Singularities of Fano varieties of lines on singular cubic fourfolds

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Abstract

Let X be a cubic fourfold that has only simple singularities and does not contain a plane. We prove that the Fano variety of lines on X has the same analytic type of singularity as the Hilbert scheme of two points on a surface with only ADE-singularities.

2000 Mathematics Subject Classification. Primary 14J17; Secondary 14D06.
Key words and phrases. Cubic fourfolds, symplectic varieties, degeneration.

1 Introduction

Irreducible holomorphic symplectic manifolds (IHS manifolds for short) are building blocks of compact Kähler manifolds with trivial canonical bundles and have been studied intensively by many authors. The only known IHS manifolds up to deformation equivalence are the Hilbert schemes of points on a $K3$ surface, the generalized Kummer varieties, and the two sporadic examples constructed by O'Grady.

One method to construct IHS is to consider moduli spaces of subschemes, sheaves or more generally objects in the derived categories of smooth cubic fourfolds. For example, Beauville and Donagi showed that the Fano variety of lines on a smooth cubic fourfold is an irreducible holomorphic symplectic manifold which is deformation equivalent to the Hilbert square of a $K3$ surface [BD]. Other examples are given in [LLSvS], [LMS], [Ou], and so on.

The aim of this article is to study what happens on the Fano variety of lines if cubic fourfolds mildly degenerate. This is an important problem when one considers completion of moduli the space of IHS manifolds. We will consider cubic fourfolds having at worst simple singularities. Such cubic fourfolds are characterized as stable ones in the moduli space of (possibly singular) cubic fourfolds with minor exceptions [La1]. A study of a similar problem for another type of degenerations can be found in [vdB], where the author considers determinantal cubic fourfolds. Also, more general degeneration of IHS manifolds are studied in [KLSV] and [Nag].

It is already shown in [Le] that the Fano variety $F(X)$ of lines on a cubic fourfold X that has only simple singularities has symplectic singularities in the sense of [B] (see §2). In fact, the singular cubic fourfolds satisfying this assumption can be regarded as exactly the degenerations of smooth ones whose Fano varieties of lines have symplectic singularities (Proposition 4.1). In this article we will determine the precise singularity type of the Fano variety $F(X)$ of a cubic fourfold X that has at worst simple singularities under the assumption that X does not contain a plane. This assumption does not seem essential but the author could not remove it unfortunately (cf. Remark 3.3). The main theorem states that such $F(X)$ has the same singularity type as the Hilbert scheme $\mathrm{Hilb}^2(S)$ of 2 points on a surface S with only ADE-singularities (Theorems 1.1 and 1.2). As a corollary, we see that the germs of singular points in $F(X)$ only depend on the ADE-types of the singular points of X. This seems interesting since even local structure of $F(X)$ is a priori determined by global structure of X. In fact this coincidence between the ADE-types of the singularities of X and $F(X)$ can be explained through comparison of the monodromy actions on the cohomology groups of X and $F(X)$ (see §4.2).

To state the main theorem more precisely, we should note that germs of points of singular symplectic varieties have relatively simple structures. Kaledin showed that any symplectic variety has natural stratification by locally closed connected symplectic submanifolds called symplectic leaves [K, Theorem 2.3]. In our case $F(X)$ is 4-dimensional and thus Kaledin's result particularly implies that the singular locus $\mathrm{Sing}(F(X))$ of $F(X)$ consists of a number of 2-dimensional components, which are symplectic, and a bunch of points. In fact, as we will see in Section 2, $\mathrm{Sing}(F(X))$ has pure dimension 2 and is divided into finitely many "bad" points $q_1, \ldots, q_k$ and the complementary open subset U. The theorem of Kaledin also tells us that the local structure of $F(X)$ at each point in U is the same as the product $\mathbb{C}^2 \times S$ where S is a surface with an ADE-singular point. Therefore, the problem is to study the structure near q_i's and relate it to the geometry of the singular cubic fourfold.

For a singular cubic fourfold X, projection from a singular point $p \in X$ gives a birational map $X \dashrightarrow \mathbb{P}^4$ (implying X is rational) and identifies the set of lines contained in X passing through p, which we denote by S_p, with a (2, 3) complete intersection in $\mathbb{P}^4$. It is also known that $F(X)$ is birational to the Hilbert scheme $\mathrm{Hilb}^2(S_p)$ (cf. Theorem 2.2). We will see that S_p's ($p \in \mathrm{Sing}(X)$) as subsets of $F(X)$ are exactly the irreducible components of $\mathrm{Sing}(F(X))$ (Proposition 2.3).

If X is smooth outside p, then S_p is a $K3$ surface of degree 6. However, if X has more than one (at worst simple) singular points, then S_p has ADE singularities and these singular points are the "bad" points $q_1, \ldots, q_k$ of $F(X)$. By a classical result, the ADE types of the these singular points in S_p can be read from the ADE types of singular points of X (Proposition 2.1). The aim of this paper is to determine the local structure near the singular points q_i of $F(X)$ (rather than of S_p).

In [Y], it is shown that $\mathrm{Hilb}^2(S)$ for a symplectic surface S with an ADE-singular point is also a symplectic variety and thus it has stratification as well. We will show that for each $q_i \in F(X)$ there is some bad point (namely, a 0-dimensional symplectic leaf) q in the singular locus of $\mathrm{Hilb}^2(S)$ such that the local structures near q_i and q are the same, i.e., the formal completions of the local rings at these points are isomorphic. The bad points q_i of $F(X)$ fall into two types: roughly speaking, type 1 comes from singularities of the blow-up, while type 2 corresponds to lines joining two singularities. Accordingly we will state two theorems (Theorems 1.1 and 1.2).

