3 uses the construction of the Von Koch curve. The details of the proof are a bit involved, but the basic idea is simple. We first describe the idea and then present a rigorous proof in Subsections 3.1 and 3.2.

We recall the terminology **quasi-circle** (or **quasiconformal curve**) [8, Chapter II, Section 8.2]: A curve $\mathcal{C} \subset \mathbb{C}$ is called a quasi-circle if it is a image of a line under a quasiconformal homeomorphism of the plane. Many interesting *fractal* curves (e.g. Von Koch curve) are known to be quasi-circles.

Let $1 < \alpha < 2$. By imitating the Von Koch curve[1], we construct a quasi-circle $\mathcal{C} \subset \mathbb{C}$ such that, for any $R > 1$, the length of the curve segment $\mathcal{C} \cap D_R$ is approximately proportional to R^α. (The case $\alpha = \ln(4)/\ln(3)$ corresponds to the standard Von Koch curve, and the construction becomes easier to understand. We will mainly discuss this case later.) We find a quasiconformal homeomorphism G of the plane such that $\mathcal{C}$ is equal to $G(\mathbb{R})$ (the image of the real axis under the map G). Moreover we adjust the map G so that for any $R > 1$

$$G\left([-R^\alpha, R^\alpha]\right) \approx \mathcal{C} \cap D_R. \tag{3.1}$$

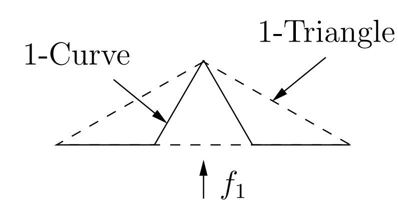
We define a quasi-conformal mapping $f : \mathbb{C} \to \mathbb{CP}^1 \setminus \{0, \infty\}$ by $f(z) = \exp(iG(z))$. We try to show that this map is uniformly continuous and its energy growth is approximately proportional to R^α.

The uniform continuity is more or less a direct consequence of the Schwarz lemma for quasiconformal mappings [7]. For investigating the energy growth, we recall that the exponential function $\exp : \mathbb{C} \to \mathbb{CP}^1$ has the order one and that its energy is concentrated around the imaginary axis. From (3.1) the image $iG(D_R)$ covers a neighborhood of $i[-R^\alpha, R^\alpha]$ and hence

$$\int_{D_R} \exp(iG)^*\omega_{FS} \approx \text{const} \cdot R^\alpha.$$

This proves Theorem 0.3.

3.1. **Construction of a quasisymmetric function on $\mathbb{R}$.** Now we start a rigorous argument. The first step consists in gluing four segments of length one as follows.

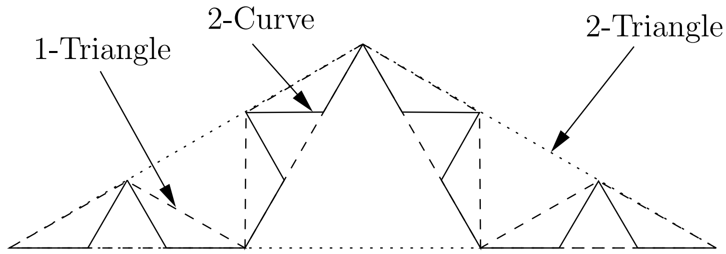


Step 1

[1]The curve $\mathcal{C}$ constructed below is a piecewise-linear curve. So it is not a fractal in the ordinary sense. But its *large scale structure* resembles the Von Koch curve and has a fractal nature.

We call the corresponding piecewise smooth curve "1-Curve" and the associated triangle "1-Triangle". Notice that a 1-Triangle is just the convex hull of a 1-Curve and that a 1-Curve has length equal to 4. Finally we denote by $f_1 : [-2, 2] \to \mathbb{C}$ a piecewise smooth surve such that $|f_1'| = 1$ almost everywhere and $f_1([-2, 2]) = \mathcal{C}_1$ where $\mathcal{C}_1$ is the initial 1-Curve.

Step 2 consists in gluing four 1-Curves, namely we replace each of the four segments in Step 1 by a 1-Curve. We call the corresponding curve a 2-Curve and its convex hull a 2-Triangle. We point out that the interior of the 2-Triangle contains the union of the four 1-Triangles that are the convex hulls of the corresponding four 1-Curves. Moreover the length of a 2-Curve is equal to 4^2. We denote by $\mathcal{C}_2$ the initial 2-Curve and by $f_2 : [-8, 8] \to \mathbb{C}$ a piecewise smooth curve such that $|f_2'| = 1$ almost everywhere and $f_2([-8, 8]) = \mathcal{C}_2$.



