ILL-POSEDNESS FOR THE COMPRESSIBLE NAVIER-STOKES EQUATIONS UNDER BAROTROPIC CONDITION IN LIMITING BESOV SPACES

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Abstract. We consider the compressible Navier–Stokes system in the critical Besov spaces. It is known that the system is (semi-)well-posed in the scaling semi-invariant spaces of the homogeneous Besov spaces $\dot{B}^{\frac{n}{p}}_{p,1} \times \dot{B}^{\frac{n}{p}-1}_{p,1}$ for all $1 \le p < 2n$. However, if the data is in a larger scaling invariant class such as $p > 2n$, then the system is not well-posed. In this paper, we demonstrate that for the critical case $p = 2n$ the system is ill-posed by showing that a sequence of initial data is constructed to show discontinuity of the solution map in the critical space. Our result indicates that the well-posedness results due to Danchin [10] and Haspot [18] are indeed sharp in the framework of the homogeneous Besov spaces.

1. Introduction

We consider the ill-posedness issue of the Cauchy problem for compressible Navier-Stokes equations under the barotropic condition:

$$\begin{cases} \partial_t \tilde\rho + \operatorname{div}\big(\tilde\rho u\big) = 0, & t>0, x\in\mathbb{R}^n,\\ \tilde\rho\partial_t u + \tilde\rho(u\cdot\nabla)u + \nabla P(\tilde\rho) = \mathcal{L}u, & t>0, x\in\mathbb{R}^n,\\ \tilde\rho(0,x) = \tilde\rho_0(x),\ u(0,x) = u_0(x), & x\in\mathbb{R}^n, \end{cases} \tag{1.1}$$

where $\mathcal{L}$ is given by $\mathcal{L} \equiv \mu\Delta + (\mu+\lambda)\nabla\mathrm{div}$ with the Lamé constants λ, $\mu > 0$, $2\mu + \lambda > 0$. The unknown functions are $\tilde\rho = \tilde\rho(t,x) : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ and $u = u(t,x) : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ that denote the density and velocity for the fluid, respectively, and P is the pressure given by a function of $\tilde\rho$ as $P = P(\tilde\rho) = \tilde\rho^{\gamma}$ with $\gamma > 1$. We assume that the density of the fluid $\tilde\rho = \tilde\rho(t,x) : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ satisfies the non-vacuum condition, namely $\tilde\rho(t,x) = 1 + \rho(t,x)$ and ρ belongs to the critical Lebesgue or Besov spaces and we assume that $\rho(t,x) \ge \underline{\nu} > -1$. Under such setting the barotropic model (1.1) can be reduced to the following Cauchy problem of the compressible Navier-Stokes equations:

$$\begin{cases} \partial_t \rho + \operatorname{div}\big((1+\rho)u\big) = 0, & t>0, x\in\mathbb{R}^n,\\ \partial_t u - \mathcal{L}u + (u\cdot\nabla)u + \dfrac{\nabla P(1+\rho)}{1+\rho} = -\dfrac{\rho}{1+\rho}\mathcal{L}u, & t>0, x\in\mathbb{R}^n,\\ \rho(0,x) = \rho_0(x),\ u(0,x) = u_0(x), & x\in\mathbb{R}^n. \end{cases} \tag{1.2}$$

Then naturally one can consider the existence theory of the strong solution under the uniformly parabolic setting for the velocity equation.

The existence and the well-posedness for the compressible Navier-Stokes equation have been considered by several researchers (Serrin [28], Nash [26], Itaya [21], Matsumura-Nishida [24], Tani

AMS Subject Classification: primary 35Q30, 76D05, secondary 76N10, 47J06 .

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[31], Hoff-Zumbrun [19]). Danchin [10] first considered the problem (1.1) in the scaling critical function spaces $\dot{B}^{\frac{n}{2}}_{2,1}(\mathbb{R}^n)\times\dot{B}^{\frac{n}{2}-1}_{2,1}(\mathbb{R}^n)$ (see [11]–[13] and also Chen-Miao-Zhang [6]). Haspot [18] generalized the results into a larger class of homogeneous Besov spaces $\dot{B}^{\frac{n}{p_0}}_{p_0,1}(\mathbb{R}^n)\times\dot{B}^{\frac{n}{p_1}-1}_{p_1,1}(\mathbb{R}^n)$ for $1/p_0+1/p_1>1/n$ with $0\le 1/p_1-1/p_0\le 1/n$. Those results suggest that limiting Banach scale allows to obtain the strong solution to (1.2) in the scaling-invariant framework. In particular the space $\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n)\times\dot{B}^{\frac{n}{p}-1}_{p,1}(\mathbb{R}^n)$ with $p=2n$ is the threshold space. Indeed, when $n=3$, Chen-Miao-Zhang [8] showed that the system is ill-posed for the case $\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n)\times\dot{B}^{\frac{n}{p}-1}_{p,1}(\mathbb{R}^n)$ with $p>6$. Chen-Wan [9] proved the ill-posedness under the partially critical setting. Namely either the density or the velocity belongs to the critical space with $p=6$ for the $n=3$. In this paper, we consider all the critical setting $p=2n$ with $n\ge 2$ in both the density and velocity functions spaces, and extract the least regular part from the nonlinear interactions. Indeed, the worst term in our setting is essentially different from the known worst term, i.e., the convection term $(u\cdot\nabla)u$.

Here the homogeneous Besov spaces are defined by

$$\dot{B}^s_{p,q}:=\{f\in\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)\,|\,\|f\|_{\dot{B}^s_{p,q}}<\infty\},\quad \|f\|_{\dot{B}^s_{p,\sigma}}\equiv\left(\sum_{j\in\mathbb{Z}}2^{j\sigma s}\|\phi_j*f\|_p^\sigma\right)^{1/\sigma} \tag{1.3}$$

where $1\le p,\sigma\le\infty$, $\mathcal{P}(\mathbb{R}^n)$ is the set of all polynomials and $\{\phi_j\}_{j\in\mathbb{Z}}$ denotes the Littlewood–Paley dyadic decomposition. It is well-known that if $s<n/p$ or $(s,q)=(n/p,1)$, then the space is characterized subspaces of $\mathcal{S}'(\mathbb{R}^n)$, namely,

$$\dot{B}^s_{p,q}:=\Big\{f\in\mathcal{S}'(\mathbb{R}^n)\,\Big|\,\|f\|_{\dot{B}^s_{p,q}}<\infty, f=\sum_{j\in\mathbb{Z}}\phi_j*f\text{ in }\mathcal{S}'(\mathbb{R}^n)\Big\}.$$

The spaces above are considered as the scaling invariant classes under the following scaling: For any $\alpha>0$,

$$\begin{cases}\rho_\alpha(t,x)=\rho(\alpha^2t,\alpha x),\\ u_\alpha(t,x)=\alpha u(\alpha^2t,\alpha x).\end{cases}$$

We herein consider the strong discontinuity of a strong solution for the Cauchy problem (1.2) in the threshold spaces $\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n)\times\dot{B}^{\frac{n}{p}-1}_{p,1}(\mathbb{R}^n)$ with $p=2n$ and demonstrate the ill-posedness by constructing a sequence of initial data that are smooth and belong to the limiting class $\dot{B}^{\frac{n}{p}-1}_{p,1}(\mathbb{R}^n)$ with $p=2n$ such that their norms decrease while those of the solutions increase in the same class at a time near $t=0$. The result does not answer the question pertaining to the blow-up but clarifies the optimal regularity of the data between the stability and instability of solutions. It is noteworthy that we shall show that the norm inflation stems from both the convection and viscosity terms.

1.1. **Mild solution and main result.** We introduce the mild solution to the problem (1.2).

Definition. Given initial data (ρ_0,u_0), we call (ρ,u) a mild solution to the Cauchy problem of (1.2) if

$$\begin{cases}\rho(t)=\rho_0-\displaystyle\int_0^t\operatorname{div}\Big((1+\rho(s))u(s)\Big)ds,\quad t>0,\\ u(t)=e^{t\mathcal{L}}u_0-\displaystyle\int_0^t e^{(t-\tau)\mathcal{L}}\Big((u\cdot\nabla)u+\frac{\nabla P(1+\rho)}{1+\rho}+\frac{\rho}{1+\rho}\mathcal{L}u\Big)d\tau,\quad t>0,\end{cases} \tag{1.4}$$

in the Besov space $C\big([0,T);\dot{B}^{\frac{n}{p}}_{p,\sigma}\big)\times C\big([0,T);\dot{B}^{\frac{n}{p}-1}_{p,\sigma}\big)$, where $e^{t\mathcal{L}}$ denotes the semi-group generated by the elliptic operator $\mathcal{L}$. We remark that the mild solution (ρ,u) belonging to this space is a strong solution of (1.4). To prove the ill-posedness of the problem (1.2), in particular the strong discontinuity of the solution near the initial time, we introduce a parameter $N \in \mathbb{N}$ and the time sequence $t_N \simeq 2^{-2N}$ such that the critical Besov norm of the velocity fields $\|u(t_N)\|_{\dot{B}^{-\frac{1}{2}}_{2n,\sigma}}$ diverges as $N \to \infty$.

Theorem 1.1. *Let $n \geq 2$. There exist $\{u_{0,N}\}_{N=1}^{\infty} \subset \big(\dot{B}^{-\frac{1}{2}}_{2n,1}(\mathbb{R}^n)\big)^n$, $\{T_N\}_{N=1}^{\infty}$ with $T_N \to 0$ as $N \to \infty$, and $\{(\rho_N, u_N)\}_{N=1}^{\infty}$, that is a sequence of solutions to* (1.2) *with the initial data $(\rho_N(0), u_N(0)) = (0, u_{0,N})$ such that*

$$\lim_{N\to\infty} \|u_{0,N}\|_{\dot{B}^{-\frac{1}{2}}_{2n,1}} \to 0,$$

and

$$\lim_{N\to\infty} \|u_N(T_N)\|_{\dot{B}^{-\frac{1}{2}}_{2n,1}} \to \infty.$$

The main argument showing the theorem above originally goes back to Bourgain-Pavlović [3] (see also a recent paper by Wang [33]). Chen-Miao-Zhang [8] showed that the worst contribution on regularity in a short time stems from the nonlinear interaction of the system, in particular, the convection term $u\cdot\nabla u$. For the limiting case, one may also expect the convection term to provide the lowest regularity and a strong discontinuity may be shown from such an interaction term. However, the threshold case $p = 2n$ is much different from the case $p > 2n$. The quasi-linear dissipation term

$$\frac{\rho}{1+\rho}\mathcal{L}u$$

appearing in (1.2) also has a comparable order of the singularity in a short time; furthermore, the effect from the convection term and the quasi-linear dissipation term above cancel each other. This can be understood by investigating the high–high frequency interaction to the low frequency of the second iteration, and the crucial point is to obtain better regularity by the cancellation (see Proposition A.1, Proposition 2.2, and Lemma 2.3 with Remarks below). Therefore, a more detailed analysis is required to extract the lowest regularity for the solution in the critical scaling space. We employ a sharper example to prove the strong discontinuity with the time sequence $T_N \to 0$ as $N \to \infty$ that was developed in [22]. It is noteworthy that for the drift–diffusion system, a similar limitation to the solvability in the critical Besov space $\dot{B}^{\frac{n}{p}-2}_{p,\sigma}(\mathbb{R}^n)$ can be observed in a monopolor case and a bipolar case ([22]).

We first consider the strong solution $u(t)$ by the integral form (1.4). To illustrate our main idea, we introduce the first approximation of the density ρ and the velocity fields u of the strong solution to (1.4), by the following linearlized equation: Let (ρ_1, U) solve

$$\begin{cases} \partial_t \rho_1 + \operatorname{div}\big((1+\rho_0)U\big) = 0, & t > 0, x \in \mathbb{R}^n, \\ \partial_t U - \mathcal{L}U = -\dfrac{\rho_0}{1+\rho_0}\mathcal{L}U, & t > 0, x \in \mathbb{R}^n, \\ \rho_1(0,x) = \rho_0(x) \equiv 0, \quad U(0,x) = u_0(x), & x \in \mathbb{R}^n. \end{cases} \tag{1.5}$$

By using (ρ_1, U), we extract the worst term of the velocity fields that primarily affects the discontinuity to the original solution as $N \to \infty$.

We introduce the Helmholtz decomposition of the vector fields in $L^p(\mathbb{R}^n;\mathbb{R}^n)$.

__*Definition.*__ For any $f \in L^p(\mathbb{R}^n;\mathbb{R}^n)$, let the operators P_σ, P_π be defined by projections into the closed subspaces:

$$L^p_\sigma(\mathbb{R}^n;\mathbb{R}^n) \equiv \big\{f \in L^p(\mathbb{R}^n;\mathbb{R}^n); \operatorname{div} f = 0 \text{ in } \mathcal{D}^*\big\},$$
$$L^p_\pi(\mathbb{R}^n;\mathbb{R}^n) \equiv \big\{f \in L^p(\mathbb{R}^n;\mathbb{R}^n); \operatorname{rot} f = 0 \text{ in } \mathcal{D}^*\big\}.$$

It is well known that for any $1 < p < \infty$, the projection operators P_σ, P_π are bounded from $L^p(\mathbb{R}^n;\mathbb{R}^n)$ into $L^p_\sigma(\mathbb{R}^n;\mathbb{R}^n)$ and $L^p_\pi(\mathbb{R}^n;\mathbb{R}^n)$, respectively, and $L^p(\mathbb{R}^n;\mathbb{R}^n)$ can be decomposed by the direct sum of those subspaces as

$$L^p(\mathbb{R}^n;\mathbb{R}^n) = L^p_\sigma(\mathbb{R}^n;\mathbb{R}^n) \oplus L^p_\pi(\mathbb{R}^n;\mathbb{R}^n).$$

Those operators are realized by

$$\begin{aligned} P_\sigma &\equiv 1 + (-\Delta)^{-1}\nabla\mathrm{div}\,,\\ P_\pi &\equiv -(-\Delta)^{-1}\nabla\mathrm{div}\,. \end{aligned} \tag{1.6}$$

1.2. **Worst term from nonlinear interactions.** To extract the worst regularity term from the fluid vector u, we examine each term appearing in the right-hand side of the integral equation (1.4). If we assume that the pressure term ∇P is smoother compared with the other terms, then the candidates of the worst term are the remaining two terms:

$$(u\cdot\nabla)u, \qquad \frac{\rho}{1+\rho}\mathcal{L}u. \tag{1.7}$$

For the critical space, the regularity difference of the density ρ and the velocity u is -1 and they appear as comparable terms. To see it more precisely, we decompose the convection term $u\cdot\nabla u$ and the quasilinear term $-\frac{\rho}{1+\rho}\mathcal{L}u$ as follows. We derive the worst term in the convection term by the decomposition.

$$U = P_\sigma U + P_\pi U,$$

where P_σ and P_π are given in (1.6). On the first term of (1.7), we have

$$\begin{aligned} u\cdot\nabla u =& U\cdot\nabla U + l.o.t.\\ =&(P_\sigma U + P_\pi U)\cdot\nabla\big(P_\sigma U + P_\pi U\big) + l.o.t.\\ =&P_\sigma U\cdot\nabla(P_\sigma U + P_\pi U) + P_\pi U\cdot\nabla P_\sigma U + P_\pi U\cdot\nabla P_\pi U + l.o.t.\\ =&\mathrm{div}\,\big(P_\sigma U\otimes(P_\sigma U + P_\pi U)\big) + P_\pi U\cdot\nabla P_\sigma U + \frac{1}{2}\nabla|P_\pi U|^2 + l.o.t., \end{aligned} \tag{1.8}$$

where *l.o.t.* denotes the lower-order terms. We notice that the j-th component of the last term on the right-hand side can be represented by

$$\begin{aligned} (P_\pi U)_k\cdot\nabla_k\big(P_\pi U\big)_j =&\big(\nabla_k(-\Delta)^{-1}\mathrm{div}\,U\big)\nabla_j\big(\nabla_k(-\Delta)^{-1}\mathrm{div}\,U\big)\\ =&\frac{1}{2}\nabla_j\big|\nabla(-\Delta)^{-1}\mathrm{div}\,U\big|^2. \end{aligned} \tag{1.9}$$

Hence the first and third terms in (1.8) exhibit divergence form structures and they behave relatively well from the regularity viewpoint. Hence, the worst term is now identified by

$$u\cdot\nabla u =\big(P_\pi U\cdot\nabla\big)P_\sigma U + l.o.t. \tag{1.10}$$

Meanwhile, using the linear approximation ρ_1 for ρ defined in (1.5):

$$\frac{\rho}{1+\rho}\mathcal{L}u = \frac{\rho_1}{1+\rho_1}\mathcal{L}U + \frac{\rho_1}{1+\rho_1}(\mathcal{L}u - \mathcal{L}U) + \Big(\frac{\rho}{1+\rho} - \frac{\rho_1}{1+\rho_1}\Big)\mathcal{L}u. \tag{1.11}$$

One can regard that the first term on the right hand side of the identity (1.11) above is the leading term and we apply the Helmholtz decomposition $U = P_\sigma U + P_\pi U$ and change into the divergence form as follows:

$$\begin{aligned} \frac{\rho_1}{1+\rho_1}\mathcal{L}U =& \frac{\rho_1}{1+\rho_1}\mu\Delta P_\sigma U + \frac{\rho_1}{1+\rho_1}(2\mu+\lambda)\Delta P_\pi U\\ =& \mu\nabla\cdot\Big(\frac{\rho_1}{1+\rho_1}\nabla P_\sigma U\Big) - \underbrace{\mu\Big(\frac{\nabla\rho_1}{(1+\rho_1)^2}\cdot\nabla\Big)P_\sigma U}_{\text{non-divergence term I}}\\ &+ (2\mu+\lambda)\nabla\cdot\Big(\frac{\rho_1}{1+\rho_1}\nabla P_\pi U\Big) - \underbrace{(2\mu+\lambda)\Big(\frac{\nabla\rho_1}{(1+\rho_1)^2}\cdot\nabla\Big)P_\pi U}_{\text{``non-divergence term II''}} \end{aligned} \tag{1.12}$$

Apply the first intergral equation of (1.5) to the second term on the right-hand side of (1.12); "non-divergence term I" under the condition $\rho_0 = 0$,

$$
\begin{aligned}
\nabla\rho_1(t) =&\nabla\rho_0 - \int_0^t \nabla\Big(\mathrm{div}\,(1+\rho_0(s))U(s)\Big)ds\\
=&-\int_0^t \nabla\mathrm{div}\,U(s)ds,
\end{aligned}
\tag{1.13}
$$

and then approximate u by U in (1.5) with $\nabla\mathrm{div}\,U = \Delta P_\pi U$,

$$
\begin{aligned}
&-\mu\Big(\frac{\nabla\rho_1}{(1+\rho_1)^2}\cdot\nabla\Big)P_\sigma U\\
&=-\mu\frac{1}{(1+\rho_1)^2}\Big(-\int_0^t \nabla\mathrm{div}\,U ds\cdot\nabla\Big)P_\sigma U\\
&=\mu\frac{1}{(1+\rho_1)^2}\Big(\int_0^t \Delta P_\pi U ds\cdot\nabla\Big)P_\sigma U + l.o.t.
\end{aligned}
\tag{1.14}
$$

Here, we have neglected all the terms involving the initial data ρ_0 as we assumed that $\rho_0 = 0$. For the fourth term on the right-hand side in (1.12) i.e., "non-divergence term II", we can show that

$$
\begin{aligned}
&-(2\mu+\lambda)\Big(\frac{\nabla\rho_1}{(1+\rho_1)^2}\cdot\nabla\Big)P_\pi U\\
&=(2\mu+\lambda)\frac{1}{(1+\rho_1)^2}\Big(\int_0^t \Delta P_\pi U ds\Big)\cdot\nabla P_\pi U + l.o.t.
\end{aligned}
\tag{1.15}
$$

By noting

$$
P_\pi\partial_t U(t) = P_\pi\mathcal{L}U(t) = (2\mu+\lambda)\Delta P_\pi U(t),
\tag{1.16}
$$

substituting (1.16) into (1.15), it follows that

$$
\begin{aligned}
&-(2\mu+\lambda)\Big(\frac{\nabla\rho_1}{(1+\rho_1)^2}\cdot\nabla\Big)P_\pi U\\
&=\frac{1}{(1+\rho_1)^2}\Big(P_\pi(U(t)-u_0)\Big)\cdot\nabla P_\pi U + l.o.t.\\
&=\frac{1}{(1+\rho_1)^2}\Big(\frac{1}{2}\nabla\big|P_\pi U\big|^2 - P_\pi u_0\cdot\nabla P_\pi u_0\Big) + l.o.t.\\
&=\frac{1}{(1+\rho_1)^2}\Big(\frac{1}{2}\nabla\big|P_\pi U\big|^2 - \frac{1}{2}\nabla\big|P_\pi u_0\big|^2\Big) + l.o.t.,
\end{aligned}
\tag{1.17}
$$

where we regard $U(t) \simeq u_0$ if $t \ll 1$. Assuming that $\rho_0 \sim 0$, we neglect all the terms involving ρ_0 such that the following term can be regarded as the top term of the quasilinear dissipation:

$$
\begin{aligned}
-\frac{\rho}{1+\rho}\mathcal{L}u =& -\frac{\mu}{1+\rho_1}\int_0^t \Delta P_\pi U ds\cdot\nabla P_\sigma U + l.o.t.\\
=& -\frac{1}{1+\rho_1}\int_0^t P_\pi\Big(\mathcal{L}U\Big)ds\cdot\nabla P_\sigma U + l.o.t\\
=& -\frac{1}{1+\rho_1}P_\pi\big(U(s)-u_0\big)\cdot\nabla P_\sigma U + l.o.t.
\end{aligned}
\tag{1.18}
$$

From (1.10) and (1.18), the worst term for regularity of approximated solution can be extracted from the term of the convection and the quasilinear dissipation terms as follows:

$$
\begin{aligned}
u(t) &= e^{t\mathcal{L}}u_0 - \int_0^t e^{(t-s)\mathcal{L}}\Big(u(s)\cdot\nabla u(s) + \frac{\nabla P(1+\rho(s))}{1+\rho(s)} + \frac{\rho(s)}{1+\rho(s)}\mathcal{L}u(s)\Big)ds \\
&= -\int_0^t e^{(t-s)\mathcal{L}}\Big(u(s)\cdot\nabla u(s) + \frac{\rho(s)}{1+\rho(s)}\mathcal{L}u(s)\Big)ds + l.o.t \\
&= -\int_0^t e^{(t-s)\mathcal{L}}\bigg(\big(P_\pi U\cdot\nabla\big)P_\sigma U + \frac{\mu}{2\mu+\lambda}P_\pi\big(U(s)-u_0\big)\nabla P_\sigma U(s)\bigg)ds + l.o.t.
\end{aligned}
\tag{1.19}
$$

Since $U(t) = e^{t\mathcal{L}}u_0$ and

$$
e^{(t-s)\mathcal{L}}h = e^{\mu(t-s)\Delta}P_\sigma h + e^{(2\mu+\lambda)(t-s)\Delta}P_\pi h,
$$

we deduce the right-hand side of (1.19) by

$$
I[u_0] \equiv -\int_0^t e^{(t-s)\mathcal{L}}\Big\{e^{s\mathcal{L}}P_\pi u_0\cdot\nabla e^{s\mathcal{L}}P_\sigma u_0 + \frac{\mu}{2\mu+\lambda}\big(e^{s\mathcal{L}}P_\pi u_0 - P_\pi u_0\big)\nabla e^{s\mathcal{L}}P_\sigma u_0\Big\}ds, \tag{1.20}
$$

which is the crucial top term for proving the ill-posedness. A striking difference between our critical case $p = 2n$ and non-critical case $p > 2n$ is that both the convection and quasilinear viscosity terms share the worst singularity and they are canceled with each other. We show those arguments in a rigorous way in the following section working in the Fourier space and use the cancelling property in Lemma 2.3 below.