Each bad point of the first type corresponds to a certain line in X which passes through just one simple singular point of X. For each simple singular point p of X, the bad points of the first type whose corresponding lines pass through p are in bijection with the singular points of the blow-up of the type-T surface singularity S_T where T is the type of p. The bad points of the Hilbert scheme $\mathrm{Hilb}^2(S_T)$ are also in bijection with the singular points of the blow-up of S_T. When $T = \mathbf{A}_n\,(n \geq 3)$ or $\mathbf{E}_n\,(n = 6, 7, 8)$, the bad points are unique, and we denote the germ of these bad points in $\mathrm{Hilb}^2(S_T)$ by $\bar{\mathbf{A}}_n$ and $\bar{\mathbf{E}}_n$ respectively. When $T = \mathbf{D}_n\,(n \geq 5)$, the blow-up has one $\mathbf{A}_1$-singular point and one $\mathbf{D}_{n-2}$-singular point, and we denote the corresponding germs by $\bar{\mathbf{D}}_{n,\mathrm{I}}$ and $\bar{\mathbf{D}}_{n,\mathrm{II}}$ respectively. When $T = \mathbf{D}_4$, the blow-up has three $\mathbf{A}_1$-singular points. However, the germs of the corresponding points in $\mathrm{Hilb}^2(S_T)$ are the same, and we denote it by $\bar{\mathbf{D}}_{4,\mathrm{I}}$.

We show that the bad points of $F(X)$ of the first type have the same singularities as $\mathrm{Hilb}^2(S_T)$:

Theorem 1.1. *(=Theorem 2.5) Let $X \subset \mathbb{P}^5$ be a cubic hypersurface which does not contain a plane and has at worst simple singularities. Assume X has a simple singular point p of type T_n where $T = \mathbf{A}$, $\mathbf{D}$ or $\mathbf{E}$. If $\ell \in S_p \subset F(X)$ is a 0-dimensional symplectic leaf of $F(X)$ and if the corresponding line does not pass through any other singular points of X, then we have*

$$(F(X), \ell) = \begin{cases} \bar{\mathbf{A}}_n & \text{if} \quad T = \mathbf{A} \\ \bar{\mathbf{D}}_{n,\mathrm{I}} \text{ or } \bar{\mathbf{D}}_{n,\mathrm{II}} & \text{if} \quad T = \mathbf{D} \\ \bar{\mathbf{E}}_n & \text{if} \quad T = \mathbf{E}. \end{cases}$$

where whether being $\bar{\mathbf{D}}_{n,\mathrm{I}}$ or $\bar{\mathbf{D}}_{n,\mathrm{II}}$ depends on whether the germ (S_p, ℓ) is of type $\mathbf{A}_1$ or $\mathbf{D}_{n-2}$ respectively.

Each bad point of the second type corresponds to the line connecting two simple singular points p and p', or in other words it is the intersection $S_p \cap S_{p'}$. In this case the germs of such bad points are the same as the products of ADE-surface singularities:

Theorem 1.2. *(=Theorem 2.4) Let $X \subset \mathbb{P}^5$ be a cubic hypersurface not containing a plane. Assume X has two simple singular points p_1 and p_2 of type T_{n_1} and T_{n_2} respectively where $T = \mathbf{A}$, $\mathbf{D}$ or $\mathbf{E}$. Let $\ell \in F(X)$ be the point corresponding to the line connecting p_1 and p_2. Then the singularity type of $F(X)$ at ℓ is the same as that of the germ $(S_1 \times S_2, (o_1, o_2))$ where $(S_i, o_i)\,(i = 1, 2)$ is a germ of ADE surface singularity of type T_{n_i}.*

The main theorems above will be shown as a corollary to a slightly more general result (Theorem 3.1). This theorem says that a symplectic variety Z whose unique symplectic resolution is a symplectic manifold of $K3^{[2]}$-type has the same singularity type as $\mathrm{Hilb}^2(S)$. In the situation studied here, the assumption of containing no plane guarantees the uniqueness of symplectic resolution, and thus Theorem 3.1 applies. We expect that this assumption can be removed (Remark 3.3), but it seems that much more effort is necessary to drop this assumption.

To prove Theorem 3.1, we use the Bayer-Macrì theory about moduli spaces of Bridgeland stable objects on a twisted $K3$ surface (cf. [BM]). Then we can determine the structure of a special fiber of a symplectic resolution of Z and show that it is isomorphic to a special fiber for $\mathrm{Hilb}^2(S)$. By using the result in [Y], we deduce that Z and $\mathrm{Hilb}^2(S)$ have the same singularities.

The main theorems (Theorems 2.4 and 2.5) particularly imply that the local structure of singularities of $F(X)$ only depends on the ADE-types of simple singularities of X. In fact the correspondence of ADE-types of singularities between X and $F(X)$ can be regarded as a correspondence of the local monodromy groups associated to a degeneration of a smooth cubic fourfold to singular X. More precisely, such a degeneration gives finite monodromy groups both in the period domain of cubic fourfolds and in the deformation space of symplectic manifolds of the same numerical type as $F(X)$, and we will see that these monodromy groups coincide (Proposition 4.3).

This article is organized as follows. In Section 2 we review the general facts about $F(X)$ and in particular explain the structure of the singular locus of $F(X)$. In Section 3 we give proofs of the main theorems (Theorems 2.4 and 2.5). Finally, in Section 4, we discuss the period maps for cubic fourfolds and polarized $K3^{[2]}$-type manifolds, and review the main results from the viewpoint of degeneration of smooth varieties.

Acknowledgements
The author would like to thank Manfred Lehn for suggesting topics in this article, and also thank Yoshinori Namikawa for giving helpful advice. The author is also grateful to the anonymous referee for many suggestions to improve this article. This work was supported by JSPS KAKENHI Grant Number JP16J04485 and JP19K14504, and by World Premier International Research Center Initiative (WPI), MEXT, Japan.

2 Generalities on $F(X)$

Throughout this article all varieties are defined over the complex number field $\mathbb{C}$. By the *singularity type* (or the *analytic type*) of a variety at a point, we mean the isomorphism class of the formal completion of the variety at the point.

Let $X \subset \mathbb{P}^5$ be a cubic hypersurface. In this section we assume that X has only simple singularities in the sense of Arnold [Ar], i.e., every singular point p of X is an isolated singularity defined by one of the following equations

$$\begin{aligned}
\mathbf{A}_n &: x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^{n+1} \ (n \geq 1) \\
\mathbf{D}_n &: x_1^2 + x_2^2 + x_3^2 + x_4^2 x_5 + x_5^{n-1} \ (n \geq 4) \\
\mathbf{E}_6 &: x_1^2 + x_2^2 + x_3^2 + x_4^3 + x_5^4 \\
\mathbf{E}_7 &: x_1^2 + x_2^2 + x_3^2 + x_4^3 + x_4 x_5^3 \\
\mathbf{E}_8 &: x_1^2 + x_2^2 + x_3^2 + x_4^3 + x_5^5
\end{aligned}$$

for some analytically local coordinates $x_1, x_2, \ldots, x_5$ near p.