Step 2

We proceed recursively, obtaining the $(n+1)$-Curve by gluing four n-Curves as before. We denote by $(n+1)$-Triangle its convex hull; it contains the union of the four n-Triangles that are the convex hulls of the n-Curves. Finally the length of a n-Curve is equal to 4^n. We denote by $\mathcal{C}_n$ the initial n-Curve constructed above and by $f_n : [-2 \times 4^{n-1}, 2 \times 4^{n-1}] \to \mathbb{C}$ a piecewise smooth curve such that $|f_n'| = 1$ almost everywhere and $f_n([-2 \times 4^{n-1}, 2 \times 4^{n-1}]) = \mathcal{C}_n$.

Finally we denote by $\mathcal{C}$ the piecewise smooth curve obtained as the limit of the n-Curve $\mathcal{C}_n$ when n goes to infinity. Since for every $n \geq 1$, $f_{n+1|[-2\times 4^{n-1}, 2\times 4^{n-1}]} = f_n$ we may consider $f := \lim_{n\to\infty} f_n$. Then $f : \mathbb{R} \to \mathbb{C}$ is a piecewise smooth map, with $|f'| = 1$ almost everywhere and $f(\mathbb{R}) = \mathcal{C}$.

The curve $\mathcal{C}$ satisfies the following well-known estimates which means that the Hausdorff dimension of the Von Koch curve is equal to $\ln(4)/\ln(3)$:

Lemma 3.1. *There exists a positive constant c such that for every sufficiently large R we have:* $\frac{1}{c} R^{\frac{\ln(4)}{\ln(3)}} \leq length(\mathcal{C} \cap D_R) \leq c R^{\frac{\ln(4)}{\ln(3)}}$.

Here $length(\mathcal{C} \cap D_R)$ denotes the Euclidean length of the intersection between $\mathcal{C}$ and D_R.

We recall that a map $\varphi : \mathbb{R} \to \mathbb{R}^2$ is quasisymmetric if there exists a constant $L > 0$ such that for any real numbers $x_1 < x_2 < x_3$ satisfying $x_3 - x_2 = x_2 - x_1$ we have:

$$\frac{1}{L} \leq \frac{|\varphi(x_3) - \varphi(x_2)|}{|\varphi(x_2) - \varphi(x_1)|} \leq L. \tag{3.2}$$

We have the following

Proposition 3.2. *The map f is quasisymmetric.*

Proof of Proposition 3.2. Consider real numbers $x_1 < x_2 < x_3$ such that $x_3 - x_2 = x_2 - x_1$. Let $p_j = f(x_j)$ for $j = 1, 2, 3$.

We start with the following key result:

Lemma 3.3. *For $n \geq 2$, let T_n be a n-Triangle. We denote from the left to the right by $T_{n,1}$ and $T_{n,2}$ the two n-Triangles following T_n. Then:*

(i) *$dist(T_n, T_{n,2}) = \frac{\sqrt{3}}{2} R_n$, where R_n is the length of the largest side of T_n,*
(ii) *If T'_n is any n-Triangle on the right side of $T_{n,2}$ then $dist(T_n, T'_n) \geq dist(T_n, T_{n,2})$.*

Proof. (i) The distance between T_n and $T_{n,2}$ is equal to the infimum of the distance between the intersection point p^n of $T_{n,2}$ and of $T_{n,1}$ and the projection of p^n on the base of T_n, and the of the distance between the intersection point q^n of T_n and of $T_{n,1}$ and the projection of q^n on the base of $T_{n,2}$. The two distances are equal to $\frac{\sqrt{3}}{2} R_n$.

(ii) This is immediate. □

Case 1. Points p_1 and p_2 are in the interior of the same 1-Triangle, denoted by T_1. If p_1 and p_2 belong to the same segment then $|p_2 - p_1| = x_2 - x_1$. Assume now that p_1 and p_2 belong to two consecutive segments. Since for fixed $x_2 - x_1$ the smallest value of $|p_2 - p_1$ is reached for $|p_1 - q| = |p_2 - q|$ where q is the intersection point of the two segments containing p_1 and p_2, we have:

$$|p_2 - p_1| \leq |x_2 - x_1| \leq 2|p_2 - p_1|. \tag{3.3}$$

If p_1 and p_2 belong to the first and the fourth segments on the 1-Curve, then Condition (3.3) is still valid. In the last case, let p denote the third intersection point between the segment joining p_1 to p_2 and the 1-Curve, with $p = f(x)$ (p_1 and p_2 are the tow other intersection points). Then $x_1 < x < x_2$ and Condition (3.3) gives:

$$|p_2 - p| \leq |x_2 - x| \leq 2|p_2 - p| \quad \text{and} \quad |p - p_1| \leq |x - x_1| \leq 2|p - p_1|.$$

Hence Condition (3.3) is satisfied for p_1 and p_2.