This paper is organized as follows. In section 2, we consider the discontinuity of the crucial part (1.20) on the initial data contained in the integral equation for the velocity u, and we will provide a counter example of the initial data to the continuity property. In section 3, we prove Theorem 1.1 by decomposing the velocity u into the top term observed in section 2 and the remainder part.

Hereafter we use the following notation: $\dot{B}^s_{p,\sigma}$ is the homogeneous Besov space defined in (1.3). The intersection space $\dot{B}^\alpha_{[n,\infty],1} := \dot{B}^\alpha_{n,1}(\mathbb{R}^n)\cap\dot{B}^\alpha_{\infty,1}(\mathbb{R}^n)$ is used in the section 3.

2. Crucial bilinear estimate for the ill-posedness

We discuss the bilinear estimates for scalar valued functions in Proposition A.1 and prove the discontinuity of the term (1.20) at the origin in the Besov spaces $\dot{B}^{-\frac{1}{2}}_{2n,\sigma}(\mathbb{R}^n)$ with $1 \le \sigma < n$ in Proposition 2.1.

2.1. Choice of initial data.

The choice of initial data for showing the discontinuity is the most influencing. For the critical homogeneous Besov spaces, $\dot{B}^{\frac{1}{2}}_{2n,\sigma}(\mathbb{R}^n)\times\dot{B}^{-\frac{1}{2}}_{2n,\sigma}(\mathbb{R}^n)$ $(\sigma > 2)$ and a parameter $N \in \mathbb{N}$ we chose the initial data for the velocity fields as follows:

Definition. For some fixed $0 < \delta < 1$ which is determined later and for any $N \in \mathbb{N}$, we define $(\rho_0, u_0) = (\rho_{0,N}, u_{0,N})$ by

$$
\begin{cases}
\rho_0(x) = 0 \\
u_0(x) = \Big(f_N(x), g_N(x), 0, \cdots, 0\Big),
\end{cases}
\tag{2.1}
$$

for all $N = 1, 2, \cdots$, where f_N, g_N [3] are defined by

$$\begin{cases} f_N(x) = R\dfrac{2^{\frac{N}{2}}}{N^{\frac{1}{2n}}} \displaystyle\sum_{-\delta N \le j \le 0} 2^{-(n-\frac{1}{2})j} \phi_j\big(x - 2^{|j|+2N} e_1\big) \sin\big(2^N x_1\big), \\ g_N(x) = R\dfrac{2^{\frac{N}{2}}}{N^{\frac{1}{2n}}} \displaystyle\sum_{-\delta N \le j \le 0} 2^{-(n-\frac{1}{2})j} \phi_j\big(x - 2^{|j|+2N} e_1\big) \cos\big(2^N x_1\big), \end{cases} \tag{2.2}$$

e_1 is the unit vector for x_1 and $R > 0$ is a constant.

It is straightforward that $f_N, g_N \in \dot{B}^{-\frac{1}{2}}_{2n,1}(\mathbb{R}^n)$, since $f_N, g_N \in \mathcal{S}(\mathbb{R}^n)$.

The following proposition is crucial for the proof of the discontinuity of solutions.

Proposition 2.1. *Let u_0 be defined by* (2.1)*. Then there exists $C > 0$ independent of N such that*

$$\|u_0\|_{\dot{B}^{-\frac{1}{2}}_{2n,\sigma}} \le CR.$$

Furthermore, there exist c , $\varepsilon_0 > 0$ and $N_0 \in \mathbb{N}$ such that for any $N \ge N_0$, $1 \le \sigma \le \infty$ and for $T_N = \varepsilon_0 2^{-2N}$

$$\Big\| I[u_0](t) \Big\|_{\dot{B}^{-\frac{1}{2}}_{2n,\sigma}} \Big|_{t=T_N} \ge cN^{\frac{1}{\sigma}-\frac{1}{n}} R^2 - CR^2, \tag{2.3}$$

where $I[u_0]$ is defined by (1.20)*.*

Remark. If we take $R = (\log N)^{-1}$ and $\sigma = 1$, the left hand side in (2.3) diverges to infinity while the initial data u_0 tends to 0. Therefore one can expect to obtain the ill-posedness result for the compressible Navier-Stokes equations in the Besov space $\dot{B}^{-\frac{1}{2}}_{2n,1}(\mathbb{R}^n)$.

To prove the estimate (2.3), we decompose $I[u_0]$ into a term with the product of f_N, g_N, which cause the discontinuity, and the remainder term. Let us introduce a bilinear form $\widetilde{I}[f, g]$ for scalar valued functions f and g, where f is an approximating function of $P_\pi u_0$ and g is an approximated function of $P_\sigma u_0$, respectively (cf. (1.20)).

Definition. Let $A = 2\mu + \lambda$, $B = \mu$, c be either $2\mu + \lambda$ or μ and f, $g \in \mathcal{S}(\mathbb{R}^n)$. Define the quadratic form $\widetilde{I}[\cdot, \cdot]$ by

$$\widetilde{I}[f, g] = \widetilde{I}[f, g](t) \equiv -\int_0^t e^{c(t-s)\Delta} \Big\{ e^{As\Delta} f \nabla e^{Bs\Delta} g + \frac{B}{A} \big(e^{As\Delta} f - f\big) \nabla e^{Bs\Delta} g \Big\} ds, \tag{2.4}$$

We will choose the parameter $R > 0$ such that f, g are arbitrarily small in the critical space $\dot{B}^{-\frac{1}{2}}_{2n,\sigma}$, while $\widetilde{I}[f, g]$ is large. The following proposition shows the discontinuity for $\widetilde{I}[f, g]$

Proposition 2.2. *For $R > 0$, let $\{f_N\}_{N\in\mathbb{N}}$, $\{g_N\}_{N\in\mathbb{N}}$ be defined in* (2.2)*. Then there exists C_n independet of N such that*

$$\|f_N\|_{\dot{B}^{-\frac{1}{2}}_{2n,\sigma}} \le C_n R, \qquad \|g_N\|_{\dot{B}^{-\frac{1}{2}}_{2n,\sigma}} \le C_n R. \tag{2.5}$$

Furthermore, there exists $c > 0$ independent of N such that for sufficiently small $\varepsilon_0 > 0$ and $T_N = \varepsilon_0 2^{-2N}$,

$$\big\| \widetilde{I}[f_N, g_N](T_N) \big\|_{\dot{B}^{-\frac{1}{2}}_{2n,\sigma}} \ge cN^{\frac{1}{\sigma}-\frac{1}{n}} R^2. \tag{2.6}$$

[3] the example here is inspired from the definition of the modulation space

Remark. The condition of σ is indeed sharp. When $1 \le \sigma < n$, the estimates (2.5) and (2.6) with $R = (\log N)^{-1}$ and $N \to \infty$ imply the discontinuity of the map

$$\dot{B}^{-\frac{1}{2}}_{2n,\sigma} \times \dot{B}^{-\frac{1}{2}}_{2n,\sigma} \ni (f,g) \mapsto \widetilde{I}[f,g] \in C([0,1]; \dot{B}^{-\frac{1}{2}}_{2n,\sigma}).$$

On the other hand, the continuity holds when $n \le \sigma \le \infty$, which will be discussed in the appendix (see Proposition A.1). Such a situation for the bilinear estimate for the nonlinear terms of the compressible Navier-Stokes system is peculiar since the better regularity provides the bilinear estimate failure and the other case makes the estimate valid.

We prepare some lemmas. Lemma 2.3 gives a decomposition of the Fourier transform of $\widetilde{I}[f,g]$ into two parts of divergence and non-divergence form structure. After that we will show inequalities from below in Lemmas 2.4, 2.5, and then we will prove Proposition 2.2, which will be applied to the proof of Proposition 2.1.

Lemma 2.3. *Let $\widetilde{I}[f,g]$ be defined by* (2.4)*. The Fourier transform of $\widetilde{I}[f,g]$ satisfies*

$$\widehat{\widetilde{I}[f,g]}(t) = -\frac{B}{A}\int_{\mathbb{R}^n} \frac{1-e^{-At|\xi-\eta|^2}}{B|\eta|^2 - c|\xi|^2} e^{-Bt|\eta|^2} i\eta \widehat{f}(\xi-\eta)\widehat{g}(\eta)d\eta + \widehat{R[f,g]}(t), \tag{2.7}$$

where $\widehat{R[f,g]}$ is defined by

$$\widehat{R[f,g]}(t) := \int_{\mathbb{R}^n} \left\{ \frac{\{(B+c)|\xi|^2 - 2B\xi\cdot\eta\}(e^{-ct|\xi|^2} + e^{-At|\xi-\eta|^2 - Bt|\eta^2|})}{(A|\xi-\eta|^2 + B|\eta|^2 - c|\xi|^2)(B|\eta|^2 - c|\xi|^2)} \right\} i\eta\widehat{f}(\xi-\eta)\widehat{g}(\eta)d\eta. \tag{2.8}$$

Remark. Let us investigate by focusing on the high–high frequency interaction to the low frequency, namely,

$$|\xi| \simeq 1, \quad \operatorname{supp}\widehat{f}, \operatorname{supp}\widehat{g} \subset \{\xi \in \mathbb{R}^n \mid |\xi| \simeq 2^N\}, \quad N \gg 1$$

to compare with the convection term. One can deduce from (2.7), $|\xi| \le 1$, $|\xi - \eta| \simeq 2^N$, $|\eta| \simeq 2^N$ that

$$\begin{aligned}
\widehat{\widetilde{I}[f,g]}(t) &\simeq -\frac{B}{A}\int_{\mathbb{R}^n} \frac{1-e^{-At2^{2N}}}{B\cdot 2^{2N} - c\cdot 1^2} e^{-Bt2^{2N}} i2^N \widehat{f}(\xi-\eta)\widehat{g}(\eta)d\eta \\
&\simeq -\frac{B}{A}\int_{\mathbb{R}^n} \frac{1}{2^{2N}} e^{-Bt2^{2N}} i2^N \widehat{f}(\xi-\eta)\widehat{g}(\eta)d\eta,
\end{aligned}$$

if $t \simeq 2^{-2N}$. The term $e^{-Bt2^{2N}}$ above can provide a certain regularity to obtain the estimate for $n \le \sigma \le \infty$ in Proposition A.1. Meanwhile, the convection term $(u\cdot\nabla)u$ will exhibit less regularity than that for $(u\cdot\nabla)u + \frac{\rho}{1+\rho}\mathcal{L}$. In fact,

$$\begin{aligned}
&\mathcal{F}\Big[\int_0^t e^{c(t-s)\Delta} e^{As\Delta} f \nabla e^{Bs\Delta} g ds\Big](\xi) \\
=&e^{-ct|\xi|^2}\int_{\mathbb{R}^n} \frac{1-e^{-At|\xi-\eta|^2 - Bt|\eta|^2 + ct|\xi|^2}}{A|\xi-\eta|^2 + B|\eta|^2 - c|\xi|^2} i\eta\widehat{f}(\xi-\eta)\widehat{g}(\eta)d\eta \\
\simeq&\int_{\mathbb{R}^n} \frac{1}{2^{2N}} i2^N \widehat{f}(\xi-\eta)\widehat{g}(\eta)d\eta
\end{aligned}$$

for $2^{-2N} \le t \le 1$, and the term $e^{-Bt2^{2N}}$ does not appear. Hence, the observations above suggest that $(u\cdot\nabla)u$ and $\frac{\rho}{1+\rho}\mathcal{L}u$ would exhibit comparable regularity and some components are canceled out.

Proof of Lemma 2.3. By taking the Fourier transform of $\widetilde{I}[f,g]$, it holds that

$$\begin{aligned}
&\widehat{\widetilde{I}[f,g]}(t) \\
&= -\int_0^t e^{-c(t-s)|\xi|^2} \int_{\mathbb{R}^n} \Big\{ e^{-As|\xi-\eta|^2} \widehat{f}(\xi-\eta) i\eta e^{-Bs|\eta|^2} \widehat{g}(\eta) \\
&\qquad\qquad + \frac{B}{A}\Big(e^{-As|\xi-\eta|^2}\widehat{f}(\xi-\eta) - \widehat{f}(\xi-\eta)\Big) i\eta e^{-Bs|\eta|^2}\widehat{g}(\eta)\Big\} d\eta ds \qquad (2.9)\\
&= -\int_{\mathbb{R}^n} e^{-ct|\xi|^2} \int_0^t e^{cs|\xi|^2} \Big\{ e^{-s(A|\xi-\eta|^2+B|\eta|^2)} + \frac{B}{A}\Big(e^{-As|\xi-\eta|^2}-1\Big) e^{-Bs|\eta|^2}\Big\} ds \\
&\qquad \times \widehat{f}(\xi-\eta) i\eta \widehat{g}(\eta) d\eta.
\end{aligned}$$

Following the integral on variable $s \in (0,t)$,

$$\begin{aligned}
&\int_0^t e^{cs|\xi|^2} \Big\{ e^{-s(A|\xi-\eta|^2+B|\eta|^2)} + \frac{B}{A}\Big(e^{-As|\xi-\eta|^2}-1\Big) e^{-Bs|\eta|^2}\Big\} ds \\
&= \int_0^t \Big\{ \frac{A+B}{A} e^{-s(A|\xi-\eta|^2+B|\eta|^2-c|\xi|^2)} - \frac{B}{A} e^{-s(B|\eta|^2-c|\xi|^2)}\Big\} ds \\
&= \frac{A+B}{A}\cdot \frac{1-e^{-t(A|\xi-\eta|^2+B|\eta|^2-c|\xi|^2)}}{A|\xi-\eta|^2+B|\eta|^2-c|\xi|^2} - \frac{B}{A}\cdot\frac{1-e^{-t(B|\eta|^2-c|\xi|^2)}}{B|\eta|^2-c|\xi|^2} \qquad (2.10)\\
&= \Big\{ \frac{A+B}{A}\cdot\frac{1}{A|\xi-\eta|^2+B|\eta|^2-c|\xi|^2} - \frac{B}{A}\cdot\frac{1}{B|\eta|^2-c|\xi|^2}\Big\} \\
&\quad + e^{ct|\xi|^2}\Big\{ \frac{A+B}{A}\cdot\frac{-e^{-t(A|\xi-\eta|^2+B|\eta|^2)}}{A|\xi-\eta|^2+B|\eta|^2-c|\xi|^2} - \frac{B}{A}\cdot\frac{-e^{-tB|\eta|^2}}{B|\eta|^2-c|\xi|^2}\Big\} \\
&=: J_1 + e^{ct|\xi|^2} J_2.
\end{aligned}$$

By reducing the two terms of J_1 to a common denominator, we have

$$\begin{aligned}
J_1 &= \frac{1}{A}\cdot\frac{(A+B)(B|\eta|^2-c|\xi|^2) - B(A|\xi-\eta|^2+B|\eta|^2-c|\xi|^2)}{(A|\xi-\eta|^2+B|\eta|^2-c|\xi|^2)(B|\eta|^2-c|\xi|^2)} \\
&= \frac{1}{A}\cdot\frac{-A\big\{(B+c)|\xi|^2-2B\xi\cdot\eta\big\}}{(A|\xi-\eta|^2+B|\eta|^2-c|\xi|^2)(B|\eta|^2-c|\xi|^2)} \qquad (2.11)\\
&= -\frac{(B+c)|\xi|^2-2B\xi\cdot\eta}{(A|\xi-\eta|^2+B|\eta|^2-c|\xi|^2)(B|\eta|^2-c|\xi|^2)}.
\end{aligned}$$

By considering the difference of $-e^{-t(A|\xi-\eta|^2+B|\eta|^2)}$ and $-e^{-tB|\eta|^2}$ for the second term of J_2, we obtain

$$\begin{aligned}
J_2 &= \Big\{ \frac{A+B}{A}\cdot\frac{-e^{-t(A|\xi-\eta|^2+B|\eta|^2)}}{A|\xi-\eta|^2+B|\eta|^2-c|\xi|^2} - \frac{B}{A}\cdot\frac{-e^{-t(A|\xi-\eta|^2+B|\eta|^2)}}{B|\eta|^2-c|\xi|^2}\Big\} \\
&\quad + \Big\{ \frac{B}{A}\cdot\frac{-e^{-t(A|\xi-\eta|^2+B|\eta|^2)}}{B|\eta|^2-c|\xi|^2} - \frac{B}{A}\cdot\frac{-e^{-tB|\eta|^2}}{B|\eta|^2-c|\xi|^2}\Big\} \qquad (2.12)\\
&= -e^{-t(A|\xi-\eta|^2+B|\eta|^2)} J_1 + \frac{B}{A}\cdot\frac{-e^{-tA|\xi-\eta|^2}+1}{B|\eta|^2-c|\xi|^2}\cdot e^{-tB|\eta|^2}.
\end{aligned}$$

Therefore, the equality (2.7) is obtained by (2.9), (2.10), (2.11) and (2.9) because the first term in the right hand side of (2.7) corresponds to the second term in the last right hand side of (2.12), and the remainder term $R[f,g]$ consists of J_1 from (2.11) and the first term in the last right hand side of (2.12). □

Lemma 2.4. *Let φ satisfy $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $\operatorname{supp}\widehat{\varphi} \subset \{|\xi| \le 1\}$, and let $\{f_N\}_N$ and $\{g_N\}_N$ be defined by (2.2). Then there exists $N_0 \in \mathbb{N}$ such that for any σ with $1 \le \sigma \le \infty$ and $N \in \mathbb{N}$ with*

$N \geq N_0$

$$2^{-2N}\|\varphi * (f_N\partial_{x_1}g_N)\|_{\dot{B}^{-\frac{1}{2}}_{2n,\sigma}} \geq cR^2N^{\frac{1}{\sigma}-\frac{1}{n}}. \tag{2.13}$$

Proof of Lemma 2.4. Since the norm in the left hand side of (2.13) is restricted in the low frequency part, it follows that

$$2^{-2N}\|\varphi * (f_N\partial_{x_1}g_N)\|_{\dot{B}^{-\frac{1}{2}}_{2n,\sigma}} \geq c2^{-2N}\Big\{\sum_{-\delta N\leq j\leq 0}\Big(2^{-\frac{1}{2}j}\|\phi_j * (f_N\partial_{x_1}g_N)\|_{L^{2n}}\Big)^{\sigma}\Big\}^{\frac{1}{\sigma}}. \tag{2.14}$$

For simplicity, we rewrite the initial of f_N and g_N in (2.2) by introducing

$$\Phi_j := \frac{R2^{\frac{1}{2}N}}{N^{\frac{1}{2n}}}2^{(-n+\frac{1}{2})j}\phi_j(x-2^{|j|+2N}e_1), \tag{2.15}$$

then f_N and g_N are written as

$$f_N(x) = \sum_{-\delta N\leq k\leq 0}\Phi_k(x)\sin(2^Nx_1), \quad g_N(x) = \sum_{-\delta N\leq l\leq 0}\Phi_l(x)\cos(2^Nx_1),$$

and we have the following equality:

$$\begin{aligned}
f_N\partial_{x_1}g_N &= -\sum_{-\delta N\leq k,l\leq 0}2^N\Phi_k\Phi_l\sin^2(2^Nx_1) + \sum_{-\delta N\leq k,l\leq 0}\Phi_k\Big(\partial_{x_1}\Phi_l\Big)\times\cos(2^Nx_1)\sin(2^Nx_1)\\
&= \sum_{-\delta N\leq k\leq 0}2^N\Phi_k^2\sin^2(2^Nx_1) + \sum_{-\delta N\leq k,l\leq 0,\,k\neq l}2^N\Phi_k\Phi_l\sin^2(2^Nx_1)\\
&\quad + \sum_{-\delta N\leq k,l\leq 0}\Phi_k\Big(\partial_{x_1}\Phi_l\Big)\times\cos(2^Nx_1)\sin(2^Nx_1)\\
&=: F_1+F_2+F_3.
\end{aligned} \tag{2.16}$$

Let us introduce the set A_j defined by

$$A_j := \{\, x \,|\, |x-2^{|j|+2N}e_1| \leq 2^{-j}\,\},$$

which is the main area on the support of Φ_j. In what follows we estimate F_1, F_2 and F_3 one by one.