We fix a singular point $p \in X$ and assume that $p = [0:0:0:0:0:1] \in \mathbb{P}^5$ by a linear change of coordinates. Let $x_0, \ldots, x_5$ be the homogeneous coordinates of $\mathbb{P}^5$. By the condition that p is a singular point of X, one sees that X is defined by the following cubic polynomial

$$f(x_0, \ldots, x_5) = x_5 f_2(x_0, \ldots, x_4) + f_3(x_0, \ldots, x_4) \tag{2.1}$$

where $f_2(x_0, \ldots, x_4)$ and $f_3(x_0, \ldots, x_4)$ are homogeneous polynomials of degree 2 and 3 respectively.

Let

$$F(X) = \{\ell \in \mathrm{Grass}(1, \mathbb{P}^5) \mid \ell \subset X\}$$

be the Fano variety of lines on X where $\mathrm{Grass}(1, \mathbb{P}^5)$ is the Grassmannian of lines in $\mathbb{P}^5$. $F(X)$ contains a subvariety $S_p = \{\ell \in \mathrm{Grass}(1, \mathbb{P}^5) \mid p \in \ell \subset X\}$. The natural map from S_p to the hyperplane $H = \{x_5 = 0\} \subset \mathbb{P}^5$ given by $\ell \mapsto \ell \cap H$ is an embedding and one can check that the image in $H \cong \mathbb{P}^4$ with the coordinates $x_0, \dots, x_4$ is defined by $f_2 = f_3 = 0$.

From now on we assume that X contains no planes and in particular S_p contains no lines. It is shown in [O'G, Proposition 5.8(b)] and [Le, Lemma 3.1] that S_p is a normal surface with only ADE-singularities and that its minimal resolution is a $K3$ surface. We can determine the singularity type of S_p from that of X by the result of Wall. To state the result, one should note that the exceptional divisor E of the blowing-up $\widetilde{X}$ of X at p is isomorphic to the quadric $\mathcal{Q} = \{f_2 = 0\}$ in $\mathbb{P}^4$. We put $\mathcal{C} = \{f_3 = 0\} \subset \mathbb{P}^4$.

Proposition 2.1. *(C. T. C. Wall, [Wa, Theorem 2.1]) Let $q \in S_p = \mathcal{Q} \cap \mathcal{C}$ be a singular point. Then the following holds.*
(1) Either "$q \in \mathrm{Sing}(\mathcal{Q})$ and $q \notin \mathrm{Sing}(\mathcal{C})$" or "$q \notin \mathrm{Sing}(\mathcal{Q})$" holds.
(2) If $q \in \mathrm{Sing}(\mathcal{Q})$, then there does not exist a singular point of X on the line $\overline{pq}$ (here q is regarded as a point of X) other than p and q and the ADE-type of S_p at q is the same as that of $\widetilde{X}$ at the point of E that corresponds to q.
(3) If $q \notin \mathrm{Sing}(\mathcal{Q})$, then there exists exactly one singular point p' of X on $\overline{pq}$ other than p and q and the ADE-type of S_p at q is the same as that of X at p'.

By this proposition we can explicitly describe singular points of S_p. Let $\{p, p_1, \dots, p_k\}$ (k can be 0) be the set of all singular points of X and assume that the singularity types of these points are $T_n, T_{n_1}, \dots, T_{n_k}$ respectively where $T = \mathbf{A}, \mathbf{D}$ or $\mathbf{E}$. From the equations of simple singularities one knows that the corank of the quadric $\mathcal{Q}$ and the types of singular points of $\widetilde{X}$ on E are determined as in Table 1. In the table one should understand $\mathbf{A}_0$ and $\mathbf{D}_3$ as $\emptyset$ and $\mathbf{A}_3$ respectively.

Table 1: Degeneracy of quadrics and singularities of $\widetilde{X}$

T	corank of $\mathcal{Q}$	$\mathrm{Sing}(E)$	$\mathrm{Sing}(\widetilde{X}) \cap E$
$\mathbf{A}_1$	0 (nondegenerate)	$\emptyset$	$\emptyset$
$\mathbf{A}_n$ $(n \geq 2)$	1	point	$\mathbf{A}_{n-2}$
$\mathbf{D}_4$	2	line	$3\mathbf{A}_1$
$\mathbf{D}_n$ $(n \geq 5)$	2	line	$\mathbf{A}_1 + \mathbf{D}_{n-2}$
$\mathbf{E}_6$	2	line	$\mathbf{A}_5$
$\mathbf{E}_7$	2	line	$\mathbf{D}_6$
$\mathbf{E}_8$	2	line	$\mathbf{E}_7$

Note also that any line passing through two singular points of X is always contained in X since such a line intersects X with multiplicity at least 4 but X is cubic and the line must be in X by Bézout's theorem. By combining these with Proposition 2.1, one can conclude that S_p has k singular points of types $T_{n_1}, \ldots, T_{n_k}$ coming from intersections with other S_{p_i}'s and extra singular point(s) coming from "the blowing-up of T_n." For example, if $T_n = \mathbf{D}_4$, then S_p has 3 more singular points of type $\mathbf{A}_1$ other than the k singular points of types $T_{n_1}, \ldots, T_{n_k}$.

Next we introduce the results by C. Lehn about $F(X)$. An important property of $F(X)$ is that it is a symplectic variety in the sense of [B].

Theorem 2.2. *(C. Lehn, [Le, Theorem 3.3]) $F(X)$ is a singular symplectic variety birational to $\mathrm{Hilb}^2(S_p)$, the Hilbert scheme of 2 points on S_p.*

In the proof of the above theorem in [Le], a birational map $\mathrm{Hilb}^2(S_p) \dashrightarrow F(X)$ is explicitly constructed as follows. For an element $\xi \in \mathrm{Hilb}^2(S_p)$, the intersection of X and the linear span of p and ξ (regarded as points in the hyperplane $H \subset \mathbb{P}^5$) is the union of 3 lines since X is cubic and S_p does not contain a line by assumption. Two of the lines are the cone over ξ and the residual line gives an element of $F(X)$. Note that this map is well-defined even if ξ is supported at one point. Therefore, we have a birational morphism $\pi : \mathrm{Hilb}^2(S_p) \to F(X)$ and one can check that the indeterminacy locus of the rational map π^{-1} is S_p.