By assumption, p_3 is in the interior of the 1-Triangle T'_1 just following T_1 and contained in the interior of the same 3-Triangle as p_1 and p_2. Notice that the angle between the bases of T_1 and T'_1 is equal to $2\pi/3$ and the angle between the two consecutive sides of T_1 and T'_1 is equal to $\pi/3$. Notice that for fixed $|x_3 - x_2|$ the smallest value of $|p_3 - p_2|$ is reached for $|p_3 - q'| = |q' - p_2|$ where q' denotes the intersection point between T_1 and T'_1.

We still divide the study into four cases.

- If p_2 belongs to the fourth segment of T_1 and p_3 belongs to the first segment of T'_1 then

$$|p_3 - p_2| \leq |x_3 - x_2| \leq \frac{2}{\sqrt{3}}|p_3 - p_2|$$

since the angle between theses two segments is equal to $\pi/6$.

- If p_2 belongs to the third segment of T_1 and p_3 belongs to the second segment of T'_1 then

$$|p_3 - p_2| \leq |x_3 - x_2| \leq \frac{2}{\sqrt{3}}|p_3 - p_2|$$

since the angle between theses two segments is equal to $\pi/6$.

Case 2. There exists some $n \geq 1$ such that p_2 is contained in a n-triangle and p_1 and p_3 are not in that triangle. We denote by n_0 the smallest such number n and by T_{n_0} the corresponding triangle. Then by construction the (n_0+1)-Triangle T_{n_0+1} containing T_{n_0} contains either p_1 or p_3. We assume that T_{n_0+1} contains p_3. For $n \geq 1$ we denote by R_n the length of the largest side of a triangle T_n. Since p_2 and p_3 belong to T_{n_0+1} we have $|p_2 - p_3| \leq R_{n_0+1}$. If p_1 is in the interior of T_{n_0+1} then $|p_2 - p_1| \leq R_{n_0+1}$.

If p_1 is not in the interior of T_{n_0+1} then p_1 is necessarily contained in the (n_0+1)-Triangle preceding T_{n_0+1} in Step (n_0+2). Hence $|p_2 - p_1| \leq 2R_{n_0+1}$.

Consider the four (n_0-1)-Triangles, denoted from the left to the right by $T_{n_0-1,1}$, $T_{n_0-1,2}$, $T_{n_0-1,3}$, $T_{n_0-1,4}$, contained in T_{n_0}. We point out that if $n_0 = 1$ then a 0-Triangle is just a segment with length one. Without loss of generality we may assume that p_2 is in the interior of one of the two triangles $T_{n_0-1,3}$ and $T_{n_0-1,4}$ (or on the Triangles if $n_0 = 1$).

(2.a) $p_2 \in T_{n_0-1,3}$. Since $p_3 \notin T_{n_0}$ then it follows from Lemma 3.3 that $|p_2 - p_3| \geq \frac{\sqrt{3}}{2} R_{n_0-1} \geq \frac{\sqrt{3}}{6} R_{n_0}$. Moreover since $p_1 \notin T_{n_0}$, it follows from Lemma 3.3 that $|p_2 - p_1| \geq \frac{\sqrt{3}}{2} R_{n_0-1} \geq \frac{\sqrt{3}}{6} R_{n_0}$.

(2.b) $p_2 \in T_{n_0-1,4}$. Since $p_1 \notin T_{n_0}$, it follows from Lemma 3.3 that $|p_2 - p_1| \geq \frac{\sqrt{3}}{2} R_{n_0-1} \geq \frac{\sqrt{3}}{6} R_{n_0}$. Moreover there is at least one n_0-Triangle between T_{n_0} and the n_0-Triangle containing p_3. Hence it follows from Lemma 3.3 that $|p_2 - p_1| \geq \frac{\sqrt{3}}{2} R_{n_0-1} \geq \frac{\sqrt{3}}{6} R_{n_0}$.