On the estimate of $\|\phi_j * F_1\|_{L^{2n}}$, we restrict the L^{2n}-norm in A_j and expect that the case $k=j$ in the sum in the definition of F_1 is the largest, and obtain

$$\begin{aligned}
\|\phi_j * F_1\|_{L^{2n}} &\geq \|\phi_j * F_1\|_{L^{2n}(A_j)}\\
&\geq 2^N\big\|\phi_j * \big(\Phi_j^2\sin^2(2^Nx_1)\big)\big\|_{L^{2n}(A_j)}\\
&\quad - \sum_{-\delta N\leq k\leq 0,\,k\neq j}2^N\big\|\phi_j * \big(\Phi_k^2\sin^2(2^Nx_1)\big)\big\|_{L^{2n}(A_j)}.
\end{aligned} \tag{2.17}$$

On the first term in the last right hand side of (2.17), it holds that for j with $-N\leq j\leq 0$

$$\begin{aligned}
&2^N\big\|\phi_j * \big(\Phi_j^2\sin^2(2^Nx_1)\big)\big\|_{L^{2n}(A_j)}\\
&= 2^N\Big(R2^{\frac{1}{2}N}2^{(-n+\frac{1}{2})j}N^{\frac{1}{2n}}\Big)^2\\
&\quad\times 2^{3nj}\Big\|\int_{\mathbb{R}^n}\phi_0(2^j(x-y))\Big(\phi_0\big(2^j(y-2^{|j|+2N}e_1)\big)\Big)^2\sin^2(2^Ny_1)dy\Big\|_{L^{2n}_x(A_j)}
\end{aligned}$$

Since $\sin^2(2^N y_1) = (1-\cos(2^{N+1}y_1))/2$, it follows from the triangle inequality that

$$\begin{aligned}
&\Big\| \int_{\mathbb{R}^n} \phi_0(2^j(x-y))\Big(\phi_0\big(2^j(y-2^{|j|+2N}e_1)\big)\Big)^2 \sin^2(2^N y_1)dy\Big\|_{L^{2n}_x(A_j)}\\
\geq&\frac{1}{2}\Big\| \int_{\mathbb{R}^n} \phi_0(2^j(x-y))\Big(\phi_0\big(2^j(y-2^{|j|+2N}e_1)\big)\Big)^2 dy\Big\|_{L^{2n}_x(A_j)}\\
&-\frac{1}{2}\Big\| \int_{\mathbb{R}^n} \phi_0(2^j(x-y))\Big(\phi_0\big(2^j(y-2^{|j|+2N}e_1)\big)\Big)^2 \cos(2^{N+1}y_1)dy\Big\|_{L^{2n}_x(A_j)}.
\end{aligned}$$

By change of variables, we write the first term in the right hand side

$$\begin{aligned}
&\frac{1}{2}\left\|\int_{\mathbb{R}^n} \phi_0(2^j(x-y))\Big(\phi_0\big(2^j(y-2^{|j|+2N}e_1)\big)\Big)^2 dy\right\|_{L^{2n}(A_j)}\\
=&\frac{1}{2}\cdot 2^{-nj}\cdot 2^{-\frac{1}{2}j}\left\|\int_{\mathbb{R}^n} \phi_0(x-y)\Big(\phi_0(y)\Big)^2 dy\right\|_{L^{2n}(\{|x|\leq 1\})}.
\end{aligned}$$

The second term is dealt with the integration by parts, and we have that

$$\begin{aligned}
&\frac{1}{2}\left\|\int_{\mathbb{R}^n} \phi_0(2^j(x-y))\Big(\phi_0\big(2^j(y-2^{|j|+2N}e_1)\big)\Big)^2 \cos(2^{N+1}y_1)dy\right\|_{L^{2n}(A_j)}\\
=&\frac{1}{2}\left\|\int_{\mathbb{R}^n} \partial_{y_1}\left(\phi_0(2^j(x-y))\Big(\phi_0\big(2^j(y-2^{|j|+2N}e_1)\big)\Big)^2\right)\frac{\sin(2^{N+1}y_1)}{2^{N+1}}dy\right\|_{L^{2n}(A_j)}\\
\leq& C\frac{2^j}{2^{N+1}}\times 2^{-nj}\cdot 2^{-\frac{1}{2}j}.
\end{aligned}$$

The three inequalities above imply that

$$\begin{aligned}
&2^N\big\|\phi_j * \big(\Phi_j^2\sin^2(2^N x_1)\big)\big\|_{L^{2n}(A_j)}\\
&\geq c2^N R^2 2^N 2^{(-2n+1)j}N^{-\frac{1}{n}}\times 2^{3nj}\cdot 2^{-2j}\cdot 2^{-\frac{1}{2}j}\Big(1-\frac{C2^j}{2^N}\Big)\\
&\geq c2^{2N}R^2 2^{\frac{1}{2}j}N^{-\frac{1}{n}}.
\end{aligned}\tag{2.18}$$

On the second term in the last right hand of (2.17), let α satisfy $\alpha > n$, and it follows from $\phi_0\in\mathcal{S}(\mathbb{R}^n)$ that

$$\begin{aligned}
&2^N\big\|\phi_j * \big(\Phi_k^2\sin^2(2^N x_1)\big)\big\|_{L^{2n}(A_j)}\\
&\leq 2^N\Big(R2^{\frac{1}{2}N}2^{(-n+\frac{1}{2})k}N^{-\frac{1}{2n}}\Big)^2\times 2^{2nk+nj}\Big\|\int_{\mathbb{R}^n}\phi_0(2^j(x-y))\Big(\phi_0\big(2^k(y-2^{|k|+2N}e_1)\big)\Big)^2dy\Big\|_{L^{2n}_x(A_j)}\\
&\leq C2^{2N}R^2 2^{(-2n+1)k}N^{-\frac{1}{n}}\times 2^{2nk+nj}\Big\|\int_{\mathbb{R}^n}\frac{1}{(1+2^j|x-y|)^\alpha}\frac{1}{(1+2^k|y-2^{|k|+2N}e_1|)^\alpha}dy\Big\|_{L^{2n}_x(A_j)}.
\end{aligned}\tag{2.19}$$

Since

$$2^k|x-2^{|k|+2N}e_1|\geq c2^k|2^{|j|+2N}-2^{|k|+2N}|\geq c2^k2^{2N}\quad\text{for any } j,k \text{ with } j\neq k \text{ and } x\in A_j,$$

we have from (2.19) that

$$\begin{aligned}
2^N\big\|\phi_j * \big(\Phi_k^2\sin^2(2^N x_1)\big)\big\|_{L^{2n}(A_j)} &\leq C2^{2N}R^2 2^{(-2n+1)k}N^{-\frac{1}{n}}\times 2^{2nk+nj}\Big\|\frac{2^{-nj}}{2^k2^{2N}}\Big\|_{L^{2n}_x(A_j)}\\
&\leq CR^2 2^{-\frac{1}{2}j}.
\end{aligned}\tag{2.20}$$

Therefore in the right hand side of (2.14), the term consisting of F_1 is estimated by (2.17), (2.18) and (2.20) as follows

$$
\begin{aligned}
&2^{-2N}\Big\{ \sum_{-\delta N\le j\le 0} \Big(2^{-\frac{1}{2}j}\|\phi_j * F_1\|_{L^{2n}}\Big)^{\sigma}\Big\}^{\frac{1}{\sigma}} \\
&\ge c2^{-2N}\Big\{ \sum_{-\delta N\le j\le 0} \Big(2^{-\frac{1}{2}j}2^{2N}R^2 2^{\frac{1}{2}j}N^{-\frac{1}{n}}\Big)^{\sigma}\Big\}^{\frac{1}{\sigma}} - C2^{-2N}\Big\{ \sum_{-\delta N\le j\le 0} \Big(2^{-\frac{1}{2}j}R^2 2^{-\frac{1}{2}j}\Big)^{\sigma}\Big\}^{\frac{1}{\sigma}} \qquad (2.21)\\
&\ge cR^2N^{\frac{1}{\sigma}-\frac{1}{n}} - CR^2 2^{-N}.
\end{aligned}
$$

On the term F_2, by the embedding $\dot{B}^0_{n,\sigma}(\mathbb{R}^n) \hookrightarrow \dot{B}^{-\frac{1}{2}}_{2n,\sigma}(\mathbb{R}^n)$,

$$
2^{-2N}\Big\{ \sum_{-\delta N\le j\le 0} \Big(2^{-\frac{1}{2}j}\|\phi_j * F_2\|_{L^{2n}}\Big)^{\sigma}\Big\}^{\frac{1}{\sigma}} \le C2^{-2N}\Big\{ \sum_{-\delta N\le j\le 0} \|\phi_j * F_2\|^{\sigma}_{L^n}\Big\}^{\frac{1}{\sigma}}. \qquad (2.22)
$$

In order to estimate $\|\phi_j * F_2\|_{L^n}$, let α be defined by $\alpha = \frac{n}{2}+\frac{1}{2}$, and it follows from Young's inequality and $\phi_0 \in \mathcal{S}(\mathbb{R}^n)$ that

$$
\begin{aligned}
&\|\phi_j * F_2\|_{L^n} \\
&\le \|\phi_j\|_{L^1}\|F_2\|_{L^n} \le \|\phi_0\|_{L^1} \sum_{-\delta N\le k,l\le 0,\, k\ne l} 2^N\|\Phi_k\Phi_l\|_{L^n} \\
&\le C \sum_{-\delta N\le k,l\le 0,\, k\ne l} 2^N\Big(\frac{R2^{\frac{1}{2}N}}{N^{\frac{1}{2n}}}\Big)^2 2^{(-n+\frac{1}{2})(k+l)}\|\phi_k(\cdot - 2^{|k|+2N}e_1)\phi_l(\cdot - 2^{|l|+2N}e_1)\|_{L^n} \qquad (2.23)\\
&\le C \sum_{-\delta N\le k,l\le 0,\, k\ne l} 2^{2N}R^2N^{-\frac{1}{n}}2^{(-n+\frac{1}{2})(k+l)}\cdot 2^{-2\alpha N}.
\end{aligned}
$$

Then we obtain by (2.22) and (2.23)

$$
\begin{aligned}
&2^{-2N}\Big\{ \sum_{-\delta N\le j\le 0} \Big(2^{-\frac{1}{2}j}\|\phi_j * F_2\|_{L^{2n}}\Big)^{\sigma}\Big\}^{\frac{1}{\sigma}} \\
&\le C2^{-2N}\Big\{ \sum_{-\delta N\le j\le 0} \Big(\sum_{-\delta N\le k,l\le 0,\, k\ne l} 2^{2N}R^2N^{-\frac{1}{n}}2^{(-n+\frac{1}{2})(k+l)}\cdot 2^{n(k+l)}2^{-2\alpha N}\Big)^{\sigma}\Big\}^{\frac{1}{\sigma}} \qquad (2.24)\\
&\le CR^2N^{-\frac{1}{n}}2^{-2\alpha N}\Big\{ \sum_{-\delta N\le j\le 0} \Big(\sum_{-N\le k,l\le 0,\, k\ne l} 2^{\frac{1}{2}(k+l)}\Big)^{\sigma}\Big\}^{\frac{1}{\sigma}} \\
&\le CR^2N^{-\frac{1}{n}+\frac{1}{\sigma}}2^{-2\alpha N},
\end{aligned}
$$

where C depends on δ.

The term F_3 can be treated as in the similar way for F_2, we estimate the right hand side of (2.22) for F_3 instead of F_2 by Young's inequality and Hölder's inequality that

$$
\begin{aligned}
\|\phi_j * F_3\|_{L^n} &\le \|\phi_j\|_{L^1}\|F_3\|_{L^n} \le \|\phi_0\|_{L^1} \sum_{-\delta N\le k,l\le 0} \Big\|\Phi_k\Big(\partial_{x_1}\Phi_l\Big)\Big\|_{L^n} \\
&\le C \sum_{-\delta N\le k,l\le 0} \|\Phi_k\|_{L^{2n}}\|\partial_{x_1}\Phi_l\|_{L^{2n}} \\
&\le C \sum_{-\delta N\le k,l\le 0} \Big(\frac{R2^{\frac{1}{2}N}}{N^{\frac{1}{2n}}}\Big)^2 2^{(-n+\frac{1}{2})(k+l)}\|\phi_k\|_{L^{2n}}2^l\|\phi_l\|_{L^{2n}} \qquad (2.25)\\
&\le CR^2 2^N N^{-\frac{1}{n}} \sum_{-\delta N\le k,l\le 0} 2^l \\
&\le CR^2 2^N N^{1-\frac{1}{n}}.
\end{aligned}
$$

Then we have from the corresponding estimate for F_3 of (2.22) and (2.25)

$$2^{-2N}\Big\{\sum_{-\delta N\le j\le 0}\Big(2^{-\frac{1}{2}j}\|\phi_j * III\|_{L^{2n}}\Big)^{\sigma}\Big\}^{\frac{1}{\sigma}} \le C2^{-2N}\Big\{\sum_{-\delta N\le j\le 0}\Big(R^2 2^N N^{1-\frac{1}{n}}\Big)^{\sigma}\Big\}^{\frac{1}{\sigma}} \le C2^{-N}N^{1-\frac{1}{n}+\frac{1}{\sigma}}R^2. \tag{2.26}$$

Therefore we obtain by (2.14), (2.16), (2.21), (2.24) and (2.26)

$$\begin{aligned}
&2^{-2N}\|\varphi*(f\partial_{x_1}g)\|_{\dot{B}^{-\frac{1}{2}}_{2n,\sigma}}\\
&\ge cR^2N^{\frac{1}{\sigma}-\frac{1}{n}} - CR^2 2^{-N} - CR^2N^{-\frac{1}{n}+\frac{1}{\sigma}}2^{-2\alpha N} - C2^{-N}N^{1-\frac{1}{n}+\frac{1}{\sigma}}R^2\\
&= R^2\Big(cN^{\frac{1}{\sigma}-\frac{1}{n}} - C2^{-N} - CN^{-\frac{1}{n}+\frac{1}{\sigma}}2^{-2\alpha N} - C2^{-N}N^{1-\frac{1}{n}+\frac{1}{\sigma}}\Big).
\end{aligned}$$

The proof of the estimate (2.13) for sufficiently large $N\in\mathbb{N}$ is completed. ❑

Next, we show a lemma on the bilinear estimate, which is concerned with the top term of (2.7).

Lemma 2.5. *Let f_N, g_N be defined by (2.2). For sufficiently small $\varepsilon_0>0$ and $t=\varepsilon_0 2^{-2N}$, it holds that for any large $N\in\mathbb{N}$*

$$\bigg\|\mathcal{F}^{-1}\Big[\frac{B}{A}\int_{\mathbb{R}^n}\frac{1-e^{-At|\xi-\eta|^2}}{B|\eta|^2-c|\xi|^2}e^{-Bt|\eta|^2}i\eta\widehat{f_N}(\xi-\eta)\widehat{g_N}(\eta)d\eta\Big]\bigg\|_{\dot{B}^{-\frac{1}{2}}_{2n,\sigma}} \ge c\varepsilon_0 N^{\frac{1}{\sigma}-\frac{1}{n}}R^2. \tag{2.27}$$

Proof of Lemma 2.5. Let $\varphi\in\mathcal{S}(\mathbb{R}^n)$ satisfy $\operatorname{supp}\widehat{\varphi}\subset\{\xi\,|\,|\xi|\le 1\}$ and we consider the restriction to low frequency part:

$$\begin{aligned}
&\bigg\|\mathcal{F}^{-1}\Big[\frac{B}{A}\int_{\mathbb{R}^n}\frac{1-e^{-At|\xi-\eta|^2}}{B|\eta|^2-c|\xi|^2}e^{-Bt|\eta|^2}i\eta\widehat{f}_N(\xi-\eta)\widehat{g}_N(\eta)d\eta\Big]\bigg\|_{\dot{B}^{-\frac{1}{2}}_{2n,\sigma}}\\
&\ge c\bigg\|\mathcal{F}^{-1}\Big[\widehat{\varphi}(\xi)\int_{\mathbb{R}^n}\frac{1-e^{-At|\xi-\eta|^2}}{B|\eta|^2-c|\xi|^2}e^{-Bt|\eta|^2}i\eta\widehat{f}_N(\xi-\eta)\widehat{g}_N(\eta)d\eta\Big]\bigg\|_{\dot{B}^{-\frac{1}{2}}_{2n,\sigma}}.
\end{aligned}\tag{2.28}$$

Noting our frequency $|\xi|\le 1$, $|\xi-\eta|\simeq 2^N$, $|\eta|\simeq 2^N$ and $t=\varepsilon_0 2^{-2N}$ we consider the expansion

$$\begin{aligned}
1-e^{-At|\xi-\eta|^2} &= -\sum_{m_1=1}^{\infty}\frac{(-At|\xi-\eta|^2)^{m_1}}{m_1!}\\
&= At|\eta|^2 + At(-2\xi\cdot\eta+|\xi|^2) - \sum_{m_1=2}^{\infty}\frac{(-At|\xi-\eta|^2)^{m_1}}{m_1!}\Big)\\
&= At|\eta|^2 + O\Big(t2^N\Big) + O\Big(\varepsilon_0^2\Big),\\
\frac{1}{B|\eta|^2-c|\xi|^2} &= \frac{1}{B|\eta|^2}\frac{1}{1-\frac{c|\xi|^2}{B|\eta|^2}} = \frac{1}{B|\eta|^2}\Big(1+\sum_{m_2=1}^{\infty}\Big(\frac{c|\xi|^2}{B|\eta|^2}\Big)^{m_2}\Big)\\
&= \frac{1}{B|\eta|^2}(1+O(2^{-2N})),\\
e^{-Bt|\eta|^2} &= 1+\sum_{m_3=1}^{\infty}\frac{(-Bt|\eta|^2)^{m_3}}{m_3!}\\
&= 1+O(\varepsilon_0),
\end{aligned}$$

and the leading term should be $\frac{A}{B}t$, since

$$\frac{1-e^{-At|\xi-\eta|^2}}{B|\eta|^2-c|\xi|^2}e^{-Bt|\eta|^2} = \frac{A}{B}t + O\Big(\frac{\varepsilon_0^2}{2^{2N}}\Big), \quad N\to\infty \tag{2.29}$$

for sufficiently large N. On the leading term, the inequality (2.13) and $t=\varepsilon_0 2^{-2N}$ imply

$$\Big\|\mathcal{F}^{-1}\Big[\widehat{\varphi}(\xi)\int_{\mathbb{R}^n}\Big(\frac{A}{B}t\Big)i\eta\widehat{f}_N(\xi-\eta)\widehat{g}_N(\eta)d\eta\Big]\Big\|_{\dot{B}^{-\frac{1}{2}}_{2n,\sigma}}\geq c\varepsilon_0 N^{\frac{1}{\sigma}-\frac{1}{n}}R^2.$$