We can describe the singular locus of $F(X)$ by the following lemma.

Lemma 2.3. *The singular locus of $F(X)$ is $\bigcup_{p \in \mathrm{Sing}(X)} S_p$. Any two of the irreducible components S_p intersect at one point but no three of them intersect at one point.*

Proof. We first show that any line $\ell \in S_p$ for $p \in \mathrm{Sing}(X)$ is a singular point of $F(X)$. Since $F(X)$ is 4-dimensional, it suffices to show that $\dim T_\ell F(X) > 4$. By a linear change of coordinates, we may assume that ℓ is defined as $\{x_0 = x_1 = x_2 = x_3 = 0\}$ and $p = [0\!:\!0\!:\!0\!:\!0\!:\!0\!:\!1]$ in $\mathbb{P}^5$. Let $f \in H^0(\mathcal{O}_{\mathbb{P}^5}(3))$ be the defining equation of X. Then there exist unique $g_i \in H^0(\mathcal{O}_\ell(2))$, $i = 0, 1, 2, 3$ satisfying

$$f = \sum_{i=0}^{3} x_i g_i + h$$

where $h \in (x_0, x_1, x_2, x_3)^2$. We have the following exact sequence (see [EH, Chapter 6]):

$$0 \to \mathcal{N}_{\ell/X} \to \mathcal{O}_\ell^4(1) \xrightarrow{\theta} \mathcal{O}_\ell(3)$$

where $\theta = (g_0, g_1, g_2, g_3)$. Since p is a singular point, each g_i is in $x_4 \cdot H^0(\mathcal{O}_\ell(1))$ and thus $H^0(\theta)$ is not surjective. Therefore,

$$\dim T_\ell F(X) = \dim H^0(\mathcal{N}_{\ell/X}) > \dim H^0(\mathcal{O}_\ell^4(1)) - \dim H^0(\mathcal{O}_\ell(3)) = 8 - 4 = 4.$$

By the construction of $\pi : \mathrm{Hilb}^2(S_p) \to F(X)$, one sees that

$$U = \pi^{-1}(F(X) \setminus \bigcup_{p \in \mathrm{Sing}(X)} S_p)$$

consists of elements of $\mathrm{Hilb}^2(S_p)$ whose support does not intersect with singular points in S_p. Since the Hilbert scheme of points on a smooth surface is again smooth [Fo, Theorem 2.4], U and hence $\pi(U)$ are also smooth.

The last claims follow from the fact that the line passing through two singular points of X is contained X and that no three singular points of X are on the same line (see Proposition 2.1 and the argument in the paragraph after it). □

One purpose of this article is to investigate the germs of singular points of $F(X)$. In general Kaledin showed that taking singular loci of symplectic varieties gives stratification by symplectic submanifolds which we call symplectic leaves [K, Theorem 2.3]. The analytic type of a point ℓ in the 2-dimensional symplectic leaf (or, in other words, the smooth part of $\mathrm{Sing}(F(X))$) is just the same as the product of $\mathbb{C}^2$ and an ADE-singularity. However, when $\ell \in \mathrm{Sing}(S_p)$, the situation near ℓ in $F(X)$ is not so obvious. The main theorems concern the singularity type near ℓ. If ℓ is the intersection point of S_p and another S_{p_i}, the singularity type near ℓ is in fact the product of ADE-singularities. This is formulated as follows.

Theorem 2.4. *Let $X \subset \mathbb{P}^5$ be a cubic hypersurface not containing a plane. Assume X has two simple singular points p_1 and p_2 of type T_{n_1} and T_{n_2} respectively where $T = \mathbf{A}$, $\mathbf{D}$ or $\mathbf{E}$. Set $\{l\} = S_{p_1} \cap S_{p_2} \subset F(X)$. Then the singularity type of $F(X)$ at ℓ is the same as that of the germ $(S_1 \times S_2, (o_1, o_2))$ where (S_i, o_i) $(i = 1, 2)$ is a germ of ADE surface singularity of type T_{n_i}.*

This will be proved in the next section. If ℓ is not the intersection point, the singularity is a little more complicated. In this case the singularity of $\ell \in F(X)$ is the same as that of a certain point in the Hilbert scheme of 2 points on a singular surface. As we saw above, ℓ comes from the "blowing-up" of some T_n. Let S be a surface that has a unique singular point of type T_n. Then the singular locus of the Hilbert scheme $\mathrm{Hilb}^2(S)$ is isomorphic to the blowing-up of S at the singular point (cf. [Y, §2]). Thus $\mathrm{Hilb}^2(S)$ has a unique 0-dimensional symplectic leaf if $T = \mathbf{A}$ or $\mathbf{E}$. We denote the singularity type of this leaf by $\bar{\mathbf{A}}_n$ $(n \geq 3)$ or $\bar{\mathbf{E}}_n$ $(n = 6, 7, 8)$ respectively. When $T_n = \mathbf{D}_4$, there are three 0-dimensional symplectic leaves which have the same singularity types in $\mathrm{Hilb}^2(S)$. We denote it by $\bar{\mathbf{D}}_{4,\mathrm{I}}$. When $T_n = \mathbf{D}_n$ with $n \geq 5$, there are two 0-dimensional symplectic leaves. One is of type $\mathbf{A}_1$ in $\mathrm{Sing}(\mathrm{Hilb}^2(S))$ and the other of type $\mathbf{D}_{n-2}$. We denote the singularity types of them in $\mathrm{Hilb}^2(S)$ by $\bar{\mathbf{D}}_{n,\mathrm{I}}$ and $\bar{\mathbf{D}}_{n,\mathrm{II}}$ respectively. Thus the singularity types which can appear in $\mathrm{Hilb}^2(S)$ are listed as follows.

$$\bar{\mathbf{A}}_n\,(n \geq 3),\ \bar{\mathbf{D}}_{n,\mathrm{I}}\,(n \geq 4),\ \bar{\mathbf{D}}_{n,\mathrm{II}}\,(n \geq 5),\ \text{and}\ \bar{\mathbf{E}}_n\,(n = 6, 7, 8). \tag{2.2}$$

The claim about the singularity type of ℓ is stated as follows.