In each case we obtain:

$$\frac{\sqrt{3}}{6} \leq \frac{|p_3 - p_2|}{|p - 2 - p_1|} \leq \frac{6}{\sqrt{3}}.$$

This proves that f is quasisymmetric. □

3.2. **Construction of a quasiconformal map.** Let $\mathcal{C}^+$ and $\mathcal{C}^-$ be the curves constructed as follows. We denote by S_{-2}, S_{-1}, S_1 and S_2 the four segments of length one of $\mathcal{C}_1$, from the left to the right, and by m_{-2} the middle point of the first (horizontal) segment S_{-2}. Let L_{-2}^+ be the line parallel to S_{-2}, at a distance equal to $1/2$ from S_{-2}, such that the intersection point m_{-2}^+ between L_{-2}^+ and the orthogonal line to S_{-2} passing through m_{-2} satisfies $Im(m_{-2}^+) > Im(m_{-2})$. Then we construct the real line L_{-1}^+ (respectively L_1^+ and L_2^+), parallel to S_{-1} (respectively to S_1 and to S_2), at a distance equal to $1/2$ from S_{-1} (respectively from S_1 and S_2), and such that the intersection point q_{-1}^+ (respectively q_0^+ and q_1^+) between L_{-2}^+ and L_{-1}^+ (respectively between L_{-1}^+ and L_1^+ and between L_1^+ and L_2^+) is in the connected component of $\mathbb{C}\backslash\mathcal{C}$ containing m_{-2}^+. We construct inductively a sequence of points $(q_j^+)_{j\in\mathbb{Z}}$ by following that procedure. The curve $\mathcal{C}^+$ is then defined by $\mathcal{C}^+ := \cup_{j\in\mathbb{Z}}[q_j^+, q_{j+1}^+]$.

The curve $\mathcal{C}^-$ is constructed by considering the intersection point m_{-2}^- between the line L_{-2}^- parallel to S_{-2}, at a distance equal to $1/2$ from S_{-2}, and the orthogonal line to S_{-2} passing through m_{-2}, such that $Im(m_{-2}^-) < Im(m_{-2})$, and by repeating the previous construction. We construct inductively a sequence of points $(q_j^-)_{j\in\mathbb{Z}}$ by following that procedure. The curve $\mathcal{C}^-$ is then defined by $\mathcal{C}^- := \cup_{j\in\mathbb{Z}}[q_j^-, q_{j+1}^-]$.

Then the curves $\mathcal{C}^+$ and $\mathcal{C}^-$ are the boundaries of a neigborhood $\mathcal{N}_{\mathcal{C}}$ of $\mathcal{C}$. Consider the tubular neighborhood $\mathcal{N}_{\mathbb{R}} := \{\zeta \in \mathbb{C}/ \ -1 < Im(\zeta) < 1\}$. We first prove the following

Lemma 3.4. *There is some $k' > 0$ and some uniformly continuous k'-quasiconformal homeomorphism g from $\mathcal{N}_{\mathcal{C}}$ to $\mathcal{N}_{\mathbb{R}}$, that extends as a homeomorphism from $\overline{\mathcal{N}_{\mathcal{C}}}$ to $\overline{\mathcal{N}_{\mathbb{R}}}$, such that the restrictions of g to $\mathcal{C}^+$ and to $\mathcal{C}^-$ are quasisymmetric maps.*

Proof of Lemma 3.4. We first divide $\mathcal{N}_{\mathcal{C}}$ and $\mathcal{N}_{\mathbb{R}}$ into elementary blocks $B_j^{\mathcal{C}}$ and $B_j^{\mathbb{R}}$, $j \in \mathbb{Z}$, as follows (see Figure 3). For every $j \in \mathbb{Z}$, the block $B_j^{\mathcal{C}}$ is obtained by joining the middle points of the segments $[q_{j-1}^+, q_j^+]$ and $[q_{j-1}^-, q_j^-]$ on one side, and the middle points of the segments $[q_j^+, q_{j+1}^+]$ and $[q_j^-, q_{j+1}^-]$ on the other side. Notice that $B_2^{\mathcal{C}}$ is obtained as the image of $B_{-2}^{\mathcal{C}}$ under the composition of a translation and of a rotation with angle $\pi/3$. Then, by the self similarity property of $\mathcal{N}_{\mathcal{C}}$, each elementary block $B_j^{\mathcal{C}}$ in the decomposition of $\mathcal{N}_{\mathcal{C}}$ is obtained as the image of $B_{j-4}^{\mathcal{C}}$ under the composition of a translation and of a rotation with angle $\pm\pi/3$, for $j \in \mathbb{Z}$. Finally the blocks in the decomposition of $\mathcal{N}_{\mathbb{R}}$ are squares of length equal to 2 (see Figure 3 below).