The remainder can be estimated in a similar manner to the proof of Lemma 2.4, where the main part there was the estimate from below but one can prove the bounds from above. In fact, it follows by the boundedness of the Fourier transform in Lebesgue spaces and (2.29) that

$$\begin{aligned}
&\Big\|\mathcal{F}^{-1}\Big[\widehat{\varphi}(\xi)\int_{\mathbb{R}^n}\Big(\frac{1-e^{-At|\xi-\eta|^2}}{B|\eta|^2-c|\xi|^2}e^{-Bt|\eta|^2}-\frac{A}{B}t\Big)i\eta_1\widehat{f}_N(\xi-\eta)\widehat{g}_N(\eta)d\eta\Big]\Big\|_{\dot{B}^{-\frac{1}{2}}_{2n,\sigma}}\\
\leq& C\Big\{\sum_{j\leq 0}\Big(2^{-\frac{1}{2}j}\Big\|\widehat{\phi}_j\int_{\mathbb{R}^n}\Big(\frac{\varepsilon_0^2}{2^{2N}}\Big)i\eta_1\widehat{f_N}(\xi-\eta)\widehat{g_N}(\eta)d\eta\Big\|_{L^{\frac{2n}{2n-1}}}\Big)^{\sigma}\Big\}^{\frac{1}{\sigma}}\\
=&C\Big\{\sum_{j\leq 0}\Big(2^{-\frac{1}{2}j}\frac{\varepsilon_0^2}{2^{2N}}\Big\|\widehat{\phi}_j\mathcal{F}\Big[f_N\partial_{x_1}g_N\Big]\Big\|_{L^{\frac{2n}{2n-1}}}\Big)^{\sigma}\Big\}^{\frac{1}{\sigma}}\\
=&C\Big\{\sum_{j\leq 0}\Big(2^{-\frac{1}{2}j}\frac{\varepsilon_0^2}{2^{2N}}\Big\|\widehat{\phi}_j\mathcal{F}\Big[F_1+F_2+F_3\Big]\Big\|_{L^{\frac{2n}{2n-1}}}\Big)^{\sigma}\Big\}^{\frac{1}{\sigma}},
\end{aligned}\tag{2.30}$$

where we employ the decomposition (2.16). We then focus on the worst term F_1 which is given by

$$\begin{aligned}
&2^{-\frac{1}{2}j}\frac{\varepsilon_0^2}{2^{2N}}\Big\|\widehat{\phi}_j\mathcal{F}\Big[F_1\Big]\Big\|_{L^{\frac{2n}{2n-1}}}\\
&=2^{-\frac{1}{2}j}\Big\|\widehat{\phi}_j\int_{\mathbb{R}^n}\frac{\varepsilon_0^2}{2^{2N}}\cdot 2^N\Big|\sum_{-\delta N\leq k\leq 0}\mathcal{F}\big[\Phi_k\sin(2^Nx_1)\big](\xi-\eta)\mathcal{F}\big[\Phi_k\sin(2^Nx_1)\big](\eta)\Big|d\eta\Big\|_{L^{\frac{2n}{2n-1}}}\\
&\leq C2^{-\frac{1}{2}j}\frac{\varepsilon_0^2}{2^N}\sum_{\max\{j-3,-\delta N\}\leq k\leq 0}\Big(\frac{R2^{\frac{1}{2}N}}{N^{\frac{1}{2n}}}2^{(-n+\frac{1}{2})k}\Big)^2\Big\|\widehat{\phi}_j\int_{\mathbb{R}^n}\Big|\widehat{\phi_k}(\xi-\eta)\widehat{\phi_k}(\eta)\Big|d\eta\Big\|_{L^{\frac{2n}{2n-1}}}\\
&\leq C2^{-\frac{1}{2}j}\varepsilon_0^2\sum_{\max\{j-3,-\delta N\}\leq k\leq 0}\frac{R^2}{N^{\frac{1}{n}}}2^{(-2n+1)k}\cdot 2^{(n-\frac{1}{2})j}2^{nk}\\
&\leq C\varepsilon_0^2\sum_{\max\{j-3,-\delta N\}\leq k\leq 0}\frac{R^2}{N^{\frac{1}{n}}}2^{(-n+1)k}\cdot 2^{(n-1)j},
\end{aligned}\tag{2.31}$$

where Φ_k is given by (2.15) as $\Phi_k := \frac{R2^{\frac{1}{2}N}}{N^{\frac{1}{2n}}} 2^{(-n+\frac{1}{2})k}\phi_k(x - 2^{|k|+2N}e_1)$ and the range of k is restricted by the support of $\widehat{\phi}_j$ as $j-3 \le k$. We then have that

$$
\begin{aligned}
&\left\{ \sum_{j\le 0} \left(2^{-\frac{1}{2}j} \frac{\varepsilon_0^2}{2^{2N}} \left\| \widehat{\phi}_j \mathcal{F}\left[F_1\right] \right\|_{L^{\frac{2n}{2n-1}}} \right)^{\sigma} \right\}^{\frac{1}{\sigma}} \\
&\qquad \le \left\{ \sum_{j\le -\delta N} \left(2^{-\frac{1}{2}j} \frac{\varepsilon_0^2}{2^{2N}} \left\| \widehat{\phi}_j \mathcal{F}\left[F_1\right] \right\|_{L^{\frac{2n}{2n-1}}} \right)^{\sigma} \right\}^{\frac{1}{\sigma}} \\
&\qquad\qquad + \left\{ \sum_{-\delta N\le j\le 0} \left(2^{-\frac{1}{2}j} \frac{\varepsilon_0^2}{2^{2N}} \left\| \widehat{\phi}_j \mathcal{F}\left[F_1\right] \right\|_{L^{\frac{2n}{2n-1}}} \right)^{\sigma} \right\}^{\frac{1}{\sigma}} \\
&\qquad \le C\varepsilon_0^2 \frac{R^2}{N^{\frac{1}{n}}} \left\{ \sum_{j\le -\delta N} \left(2^{(n-1)j} \sum_{-\delta N\le k\le 0} 2^{(-n+1)k} \right)^{\sigma} \right\}^{\frac{1}{\sigma}} \\
&\qquad\qquad + C\varepsilon_0^2 \frac{R^2}{N^{\frac{1}{n}}} \left\{ \sum_{-\delta N\le j\le 0} \left(2^{(n-1)j} \sum_{j-3\le k\le 0} 2^{(-n+1)k} \right)^{\sigma} \right\}^{\frac{1}{\sigma}} \\
&\qquad \le C\varepsilon_0^2 \frac{R^2}{N^{\frac{1}{n}}} (1 + N^{\frac{1}{\sigma}}),
\end{aligned}
\tag{2.32}
$$

while the other terms involving $F_2 + F_3$ in the right hand side of (2.30) can be estimated by (2.24) and (2.26) which are lower order terms when $N \to \infty$. Besides all the other terms involving the derivatives $f_N \partial_{x_2} g_N$, $\cdots$, $f_N \partial_{x_n} g_N$ corresponding to (2.30) have an extra order 2^{-N}. Hence we conclude that

$$
\left\| \mathcal{F}^{-1}\left[\widehat{\varphi}(\xi) \int_{\mathbb{R}^n} \Big(\frac{1-e^{-At|\xi-\eta|^2}}{B|\eta|^2 - c|\xi|^2} e^{-Bt|\eta|^2} - \frac{A}{B} t \Big) i\eta \widehat{f}_N(\xi-\eta) \widehat{g}_N(\eta) d\eta \right] \right\|_{\dot{B}^{-\frac{1}{2}}_{2n,\sigma}} \le C\varepsilon_0^2 N^{\frac{1}{\sigma}-\frac{1}{n}} R^2.
$$

Therefore, we obtained (2.27). $\square$

Proof of Proposition 2.2. We firstly prove the estimate (2.5) and we only treat f_N since g_N can be treated analogously. Since the support of $\widehat{f}_N$ satisfies

$$
\operatorname{supp} \widehat{f_N} \subset \{\xi \in \mathbb{R}^n \,;\, 2^N - 2 \le |\xi| \le 2^N + 2\},
$$

we have

$$
\|f_N\|_{\dot{B}^{-\frac{1}{2}}_{2n,q}} \le C 2^{-\frac{1}{2}N} \|f_N\|_{L^{2n}}. \tag{2.33}
$$

By $|\cos(2^N x_1)| \le 1$,

$$
2^{-\frac{1}{2}N} \|f_N\|_{L^{2n}} \le \Big(\int_{\mathbb{R}^n} \Big| \sum_{-\delta N\le j\le 0} R N^{-\frac{1}{2n}} 2^{(-n+\frac{1}{2})j} \phi_j(x - 2^{|j|+2N}e_1) \Big|^{2n} dx \Big)^{\frac{1}{2n}}. \tag{2.34}
$$

For $j = j_1, j_2$ with $j_1 \ne j_2$, the main area of the support of the function $\phi_j(x - 2^{|j|+2N}e_1) = 2^{nj}\phi_0\Big(2^j(x - 2^{|j|+2N}e_1)\Big)$ is the set $A_j := \{\, x \,|\, |x - 2^{|j|+2N}e_1| \le 2^{-j} \,\}$, and A_{j_1} and A_{j_2} are away sufficiently if $j_1 \ne j_2$, therefore one can expect that the main part of $|\sum_j 2^{(-n+\frac{1}{2}j)}\phi_j|^{2n}$ is $\sum_j |2^{(-n+\frac{1}{2}j)}\phi_j|^{2n}$. For simplicity, let Φ_j be defined by $\Phi_j(x) := 2^{\frac{1}{2}j}\phi_0\Big(2^j(x - 2^{|j|+2N}e_1)\Big)$, and we have

$$\begin{aligned}&\Big|\sum_{-\delta N\le j\le 0} R\frac{1}{N^{\frac{1}{2n}}}2^{(-n+\frac{1}{2})j}\phi_j(x-2^{|j|+2N}e_1)\Big|^{2n}\\&\le \frac{R^{2n}}{N}\sum_{-\delta N\le j\le 0}|\Phi_j(x)|^{2n}+\frac{R^{2n}}{N}\sum_{(j_1,\cdots,j_{2n})\in J}\Big|\Phi_{j_1}(x)\cdots\Phi_{j_n}(x)\overline{\Phi_{j_{n+1}}(x)\cdots\Phi_{j_{2n}}(x)}\Big|,\end{aligned}\tag{2.35}$$

where J is defined by

$$J:=\Big\{(j_1,j_2,\cdots,j_{2n})\,;\text{ there exist } j_{l_1},j_{l_2} \text{ such that } j_{l_1}\ne j_{l_2},\ -\delta N\le j_l\le 0\ (1\le^{\forall} l\le 2n)\Big\}.$$

On the integral of the first term in the right hand side of (2.35), it holds that

$$\int_{\mathbb{R}^n}\frac{R^{2n}}{N}\sum_{-\delta N\le j\le 0}|\Phi_j(x)|^{2n}dx=\frac{R^{2n}}{N}\sum_{-\delta N\le j\le 0}\int_{\mathbb{R}^n}|2^{\frac{1}{2}j}\phi_0(2^jx)|^{2n}dx=C\frac{R^{2n}}{N}\cdot N=CR^{2n}.\tag{2.36}$$

On the integral of the second term in the right hand side of (2.35), we utilize the following estimate: Let α be a positive constant satisfying $\alpha>n$, it follows from $j_1\ne j_2$ and $|\Phi_j(x)|\le C2^{\frac{1}{2}j}(1+2^j|x-2^{|j|+2^{2N}}e_1|)^{-\alpha}$ that

$$\int_{\mathbb{R}^n}|\Phi_{j_1}(x)\Phi_{j_2}(x)\Phi_{j_3}(x)\cdots\Phi_{j_{2n}}(x)|dx\le C2^{\frac{1}{2}(j_1+j_2+j_3+\cdots+j_{2n})}.$$

By the above estimate,

$$\begin{aligned}&\frac{R^{2n}}{N}\sum_{(j_1,\cdots,j_{2n})\in J}\int_{\mathbb{R}^n}|\Phi_{j_1}(x)\Phi_{j_2}(x)\times\Phi_{j_3}(x)\cdots\Phi_{j_{2n}}(x)|dx\\&\le C\frac{R^{2n}}{N}\sum_{-N\le j_1,\cdots,j_{2n}\le 0}2^{\frac{1}{2}(j_1+j_2+j_3+\cdots+j_{2n})}\le C\frac{R^{2n}}{N}.\end{aligned}\tag{2.37}$$

Therefore, we obtain (2.5) by (2.33), (2.34), (2.36) and (2.37).

We secondly show the crucial estimate (2.6). By the equality (2.7) and the triangle inequality, it holds that

$$\begin{aligned}\big\|I\big[f_N,g_N\big](T_N)\big\|_{\dot B^{-\frac{1}{2}}_{2n,\sigma}}\ge&\left\|\mathcal{F}^{-1}\left[\frac{B}{A}\int_{\mathbb{R}^n}\frac{1-e^{-At|\xi-\eta|^2}}{B|\eta|^2-c|\xi|^2}e^{-Bt|\eta|^2}i\eta\widehat{f}_N(\xi-\eta)\widehat{g}_N(\eta)d\eta\right]\right\|_{\dot B^{-\frac{1}{2}}_{2n,\sigma}}\\&-\big\|R\big[f_N,g_N\big]\big\|_{\dot B^{-\frac{1}{2}}_{2n,\sigma}}.\end{aligned}\tag{2.38}$$

In order to estimate the first term in the right hand side of (2.38), we apply Lemma 2.5 (2.27) and then obtain that

$$\begin{aligned}&\left\|\mathcal{F}^{-1}\left[\frac{B}{A}\int_{\mathbb{R}^n}\frac{1-e^{-At|\xi-\eta|^2}}{B|\eta|^2-c|\xi|^2}e^{-Bt|\eta|^2}i\eta\widehat{f}_N(\xi-\eta)\widehat{g}_N(\eta)d\eta\right]\right\|_{\dot B^{-\frac{1}{2}}_{2n,\sigma}}\\&\ge cN^{\frac{1}{\sigma}-\frac{1}{n}}R^2.\end{aligned}\tag{2.39}$$

We next consider the estimate of the remainder term $R[f_N,g_N]$ of (2.38). Taking advantage of the divergence form structure due to (2.8) analogously to (A.10) and (A.11), we have from the embedding $\dot B^{-\frac{1}{2}}_{2n,\sigma}(\mathbb{R}^n)\hookrightarrow\dot B^{-\frac{1}{2}}_{2n,2\sigma}(\mathbb{R}^n)$ and (2.5)

$$\big\|R\big[f_N,g_N\big]\big\|_{\dot B^{-\frac{1}{2}}_{2n,\sigma}}\le C\|f_N\|_{\dot B^{-\frac{1}{2}}_{2n,2\sigma}}\|g_N\|_{\dot B^{-\frac{1}{2}}_{2n,2\sigma}}\le C\|f_N\|_{\dot B^{-\frac{1}{2}}_{2n,\sigma}}\|g_N\|_{\dot B^{-\frac{1}{2}}_{2n,\sigma}}\le CR^2.\tag{2.40}$$

Therefore, we obtain (2.6) by (2.38), (2.39) and (2.40), and the proof of Proposition 2.2 is completed. ❑

Before proving Proposition 2.1, we recall a Fourier multiplier theorem.

Lemma 2.6. ([5]) *Let $m : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ be a smooth function such that*

$$|\partial^\alpha_{\xi,\eta} m(\xi,\eta)| \le C_\alpha(|\xi| + |\eta|)^{-|\alpha|}$$

for all multi-index α and some positive constant C_α depending on α. Suppose that $1 < p, p_1, p_2 < \infty$ and $1/p = 1/p_1 + 1/p_2$. Then

$$\left\| \mathcal{F}^{-1}\left[\int_{\mathbb{R}^n} m(\xi-\eta,\eta)\widehat{h_1}(\xi-\eta)\widehat{h_2}(\eta)d\eta \right] \right\|_{L^p} \le C\|h_1\|_{L^{p_1}}\|h_2\|_{L^{p_2}}.$$

Proof of Lemma 2.6. This is a multilier Fourier multiplier theorem for bilinear case. We refer to a paper by Coifman-Meyer [5] for its proof. ❑

Proof of Proposition 2.1. The estimate of the initial data $\|u_0\|_{\dot{B}^{-\frac{1}{2}}_{2n,\sigma}} \le CR$ is obtained by (2.5), and it suffices to show (2.3).

We introduce an approximation of u_0 so that

$$P_\pi u_0 \simeq u_{0,\pi} := (f_N, 0, 0), \quad P_\sigma u_0 \simeq u_{0,\sigma} := (0, g_N, 0),$$

where f_N and g_N are defined by (2.2). We can see that

$$\|P_\sigma u_0 - u_{0,\sigma}\|_{L^n} + \|P_\pi u_0 - u_{0,\pi}\|_{L^n} \le CR2^{(-\frac{1}{2}+\frac{\delta}{2})N}, \tag{2.41}$$

$$\|\nabla(P_\sigma u_0 - u_{0,\sigma})\|_{L^n} + \|\nabla(P_\pi u_0 - u_{0,\pi})\|_{L^n} \le CR2^{(\frac{1}{2}+\frac{\delta}{2})N}, \tag{2.42}$$

so that leading terms of $P_\sigma u_0$, $P_\pi u_0$ can be $u_{0,\sigma}$, $u_{0,\pi}$, respectively. We also use the following estimates:

$$\|\nabla P_\sigma u_0\|_{L^{2n}} \le CR2^{\frac{3}{2}N}, \quad \|u_{0,\pi}\|_{L^{2n}} \le CR2^{\frac{1}{2}N}, \tag{2.43}$$

which are obtained by (2.5) and $\operatorname{supp}\widehat{u_0} \subset \{\xi \in \mathbb{R}^n \,|\, 2^{N-1} \le |\xi| \le 2^{N+1}\}$. We write

$$\begin{aligned}
I[u_0] = &-\int_0^t e^{(t-s)\mathcal{L}}\Big\{(e^{s(\lambda+2\mu)\Delta}u_{0,\pi}\cdot\nabla)e^{s\mu\Delta}u_{0,\sigma} \\
&\qquad + \frac{\mu}{\lambda+2\mu}\Big(\big(e^{s(\lambda+2\mu)\Delta}u_{0,\pi} - u_{0,\pi}\big)\cdot\nabla\Big)e^{s\mu\Delta}u_{0,\sigma}\Big\}ds \\
&- \int_0^t e^{(t-s)\mathcal{L}}\Big\{(e^{s(\lambda+2\mu)\Delta}P_\pi u_0\cdot\nabla)e^{s\mu\Delta}P_\sigma u_0 - (e^{s(\lambda+2\mu)\Delta}u_{0,\pi}\cdot\nabla)e^{s\mu\Delta}u_{0,\sigma}\Big\}ds \\
&- \int_0^t e^{(t-s)\mathcal{L}}\frac{\mu}{\lambda+2\mu}\Big\{\Big(\big(e^{s(\lambda+2\mu)\Delta}P_\pi u_0 - P_\pi u_0\big)\cdot\nabla\Big)e^{s\mu\Delta}P_\sigma u_0 \\
&\qquad - \Big(\big(e^{s(\lambda+2\mu)\Delta}u_{0,\pi} - u_{0,\pi}\big)\cdot\nabla\Big)e^{s\mu\Delta}u_{0,\sigma}\Big\}ds \\
=:& \, K_1 + K_2 + K_3.
\end{aligned}$$

By taking $A = \lambda + 2\mu$, $B = \mu$, and $c = \lambda + 2\mu$, μ, it follows from (2.6) in Proposition 2.2 that for $t = 2^{-2N}$

$$\|K_1\|_{\dot{B}^{-\frac{1}{2}}_{2n,\sigma}} \ge cN^{\frac{1}{\sigma}-\frac{1}{n}}R^2. \tag{2.44}$$

On the estimate of K_2, we have from the embedding $\dot{B}^{\frac{1}{2}}_{\frac{2n}{3},1}(\mathbb{R}^n) \hookrightarrow \dot{B}^{-\frac{1}{2}}_{2n,1}(\mathbb{R}^n)$, the linear estimate of $e^{(t-s)\mathcal{L}}$ from $L^{\frac{2n}{3}}(\mathbb{R}^n)$ to $\dot{B}^{\frac{1}{2}}_{\frac{2n}{3},1}(\mathbb{R}^n)$, the Hölder inequality, (2.41), (2.42) and (2.43)

$$\begin{aligned}
&\|K_2\|_{\dot{B}^{-\frac{1}{2}}_{2n,1}} \le C\|K_2\|_{\dot{B}^{\frac{1}{2}}_{\frac{2n}{3},1}}\\
&\le C\int_0^t (t-s)^{-\frac{1}{4}}\Big\{\big\|\big(e^{s(\lambda+2\mu)\Delta}(P_\pi u_0 - u_{0,\pi})\cdot\nabla\big)e^{s\mu\Delta}P_\sigma u_0\big\|_{L^{\frac{2n}{3}}}\\
&\qquad\qquad + \big\|\big(e^{s(\lambda+2\mu)\Delta}u_{0,\pi}\cdot\nabla\big)e^{s\mu\Delta}(P_\sigma u_0 - u_{0,\sigma})\big\|_{L^{\frac{2n}{3}}}\Big\}ds\\
&\le Ct^{\frac{3}{4}}\Big\{\|P_\pi u_0 - u_{0,\pi}\|_{L^n}\|\nabla P_\sigma u_0\|_{L^{2n}} + \|u_{0,\pi}\|_{L^{2n}}\|\nabla(P_\sigma u_0 - u_{0,\sigma})\|_{L^n}\Big\}\\
&\le CR^2(2^{-2N})^{\frac{3}{4}}\Big(2^{(-\frac{1}{2}+\frac{\delta}{2})N}\cdot 2^N 2^{\frac{1}{2}N} + 2^{\frac{1}{2}N}\cdot 2^N 2^{(-\frac{1}{2}+\frac{\delta}{2})N}\Big)\\
&= CR^2 2^{(-\frac{1}{2}+\frac{\delta}{2})N}.
\end{aligned}$$

The estimate for K_3 follows analogously to K_2. Therefore we obtain by (2.44)

$$\|I[u_0](t)\|_{\dot{B}^{-\frac{1}{2}}_{2n,\sigma}} \ge cN^{\frac{1}{\sigma}-\frac{1}{n}}R^2 \quad \text{for } t = 2^{-2N}, \tag{2.45}$$

for sufficiently large N. ❑

At the end of this section, we consider the difference of $I[u_0]$ and the sum of the second iterate for the convection term and the quasi-linear term to show that they are small.

Lemma 2.7. *Let u_0 be defined by* (2.1). *If $t \le 2^{-2N}$, then*

$$\begin{aligned}
&\Big\|I[u_0] - \Big\{-\int_0^t e^{(t-s)\mathcal{L}}(e^{s\mathcal{L}}u_0\cdot\nabla)e^{s\mathcal{L}}u_0 ds + \int_0^t e^{(t-s)\mathcal{L}}\Big(\int_0^s \operatorname{div} e^{\tilde{s}\mathcal{L}}u_0 d\tilde{s}\Big)\mathcal{L}e^{s\mathcal{L}}u_0 ds\Big\}\Big\|_{\dot{B}^{-\frac{1}{2}}_{2n,1}}\\
&\le CR^2,
\end{aligned}\tag{2.46}$$

where $I[u_0]$ is defined by (1.20).