Theorem 2.5. *Let $X \subset \mathbb{P}^5$ be a cubic hypersurface not containing a plane. Assume X has a simple singular point p of type T_n where $T = \mathbf{A}$, $\mathbf{D}$ or $\mathbf{E}$. If $\ell \in S_p$ is a 0-dimensional symplectic leaf that is not contained in any other irreducible component of $\mathrm{Sing}(F(X))$, then we have*

$$(F(X), \ell) = \begin{cases} \bar{\mathbf{A}}_n & \text{if} \quad T = \mathbf{A} \\ \bar{\mathbf{D}}_{n,\mathrm{I}} \text{ or } \bar{\mathbf{D}}_{n,\mathrm{II}} & \text{if} \quad T = \mathbf{D} \\ \bar{\mathbf{E}}_n & \text{if} \quad T = \mathbf{E}. \end{cases}$$

where being whether $\bar{\mathbf{D}}_{n,\mathrm{I}}$ *or* $\bar{\mathbf{D}}_{n,\mathrm{II}}$ *depends on whether the germ* (S_p, ℓ) *is of type* $\mathbf{A}_1$ *or* $\mathbf{D}_{n-2}$ *respectively.*

This will also be proved in the next section.

3 Proof of the main theorems

In this section we will prove Theorems 2.4 and 2.5. We consider a slightly more general situation. Let Y be a projective irreducible holomorphic symplectic manifold of $K3^{[2]}$-type, i.e. deformation equivalent to the Hilbert scheme of two points on a $K3$ surface (see e.g. [GHJ, Part III]). We assume that we are given a projective birational morphism $\pi : Y \to Z$ where Z is a normal variety and $\pi_*\mathcal{O}_Y = \mathcal{O}_Z$. Then Z is a symplectic variety by definition. In addition, we assume π is a unique symplectic resolution (i.e. a resolution by a symplectic manifold) of Z.

Remark 3.1. As is well-known, symplectic resolutions of ADE singularities are the minimal resolutions and hence unique. However, in higher dimensions, symplectic resolutions are not unique in general. For example, the symmetric square of an ADE singularity is a 4-dimensional symplectic singularity having non-unique symplectic resolutions (see e.g. [Y]). In dimension 4, it is known that two different symplectic resolutions are related by a sequence of Mukai flops [WW, Theorem 1.2]. □

We will deduce Theorems 2.4 and 2.5 as a corollary to the following.

Theorem 3.1. *Let* $\pi : Y \to Z$ *be a contraction of an IHS* Y *of* $K3^{[2]}$*-type such that* Z *has a unique symplectic resolution as explained above. If a fiber* $\pi^{-1}(z)$ *of* $z \in Z$ *is two-dimensional, then the germ* (Z, z) *is isomorphic to* $(\mathrm{Hilb}^2(S), q)$ *for some* $q \in \mathrm{Hilb}^2(S)$ *where* S *is a surface with at worst (possibly more than one) ADE-singularities.*

The proof of this theorem will be done in this section by taking several steps. The important observation is that Y is a moduli space of Bridgeland stable objects in the derived category of a (twisted) $K3$ surface and thus its contractions are described in the framework of the Bayer-Macrì theory [BM].

Let us recall briefly the theory of Bridgeland moduli spaces over $K3$ surfaces. We refer to [BM, §2] for details. Let S be an algebraic $K3$ surface and $\alpha \in \mathrm{Br}(S)$ be a Brauer class. The pair (S, α) is called a twisted $K3$ surface. The cohomology group

$$H^*(S, \mathbb{Z}) = H^0(S, \mathbb{Z}) \oplus H^2(S, \mathbb{Z}) \oplus H^4(S, \mathbb{Z})$$

admits a weight-2 Hodge structure (which is the usual one on S when α is trivial) and we denote the integral $(1,1)$-part by $H^*_{\mathrm{alg}}(S, \alpha, \mathbb{Z})$. Every object A of $D^b(S, \alpha)$ defines its Mukai vector $v(A) \in H^*_{\mathrm{alg}}(S, \alpha, \mathbb{Z})$. $H^*_{\mathrm{alg}}(S, \alpha, \mathbb{Z})$ admits a symmetric bilinear form called the Mukai pairing. With this pairing, $H^*_{\mathrm{alg}}(S, \alpha, \mathbb{Z})$ becomes an even lattice of signature $(2, \rho)$ where ρ is the Picard number of S.

For a generic stability condition $\sigma \in \mathrm{Stab}^\dagger(S)$ and a primitive element v of $H^*_{\mathrm{alg}}(S, \alpha, \mathbb{Z})$, we can construct a moduli space $M_\sigma(v)$ of σ-stable objects in $D^b(S, \alpha)$ whose Mukai vectors are v. If $v^2 \geq -2$, then $M_\sigma(v)$ is of dimension $2n = v^2 + 2$ and is a projective holomorphic symplectic manifold of $K3^{[n]}$-type. The Néron-Severi group $NS(M_\sigma(v))$ with the Beauville-Bogomorov form can be identified with the orthogonal complement $v^\perp \subset H^*_{\mathrm{alg}}(S, \alpha, \mathbb{Z})$ as a lattice.

In general, a $K3^{[n]}$-type manifold M with $n \geq 2$ admits a lattice $\widetilde{\Lambda}(M) \supset H^2(M, \mathbb{Z})$ carrying a natural extension of the Hodge structure of $H^2(M, \mathbb{Z})$ such that the orthogonal complement $H^2(M, \mathbb{Z})^\perp$ is generated by a primitive element v' which is in the $(1,1)$-part with $v'^2 = \dim M - 2$ [Ma1, §9]. It is also known that two $K3^{[n]}$-type manifolds M and M' are birational if and only if there is a Hodge isometry of $\widetilde{\Lambda}(M)$ and $\widetilde{\Lambda}(M')$ which maps $H^2(M, \mathbb{Z})$ isomorphically to $H^2(M', \mathbb{Z})$. If M is a moduli space of Bridgeland stable objects on (S, α), then we have an isomorphism $H^*(S, \alpha, \mathbb{Z}) \cong \widetilde{\Lambda}(M)$ which sends the Mukai vector to v'.