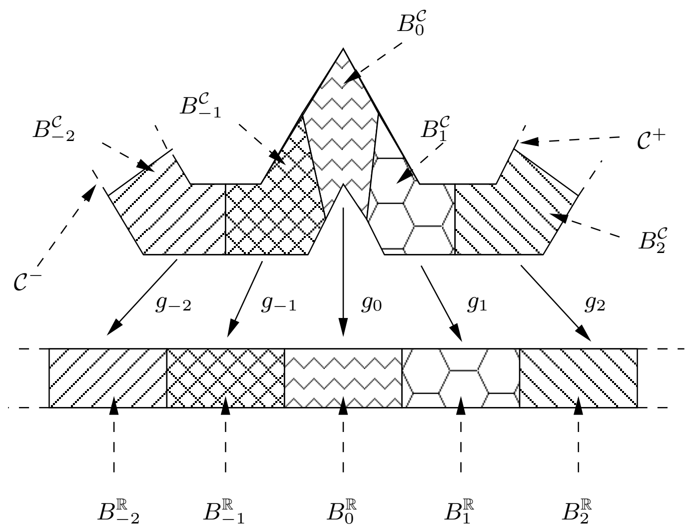


Figure 3

We have the following

Lemma 3.5. *There exists $k' > 0$ such that for every $j \in \mathbb{Z}$, there is a k'-quasiconformal homeomorphism g_j from $B_j^{\mathcal{C}}$ to $B_j^{\mathbb{R}}$, continuous up to $\partial B_j^{\mathcal{C}}$, such that g_j and g_{j+1} coincide on the intersection $\partial B_j^{\mathcal{C}} \cap \partial B_{j+1}^{\mathcal{C}}$.*

Proof of Lemma 3.5. We point out that for every $j \in \mathbb{Z}$, $B_{j+4}^{\mathcal{C}}$ is the image of $B_j^{\mathcal{C}}$ under the composition of a translation and of a rotation with angle $\pm\pi/3$ and $B_{j+4}^{\mathbb{R}}$ is the image of $B_j^{\mathbb{R}}$ under a translation. For $j = -2, \ldots, 1$, we denote by $S_j^1, S_j^2, S_j^3, S_j^4, S_j^5$ and S_j^6 the six line segments composing $\partial B_j^{\mathcal{C}}$, numbered clockwise, where S_j^6 is the intersection between $\partial B_j^{\mathcal{C}}$ and $\partial B_{j+1}^{\mathcal{C}}$. For $j = -2, \ldots, 1$, let $F_j^{\mathcal{C}}$ be a biholomorphism from $B_j^{\mathcal{C}}$ to D_1, that extends as a homeomorphism from $\overline{B_j^{\mathcal{C}}}$ to $\overline{D_1}$. We denote by T_j^1, T_j^2, T_j^3 and T_j^4 the four line segments of $B_j^{\mathbb{R}}$, numbered clockwise, where T_j^4 is the intersection between $\partial B_j^{\mathbb{R}}$ and $\partial B_{j+1}^{\mathbb{R}}$. Let $F_j^{\mathbb{R}}$ be a biholomorphism from $B_j^{\mathbb{R}}$ to D_1, that extends as a homeomorphism from $\overline{B_j^{\mathbb{R}}}$ to $\overline{D_1}$. We consider a quasisymmetric homeomorphism μ_j from the unit circle to itself such that:

$$\begin{cases} \mu_j(F_j^{\mathcal{C}}(S_j^1 \cup S_j^2)) &= F_j^{\mathbb{R}}(T_j^1), \\ \mu_j(F_j^{\mathcal{C}}(S_j^3)) &= F_j^{\mathbb{R}}(T_j^2), \\ \mu_j(F_j^{\mathcal{C}}(S_j^4 \cup S_j^5)) &= F_j^{\mathbb{R}}(T_j^3), \\ \mu_j(F_j^{\mathcal{C}}(S_j^6)) &= F_j^{\mathbb{R}}(T_j^4). \end{cases}$$