Proof of Lemma 2.7. By the equalities from (1.8) to (1.9), the terms in the left hand side of (2.46) is decomposed as follows:

$$\begin{aligned}
&I[u_0] - \Big\{-\int_0^t e^{(t-s)\mathcal{L}}(e^{s\mathcal{L}}u_0\cdot\nabla)e^{s\mathcal{L}}u_0 ds + \int_0^t e^{(t-s)\mathcal{L}}\Big(\int_0^s \operatorname{div} e^{\tilde{s}\mathcal{L}}u_0 d\tilde{s}\Big)\mathcal{L}e^{s\mathcal{L}}u_0 ds\Big\}\\
&= \widetilde{K}_1 + \widetilde{K}_2
\end{aligned}$$

where

$$\begin{aligned}
\widetilde{K}_1 &:= \int_0^t e^{(t-s)\mathcal{L}}\Big\{\nabla\cdot(e^{s\mu\Delta}P_\sigma u_0\otimes e^{s\mathcal{L}}u_0) + \frac{1}{2}\nabla|e^{s(\lambda+2\mu)\Delta}P_\pi u_0|^2\Big\}ds\\
&\quad - \int_0^t e^{(t-s)\mathcal{L}}\nabla\cdot\Big\{\Big(\frac{(-\Delta)^{-1}}{2\mu+\lambda}\operatorname{div}\big(e^{s(\lambda+2\mu)\Delta}P_\pi u_0 - P_\pi u_0\big)\Big)\\
&\qquad\qquad \times\nabla\big(\mu e^{s\mu\Delta}P_\sigma u_0 + (\lambda+2\mu)e^{s(\lambda+2\mu)\Delta}P_\pi u_0\big)\Big\}ds,\\
\widetilde{K}_2 &:= \int_0^t e^{(t-s)\mathcal{L}}\Big(\big(e^{s(\lambda+2\mu)\Delta}P_\pi u_0 - P_\pi u_0\big)\cdot\nabla\Big)e^{s(\lambda+2\mu)\Delta}P_\pi u_0 ds.
\end{aligned}$$

We apply the embedding $\dot{B}^0_{n,1}(\mathbb{R}^n) \hookrightarrow \dot{B}^{-\frac{1}{2}}_{2n,1}(\mathbb{R}^n)$, the boundedness of $e^{(t-s)\mathcal{L}}$ from $\dot{B}^{-1}_{n,\infty}(\mathbb{R}^n)$ to $\dot{B}^0_{n,1}(\mathbb{R}^n)$, the embedding $L^n(\mathbb{R}^n) \hookrightarrow \dot{B}^0_{n,\infty}(\mathbb{R}^n)$, and the Hölder inequality to $\widetilde{K}_1$ to obtain

$$
\begin{aligned}
&\|\widetilde{K}_1\|_{\dot{B}^0_{n,1}} \\
\le& C\int_0^t (t-s)^{-\frac{1}{2}} \left\| \nabla \cdot (e^{s\mu\Delta} P_\sigma u_0 \otimes e^{s\mathcal{L}} u_0) + \frac{1}{2}\nabla |e^{s(\lambda+2\mu)\Delta} P_\pi u_0|^2 \right\|_{\dot{B}^{-1}_{n,\infty}} ds \\
&+ C\int_0^t (t-s)^{-\frac{1}{2}} \Bigg\| \nabla \cdot \Bigg\{ (-\Delta)^{-1}\mathrm{div} \left(\frac{(-\Delta)^{-1}}{2\mu+\lambda} \mathrm{div} \left(e^{s(\lambda+2\mu)\Delta} P_\pi u_0 - P_\pi u_0 \right) \right) \\
&\qquad\qquad \times \nabla \big(\mu e^{s\mu\Delta} P_\sigma u_0 + (\lambda+2\mu) e^{s(\lambda+2\mu)\Delta} P_\pi u_0 \big) \Bigg\} \Bigg\|_{\dot{B}^{-1}_{n,\infty}} ds \\
\le& C\int_0^t (t-s)^{-\frac{1}{2}} \Big\{ \big\| e^{s\mu\Delta} P_\sigma u_0 \otimes e^{s\mathcal{L}} u_0 \big\|_{L^n} + \big\| |e^{s(\lambda+2\mu)\Delta} P_\pi u_0|^2 \big\|_{L^n} \Big\} ds \\
&+ C\int_0^t (t-s)^{-\frac{1}{2}} \Big\{ \Big\| \Big((-\Delta)^{-1}\mathrm{div} \left(e^{s(\lambda+2\mu)\Delta} P_\pi u_0 - P_\pi u_0 \right) \Big) \\
&\qquad\qquad \times \nabla \big(\mu e^{s\mu\Delta} P_\sigma u_0 + (\lambda+2\mu) e^{s(\lambda+2\mu)\Delta} P_\pi u_0 \big) \Big\|_{L^n} \Big\} ds \\
\le& Ct^{\frac{1}{2}} \|u_0\|^2_{L^{2n}} + Ct^{\frac{1}{2}} \|(-\Delta)^{-1} \,\mathrm{div}\, u_0\|_{L^{2n}} \|\nabla u_0\|_{L^{2n}} \\
\le& C(2^{-2N})^{\frac{1}{2}} R^2 2^N \\
=& CR^2,
\end{aligned}
\tag{2.47}
$$

where the estimate $\|\nabla^\alpha u_0\|_{L^{2n}} \le CR2^{(\alpha+\frac{1}{2})N}$ $(\alpha = -1, 0, 1)$ is obtained by (2.5) and $\mathrm{supp}\,\widehat{u_0} \subset \{\xi \in \mathbb{R}^n \,|\, 2^{N-1} \le |\xi| \le 2^{N+1}\}$.

We consider the estimate of $\widetilde{K}_2$. Although the term $(P_\pi u_0 \cdot \nabla) e^{s(\lambda+2\mu)\Delta} P_\pi u_0$ can not be written by the divergence form directly, it is possible to show the smallness of $(P_\pi u_0 \cdot \nabla) e^{s(\lambda+2\mu)\Delta} P_\pi u_0$ by means of

$$
(e^{s(\lambda+2\mu)\Delta} P_\pi u_0 \cdot \nabla) e^{s(\lambda+2\mu)\Delta} P_\pi u_0 = \frac{1}{2}\nabla |e^{s(\lambda+2\mu)\Delta} P_\pi u_0|^2.
$$

In fact, we have

$$
\begin{aligned}
\widetilde{K}_2 =& -\int_0^t e^{(t-s)\mathcal{L}} \frac{1}{2} \nabla \big| e^{s(\lambda+2\mu)\Delta} P_\pi u_0 \big|^2 ds \\
&+ \int_0^t e^{(t-s)\mathcal{L}} \Big\{ \big(P_\pi u_0 \cdot \nabla \big) e^{s(\lambda+2\mu)\Delta} P_\pi u_0 - \big(e^{\frac{1}{2}s(\lambda+2\mu)\Delta} P_\pi u_0 \cdot \nabla \big) e^{\frac{1}{2}s(\lambda+2\mu)\Delta} P_\pi u_0 \Big\} ds \\
&+ \int_0^t e^{(t-s)\mathcal{L}} \frac{1}{2} \nabla \big| e^{\frac{1}{2}s(\lambda+2\mu)\Delta} P_\pi u_0 \big|^2 ds \\
=:& \widetilde{K}_{21} + \widetilde{K}_{22} + \widetilde{K}_{23}.
\end{aligned}
$$

Since $\widetilde{K}_{21}$ and $\widetilde{K}_{23}$ have the divergence form structure, it follows from the same method as (2.47) that

$$
\|\widetilde{K}_{21}\|_{\dot{B}^{-\frac{1}{2}}_{2n,1}} + \|\widetilde{K}_{23}\|_{\dot{B}^{-\frac{1}{2}}_{2n,1}} \le CR^2. \tag{2.48}
$$

On the estimate of $\widetilde{K}_{22}$, we consider the following Fourier transform and have from $e^{s(\lambda+2\mu)\Delta} = e^{\frac{1}{2}s(\lambda+2\mu)\Delta} e^{\frac{1}{2}s(\lambda+2\mu)\Delta}$

$$
\begin{aligned}
&\mathcal{F}\Big[\big(P_\pi u_0 \cdot \nabla \big) e^{s(\lambda+2\mu)\Delta} P_\pi u_0 - \big(e^{\frac{1}{2}s(\lambda+2\mu)\Delta} P_\pi u_0 \cdot \nabla \big) e^{\frac{1}{2}s(\lambda+2\mu)\Delta} P_\pi u_0 \Big](\xi) \\
&= \int_{\mathbb{R}^n} \Big(e^{-\frac{1}{2}s(2\mu+\lambda)|\eta|^2} - e^{-\frac{1}{2}s(2\mu+\lambda)|\xi-\eta|^2} \Big) \Big(\widehat{P_\pi u_0}(\xi-\eta) \cdot i\eta \Big) e^{-\frac{1}{2}s(2\mu+\lambda)|\eta|^2} \widehat{P_\pi u_0}(\eta) d\eta.
\end{aligned}
\tag{2.49}
$$

On the term $e^{-\frac{1}{2}s(2\mu+\lambda)|\eta|^2} - e^{-\frac{1}{2}s(2\mu+\lambda)|\xi-\eta|^2}$, it holds that

$$\begin{aligned} e^{-\frac{1}{2}s(2\mu+\lambda)|\eta|^2} - e^{-\frac{1}{2}s(2\mu+\lambda)|\xi-\eta|^2} &= \int_0^1 \partial_\theta e^{-\frac{1}{2}s(2\mu+\lambda)|\eta-\theta\xi|^2} d\theta \\ &= (2\mu+\lambda)\xi \cdot \int_0^1 (\eta-\theta\xi) s e^{-\frac{1}{2}s(2\mu+\lambda)|\eta-\theta\xi|^2} d\theta. \end{aligned}$$

Therefore (2.49) can be regarded as the term with the divergence form structure and we have from the embedding $\dot{B}^0_{n,1}(\mathbb{R}^n) \hookrightarrow \dot{B}^{-\frac{1}{2}}_{2n,1}(\mathbb{R}^n)$, the linear estimate of $e^{(t-s)\mathcal{L}}$ from $\dot{B}^{-1}_{n,\infty}(\mathbb{R}^n)$ to $\dot{B}^0_{n,1}(\mathbb{R}^n)$, (2.49), the above equality,

$$\begin{aligned} \|\widetilde{K}_{22}\|_{\dot{B}^{-\frac{1}{2}}_{2n,1}} &\le \|\widetilde{K}_{22}\|_{\dot{B}^0_{n,1}} \\ &\le C\int_0^t (t-s)^{-\frac{1}{2}} \Big\| \nabla\cdot\mathcal{F}^{-1}\Big[\int_{\mathbb{R}^n}\int_0^1 (\eta-\theta\xi) s e^{-\frac{1}{2}s(2\mu+\lambda)|\eta-\theta\xi|^2} d\theta \\ &\qquad \times \Big(\widehat{P_\pi u_0}(\xi-\eta)\cdot i\eta\Big) e^{-\frac{1}{2}s(2\mu+\lambda)|\eta|^2} \widehat{P_\pi u_0}(\eta) d\eta\Big]\Big\|_{\dot{B}^{-1}_{n,\infty}} ds \\ &\le C\int_0^1\int_0^t (t-s)^{-\frac{1}{2}} \Big\| \mathcal{F}^{-1}\Big[\int_{\mathbb{R}^n} (\eta-\theta\xi) s e^{-\frac{1}{2}s(2\mu+\lambda)|\eta-\theta\xi|^2} \\ &\qquad \times \Big(\widehat{P_\pi u_0}(\xi-\eta)\cdot i\eta\Big) e^{-\frac{1}{2}s(2\mu+\lambda)|\eta|^2} \widehat{P_\pi u_0}(\eta) d\eta\Big]\Big\|_{L^n} ds d\theta \end{aligned}$$

Here let us consider the Fourier multiplier

$$m(\xi-\eta,\eta) = \frac{(\eta-\theta\xi)}{2^N} s e^{-\frac{1}{2}s(2\mu+\lambda)|\eta-\theta\xi|^2} \frac{i\eta}{2^N} e^{-\frac{1}{2}s(2\mu+\lambda)|\eta|^2} \widehat{\Phi_N}(\xi-\eta)\widehat{\Phi_N}(\eta),$$

where $\Phi_N := \phi_{N-1} + \phi_N + \phi_{N+1}$. Noting that $P_\pi u_0 = \Phi_N * P_\pi u_0$, we apply Lemma 2.6 and then have that

$$\begin{aligned} &\Big\| \mathcal{F}^{-1}\Big[\int_{\mathbb{R}^n} (\eta-\theta\xi) s e^{-\frac{1}{2}s(2\mu+\lambda)|\eta-\theta\xi|^2} \Big(\widehat{P_\pi u_0}(\xi-\eta)\cdot i\eta\Big) e^{-\frac{1}{2}s(2\mu+\lambda)|\eta|^2} \widehat{P_\pi u_0}(\eta) d\eta\Big]\Big\|_{L^n} \\ &\le C 2^{2N} \|P_\pi u_0\|^2_{L^{2n}}, \end{aligned}$$

which proves that

$$\|\widetilde{K}_{22}\|_{\dot{B}^{-\frac{1}{2}}_{2n,1}} \le C t^{\frac{3}{2}} 2^{2N} \|P_\pi u_0\|^2_{L^{2n}} \le CR^2,$$

where the estimate $\|u_0\|_{L^{2n}} \le CR2^{\frac{1}{2}N}$ is obtained by (2.5) and $\operatorname{supp}\widehat{u_0} \subset \{\xi \in \mathbb{R}^n \,|\, 2^{N-1} \le |\xi| \le 2^{N+1}\}$. Then it follows from (2.48) and the above inequality that

$$\|\widetilde{K}_{22}\|_{\dot{B}^{-\frac{1}{2}}_{2n,1}} \le CR^2.$$

Therefore we obtain (2.46). ❑

3. Proof of Theorem 1.1

Let the initial data (ρ_0, u_0) be defined by

$$\rho_0(x) = 0, \qquad u_0(x) = (f_N(x), g_N(x), 0, \cdots, 0), \tag{3.1}$$

where f_N and g_N are defined by (2.2). To prove the ill-posedness, we first prove the existence of a local solution of the compressible Navier-Stokes equations for the initial data (3.1).

Proposition 3.1 ([14]). *Let $1 < p \le n$. For $\rho_0 \in \dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n)$ and $u_0 \in \dot{B}^{\frac{n}{p}-1}_{p,1}(\mathbb{R}^n)$ with*

$$\|\rho_0\|_{\dot{B}^{\frac{n}{p}}_{p,1}} \le \eta,$$

for certain $\eta > 0$, there exists $T > 0$ such that the equations (1.1) *have a unique local-in-time solution*

$$\rho \in C([0,T]; \dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n)), \quad u \in C([0,T]; \dot{B}^{\frac{n}{p}-1}_{p,1}(\mathbb{R}^n)) \cap L^1(0,T; \dot{B}^{\frac{n}{p}+1}_{p,1}(\mathbb{R}^n)). \tag{3.2}$$

We refer to the paper by Danchin [14] for the proof.

Since our initial data $\rho_0 = 0$ and $u_0 \in \mathcal{S}(\mathbb{R}^n)$ satisfy the assumption of Proposition 3.1, the solution to (1.1) exists on $[0,T]$ by Proposition 3.1. Here we need to show $T \geq T_N \equiv \varepsilon_0 2^{-2N}$ with $\varepsilon_0 < 1$ (cf. Proposition 2.1). To this end, we show the following proposition.

Proposition 3.2. *Let the initial data u_0 be defined by* (3.1)*. For some small $\delta < 1/n$, there exists N_0 such that for every $N \geq N_0$ there exists a local solution (ρ, u) of* (1.2) *such that*

$$\begin{aligned} &\rho \in C([0,T], \dot{B}^0_{2,1} \cap \dot{B}^0_{\infty,1}) \cap L^\infty([0,T], \dot{B}^1_{2,1} \cap \dot{B}^1_{\infty,1}), \\ &u \in C([0,T], \dot{B}^0_{2,1} \cap \dot{B}^0_{\infty,1}) \cap L^1(0,T; \dot{B}^2_{2,1} \cap \dot{B}^2_{\infty,1}), \end{aligned} \tag{3.3}$$

where $T = 2^{-2N}$, and they satisfy the following:

$$\sup_{t\in[0,T]} \|\rho(t)\|_{\dot{B}^0_{2,1}\cap\dot{B}^0_{\infty,1}} \leq C 2^{(-\frac{1}{2}+\frac{n\delta}{2})N}, \tag{3.4}$$

$$\sup_{t\in[0,T]} \|\rho(t)\|_{\dot{B}^1_{2,1}\cap\dot{B}^1_{\infty,1}} \leq C 2^{(\frac{1}{2}+\frac{n\delta}{2})N}, \tag{3.5}$$

$$\sup_{t\in[0,T]} \|u(t)\|_{\dot{B}^0_{2,1}\cap\dot{B}^0_{\infty,1}} + \|u\|_{L^1(0,T;\dot{B}^2_{2,1}\cap\dot{B}^2_{\infty,1})} \leq C 2^{(\frac{1}{2}+\frac{n\delta}{2})N}. \tag{3.6}$$

For the proof of Proposition 3.2, we will use maximal regularity estimates to control the L^∞-norm of density and the quasi-linear term.

Lemma 3.3 ([27])**.** *Assume that $u_0 \in \dot{B}^0_{p,1}(\mathbb{R}^n)$ and $f \in L^1(0,\infty; \dot{B}^0_{p,1}(\mathbb{R}^n))$ with $1 \leq p \leq \infty$ and u satisfies*

$$\partial_t u - \mathcal{L}u = f, \quad u(0) = u_0.$$

Then there exists a positive number C independent of u_0, f such that

$$\|\partial_t u\|_{L^1(0,\infty;\dot{B}^0_{p,1})} + \|\mathcal{L}u\|_{L^1(0,\infty;\dot{B}^0_{p,1})} \leq C\|u_0\|_{\dot{B}^0_{p,1}} + C\|f\|_{L^1(0,\infty;\dot{B}^0_{p,1})}.$$

We refer to several papers [10, 11, 14, 27] for maximal regularity and its application to compressible Navier-Stokes equations.

We also use the bilinear estimates below.

Lemma 3.4. *Let $\dot{B}^\alpha_{[2,\infty],1} := \dot{B}^\alpha_{2,1}(\mathbb{R}^n) \cap \dot{B}^\alpha_{\infty,1}(\mathbb{R}^n)$. For α and ε with $\alpha > 0$ and $0 < \varepsilon < 1/8$, we have*

$$\|fg\|_{\dot{B}^0_{[2,\infty],1}} \leq C \min\Big\{\|f\|_{\dot{B}^0_{\infty,1}\cap\dot{B}^\varepsilon_{[2,\infty],1}}\|g\|_{\dot{B}^0_{[2,\infty],1}},\ \|f\|_{\dot{B}^0_{[2,\infty],1}}\|g\|_{\dot{B}^0_{\infty,1}\cap\dot{B}^\varepsilon_{[2,\infty],1}}\Big\}, \tag{3.7}$$

$$\|fg\|_{\dot{B}^\alpha_{[2,\infty],1}} \leq C\Big\{\|f\|_{\dot{B}^\alpha_{[2,\infty],1}}\|g\|_{\dot{B}^0_{[2,\infty],1}} + \|f\|_{\dot{B}^0_{[2,\infty],1}}\|g\|_{\dot{B}^\alpha_{[2,\infty],1}}\Big\}. \tag{3.8}$$

The proof will be given in Appendix C. We note that $\dot{B}^0_{[2,\infty],1} \cap \dot{B}^\alpha_{[2,\infty],1}$ is a Banach algebra.

Proof of Proposition 3.2. Since $u_0 \in \mathcal{S}(\mathbb{R}^n)$, we know from the existing result Proposition 3.1 that there exist $T_0 > 0$ and a unique local solution such that

$$\rho \in C([0,T_0]; \dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n)), \quad u \in C([0,T_0]; \dot{B}^{\frac{n}{p}-1}_{p,1}(\mathbb{R}^n)) \cap L^1(0,T_0; \dot{B}^{\frac{n}{p}+1}_{p,1}(\mathbb{R}^n)),$$

where $1 < p \leq n$. We notice that the solution may also be in our solution space (3.3) since the data is in the Schwartz class. More precisely, followig Proposition 3.1, one can construct solution

(ρ, u) and can deduce that it belongs to not only the scaling critical class but also smoother class (3.2)

$$\rho \in C([0,T_0], \dot B^0_{2,1} \cap \dot B^0_{\infty,1}) \cap L^\infty([0,T_0], \dot B^1_{2,1} \cap \dot B^1_{\infty,1}),$$
$$u \in C([0,T_0], \dot B^0_{2,1} \cap \dot B^0_{\infty,1}) \cap L^1(0,T_0; \dot B^2_{2,1} \cap \dot B^2_{\infty,1}).$$

We also note that T_0 can be taken as in the case when $p = n$ in Proposition 3.1, since $\dot B^0_{2,1} \cap \dot B^0_{\infty,1} \hookrightarrow \dot B^0_{n,1}$ and the existence time can be chosen from the argument in the scaling critical case. To obtain the solution on the time interval $[0, 2^{-2N}]$, we will derive a prori estimates (3.4), (3.5) and (3.6).