Now we return to the situation in Theorem 3.1.

Lemma 3.2. *Y is a moduli space of Bridgeland stable objects on a twisted $K3$ surface (S, α) for some Mukai vector $v \in H^*_{\mathrm{alg}}(S, \alpha, \mathbb{Z})$ with $v^2 = 2$ (and for a suitable stability condition).*

Proof. By [BM, Theorem 1.2(c)], it suffices to show that Y is birational to a moduli space of Bridgeland stable objects. Let $\widetilde{\Lambda}_{\mathrm{alg}}(M)$ be the integral $(1,1)$-part of $\widetilde{\Lambda}(M)$. By [Hu1, Lemma 2.6] and the argument of the proof of [Hu1, Proposition 4.1] (see also [Ad, Proposition 4]), we only have to show that $\widetilde{\Lambda}_{\mathrm{alg}}(Y)$ contains a sublattice that is isomorphic to a nonzero multiple of the hyperbolic plane $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

We first claim that π is a divisorial contraction. Otherwise π would contract a $\mathbb{P}^2$ [WW, Theorem 1.1]. However, this is contrary to the uniqueness assumption of symplectic resolutions since any $\mathbb{P}^2$ can be flopped to give a different symplectic resolution.

Let $\delta' \in NS(Y) \subset \widetilde{\Lambda}_{\mathrm{alg}}(Y)$ be the class of a prime π-exceptional divisor E. Note that δ' is orthogonal to the pullbacks of the divisors of Z. Since the signature of the Beauville-Bogomorov form on $NS(Y)$ is $(1, \rho(Y) - 1)$ and the pullback H of an ample divisor of Z satisfies $H^2 > 0$, we have $\delta'^2 < 0$. Write $\delta' = m\delta$ with $m \in \mathbb{N}$ and primitive $\delta \in \widetilde{\Lambda}_{\mathrm{alg}}(Y)$. Then $\delta^2 = -2$ by [Ma2, Theorem 1.2]. We can take a primitive generator v' of $H^2(Y, \mathbb{Z})^\perp$ in $\widetilde{\Lambda}(Y)$ as mentioned above. Then $v' + \delta$ and $v' - \delta$ generate a lattice which is isomorphic to $U(4)$. □

Let $E_1, \ldots, E_n \subset Y$ be the irreducible exceptional divisors of π.

Lemma 3.3. *The relative Picard number of π is equal to n (=the number of the exceptional divisors).*

Proof. Since π is a projective symplectic resolution, Y is a Mori dream space relative to Z [AW, §3]. Although the case when Z is affine is treated there, this difference will give no effect on the results except that π can have more than one 2-dimensional fibers.

By [AW, Theorem 4.1], the classes of $E_1, \ldots, E_n$ in the relative Picard group $N^1(Y/Z)$ are linearly independent. Since π is the unique symplectic resolution, the (relative) nef cone and the movable cone of π are the same. Therefore, the movable cone is strictly convex and the exceptional divisors generate $N^1(Y/Z)$ [AW, Theorem 3.6]. Thus the claim follows. □

Lemma 3.4. *Each E_i is a $\mathbb{P}^1$-bundle over the $K3$ surface S in Lemma 3.2. Here a $\mathbb{P}^1$-bundle means that every fiber of $E_i \to S$ is isomorphic to $\mathbb{P}^1$.*

Proof. Let $e_i \in N_1(Y/Z)$ be the numerical class of a general fiber of $\pi|_{E_i}$. Then the facets of the (relative) movable cone of π are formed by the orthogonal hyperplanes of e_i's with respect to the intersection pairing [AW, Theorem 3.6]. Since the nef cone coincides with the movable cone, we can choose a π-nef divisor D such that $(D.e_i) = 0$ and $(D.e_j) \neq 0$ for $j \neq i$. Then the linear system $|mD|$ for $m \gg 0$ gives a contraction $\pi_i : Y \to Z_i$ which contracts the single divisor E_i.

As shown in Lemma 3.2, Y is a Bridgeland moduli space for a $K3$ surface S for some Mukai vector v, and therefore any elementary (meaning relative Picard number 1) contractions are realized by "wall-crossing." The complex manifold $\mathrm{Stab}^\dagger(S)$ has wall-and-chamber structure which governs the birational geometry of Y. Any wall $\mathcal{W}$ is a codimension one real submanifold of $\mathrm{Stab}^\dagger(S)$ and is associated to a rank-two sublattice $\mathcal{H}$ of $H^*_{\mathrm{alg}}(S, \alpha, \mathbb{Z})$ (see [BM, §5] for details). Elementary divisorial contractions are classified in [BM, Theorem 5.7] in terms of $\mathcal{H}$. One can check that, in the 4-dimensional case, there are two types depending on whether or not $\mathcal{H}$ contains an isotropic element w (i.e. $w^2 = 0$) such that $(v.w) = 1$ [BM, Lemma 8.7, 8.8] (see also [HT, §5.1]). In both cases, the exceptional divisor is a $\mathbb{P}^1$-bundle over S. □

Proposition 3.5. *Any 2-dimensional fiber $\pi^{-1}(z)$ is isomorphic to a 2-dimensional fiber of a unique symplectic resolution of* $\mathrm{Hilb}^2(S)$ *for some S as in Theorem 3.1.*

Proof. We first consider the structure of a two-dimensional fiber for $\mathrm{Hilb}^2(S)$.

When S has one singular point, a symplectic resolution of $\mathrm{Hilb}^2(S)$ are studied and shown to be unique in [Y, §2]. Also, the fibers of the 0-dimensional symplectic leaves, which are listed in (2.2), are explicitly described there. (They are isomorphic to some Springer fibers, see [Y, §4] and [Lor].) In any case the fiber consists of irreducible components which are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ or the second Hirzebruch surface $\Sigma_2 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2))$.