It follows from the Douady-Earle extension theorem [4] that there is a positive constant k_j such that the map μ_j extends as a k_j-quasiconformal diffeomorphism, denoted $\tilde{\mu}_j$, of the unit disc, such that $\tilde{\mu}_j$ is a homeomorphism of $\overline{D_1}$. Hence the map $g_j := (F_j^{\mathbb{R}})^{-1} \circ \tilde{\mu}_j \circ F_j^{\mathcal{C}}$ is a k_j-quasiconformal diffeomorphism from $B_j^{\mathcal{C}}$ to $B_j^{\mathbb{R}}$ that extends as a homeomorphism from $\overline{B_j^{\mathcal{C}}}$ to $\overline{B_j^{\mathbb{R}}}$. For $j = -2, \ldots, 1$ and for $j' = j + 4l$, with $l \in \mathbb{Z}\backslash\{0\}$, let $g_{j'} := (h_{j'}^{\mathbb{R}})^{-1} \circ g_j \circ h_{j'}^{\mathcal{C}}$ where $h_{j'}^{\mathcal{C}}$ is the composition of a translation and of a rotation with angle $\pm\pi/3$ such that $h_{j'}^{\mathcal{C}}(B_{j'}^{\mathcal{C}}) = B_j^{\mathcal{C}}$, and $h_{j'}^{\mathbb{R}}$ is the translation such that $h_{j'}^{\mathbb{R}}(B_{j'}^{\mathbb{R}}) = B_j^{\mathbb{R}}$. Then $g_{j'}$ is a k_j-quasiconformal diffeomorphism from $B_{j'}^{\mathcal{C}}$ to $B_{j'}^{\mathbb{R}}$ that extends as a homeomorphism from $\overline{B_{j'}^{\mathcal{C}}}$ to $\overline{B_{j'}^{\mathbb{R}}}$. Finally, if $k' := \sup\{k_j,\ j = -2, \dots, 1\}$ then, for every $j \in \mathbb{Z}$, the map g_j is a k'-quasiconformal map. □

Let $g : \mathcal{N}_{\mathcal{C}} \to \mathcal{N}_{\mathbb{R}}$ be defined by $g = g_j$ on $\overline{B_j^{\mathcal{C}}}$. By the construction of $\mathcal{N}_{\mathcal{C}}$, of $\mathcal{N}_{\mathbb{R}}$ and of g, it is clear that g is uniformly continuous on $\overline{\mathcal{N}_{\mathcal{C}}}$ and that g is a k'-quasiconformal diffeomorphism from $\mathcal{N}_{\mathcal{C}}$ to $\mathcal{N}_{\mathbb{R}}$. Finally the restrictions of g to $\mathcal{C}^+$ and to $\mathcal{C}^-$ are quasisymmetric. This proves Lemma 3.4. □

Lemma 3.6. *There exists $k'' > 0$ and a k''-quasiconformal homeomorphism $G : \mathbb{C} \to \mathbb{C}$ whose restriction to $\mathcal{N}_{\mathcal{C}}$ is equal to g.*

Proof of Lemma 3.6. Let g^+ denote the restriction of g to $\mathcal{C}^+$. Since g^+ is a quasisymmetric map, it follows from the Beurling-Ahlfors extension Theorem [2] (or using [8, Chapter II, Theorem 8.3]) that g^+ extends as a quasiconformal homeomorphism G^+ from the connected component $\mathcal{N}_{\mathcal{C}}^+$ of $\mathbb{C}\backslash\mathcal{N}_{\mathcal{C}}$ containing the point $10i$ to the upper half plane $\mathbb{H}_1^+ := \{z \in \mathbb{C}/\ Im(z) > 1\}$.

If we denote by g^- the restriction of g to $\mathcal{C}^-$ then equivalently there is a quasiconformal homeomorphism G^- from the connected component $\mathcal{N}_{\mathcal{C}}^-$ of $\mathbb{C}\backslash\mathcal{N}_{\mathcal{C}}$ containing the point $-10i$ to the lower half plane $\mathbb{H}_1^- := \{z \in \mathbb{C}/\ Im(z) < -1\}$. Notice that $\mathbb{C} = \overline{\mathcal{N}_{\mathcal{C}}} \cup \mathcal{N}_{\mathcal{C}}^+ \cup \mathcal{N}_{\mathcal{C}}^-$.

Finally let G be defined by:

$$G(z) = \begin{cases} g(z) & \text{if } z \in \overline{\mathcal{N}_{\mathcal{C}}}, \\ G^+(z) & \text{if } z \in \mathcal{N}_{\mathcal{C}}^+, \\ G^-(z) & \text{if } z \in \mathcal{N}_{\mathcal{C}}^-. \end{cases}$$

Since G is continuous by construction and since the maps g, G^+ and G^- are quasiconformal, then G is a quasiconformal homeomorphism from $\mathbb{C}$ to $\mathbb{C}$. □

We recall that we endow $\mathbb{CP}^1$ with the spherical distance d_{spher}.