The following estimate is obtained by Danchin [14] and we utilized it for our purpose. Let $\alpha = 0$, 1. It holds that

$$\begin{aligned}&\|\rho(t)\|_{\dot B^\alpha_{[2,\infty],1}}\\ &\le e^{C\int_0^t \|\nabla u(\mu)\|_{\dot B^0_{\infty,1}} d\mu} \left\{ \|\rho_0\|_{\dot B^\alpha_{[2,\infty],1}} + \int_0^t e^{-C\int_0^s \|\nabla u(\mu)\|_{\dot B^0_{\infty,1}} d\mu} \|(\rho+1)\operatorname{div} u\|_{\dot B^\alpha_{[2,\infty],1}} ds \right\}.\end{aligned} \tag{3.9}$$

We apply the bilinear estimates (3.7) and (3.8) in Lemma 3.4 to the second term of the right hand side of (3.9) to obtain

$$\|\rho(t)\|_{\dot B^0_{[2,\infty],1}} \le C\int_0^t e^{C\int_s^t \|\nabla u(\mu)\|_{\dot B^0_{\infty,1}} d\mu} (\|\rho\|_{\dot B^0_{[2,\infty],1}} \|u\|_{\dot B^1_{[2,\infty],1} \cap \dot B^{1+\varepsilon}_{[2,\infty],1}} + \|u\|_{\dot B^1_{[2,\infty],1}}) ds,$$
$$\|\rho(t)\|_{\dot B^1_{[2,\infty],1}} \le C\int_0^t e^{C\int_s^t \|\nabla u(\mu)\|_{\dot B^0_{\infty,1}} d\mu} \Big(\|\rho\|_{\dot B^1_{[2,\infty],1}} \|u\|_{\dot B^1_{[2,\infty],1}} + (1 + \|\rho\|_{\dot B^0_{[2,\infty],1}}) \|u\|_{\dot B^2_{[2,\infty],1}} \Big) ds.$$

Then the Gronwall inequality (see Lemma B.1) implies

$$\begin{cases} \|\rho(t)\|_{\dot B^0_{[2,\infty],1}} \le C\displaystyle\int_0^t e^{C\int_s^t (\|\nabla u(\mu)\|_{\dot B^0_{\infty,1}} + \|u\|_{\dot B^1_{[2,\infty],1} \cap \dot B^{1+\varepsilon}_{[2,\infty],1}}) d\mu} \|u\|_{\dot B^1_{[2,\infty],1}}\, ds, \\ \|\rho(t)\|_{\dot B^1_{[2,\infty],1}} \le C\displaystyle\int_0^t e^{C\int_s^t \|\nabla u(\mu)\|_{\dot B^0_{\infty,1}} d\mu} (1 + \|\rho\|_{\dot B^0_{[2,\infty],1}}) \|u\|_{\dot B^2_{[2,\infty],1}}\, ds. \end{cases} \tag{3.10}$$

On the other hand, the velocity can be treated by applying Maximal regularity estimate in Lemma 3.3 and we have

$$\|\partial_t e^{t\mathcal{L}} u_0\|_{L^1(0,T;\dot B^0_{[2,\infty],1})} + \|\nabla^2 e^{t\mathcal{L}} u_0\|_{L^1(0,T;\dot B^0_{[2,\infty],1})} \le C(\|u_0\|_{\dot B^0_{2,1}} + \|u_0\|_{\dot B^0_{\infty,1}}) \le C 2^{(\frac12 + \frac{n\delta}{2})N}.$$

Note that the nonlinear term of the velocity equation is given by

$$-(u\cdot\nabla)u - \frac{\nabla P(1+\rho)}{1+\rho} - \frac{\rho}{1+\rho}\mathcal{L}u. \tag{3.11}$$

Hence it is sufficient to estimate the nonlinear terms in (3.11) by the norm in $L^1(0,T;\dot B^0_{[2,\infty],1}(\mathbb{R}^n))$, because of maximal regularity in Lemma 3.3. On the convection term, it follows from (3.7) and an interpolation inequality that

$$\Big\|(u\cdot\nabla)u\Big\|_{L^1(0,T;\dot B^0_{[2,\infty],1})} \le C(T^{\frac12} + T^{\frac{1-\varepsilon}{2}}) \|u\|^2_{L^\infty(0,T;\dot B^0_{[2,\infty],1}) \cap L^1(0,T;\dot B^2_{[2,\infty],1})}.$$

Since the pressure given by the monomial $P(1+\rho) = (1+\rho)^\gamma$ $(\gamma > 0)$ is analytic for small ρ, we have by the Taylor expansion

$$P(1+\rho) = \sum_{m=0}^\infty a_m \rho^m$$

with a bounded sequence $\{a_m\}_{m=1}^\infty$ and

$$\frac{1}{1+\rho} = \sum_{m=0}^\infty (-\rho)^m,$$

which give that

$$\frac{\nabla P(1+\rho)}{1+\rho}=\nabla\sum_{m=0}^{\infty}\widetilde{a}_m\rho^m=\nabla\sum_{m=1}^{\infty}\widetilde{a}_m\rho^m,$$

where $\{\widetilde{a}_m\}_{m=0}^{\infty}$ is also a bounded sequence. We then have that

$$\begin{aligned}\Big\|\frac{\nabla P(1+\rho)}{1+\rho}\Big\|_{L^1(0,T;\dot B^0_{[2,\infty],1})}\leq& C\sum_{m=1}^{\infty}|\widetilde{a}_m|\|\rho^m\|_{L^1(0,T;\dot B^1_{[2,\infty],1})}\\ \leq& C\Big(\sum_{m=1}^{\infty}m|\widetilde{a}_m|\|\rho\|^{m-1}_{L^\infty(0,T;\dot B^0_{\infty,1})}\Big)\|\rho\|_{L^1(0,T;\dot B^1_{[2,\infty],1})}\\ \leq& CT\Big(\sum_{m=1}^{\infty}m|\widetilde{a}_m|\|\rho\|^{m-1}_{L^\infty(0,T;\dot B^0_{\infty,1})}\Big)\|\rho\|_{L^\infty(0,T;\dot B^1_{[2,\infty],1})}.\end{aligned}$$

On the quasi-linear term, applying the expansion

$$\frac{\rho}{1+\rho}=\sum_{m=0}^{\infty}(-1)^m\rho^{m+1},$$

and the bilinear estimate (3.7), we have

$$\begin{aligned}&\Big\|\frac{\rho}{1+\rho}\mathcal{L}u\Big\|_{L^1(0,T;\dot B^0_{[2,\infty],1})}\\ \leq& C\int_0^T\Big\|\frac{\rho}{1+\rho}\Big\|_{L^\infty(0,T;\dot B^0_{[2,\infty],1}\cap\dot B^{\varepsilon}_{[2,\infty],1})}\|\mathcal{L}u\|_{\dot B^0_{[2,\infty],1}}ds\\ \leq& C\Big(\sum_{m=0}^{\infty}m\|\rho\|^m_{L^\infty(0,T;\dot B^0_{\infty,1})}\Big)\|\rho\|_{L^\infty(0,T;\dot B^0_{[2,\infty],1}\cap\dot B^{\varepsilon}_{[2,\infty],1})}\|u\|_{L^1(0,T;\dot B^2_{[2,\infty],1})}.\end{aligned}$$

Therefore we obtain that

$$\begin{aligned}&\|u\|_{L^\infty(0,T;\dot B^0_{[2,\infty],1})\cap L^1(0,T;\dot B^2_{[2,\infty],1})}\\ \leq& C2^{(\frac12+\frac{n\delta}{2})N}+C(T^{\frac12}+T^{\frac{1-\varepsilon}{2}})\|u\|^2_{L^\infty(0,T;\dot B^0_{[2,\infty],1})\cap L^1(0,T;\dot B^2_{[2,\infty],1})}\\ &+CT\Big(\sum_{m=1}^{\infty}m|a_m|\|\rho\|^{m-1}_{L^\infty(0,T;\dot B^0_{\infty,1})}\Big)\|\rho\|_{L^\infty(0,T;\dot B^1_{[2,\infty],1})}\\ &+C\Big(\sum_{m=0}^{\infty}m\|\rho\|^m_{L^\infty(0,T;\dot B^0_{\infty,1})}\Big)\|\rho\|_{L^\infty(0,T;\dot B^0_{[2,\infty],1}\cap\dot B^{\varepsilon}_{[2,\infty],1})}\|u\|_{L^1(0,T;\dot B^2_{[2,\infty],1})}.\end{aligned}\tag{3.12}$$

We then derive a priori estimates (3.4)-(3.6) for $t\in[0,2^{-2N})$ by the continuity argument. Since our initial data ρ_0,u_0 satisfy

$$\|\rho_0\|_{\dot B^0_{[2,\infty],1}\cap\dot B^1_{[2,\infty],1}}=0,\quad \sup_{t>0}\|e^{t\mathcal{L}}u_0\|_{\dot B^0_{[2,\infty],1}}+\|e^{t\mathcal{L}}u_0\|_{L^1(0,\infty;\dot B^2_{[2,\infty],1})}\leq C_02^{(\frac12+\frac{n\delta}{2})N}.$$

with some $C_0>0$, letting C_1 be the biggest constant among $2C_0$ and those in (3.10) and (3.12), we find that for some $T_1>0$, the solution satisfy for $t\in(0,T_1]$

$$\sup_{\tau\in[0,t]}\|\rho(\tau)\|_{\dot B^0_{[2,\infty],1}}\leq 2C_1^2e^{C_1}2^{(-\frac12+\frac{n\delta}{2})N},\tag{3.13}$$

$$\sup_{\tau\in[0,t]}\|\rho(\tau)\|_{\dot B^1_{[2,\infty],1}}\leq 2C_1^22^{(\frac12+\frac{n\delta}{2})N},\tag{3.14}$$

$$\sup_{\tau\in[0,t]}\|u(\tau)\|_{\dot B^0_{[2,\infty],1}}+\|u\|_{L^1(0,t;\dot B^2_{[2,\infty],1})}\leq C_12^{(\frac12+\frac{n\delta}{2})N}.\tag{3.15}$$

If $T_1 \geq 2^{-2N}$ then the proof is done. If otherwise, let

$$T^* := \sup \left\{ t > 0 \,\middle|\, (3.13), (3.14) \text{ and } (3.15) \text{ hold on the time interval } [0,t] \right\}$$

and assume that $T^* < 2^{-2N}$. Then by the continuity of obtained solution $(\rho(t), u(t))$ in the solution space (3.3), for any small $0 < \eta < 1$, there exists $0 < T_\varepsilon \ll 1$ with $T^* + T_\varepsilon < 2^{-2N}$ such that one of the following estimates hold: For $t \in (T^*, T^* + T_\varepsilon)$,

$$\sup_{\tau \in [0,t]} \|\rho(\tau)\|_{\dot{B}^0_{[2,\infty],1}} < 2(1+\eta) C_1^2 2^{(-\frac{1}{2}+\frac{n\delta}{2})N}, \tag{3.16}$$

$$\sup_{\tau \in [0,t]} \|\rho(\tau)\|_{\dot{B}^1_{[2,\infty],1}} < 2(1+\eta) C_1^2 2^{(\frac{1}{2}+\frac{n\delta}{2})N}, \tag{3.17}$$

$$\sup_{\tau \in [0,t]} \|u(\tau)\|_{\dot{B}^0_{[2,\infty],1}} + \|u\|_{L^1(0,t;\dot{B}^2_{[2,\infty],1})} < (1+\eta) C_1 2^{(\frac{1}{2}+\frac{n\delta}{2})N}. \tag{3.18}$$

In either case, all the estimates (3.16)-(3.18) are valid, we then apply (3.16) and (3.18) to (3.10) to obtain that

$$\begin{aligned}
&\|\rho(t)\|_{\dot{B}^0_{[2,\infty],1}} \\
\leq& C_1 \exp\left(C_1 (t^{\frac{1}{2}} + t^{\frac{1-\varepsilon}{2}}) \|u\|_{L^\infty(0,t;\dot{B}^0_{[2,\infty],1}) \cap L^1(0,t;\dot{B}^2_{[2,\infty],1})} \right) t^{\frac{1}{2}} \|u\|_{L^\infty(0,t;\dot{B}^0_{[2,\infty],1}) \cap L^1(0,t;\dot{B}^2_{[2,\infty],1})} \\
\leq& C_1 \exp\left(C_1 (2^{-N} + 2^{-(1-\varepsilon)N}) \cdot (1+\eta) C_1 2^{(\frac{1}{2}+\frac{n\delta}{2})N} \right) \cdot 2^{-N} \cdot (1+\eta) C_1 2^{(\frac{1}{2}+\frac{n\delta}{2})N} \\
\leq& (1+\eta) C_1^2 \exp\left((1+\eta) 2 C_1^2 2^{(-\frac{1}{2}+\varepsilon+\frac{n\delta}{2})N} \right) 2^{(-\frac{1}{2}+\frac{n\delta}{2})N} \\
\leq& 2 C_1^2 2^{(-\frac{1}{2}+\frac{n\delta}{2})N}
\end{aligned}$$

and

$$\begin{aligned}
&\|\rho(t)\|_{\dot{B}^1_{[2,\infty],1}} \\
\leq& C_1 \exp\left(C_1 T^{\frac{1}{2}} \|u\|_{L^\infty(0,T;\dot{B}^0_{[2,\infty],1}) \cap L^1(0,T;\dot{B}^2_{[2,\infty],1})} \right) \left(1 + \|\rho\|_{L^\infty(0,T;\dot{B}^0_{[2,\infty],1})} \right) \\
&\quad \times \|u\|_{L^\infty(0,T;\dot{B}^0_{[2,\infty],1}) \cap L^1(0,T;\dot{B}^2_{[2,\infty],1})} \\
\leq& (1+\eta) C_1^2 \exp\left((1+\eta) C_1^2 2^{(-\frac{1}{2}+\frac{n\delta}{2})N} \right) \left(1 + \|\rho\|_{L^\infty(0,T;\dot{B}^0_{[2,\infty],1})} \right) 2^{(\frac{1}{2}+\frac{n\delta}{2})N} \\
\leq& 2 C_1^2 2^{(\frac{1}{2}+\frac{n\delta}{2})N},
\end{aligned}$$

provided that N is sufficiently large. And applying (3.16)-(3.18) to (3.12), we also obtain that

$$
\begin{aligned}
&\|u\|_{L^\infty(0,T;\dot{B}^0_{[2,\infty],1})\cap L^1(0,T;\dot{B}^2_{[2,\infty],1})}\\
\le & C_0 2^{(\frac{1}{2}+\frac{n\delta}{2})N} + C(T^{\frac{1}{2}}+T^{\frac{1-\varepsilon}{2}})\|u\|^2_{L^\infty(0,T;\dot{B}^0_{[2,\infty],1})\cap L^1(0,T;\dot{B}^2_{[2,\infty],1})}\\
&+CT\Big(\sum_{m=1}^\infty m|a_m|\|\rho\|^{m-1}_{L^\infty(0,T;\dot{B}^0_{\infty,1})}\Big)\|\rho\|_{L^\infty(0,T;\dot{B}^1_{[2,\infty],1})}\\
&+C\Big(\sum_{m=0}^\infty m\|\rho\|^m_{L^\infty(0,T;\dot{B}^0_{\infty,1})}\Big)\|\rho\|_{L^\infty(0,T;\dot{B}^0_{[2,\infty],1}\cap\dot{B}^\varepsilon_{[2,\infty],1})}\|u\|_{L^1(0,T;\dot{B}^2_{[2,\infty],1})}\\
\le & \frac{C_1}{2}2^{(\frac{1}{2}+\frac{n\delta}{2})N} + C(2^{-N}+2^{-(1-\varepsilon)N})((1+\eta)C_1 2^{(\frac{1}{2}+\frac{n\delta}{2})N})^2 \\
&+C2^{-2N}\Big(\sum_{m=1}^\infty m|a_m|(2(1+\eta)C_1^2 2^{(-\frac{1}{2}+\frac{n\delta}{2})N})^{m-1}\Big)2(1+\eta)C_1^2 2^{(\frac{1}{2}+\frac{n\delta}{2})N}\\
&+C\Big(\sum_{m=0}^\infty m(2(1+\eta)C_1^2 2^{(-\frac{1}{2}+\frac{n\delta}{2})N})^m\Big)\\
&\quad\times 2(1+\eta)C_1^2 2^{(-\frac{1}{2}+\frac{n\delta}{2}+\varepsilon)N}\cdot(1+\eta)C_1 2^{(\frac{1}{2}+\frac{n\delta}{2})N}\\
\le & C_1 2^{(\frac{1}{2}+\frac{n\delta}{2})N}.
\end{aligned}
\tag{3.19}
$$

Therefore we obtain the three bounds (3.13), (3.14) and (3.15), which contradict to the definition of T^*, and we conclude that $T^*\ge 2^{-2N}$. ❑

In the next three lemmas below, we consider several approximation. The first lemma concerns with the approximation of the solution by the linear part.

Lemma 3.5. *Let ε satisfy $0<\varepsilon<1/8$, and $T:=2^{-2N}$. Then the solution (ρ,u) obtained in Proposition 3.2 satisfies the following:*

$$
\sup_{t\in[0,T]}\Big\|\rho(t)+\int_0^t \mathrm{div} e^{s\mathcal{L}}u_0 ds\Big\|_{\dot{B}^0_{2,1}\cap\dot{B}^0_{\infty,1}} \le C2^{(-1+\varepsilon+n\delta)N}, \tag{3.20}
$$

$$
\sup_{t\in[0,T]}\|u(t)-e^{t\mathcal{L}}u_0\|_{\dot{B}^0_{2,1}\cap\dot{B}^0_{\infty,1}} + \|u-e^{t\mathcal{L}}u_0\|_{L^1(0,T;\dot{B}^2_{2,1}\cap\dot{B}^2_{\infty,1})} \le C2^{(\varepsilon+n\delta)N}. \tag{3.21}
$$

Proof of Lemma 3.5. We put $\dot{B}^\alpha_{[2,\infty],1}:=\dot{B}^\alpha_{2,1}\cap\dot{B}^\alpha_{\infty,1}$ for simplicity. To show (3.21), we apply a similar argument to (3.19) without the linear term and then obtain

$$
\begin{aligned}
&\sup_{t\in[0,T]}\|u(t)-e^{t\mathcal{L}}u_0\|_{\dot{B}^0_{[2,\infty],1}} + \|u-e^{t\mathcal{L}}u_0\|_{L^1(0,T;\dot{B}^2_{[2,\infty],1})}\\
&\le C\Big\|\int_0^t e^{(t-s)\mathcal{L}}\Big((u\cdot\nabla)u+\frac{\nabla P(\rho)}{1+\rho}+\frac{\rho}{1+\rho}\mathcal{L}u\Big)ds\Big\|_{L^\infty(0,T;\dot{B}^0_{[2,\infty],1})\cap L^1(0,T;\dot{B}^2_{[2,\infty],1})}\\
&\le C2^{(\varepsilon+n\delta)N}.
\end{aligned}
$$

Then (3.21) is proved.

To prove (3.20), we use the following equality on the integral equation of ρ:

$$
\rho(t)+\int_0^t \mathrm{div}\, e^{s\mathcal{L}}u_0 ds = \int_0^t \Big(\mathrm{div}\,(e^{s\mathcal{L}}u_0-u)\Big)ds - \int_0^t \mathrm{div}\,(\rho u)ds.
$$

It follows from the interpolation inequality, Hölder inequality, (3.8), (3.4), (3.5), (3.6) and (3.21) that

$$\begin{aligned}
&\Big\|\rho(t)+\int_0^t \operatorname{div} e^{s\mathcal{L}}u_0 ds\Big\|_{\dot{B}^0_{[2,\infty],1}}\\
&\le C\int_0^T \|u-e^{s\mathcal{L}}u_0\|^{\frac{1}{2}}_{\dot{B}^0_{[2,\infty],1}}\|u-e^{s\mathcal{L}}u_0\|^{\frac{1}{2}}_{\dot{B}^2_{[2,\infty],1}}ds\\
&\quad + C\int_0^t \Big(\|\rho\|_{\dot{B}^1_{[2,\infty]}}\|u\|_{\dot{B}^0_{[2,\infty],1}}+\|\rho\|_{\dot{B}^0_{[2,\infty],1}}\|u\|_{\dot{B}^1_{[2,\infty]}}\Big)ds\\
&\le C(2^{-2N})^{\frac{1}{2}}2^{(\varepsilon+n\delta)N}+C2^{-2N}\cdot 2^{(\frac{1}{2}+\frac{n\delta}{2})N}\cdot 2^{(\frac{1}{2}+\frac{n\delta}{2})N}+C(2^{-2N})^{\frac{1}{2}}\cdot 2^{(-\frac{1}{2}+\frac{n\delta}{2})N}\cdot 2^{(\frac{1}{2}+\frac{n\delta}{2})N}\\
&\le C2^{(-1+\varepsilon+n\delta)N}.
\end{aligned}$$

Therefore (3.20) is obtained, and we complete the proof. ❑

In the next lemma, we approximate the nonlinear part of the solution by the second iterate of the convection term and the quasi linear term.