When S has at least two singular points p_1 and p_2 of type T_{n_1} and T_{n_2} respectively where $T = \mathbf{A}, \mathbf{D}$, or $\mathbf{E}$. Then $\mathrm{Hilb}^2(S)$ has additional 0-dimensional symplectic leaves of the form $\{p_1, p_2\}$ whose analytic type is the same as that of the product singularity $T_{n_1} \times T_{n_2}$. Thus its fiber of the unique symplectic resolution is the product of the two Dynkin trees of projective lines.

We will show that the fiber $F = \pi^{-1}(z)$ is isomorphic to one of the reducible surfaces mentioned above. To do this, we consider the intersection of two π-exceptional prime divisors.

Set $\mathrm{Exc}(\pi) = E_1 \cup E_2 \cup \cdots \cup E_n$. Then a general fiber of $\pi|_{\mathrm{Exc}(\pi)}$ is a Dynkin tree of $\mathbb{P}^1$'s. We will show that a general fiber C_i of $\pi|_{E_i}$ is irreducible, i.e., it is a single $\mathbb{P}^1$ for each i. Note that, for general symplectic resolutions, it can happen that a general fiber of a single exceptional divisor is reducible. Such a phenomenon is caused by an automorphism of a Dynkin diagram and of the corresponding general fiber. Possible cases are classified in [Wi, Theorem 1.3]. Each case is presented as a pair of the type of a Dynkin diagram and a group of automorphisms of the diagram. In our setting, the case $(\mathbf{A}_{2l}, \mathbb{Z}/2\mathbb{Z})$ cannot happen since each π_i is a $\mathbb{P}^1$-bundle by the previous lemma. Now we assume that there exists i such that C_i is reducible in order to deduce contradiction. Then π is in one of the four cases: $(\mathbf{A}_{2l+1}, \mathbb{Z}/2\mathbb{Z})$, $(\mathbf{D}_l, \mathbb{Z}/2\mathbb{Z})$, $(\mathbf{E}_6, \mathbb{Z}/2\mathbb{Z})$, and $(\mathbf{D}_4, \mathfrak{S}_3)$. Note that these four cases correspond to the root systems of simple Lie algebras of non-simply laced types i.e. $\mathbf{B}_{l+1}, \mathbf{C}_{l-1}, \mathbf{F}_4$, and $\mathbf{G}_2$ respectively. In any case there are i_1 and i_2 such that

- $C_{i_1} \cong \mathbb{P}^1$, and $C_{i_2} \cong \mathbb{P}^1 \sqcup \mathbb{P}^1$ or $\mathbb{P}^1 \sqcup \mathbb{P}^1 \sqcup \mathbb{P}^1$
- C_{i_1} intersects with each $\mathbb{P}^1$ in C_{i_2} at one point.

Let $\pi_{i_2} : Y \to Z_{i_2}$ be the contraction of E_{i_2} and $\theta : Z_{i_2} \to W$ the contraction of E_{i_1}. Such θ also exists by the same argument in the proof of Lemma 3.4. Then the restriction of θ to $\pi_{i_2}(E_{i_2}) \cong S$ is generically a covering map $S \to S'$ with the covering transformation group $G \cong \mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/3\mathbb{Z}$ according to the number of the components of C_{i_2}. Note that the G-action on an open subset of S extends to the whole of S since a $K3$ surface is a minimal surface. Thus the map $S \to S'$ can be regarded as a quotient map by G. Since E_{i_1} is also a $\mathbb{P}^1$-bundle over S, the image S' is birational to S. We have the following diagram

$$\begin{array}{ccc} & & S \\ & & \downarrow r \\ S & \xrightarrow{q} & S' = S/G \end{array}$$

where q is the quotient map and r is the minimal resolution. Let M be the sublattice of $H^2(S, \mathbb{Z})$ generated by the exceptional curves of r. Then q naturally induces the pullback $q^* : H^2(S, \mathbb{Z}) \to H^2(S, \mathbb{Z})$ whose image is the invariant part $H^2(S, \mathbb{Z})^G$ (cf. [Mo, Lemma 3.1] and [Wh, Theorem 2.1]). Note that the restriction $q^*|_{M^\perp}$ to the orthogonal complement $M^\perp \subset H^2(S, \mathbb{Z})$ is injective, although it is multiplies the intersection form by $\sharp G$. Since the quotient of S by G is birational to a $K3$ surface, G preserves the symplectic form by [Hu2, Chapter 15, Lemma 4.8]. Therefore, the transcendental lattice $T_S = NS(S)^\perp \subset H^2(S, \mathbb{Z})$ is inside the invariant part $H^2(S, \mathbb{Z})^G$. Then q^* induces an isomorphism $T_S \cong T_S \subset H^2(S, \mathbb{Z})^G$. This is absurd since the same lattices T_S would have different discriminants. Thus we have shown that every C_i is irreducible.

Next, let us take two divisors E_{i_1}, E_{i_2} and general fibers C_{i_1}, C_{i_2} of $\pi|_{E_{i_1}}$ and $\pi|_{E_{i_2}}$ respectively such that C_{i_1} and C_{i_2} intersect. Then the restriction of θ to $\pi_{i_2}(E_{i_2}) \cong S$ is birational and hence an isomorphism. This implies that $E_{i_1} \cap E_{i_2}$ is a section of both of the $\mathbb{P}^1$-bundles $\pi|_{E_{i_1}}$ and $\pi|_{E_{i_2}}$. We say that such E_{i_1} and E_{i_2} are *adjacent*. If E_i and E_j satisfy $\pi(E_i) = \pi(E_j)$ (or equivalently E_i can reach E_j via adjacent divisors), we say that E_i and E_j are in the same *bunch*.