Lemma 3.7. *The map $\exp(iG) : \mathbb{C} \to \mathbb{CP}^1\backslash\{0, \infty\}$ is uniformly continuous on $\mathbb{C}$.*

Proof of Lemma 3.7. We first point out that we may restrict our attention to pairs of points at a distance less than or equal to one from each other. Moreover since g is uniformly continuous on $\mathcal{N}_{\mathcal{C}}$ then the map $\exp(iG)$ is uniformly continuous on $\mathcal{N}_{\mathcal{C}}$. We consider $z \in \mathcal{N}_{\mathcal{C}}^+$ satisfying the two following conditions:

- the unit disk $D(z, 1)$ centered at z and of radius one (for the Euclidean distance on $\mathbb{C}$) intersects $\mathcal{N}_{\mathcal{C}}$ but does not intersect $g^{-1}(L)$ where L is the real line in $\mathbb{C}$,
- the disk $D(z, 1/2)$ intersects $\mathcal{N}_{\mathcal{C}}$.

Since $\exp(iL)$ is a circle in $\mathbb{CP}^1$ then $\mathbb{CP}^1\backslash\exp(iL) = D^+ \sqcup D^-$ where D^+ and D^- are two disks. We denote by D^+ the component of $\mathbb{CP}^1\backslash\exp(iL)$ containing $\exp(iG)(D(z, 1))$. Then according to the Schwarz Lemma for K-quasiconformal maps ([7]) there is a positive constant C_K such that for every $w \in D(z, 1)$:

$$d_{D^+}(\exp(iG(w)), \exp(iG(z))) \leq C_K(d_{D(z,1)}(w, z))^{1/K}. \tag{3.4}$$

In particular, there is a compact set $\mathcal{K}$ contained in D^+ such that $\exp(iG)(D(z, 1/2)) \subset \mathcal{K}$ for every z satisfying the previous conditions.

Moreover, by construction, there exists a compact subset $\mathcal{K}'$ of D^+ such that $\exp(iG)\,(D(z, 1/2)) \subset \mathcal{K}'$ for every $z \in \mathcal{N}_{\mathcal{C}}^+$ such that $D(z, 1/2)$ does not intersect $\mathcal{N}_{\mathcal{C}}$ and Inequality (3.4) is still valid for every $w \in D(z, 1/2)$.

Since the Kobayashi distance and the Euclidean distance define locally the same topology, there exists a positive constant c such that for every $z \in \mathcal{N}_{\mathcal{C}}^+$ and for every $w \in D(z,1/2)$:

$$\frac{1}{c} d_{D(z,1)}(z,w) \leq |z-w| \leq c d_{D(z,1)}(z,w).$$

We point out that c does not depend on point z satisfying the previous condition.

Finally, equivalently, there exists a positive constant c', depending only on the compact subset $\mathcal{K} \cup \mathcal{K}'$ of D^+ such that for all $z', w' \in \mathcal{K} \cup \mathcal{K}'$:

$$\frac{1}{c'} d_{D^+}(z',w') \leq d_{spher}(z',w') \leq c' d_{D^+}(z',w').$$

This proves that $\exp(iG)$ is uniformly continuous on $G^{-1}(\{z \in \mathbb{C}/\ Im(z) \geq 0\})$. We prove in the exact same manner that $\exp(iG)$ is uniformly continuous on $G^{-1}(\{z \in \mathbb{C}/\ Im(z) \leq 0\})$. □

We can now prove Theorem 0.3. We first start with the following special case of Theorem 0.3 that is a direct application of what preceeds.

Proposition 3.8. *For every* $\alpha < (\ln 4/\ln 3)$ *we have:*

$$\lim_{R \to +\infty} \frac{1}{\pi R^\alpha} \int_{D_R} (\exp(iG))^* \omega_{FS} = +\infty.$$

Proof of proposition 3.8. We recall that $\ln 4/\ln 3$ is the Hausdorff dimension of the VK-curve. In particular, according to Lemma 3.1, there is a positive constant c such that:

$$\forall R >> 1,\ \frac{1}{c} R^{\ln 4/\ln 3} \leq length(\mathcal{C} \cap D_R) \leq c R^{\ln 4/\ln 3}.$$

It follows that $G(\mathcal{N}_{\mathcal{C}} \cap D_R)$ contains the intersection $\mathcal{N}_{\mathbb{R}} \cap D_{c' R^{\ln 4/\ln 3}}$ for some positive constant c' independent of $R >> 1$.