Lemma 3.6. *Let ε be such that $0<\varepsilon<1/8$ and let ρ_1 be defined by*

$$\rho_1(t)=-\int_0^t \operatorname{div} e^{s\mathcal{L}}u_0 ds.$$

Then the solution u obtained in Proposition 3.2 satisfies the following:

$$\begin{aligned}
&\Big\|u(t)-e^{t\mathcal{L}}u_0-\Big\{-\int_0^t e^{(t-s)\mathcal{L}}(e^{s\mathcal{L}}u_0\cdot\nabla)e^{s\mathcal{L}}u_0 ds-\int_0^t e^{(t-s)\mathcal{L}}\frac{\rho_1}{1+\rho_1}\mathcal{L}e^{s\mathcal{L}}u_0 ds\Big\}\Big\|_{\dot{B}^0_{2,1}\cap\dot{B}^0_{\infty,1}}\\
&\le C2^{(-\frac{1}{2}+\frac{3n}{2}\delta+2\varepsilon)N}\quad \text{for } t\in[0,2^{-2N}].
\end{aligned}\tag{3.22}$$

Proof of Lemma 3.6. For simplicity, we put $\dot{B}^{\alpha}_{[2,\infty],1}:=\dot{B}^{\alpha}_{2,1}\cap\dot{B}^{\alpha}_{\infty,1}$. From the integral equation for the velocity u, we have the following:

$$\begin{aligned}
&\Big\|u(t)-e^{t\mathcal{L}}u_0-\Big\{-\int_0^t e^{(t-s)\mathcal{L}}(e^{s\mathcal{L}}u_0\cdot\nabla)e^{s\mathcal{L}}u_0 ds-\int_0^t e^{(t-s)\mathcal{L}}\frac{\rho_1}{1+\rho_1}\mathcal{L}e^{s\mathcal{L}}u_0 ds\Big\}\Big\|_{\dot{B}^0_{[2,\infty],1}}\\
&\le \Big\|\int_0^t e^{(t-s)\mathcal{L}}(u\cdot\nabla)u ds-\int_0^t e^{(t-s)\mathcal{L}}(e^{s\mathcal{L}}u_0\cdot\nabla)e^{s\mathcal{L}}u_0 ds\Big\|_{\dot{B}^0_{[2,\infty],1}}\\
&\quad+\Big\|\int_0^t e^{(t-s)\mathcal{L}}\frac{\nabla P(\rho)}{1+\rho}ds\Big\|_{\dot{B}^0_{[2,\infty],1}}\\
&\quad+\Big\|\int_0^t e^{(t-s)\mathcal{L}}\frac{\rho}{1+\rho}\mathcal{L}u ds-\int_0^t e^{(t-s)\mathcal{L}}\frac{\rho_1}{1+\rho_1}\mathcal{L}e^{s\mathcal{L}}u_0 ds\Big\|_{\dot{B}^0_{[2,\infty],1}}\\
&=: L_1+L_2+L_3,
\end{aligned}\tag{3.23}$$

and we estimate L_1, L_2 and L_3 one by one.

For the estimate of L_1, we consider the difference of u and $e^{s\mathcal{L}}u_0$, apply the bilinear estimate (3.7) to obtain

$$
\begin{aligned}
L_1 &\le C\int_0^t \Big(\|(u-e^{s\mathcal{L}}u_0)\cdot\nabla u\|_{\dot B^0_{[2,\infty],1}} + \|e^{s\mathcal{L}}u_0\cdot\nabla(u-e^{s\mathcal{L}}u_0)\|_{\dot B^0_{[2,\infty],1}}\Big)ds\\
&\le C\int_0^t \|u-e^{s\mathcal{L}}u_0\|_{\dot B^0_{[2,\infty],1}}\|\nabla u\|_{\dot B^0_{[2,\infty],1}\cap\dot B^\varepsilon_{[2,\infty],1}}ds\\
&\quad + C\int_0^t \|e^{s\mathcal{L}}u_0\|_{\dot B^0_{[2,\infty],1}}\|\nabla(u-e^{s\mathcal{L}}u_0)\|_{\dot B^0_{[2,\infty],1}\cap\dot B^\varepsilon_{[2,\infty],1}}ds\\
&=: L_{11}+L_{12}.
\end{aligned}
\tag{3.24}
$$

It follows on L_{11} from the Hölder inequality, $T=2^{-2N}$, (3.6) and (3.21) that

$$
\begin{aligned}
L_{11} &\le C\int_0^t \|u-e^{s\mathcal{L}}u_0\|_{\dot B^0_{[2,\infty],1}}(\|u\|_{\dot B^1_{[2,\infty]}}+\|u\|_{\dot B^{1+\varepsilon}_{[2,\infty]}})ds\\
&\le C\|u-e^{s\mathcal{L}}u_0\|_{L^\infty(0,T;\dot B^0_{[2,\infty],1})}\Big(T^{\frac12}\|u\|^{\frac12}_{L^\infty(0,T;\dot B^0_{[2,\infty],1})}\|u\|^{\frac12}_{L^1(0,T;\dot B^2_{[2,\infty],1})}\\
&\qquad\qquad + T^{\frac{1-\varepsilon}{2}}\|u\|^{\frac{1-\varepsilon}{2}}_{L^\infty(0,T;\dot B^0_{[2,\infty],1})}\|u\|^{\frac{1+\varepsilon}{2}}_{L^1(0,T;\dot B^2_{[2,\infty],1})}\Big)\\
&\le C2^{(\varepsilon+n\delta)N}\times\Big((2^{-2N})^{\frac12}\cdot 2^{(\frac12+\frac{n\delta}{2})N}+(2^{-2N})^{\frac{1-\varepsilon}{2}}\cdot 2^{(\frac12+\frac{n\delta}{2})N}\Big)\\
&\le C2^{(-\frac12+\frac{3n}{2}\delta+2\varepsilon)N}.
\end{aligned}
\tag{3.25}
$$

We also have on L_{12} in the similar estimate to (3.25)

$$
\begin{aligned}
L_{12} &\le C\|e^{s\mathcal{L}}u_0\|_{L^\infty(0,T;\dot B^0_{[2,\infty],1})}\Big(T^{\frac12}\|u-e^{s\mathcal{L}}u_0\|^{\frac12}_{L^\infty(0,T;\dot B^0_{[2,\infty],1})}\|u-e^{s\mathcal{L}}u_0\|^{\frac12}_{L^1(0,T;\dot B^2_{[2,\infty],1})}\\
&\qquad\qquad + T^{\frac{1-\varepsilon}{2}}\|u-e^{s\mathcal{L}}u_0\|^{\frac{1-\varepsilon}{2}}_{L^\infty(0,T;\dot B^0_{[2,\infty],1})}\|u-e^{s\mathcal{L}}u_0\|^{\frac{1+\varepsilon}{2}}_{L^1(0,T;\dot B^2_{[2,\infty],1})}\Big)\\
&\le C2^{(\frac12+\frac{n\delta}{2})N}\times\Big((2^{-2N})^{\frac12}\cdot 2^{(n\delta+\varepsilon)N}+(2^{-2N})^{\frac{1-\varepsilon}{2}}\cdot 2^{(n\delta+\varepsilon)N}\Big)\\
&\le C2^{(-\frac12+\frac{3n}{2}\delta+2\varepsilon)N}.
\end{aligned}
\tag{3.26}
$$

Then we obtain by (3.24), (3.25) and (3.26)

$$
L_1\le C2^{(-\frac12+\frac{3n}{2}\delta+2\varepsilon)N}. \tag{3.27}
$$

On the estimate of L_2, it follows from the same estimate as the pressure estimate in (3.19) (see also the third term of the right hand side of (3.12)) that

$$
L_2\le C2^{(-\frac32+\frac{n\delta}{2})N}. \tag{3.28}
$$

On the estimate of L_3, we consider the differences $\rho-\rho_1$ and $u-e^{s\mathcal{L}}u_0$ to obtain

$$
\begin{aligned}
L_3 &\le \Big\|\int_0^t e^{(t-s)\mathcal{L}}\Big(\frac{\rho}{1+\rho}-\frac{\rho_1}{1+\rho_1}\Big)\mathcal{L}u\,ds\Big\|_{\dot B^0_{[2,\infty],1}}\\
&\quad + \Big\|\int_0^t e^{(t-s)\mathcal{L}}\frac{\rho_1}{1+\rho_1}\mathcal{L}(u-e^{s\mathcal{L}}u_0)ds\Big\|_{\dot B^0_{[2,\infty],1}}\\
&=: L_{31}+L_{32}.
\end{aligned}
\tag{3.29}
$$

By applying (3.7) to L_{31}, it holds that

$$
\begin{aligned}
L_{31} &\le C\int_0^t \Big\|\frac{\rho}{1+\rho}-\frac{\rho_1}{1+\rho_1}\Big\|_{\dot B^0_{\infty,1}\cap\dot B^\varepsilon_{[2,\infty],1}}\|\mathcal{L}u\|_{\dot B^0_{[2,\infty],1}}ds\\
&\le C\sup_{t\in[0,T]}\Big\|\frac{\rho}{1+\rho}-\frac{\rho_1}{1+\rho_1}\Big\|_{\dot B^0_{\infty,1}\cap\dot B^\varepsilon_{\infty,1}}\|u\|_{L^1(0,T;\dot B^2_{[2,\infty],1})}.
\end{aligned}
\tag{3.30}
$$

On the norm with ρ and ρ_1, we have from the embedding:

$$\dot{B}^{\varepsilon}_{[2,\infty]} = \dot{B}^{\varepsilon}_{2,1} \cap \dot{B}^{\varepsilon}_{\infty,1} \subset \dot{B}^{0}_{\infty,1} \cap \dot{B}^{\varepsilon}_{\infty,1}. \tag{3.31}$$

that

$$\Big\| \frac{\rho}{1+\rho} - \frac{\rho_1}{1+\rho_1} \Big\|_{\dot{B}^{0}_{\infty,1} \cap \dot{B}^{\varepsilon}_{\infty,1}} \le C \Big\| \frac{\rho}{1+\rho} - \frac{\rho_1}{1+\rho_1} \Big\|_{\dot{B}^{\varepsilon}_{[2,\infty],1}}. \tag{3.32}$$

Here we estimate by following geometric series:

$$\begin{aligned} \frac{\rho}{1+\rho} - \frac{\rho_1}{1+\rho_1} &= \sum_{m=1}^{\infty} (-1)^{m-1} (\rho^m - \rho_1^m) \\ &= \rho - \rho_1 + \sum_{m=2}^{\infty} (-1)^{m-1} (\rho^m - \rho_1^m). \end{aligned}$$

It follows from (3.20), (3.7), the embedding $\dot{B}^{\varepsilon}_{[2,\infty]}(\mathbb{R}^n) \hookrightarrow \dot{B}^{0}_{\infty,1}(\mathbb{R}^n)$ obtained by (3.31), (3.4) that

$$\begin{aligned} &\Big\| \frac{\rho}{1+\rho} - \frac{\rho_1}{1+\rho_1} \Big\|_{\dot{B}^{\varepsilon}_{[2,\infty],1}} \\ &\le \|\rho - \rho_1\|_{\dot{B}^{\varepsilon}_{[2,\infty],1}} + \sum_{m=2}^{\infty} (\|\rho^m\|_{\dot{B}^{\varepsilon}_{[2,\infty],1}} + \|\rho_1^m\|_{\dot{B}^{\varepsilon}_{[2,\infty],1}}) \\ &\le C 2^{(-1+n\delta+\varepsilon)N} + \sum_{m=2}^{\infty} C^{m-1} \Big(\|\rho\|^{m-1}_{\dot{B}^{0}_{[2,\infty],1}} \|\rho\|_{\dot{B}^{\varepsilon}_{[2,\infty],1}} + \|\rho_1\|^{m-1}_{\dot{B}^{0}_{[2,\infty],1}} \|\rho_1\|_{\dot{B}^{\varepsilon}_{[2,\infty],1}} \Big) \\ &\le C 2^{(-1+n\delta+\varepsilon)N} + \sum_{m=2}^{\infty} C^m (2^{(-\frac{1}{2}+\frac{n\delta}{2})N})^{m-1} \cdot 2^{(-\frac{1}{2}+\varepsilon+\frac{n\delta}{2})N} \\ &\le C 2^{(-1+n\delta+\varepsilon)N}. \end{aligned} \tag{3.33}$$

Therefore we obtain the following estimate for L_{31} by (3.30), (3.32), (3.33) and (3.6)

$$L_{31} \le C 2^{(-1+\varepsilon+n\delta)N} \cdot 2^{(\frac{1}{2}+\frac{n\delta}{2})N} = C 2^{(-\frac{1}{2}+\varepsilon+\frac{3n}{2}\delta)N}. \tag{3.34}$$

For the estimate of L_{32}, we apply (3.7) and (3.31) to see that

$$\begin{aligned} L_{32} &\le C \int_0^t \Big\| \frac{\rho_1}{1+\rho_1} \Big\|_{\dot{B}^{0}_{\infty,1} \cap \dot{B}^{\varepsilon}_{\infty,1}} \|\mathcal{L}(u - e^{s\mathcal{L}} u_0)\|_{\dot{B}^{0}_{[2,\infty],1}} ds \\ &\le C \sup_{t \in [0,T]} \Big\| \frac{\rho_1}{1+\rho_1} \Big\|_{\dot{B}^{\varepsilon}_{[2,\infty],1}} \|u - e^{s\mathcal{L}} u_0\|_{L^1(0,T;\dot{B}^{2}_{[2,\infty],1})}. \end{aligned} \tag{3.35}$$

On the norm with ρ_1, we again utilize the geometric series, and apply (3.7), the embedding $\dot{B}^{\varepsilon}_{[n,\infty]}(\mathbb{R}^n) \hookrightarrow \dot{B}^{0}_{\infty,1}(\mathbb{R}^n)$ obtained by (3.31), (3.4) to obtain

$$\begin{aligned} \Big\| \frac{\rho_1}{1+\rho_1} \Big\|_{\dot{B}^{\varepsilon}_{[2,\infty],1}} &\le \sum_{m=1}^{\infty} \|(-1)^{m-1} \rho_1^m\|_{\dot{B}^{\varepsilon}_{[2,\infty],1}} \le \sum_{m=1}^{\infty} C^m \|\rho_1\|^{m-1}_{\dot{B}^{0}_{[2,\infty],1}} \|\rho_1\|_{\dot{B}^{\varepsilon}_{[2,\infty],1}} \\ &\le C 2^{(-\frac{1}{2}+\varepsilon+\frac{n\delta}{2})N}. \end{aligned} \tag{3.36}$$

It follows from (3.35), the above estimate and (3.21) that

$$L_{32} \le C 2^{(-\frac{1}{2}+\varepsilon+\frac{n\delta}{2})N} \cdot 2^{(n\delta+\varepsilon)N} = C 2^{(-\frac{1}{2}+2\varepsilon+\frac{3n}{2}\delta)N}. \tag{3.37}$$

We have the following estimate from (3.29), (3.34) and (3.37)

$$L_3 \le C 2^{(-\frac{1}{2}+2\varepsilon+\frac{3n}{2}\delta)N}.$$

Therefore, (3.22) is obtained by by (3.23), (3.27), (3.28) and the above estimate. ❑

The last lemma justifies the approximation

$$\frac{\rho}{1+\rho}\mathcal{L}u \simeq \rho\mathcal{L}u$$

for the second iterate of the quasi-linear term.

Lemma 3.7. *On the last term in the left hand side of* (3.22) *in Lemma 3.6, it holds that*

$$\begin{aligned}&\Big\|\int_0^t e^{(t-s)\mathcal{L}}\frac{\rho_1}{1+\rho_1}\mathcal{L}e^{s\mathcal{L}}u_0ds - \int_0^t e^{(t-s)\mathcal{L}}\rho_1\mathcal{L}e^{s\mathcal{L}}u_0ds\Big\|_{\dot{B}^0_{2,1}\cap\dot{B}^0_{\infty,1}}\\&\leq C2^{(-\frac{1}{2}+\frac{3n}{2}\delta+\varepsilon)N} \quad \textit{for } t\in[0,2^{-2N}].\end{aligned} \tag{3.38}$$

Proof of Lemma 3.7. For simplicity, we put $\dot{B}^\alpha_{[2,\infty],1} = \dot{B}^\alpha_{2,1}\cap\dot{B}^\alpha_{\infty,1}$ and $T = 2^{-2N}$. It follows from $\frac{\rho_1}{1+\rho_1} - \rho_1 = -\frac{\rho_1^2}{1+\rho_1}$, the same estimate as (3.30), and (3.31) that

$$\begin{aligned}&\Big\|\int_0^t e^{(t-s)\mathcal{L}}\frac{\rho_1}{1+\rho_1}\mathcal{L}e^{s\mathcal{L}}u_0ds - \int_0^t e^{(t-s)\mathcal{L}}\rho_1\mathcal{L}e^{s\mathcal{L}}u_0ds\Big\|_{\dot{B}^0_{[2,\infty],1}}\\&=\Big\|\int_0^t e^{(t-s)\mathcal{L}}\frac{\rho_1^2}{1+\rho_1}\mathcal{L}e^{s\mathcal{L}}u_0ds\Big\|_{\dot{B}^0_{[2,\infty],1}}\\&\leq C\sup_{t\in[0,T]}\Big\|\frac{\rho_1^2}{1+\rho_1}\Big\|_{\dot{B}^\varepsilon_{[2,\infty],1}}\|u\|_{L^1(0,T;\dot{B}^2_{[2,\infty],1})}.\end{aligned} \tag{3.39}$$

On the term with ρ_1, we utilize the geometric series and estimate similarly to (3.36) to see that

$$\begin{aligned}\Big\|\frac{\rho_1^2}{1+\rho_1}\Big\|_{\dot{B}^\varepsilon_{[2,\infty],1}} &\leq \sum_{m=1}^\infty \|(-1)^{m-1}\rho_1^{m+1}\|_{\dot{B}^\varepsilon_{[2,\infty],1}} \leq \sum_{m=1}^\infty C^m(2^{(-\frac{1}{2}+\frac{n\delta}{2})N})^m 2^{(-\frac{1}{2}+\varepsilon+\frac{n\delta}{2})N}\\&\leq C2^{(-1+n\delta+\varepsilon)N}.\end{aligned} \tag{3.40}$$

Then we have from (3.39), (3.40) and (3.6) that

$$\begin{aligned}&\Big\|\int_0^t e^{(t-s)\mathcal{L}}\frac{\rho_1}{1+\rho_1}\mathcal{L}e^{s\mathcal{L}}u_0ds - \int_0^t e^{(t-s)\mathcal{L}}\rho_1\mathcal{L}e^{s\mathcal{L}}u_0ds\Big\|_{\dot{B}^0_{[2,\infty],1}}\\&\leq C2^{(-1+n\delta+\varepsilon)N}2^{(\frac{1}{2}+\frac{\delta}{2})N} = 2^{(-\frac{1}{2}+\frac{3n}{2}\delta+\varepsilon)N},\end{aligned}$$

and (3.38) is verified. ❑

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let the initial data (ρ_0, u_0) be defined by (3.1) and set $R = (\log N)^{-1}$. By Proposition 3.2, we have a solution (ρ, u) on the time interval $[0, 2^{-2N}]$. It follows from Proposition 2.1 that

$$\begin{aligned}&\|u_0\|_{\dot{B}^{-\frac{1}{2}}_{2n,1}} \leq CR = C(\log N)^{-1} \to 0\\&\Big\|I[u_0]\Big\|_{\dot{B}^{-\frac{1}{2}}_{2n,1}}\Big|_{t=\varepsilon_0 2^{-2N}} \geq cN^{1-\frac{1}{n}}(\log N)^{-2} - C(\log N)^{-2} \to \infty,\end{aligned}$$

as $N\to\infty$. Then we need to prove that

$$\sup_{0<t<2^{-2N}}\Big\|u(t) - I[u_0]\Big\|_{\dot{B}^{-\frac{1}{2}}_{2n,1}} < \infty. \tag{3.41}$$

To this end, we write

$$
\begin{aligned}
&u(t)-e^{t\mathcal{L}}u_0-I[u_0]\\
&=\Big[u(t)-\Big\{-\int_0^t e^{(t-s)\mathcal{L}}(e^{s\mathcal{L}}u_0\cdot\nabla)e^{s\mathcal{L}}u_0ds-\int_0^t e^{(t-s)\mathcal{L}}\frac{\rho_1}{1+\rho_1}\mathcal{L}e^{s\mathcal{L}}u_0ds\Big\}\Big]\\
&\quad-\Big[\int_0^t e^{(t-s)\mathcal{L}}\frac{\rho_1}{1+\rho_1}\mathcal{L}e^{s\mathcal{L}}u_0ds-\int_0^t e^{(t-s)\mathcal{L}}\rho_1\mathcal{L}e^{s\mathcal{L}}u_0ds\Big]\\
&\quad+\Big[-\int_0^t e^{(t-s)\mathcal{L}}(e^{s\mathcal{L}}u_0\cdot\nabla)e^{s\mathcal{L}}u_0ds+\int_0^t e^{(t-s)\mathcal{L}}\Big(\int_0^s \operatorname{div}e^{\tilde{s}\mathcal{L}}u_0d\tilde{s}\Big)\mathcal{L}e^{s\mathcal{L}}u_0ds-I[u_0]\Big]\\
&=: R_1+R_2+R_3.
\end{aligned}
$$

If $t\le 2^{-2N}$, it follows that by Lemmas 3.6 and 3.7

$$
\|R_1\|_{\dot{B}^{-\frac12}_{2n,1}}+\|R_2\|_{\dot{B}^{-\frac12}_{2n,1}}\le C2^{(-\frac12+\frac{3n}{2}\delta+2\varepsilon)N},
$$

and by Lemma 2.7

$$
\|R_3\|_{\dot{B}^{-\frac12}_{2n,1}}\le C(\log N)^{-2},
$$

which prove (3.41). Therefore, we have obtained a sequence of initial data such that it verifies the discontinuity of data-to-solution map. ❑

Appendix A

We discuss the sharpness in Remark after Proposition 2.2.

Proposition A.1. *For positive constants A,B, and c, let $\widetilde{I}[f,g]$ be defined by (2.4). For σ with $n\le\sigma\le\infty$, there exists $C=C_{n,\sigma}>0$ such that*

$$
\sup_{t>0}\|\widetilde{I}[f,g](t)\|_{\dot{B}^{-\frac12}_{2n,\sigma}}\le C\|f\|_{\dot{B}^{-\frac12}_{2n,\sigma}}\|g\|_{\dot{B}^{-\frac12}_{2n,\sigma}}\tag{A.1}
$$

for all $f,g\in\dot{B}^{-\frac12}_{2n,\sigma}(\mathbb{R}^n)$.