Moreover, since $\exp(iG)$ is uniformly continuous on $\mathbb{C}$ and exp is a finite covering from $\mathcal{N}_{\mathbb{R}} \cap D_{c' R^{\ln 4/\ln 3}}$ to its image, then there exist positive constants c'' and c''', independent of $R >> 1$, such that:

$$\int_{D_R} \exp(iG)^* \omega_{FS} \geq c'' Area(\mathcal{N}_{\mathbb{R}} \cap D_{c' R^{\ln 4/\ln 3}}) \geq c''' R^{\ln 4/\ln 3}.$$

Hence we get for every $\alpha < \ln 4/\ln 3$:

$$\lim_{R \to +\infty} \frac{1}{\pi R^\alpha} \int_{D_R} \exp(iG)^* \omega_{FS} = +\infty.$$

□

The proof of Theorem 0.3 follows the exact same lines. For $\frac{1}{4} \leq a < \frac{1}{2}$, consider the a-VK-curve defined as follows[2]: we replace the 1-Configuration in the construction of $\mathcal{C}$ by the union of four segments of length a, the angle between the first segment (horizontal) and the second segment being equal to $\arccos((1-2a)/2a)$ and the angle between the third and the fourth segment (horizontal) being equal to $\pi - \arccos((1-2a)/2a)$.

Then we construct $\mathcal{C}_a$, using the self-similarity property, exactly as we constructed $\mathcal{C}$. The angle $\pi/3$ is now replaced by $\arccos((1-2a)/2a)$ (see Subsection 3.1). For $R > 1$, the length of the curve segment $\mathcal{C}_a \cap D_R$ is roughly proportional to $R^{\log_a(1/4)}$. Note that $\log_a(1/4) \to 2$ as $a \to 1/2$.

The proof proceeds now exactly as for Proposition 3.8. We construct a neighborhood $\mathcal{N}_{\mathcal{C}_a}$ of $\mathcal{C}_a$ and a quasiconformal homeomorphism G_a from $\mathbb{C}$ to $\mathbb{C}$ such that $\exp(iG_a)$ is uniformly continuous from $\mathbb{C}$ to $\mathbb{CP}^1 \backslash \{0, \infty\}$ with $G_a(\mathcal{N}_{\mathcal{C}_a}) = \mathcal{N}_{\mathbb{R}}$. Then we get for every $\alpha < \log_a(1/4)$:

$$\lim_{R \to +\infty} \frac{1}{\pi R^\alpha} \int_{D_R} \exp(iG_a)^* \omega_{FS} = +\infty.$$

[2]This construction was communicated to the authors by Professor Masahiko Taniguchi.

□

References

[1] Barrett, M.; Eremenko, A. Generalization of a theorem of Clunie and Hayman. *Proc. Amer. Math. Soc.* **140** (2012), 1397-1402.
[2] Beurling,A.; Ahlfors,L.V. The boundary correspondence under quasiconformal mappings. *Acta Math.* **96** (1956), 125-142.
[3] Clunie, J.; Hayman, W. The spherical derivative of integral and meromorphic functions. *Comment. Math. Helv.* **40** (1965), 117-148.
[4] Douady, A.; Earle, C. Conformally natural extension of homeomorphisms of the circle. *Acta Math.* **157** (1986), 23-48.
[5] Duval, J. Un théorème de Green presque complexe. (French) [An almost complex Green theorem]. *Ann. Inst. Fourier* **54** (2004), 2357-2367.
[6] Green, M. Holomorphic maps into complex projective space omitting hyperplanes. *Trans. AMS* **169** (1972), 89-103.
[7] Kiernan, P. Quasiconformal mappings and Schwarz's lemma. *Trans. Amer. Math. Soc.* **148** (1970), 185-197.
[8] Lehto,O.; Virtanen,K.I. Quasiconformal mappings in the plane, Grund. der math.Wiss., Vol. **126**, Springer, Berlin, 1973
[9] Miniowitz, R. Normal families of quasimeromorphic mappings. *Proc. Amer. Math. Soc.* **84** (1982), 35-43.
[10] Mori, A. On an absolute constant in the theory of quasi-conformal mappings. *J. Math. Soc. Japan* **8** (1956), 156-166.
[11] Rickman, S. On the number of omitted values of entire quasiregular mappings. *J. Analyse Math.* **37** (1980), 100-117.
[12] Tsukamoto, M. A packing problem for holomorphic curves. *Nagoya Math. J.* **194** (2009), 33-68.
[13] Tsukamoto,M. Moduli space of Brody curves, energy and mean dimension. *Nagoya Math. J.* **192** (2008), 27-58.

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