Remark. If we consider only the convection term $(u\cdot\nabla)u$ without the quasilinear dissipation term, the corresponding estimate does not hold for any σ with $1\le\sigma\le\infty$. This indicates that cancellation occurs for the sum of the convection term and the quasi linear term.

We prepare a lemma to prove Proposition A.1.

Lemma A.2. *Let $N_0\ge 10$ be a fixed natural number. Suppose that $F_k,G_l\in L^{2n}(\mathbb{R}^n)$ be such that*

$$
\operatorname{supp}\widehat{F_k}\subset\{2^{k-1}\le|\xi|\le 2^{k+1}\},\quad \operatorname{supp}\widehat{G_l}\subset\{2^{l-1}\le|\xi|\le 2^{l+1}\}.
$$

If $|k-l|\le 2$, then

$$
\begin{aligned}
&\Big\|\sum_{j\le k-N_0}\phi_j*\Big\{\mathcal{F}^{-1}\Big[\int_{\mathbb{R}^n}\frac{1}{B|\eta|^2-c|\xi|^2}\widehat{F}_k(\xi-\eta)\widehat{G}_l(\eta)d\eta\Big]\Big\}\Big\|_{\dot{B}^{-\frac12}_{2n,\sigma}}\\
&\le C\|F_k\|_{L^{2n}}2^{-2l}\|G_l\|_{L^{2n}}.
\end{aligned}
$$

Proof of Lemma A.2. By the embedding $L^n(\mathbb{R}^n)\hookrightarrow\dot{B}^{-\frac12}_{2n,\sigma}(\mathbb{R}^n)$, it is sufficient to estimate the $L^n(\mathbb{R}^n)$ norm. Since the suppor of the Fourier transform of F_k,G_k is restricted, we write

$$
\widehat{F}_k=\widehat{\Psi_k}\widehat{F_k},\quad \widehat{G}_l=\widehat{\Psi_l}\widehat{G_l},
$$

where $\Psi_k:=\phi_{k-1}+\phi_k+\phi_{k+1}$. Let us choose $\varphi,\varphi_j\in\mathcal{S}(\mathbb{R}^n)$ such that

$$
\widehat{\varphi}(\xi)=\begin{cases}1 & \text{if } |\xi|\le 1,\\ \widehat{\phi_0}(\xi) & \text{if } |\xi|\ge 1,\end{cases}\qquad \varphi_j(x)=2^{nj}\varphi(2^jx).
$$

Then we can apply Lemma 2.6 to the Fourier multiplier

$$m(\xi-\eta,\eta)=\widehat{\varphi_{k-N_0}}(\xi)\frac{\widehat{\Psi_k}(\xi-\eta)\cdot 2^{2l}\,\widehat{\Psi_l}(\eta)}{B|\eta|^2-c|\xi|^2},$$

and then obtain

$$\Big\|\varphi_{k-N_0}*\Big\{\mathcal{F}^{-1}\Big[\int_{\mathbb{R}^n}\frac{\widehat{\Psi_k}(\xi-\eta)\cdot 2^{2l}\,\widehat{\Psi_l}(\eta)}{B|\eta|^2-c|\xi|^2}\widehat{F}_k(\xi-\eta)2^{-2l}\widehat{G}_l(\eta)d\eta\Big]\Big\}\Big\|_{L^n}\le C\|F_k\|_{L^{2n}}2^{-2l}\|G_l\|_{L^{2n}}.$$

We notice that the function in the right hand side above belongs to $L^n(\mathbb{R}^n)$ and

$$\begin{aligned}&\sum_{j\le k-N_0}\phi_j*\Big\{\mathcal{F}^{-1}\Big[\int_{\mathbb{R}^n}\frac{1}{B|\eta|^2-c|\xi|^2}\widehat{F}_k(\xi-\eta)\widehat{G}_l(\eta)d\eta\Big]\Big\}\\&=\varphi_{k-N_0}*\Big\{\mathcal{F}^{-1}\Big[\int_{\mathbb{R}^n}\frac{1}{B|\eta|^2-c|\xi|^2}\widehat{F}_k(\xi-\eta)\widehat{G}_l(\eta)d\eta\Big]\Big\}\quad\text{in } \mathcal{S}'(\mathbb{R}^n),\end{aligned}$$

which leads to the desired inequality. ❑

Proof of Proposition A.1. For simplicity, let f_k and g_l be defined by $f_k:=\phi_k*f$ and $g_l:=\phi_l*g$. By Bony's paraproduct formula [2], it holds that

$$\widetilde{I}[f,g](t)=\Big(\sum_{k\ge l+3}+\sum_{l\ge k+3}+\sum_{|k-l|\le 2}\Big)I[f_k,g_l]=:II_1+II_2+II_3. \tag{A.2}$$

The high–low and low–high interaction parts II_1 and II_2, respectively, can be treated easily. We show the estimate for II_1 and omit the estimate for II_2. It follows from the linear estimate $e^{(t-s)\Delta}:\dot{B}^{-\frac32}_{2n,\sigma}(\mathbb{R}^n)\to\dot{B}^{-\frac12}_{2n,\sigma}(\mathbb{R}^n)$ that

$$\begin{aligned}\|II_1\|_{\dot{B}^{-\frac12}_{2n,\sigma}}&\le C\int_0^t(t-s)^{-\frac12}\Big\|\sum_{k\ge l+3}(e^{As\Delta}f_k)\nabla e^{Bs\Delta}g_l+\frac{B}{A}\Big(e^{As\Delta}f_k-f_k\Big)\nabla e^{Bs\Delta}g_l\Big\|_{\dot{B}^{-\frac32}_{2n,\sigma}}ds\\&\le C\int_0^t(t-s)^{-\frac12}\Big\{\Big\|\sum_{k\ge l+3}(e^{As\Delta}f_k)\nabla e^{Bs\Delta}g_l\Big\|_{\dot{B}^{-\frac32}_{2n,\sigma}}+\Big\|\sum_{k\ge l+3}f_k\nabla e^{Bs\Delta}g_l\Big\|_{\dot{B}^{-\frac32}_{2n,\sigma}}\Big\}ds\\&=:II_{1,1}+II_{1,2}.\end{aligned}\tag{A.3}$$

Here, to take the above norm in the Besov spaces, we use the equality $\phi_j*\{(e^{As\Delta}f_k)\nabla e^{Bs\Delta}g_l\}=0$ for j,k,l with $|j-k|\ge 4$ and $k\ge l+3$ because the support of Fourier transform of the functions satisfy

$$\operatorname{supp}\mathcal{F}[\phi_j]\subset\{\xi\in\mathbb{R}^n\,|\,2^{j-1}\le|\xi|\le 2^{j+1}\}, \tag{A.4}$$

$$\operatorname{supp}\mathcal{F}\big[(e^{As\Delta}f_k)\nabla e^{Bs\Delta}g_l\big]\subset\{\xi\in\mathbb{R}^n\,|\,2^{k-2}\le|\xi|\le 2^{k+2}\}. \tag{A.5}$$

Subsequently, on the sum over $k\ge l+3$ in the definition of $II_{1,1}$ and $II_{1,2}$, it is sufficient to consider k with $|k-j|\le 3$ for each j. We have from Young's inequality, $\|\phi_j\|_{L^1}=\|\phi_0\|_{L^1}$, Hölder's inequality, the boundedness of $e^{As\Delta}$ in $L^{2n}(\mathbb{R}^n)$, and the linear estimate of $e^{Bs\Delta}$: $\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^n)\to\dot{B}^{0}_{\infty,\infty}(\mathbb{R}^n)$ and the embedding $\dot{B}^{-\frac12}_{2n,\sigma}(\mathbb{R}^n)\hookrightarrow\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^n)$

$$\begin{aligned}&\Big\|\sum_{k\ge l+3}(e^{As\Delta}f_k)\nabla e^{Bs\Delta}g_l\Big\|_{\dot{B}^{-\frac32}_{2n,\sigma}}\\&\le\Big\{\sum_{j\in\mathbb{Z}}\Big(2^{-\frac32 j}\|\phi_0\|_{L^1}\sum_{|k-j|\le3}\|f_k\|_{L^{2n}}\Big(\sum_{l\le k-3}\|\nabla e^{Bs\Delta}g_l\|_{L^\infty}\Big)\Big)^\sigma\Big\}^{\frac1\sigma}\\&\le C\Big\{\sum_{j\in\mathbb{Z}}\Big(2^{-\frac32 j}\sum_{|k-j|\le3}\|f_k\|_{L^{2n}}\cdot 2^k\Big)^\sigma\Big\}^{\frac1\sigma}\|e^{Bs\Delta}g\|_{\dot{B}^0_{\infty,\infty}}\le Cs^{-\frac12}\|f\|_{\dot{B}^{-\frac12}_{2n,\sigma}}\|g\|_{\dot{B}^{-\frac12}_{2n,\sigma}}.\end{aligned}\tag{A.6}$$

Next, it follows from (A.6) that

$$II_{1,1} \leq C \int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}} ds \|f\|_{\dot{B}^{-\frac{1}{2}}_{2n,\sigma}} \|g\|_{\dot{B}^{-\frac{1}{2}}_{2n,\sigma}} \leq C\|f\|_{\dot{B}^{-\frac{1}{2}}_{2n,\sigma}} \|g\|_{\dot{B}^{-\frac{1}{2}}_{2n,\sigma}}. \tag{A.7}$$

The same estimate as (A.7) holds for $II_{1,2}$ as we only replaced the propagator $e^{As\Delta}$ with the identity operator. We then obtain the bilinear estimates for II_1.

Next, we show the estimate for II_3 on high-high frequency interaction. The following equality $\phi_j * I[f_k, g_l] = 0$ holds for j, k, l with $|k-l| \leq 2$ and $k < j-5$ which is obtained by (A.4) and (A.5). We decompose the sum appearing in $\phi_j * II_3$ into the sum over $|k-j| \leq N_0$ and $k \geq j+N_0$:

$$\begin{aligned} \sum_{j\in\mathbb{Z}} \phi_j * II_3 =& \sum_{j\in\mathbb{Z}} \phi_j * \Big(\sum_{|k-j|\leq N_0} \sum_{|l-k|\leq 2} I[f_k, g_l]\Big) + \sum_{j\in\mathbb{Z}} \phi_j * \Big(\sum_{k\geq j+N_0} \sum_{|l-k|\leq 2} I[f_k, g_l]\Big) \\ =:& II_{3,1} + II_{3,2}, \end{aligned} \tag{A.8}$$

where $N_0 > 0$ is a sufficiently large number that will be taken later. It is possible to estimate $II_{3,1}$ analogously to the estimate of II_1 with the correspondences $|k-j| \leq 2$, $l \leq k-3$ of the summation in (A.6) and $|k-j| \leq N_0$, $|l-k| \leq 2$ of the summation in the definition of $II_{3,1}$, respectively; subsequently, one can obtain

$$\|II_{3,1}\|_{\dot{B}^{-\frac{1}{2}}_{2n,\sigma}} \leq C\|f\|_{\dot{B}^{-\frac{1}{2}}_{2n,\sigma}} \|g\|_{\dot{B}^{-\frac{1}{2}}_{2n,\sigma}}. \tag{A.9}$$

We consider the estimate of $II_{3,2}$ using the equality (2.7). We apply Lemma A.2 for sufficiently large $N_0 \in \mathbb{N}$ and the Fourier multiplier theorem, and then it holds that

$$\begin{aligned} \|II_{3,2}\|_{\dot{B}^{-\frac{1}{2}}_{2n,\sigma}} \leq& \sum_{|l-k|\leq 2} \Big\| \sum_{j\leq k-N_0} \phi_j * \Big\{ \mathcal{F}^{-1} \Big[\frac{A}{B} \int_{\mathbb{R}^n} \frac{(1-e^{-At|\xi-\eta|^2})\widehat{f}_k(\xi-\eta)}{B|\eta|^2 - c|\xi|^2} e^{-Bt|\eta|^2} i\eta \widehat{g}_l(\eta) d\eta \Big] \Big\} \Big\|_{\dot{B}^{-\frac{1}{2}}_{2n,\sigma}} \\ \leq& C \sum_{|l-k|\leq 2} \|(1-e^{At\Delta})f_k\|_{L^{2n}} 2^{-2l} \|e^{Bt\Delta}\nabla g_l\|_{L^{2n}} \\ \leq& C \sum_{|l-k|\leq 2} (1-e^{-C^{-1}At2^{2k}}) \|f_k\|_{L^{2n}} \frac{2^l \cdot e^{-C^{-1}Bt2^{2l}}}{2^{2l}} \|g_l\|_{L^{2n}}. \end{aligned}$$

An elementary calculation gives that

$$\|II_{3,2}\|_{\dot{B}^{-\frac{1}{2}}_{2n,\sigma}} \leq C \sum_{k\in\mathbb{Z}} t2^{2k} e^{-C^{-1}t2^{2k}} \|f\|_{\dot{B}^{-\frac{1}{2}}_{2n,\infty}} \|g\|_{\dot{B}^{-\frac{1}{2}}_{2n,\infty}} \leq C\|f\|_{\dot{B}^{-\frac{1}{2}}_{2n,\infty}} \|g\|_{\dot{B}^{-\frac{1}{2}}_{2n,\infty}}.$$

We next consider the estimate of the remainder term R defined by (2.8). Since $R[f_k, g_l]$ exhibits some divergence structures due to (2.8), we have from the embedding $\dot{B}^0_{n,\sigma}(\mathbb{R}^n) \hookrightarrow \dot{B}^{-\frac{1}{2}}_{2n,\sigma}(\mathbb{R}^n)$, Hölder's inequality

$$\begin{aligned} &\Big\| \sum_{j\in\mathbb{Z}} \phi_j * \Big(\sum_{k\geq j+N_0} \sum_{|l-k|\leq 2} R[f_k, g_l] \Big) \Big\|_{\dot{B}^{-\frac{1}{2}}_{2n,\sigma}} \\ &\leq C \Big\{ \sum_{j\in\mathbb{Z}} \Big(\sum_{k\geq j+N_0} \sum_{|l-k|\leq 2} 2^l \|f_k\|_{L^{2n}} \|g_l\|_{L^{2n}} \\ &\qquad \frac{\{(B+c)2^{2j} + 2B2^j \cdot 2^l\}\{e^{-C^{-1}ct2^{2j}} + e^{-C^{-1}At2^{2k} - C^{-1}Bt2^{2l}}\}}{(A2^{2k} + B2^{2l} - c2^{2j})(Be^{2l} - c2^{2j})} \Big) \Big\}^{\frac{1}{\sigma}} \\ &\leq C \Big\{ \sum_{j\in\mathbb{Z}} \Big(\sum_{k\geq j+N_0} \sum_{|l-k|\leq 2} 2^j 2^{-2l} \|f_k\|_{L^{2n}} \|g_l\|_{L^{2n}} \Big)^\sigma \Big\}^{\frac{1}{\sigma}}. \end{aligned} \tag{A.10}$$

By introducing the index r by $r = k - j$ ($r \geq N_0$) and applying Hölder's inequality, it holds that

$$\begin{aligned}
&\Big\{ \sum_{j\in\mathbb{Z}} \Big(\sum_{k\geq j+N_0} \sum_{|l-k|\leq 2} 2^j 2^{-2l} \|f_k\|_{L^{2n}} \|g_l\|_{L^{2n}} \Big)^{\sigma} \Big\}^{\frac{1}{\sigma}} \\
&= \Big\{ \sum_{j\in\mathbb{Z}} \Big(\sum_{r\geq N_0} 2^j 2^{-2l} \|f_{j+r}\|_{L^{2n}} \sum_{|l-j-r|\leq 2} \|g_l\|_{L^{2n}} \Big)^{\sigma} \Big\}^{\frac{1}{\sigma}} \qquad \text{(A.11)} \\
&\leq \sum_{r\geq N_0} 2^{-r} \Big\{ \sum_{j\in\mathbb{Z}} \Big(2^{j+r} 2^{-2l} \|f_{j+r}\|_{L^{2n}} \sum_{|l-j-r|\leq 2} \|g_l\|_{L^{2n}} \Big)^{\sigma} \Big\}^{\frac{1}{\sigma}} \leq C \|f\|_{\dot{B}^{-\frac{1}{2}}_{2n,2\sigma}} \|g\|_{\dot{B}^{-\frac{1}{2}}_{2n,2\sigma}}.
\end{aligned}$$

Therefore, the estimate for II_3 is obtained by (A.8), (A.10) and (A.11), and the proof is completed. ❑

Appendix B

We introduce an inequality of Gronwall type.

Lemma B.1. *Let f, g_1, g_2, g_3 be continuous function such that*

$$f, g_2, g_3 \geq 0, \quad f(t) \leq \int_0^t e^{\int_s^t g_1(\mu)\ d\mu} \Big(g_2(s) + g_3(s) f(s) \Big)\ ds, \quad t > 0.$$

Then

$$f(t) \leq \int_0^t e^{\int_s^t (g_1+g_3)\ d\mu} g_2(s)\ ds.$$

Proof. Let Ψ be the right hand side of the assumption.

$$\Psi(t) := \int_0^t e^{\int_s^t (g_1+g_3)\ d\mu} \Big(g_2(s) + g_3(s) f(s) \Big)\ ds.$$

Then

$$\begin{gathered}
\Psi' = g_1 \Psi + g_2 + g_3 f \leq g_1 \Psi + g_2 + g_3 \Psi, \\
\Big(e^{-\int_0^t (g_1+g_3)} \Psi \Big)' \leq e^{-\int_0^t (g_1+g_3)} g_2.
\end{gathered}$$

Integrating over $[0, t]$ gives the desired inequality. ❑

Appendix C

We give proof of the bilinear estimates in Lemma 3.4. The first estimate (3.7) is obtained by Bony's paraproduct formula and bilinear estimates. Indeed, we write

$$fg = \Big(\sum_{k>l+2} + \sum_{l>k+2} + \sum_{|k-l|\leq 2} \Big) f_k g_l,$$

where $f_k := \phi_k * f, g_l := \phi_l * g$. When $k > l + 2$, it follows that

$$\Big\| \sum_{k>l+2} f_k g_l \Big\|_{\dot{B}^0_{2,1}\cap\dot{B}^0_{\infty,1}} \leq C \|f\|_{\dot{B}^0_{\infty,1}} \|g\|_{L^2} \leq C \|f\|_{\dot{B}^0_{\infty,1}\cap\dot{B}^{\varepsilon}_{[2,\infty],1}} \|g\|_{\dot{B}^0_{[2,\infty],1}},$$

and we also have

$$\Big\| \sum_{k>l+2} f_k g_l \Big\|_{\dot{B}^0_{2,1}\cap\dot{B}^0_{\infty,1}} \leq C \|f\|_{\dot{B}^0_{2,1}\cap\dot{B}^0_{\infty,1}} \|g\|_{L^\infty} \leq C \|f\|_{\dot{B}^0_{[2,\infty],1}} \|g\|_{\dot{B}^0_{\infty,1}\cap\dot{B}^{\varepsilon}_{[2,\infty],1}},$$

which proves that

$$\Big\| \sum_{k>l+2} f_k g_l \Big\|_{\dot{B}^0_{2,1}\cap\dot{B}^0_{\infty,1}} \leq C \min \Big\{ \|f\|_{\dot{B}^0_{\infty,1}\cap\dot{B}^{\varepsilon}_{[2,\infty],1}} \|g\|_{\dot{B}^0_{[2,\infty],1}}, \|f\|_{\dot{B}^0_{[2,\infty],1}} \|g\|_{\dot{B}^0_{\infty,1}\cap\dot{B}^{\varepsilon}_{[2,\infty],1}} \Big\}.$$

The second case when $l > k + 2$ follows analogously. When $|k - l| \leq 2$, we apply the embedding

$$\dot{B}^{\varepsilon}_{p_1,1}(\mathbb{R}^n) \cap \dot{B}^{\varepsilon}_{p_2,1}(\mathbb{R}^n) \hookrightarrow \dot{B}^0_{2,1}(\mathbb{R}^n) \cap \dot{B}^0_{\infty,1}(\mathbb{R}^n), \qquad p_1 := \frac{2}{1 + \frac{2\varepsilon}{n}}, \quad p_2 := \frac{n}{\varepsilon}$$

to obtain that

$$\Big\|\sum_{|k-l|\le 2} f_k g_l\Big\|_{\dot B^0_{2,1}\cap \dot B^0_{\infty,1}} \le C\Big\|\sum_{|k-l|\le 2} f_k g_l\Big\|_{\dot B^{\varepsilon}_{p_1,1}\cap \dot B^{\varepsilon}_{p_2,1}}$$
$$\le C\min\Big\{\|f\|_{\dot B^0_{\infty,1}\cap \dot B^{\varepsilon}_{[2,\infty],1}}\|g\|_{\dot B^0_{[2,\infty],1}}, \|f\|_{\dot B^0_{[2,\infty],1}}\|g\|_{\dot B^0_{\infty,1}\cap \dot B^{\varepsilon}_{[2,\infty],1}}\Big\}.$$

The second estimate (3.8) follows by the standard bilinear estimate

$$\|fg\|_{\dot B^s_{r,1}} \le C\big(\|f\|_{\dot B^s_{r_1,1}}\|g\|_{L^{r_2}} + \|f\|_{L^{r_3}}\|g\|_{\dot B^s_{r_4,1}}\big), \quad s>0, \frac{1}{r}=\frac{1}{r_1}+\frac{1}{r_2}=\frac{1}{r_3}+\frac{1}{r_4}.$$

Acknowledgements. The authors would like to thank referees for their helpful comments. The first author was supported by JSPS Grant-in-Aid for Young Scientists (A) (No. 17H04824). The second author was supported by by JSPS Grant-in-Aid, Scientific Research (S) (No. 19H05597) and JSPS Challenging Research (Pioneering) (No. 17H06199).

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