Next we consider intersections of non-adjacent divisors. Let E_i and E_j be two prime π-exceptional divisors which are not adjacent. We show that each connected component of $V := E_i \cap E_j$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ unless it is empty. For this, take any point $y \in V$ and let $l_i \subset E_i$ and $l_j \subset E_j$ be the unique fibers of the $\mathbb{P}^1$-bundles $\pi|_{E_i}$ and $\pi|_{E_j}$ passing through y. Since E_i and E_j are not adjacent, the intersection numbers $(l_i.E_j)$ and $(l_j.E_i)$ are zero. This implies that $l_i \subset E_j$ and $l_j \subset E_i$. This also implies that V admits two different $\mathbb{P}^1$-bundle structures $\pi_i|_V$ and $\pi_j|_V$. Since every irreducible component V_0 of V is 2-dimensional, the image $C := \pi_i|_{V_0}$ is an irreducible curve. Note that C is a rational curve since a rational curve on V_0 (like l_j) surjects to C. We see that C is smooth since

$C \subset S$ is a curve in a $K3$ surface contracted by π. Therefore, V_0 is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$, namely, a Hirzebruch surface. We can apply the same argument for π_j and can conclude that $V_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$. We can also show that each connected component of V is irreducible since otherwise an intersection point of two irreducible components would have a bigger π_i-fiber than $\mathbb{P}^1$. This proves the claim.

If $E_i \cap E_j$ are empty for all pairs of non-adjacent divisors, then we have $\pi(\mathrm{Exc}(\pi)) \cong S$ and in particular $\pi(\mathrm{Exc}(\pi)) \subset Z$ has no 0-dimensional leaves. So we assume otherwise. Then there are two exceptional divisors, say E_1 and E_2, such that a 2-dimensional fiber $F = \pi^{-1}(z)$ contains a surface $P \subset E_1 \cap E_2$ which is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. The formal neighborhood of P in Y is isomorphic to that of the zero section in the cotangent bundle of $\mathbb{P}^1 \times \mathbb{P}^1$ [Y, Lemma 3.2]. Similarly to the argument in the proof of the previous lemma, taking a suitable divisor from the boundary of the movable cone of π gives a contraction ϕ of Y that contracts exactly E_1 and E_2. Then ϕ is identified with the natural contraction (affinization)

$$T^*(\mathbb{P}^1 \times \mathbb{P}^1) \cong T^*\mathbb{P}^1 \times T^*\mathbb{P}^1 \to (\mathbf{A}_1\text{-sing.}) \times (\mathbf{A}_1\text{-sing.}) \tag{3.1}$$

in the formal neighborhood of P.

Let E be another π-exceptional prime divisor that is not adjacent to both E_1 and E_2. If E intersects with P, then E must contain P since the intersection numbers of any fibers of π_1 and π_2 with E are zero. However, this is a contradiction since $E_i\,(i = 1, 2)$ and E do not share fibers of their $\mathbb{P}^1$-bundles. On the other hand, if E is adjacent to E_1 or E_2, then it always intersects with P.

Case 1. E is adjacent to both E_1 and E_2.
In this case the intersection $C = E \cap P$ is a section of P with respect to both $\pi_1|_P$ and $\pi_2|_P$. Therefore, it must be a diagonal (i.e. an irreducible curve with $C^2 = 2$) in P. By using normal bundle sequences and taking the fact that $N_{P/Y} \cong \Omega^1_P$ into consideration, one sees that $N_{C/Y} \cong \mathcal{O}(2) \oplus \mathcal{O}(-2)^{\oplus 2}$.

On the other hand E contains a $\mathbb{P}^1$-bundle $\Sigma \subset F$ over C, and Σ contains C as a negative section. By the calculation of the normal bundle above, we see that Σ is isomorphic to Σ_2.

Case 2. E is adjacent to just one divisor, say E_1, of E_1 and E_2.
In this case the intersection $C = E \cap P$ is the section of $\pi_1|_P$. Since the intersection number of any fiber of π_2 and E is zero, C must be a fiber of $\pi_2|_P$. Thus in this case $N_{C/Y} \cong \mathcal{O}(2) \oplus \mathcal{O}^{\oplus 2}$, and the corresponding $\mathbb{P}^1$-bundle $\Sigma' \subset E \cap F$ over C is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Note that Σ' is also contained in E_2. Indeed, the rulings of $\Sigma' \cong \mathbb{P}^1 \times \mathbb{P}^1$ of the other direction than $E \to S$ are numerically equivalent to $C \subset E_2$. Thus $\Sigma' \subset E_2 \cap E \cap F$ is in a similar situation to $P \subset E_1 \cap E_2 \cap F$.

Next we consider the intersection of Σ in **Case 1** and another exceptional divisor. The same argument as above shows that, if a π-exceptional prime divisor E' other than E_1, E_2 and E intersects with Σ, it is adjacent to one of E, E_1 and E_2. When E' is adjacent to E, such E' is unique because of the shapes of the Dynkin diagrams. Since the intersection number of fibers of π_1 and π_2 with E' are zero (otherwise a closed path would appear in a Dynkin diagram), the two sections C and $C' := \Sigma \cap E'$ are disjoint and thus C' is a section of $\Sigma \to \mathbb{P}^1$ with $C'^2 = 2$. By the normal bundle argument as above shows that the $\mathbb{P}^1$-bundle $\Sigma'' \subset E' \cap F$ over C' is isomorphic to Σ_2.

When E' is adjacent to E_1 or E_2, we see that the situation is similar to **Case 2** and Σ intersects along its fiber with (a connected component of) $E \cap E' \cap F$ which is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

Let us consider again the case when E' is adjacent to E. We can describe the intersections of Σ'' above with other exceptional divisors in a similar way: any π-exceptional prime divisor E'' other than E and E' which intersects with Σ'' is adjacent to one of E', E_1 and E_2. When E'' is adjacent to E', the intersection C'' is a section which is disjoint from C' with $C''^2 = 2$. E'' contains the $\mathbb{P}^1$-bundle over C'' which is isomorphic to Σ_2 and contains C'' as the (-2)-curve. When E'' is adjacent to E_i, the component Σ'' intersects along its fiber with (a connected component of) $E' \cap E'' \cap F$ which is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

Since we have considered all intersections with the exceptional divisors starting from P, every irreducible component of F will appear in the above procedures. If all exceptional divisors are in the same bunch, one can check that the resulting F, up to isomorphisms, fall into one of the types in the list (2.2) (see [Y, §2] or [Lor]). In particular, we see that every component of F which is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ is of the form $E_i \cap E_j \cap F$ for some i, j.

As an example, let us consider the case when a general fiber is of type $\mathbf{D}_n$ $(n \geq 4)$. Let $E_1, \ldots, E_n$ be the irreducible π-exceptional divisors intersecting with F such that E_i corresponds to the i-th vertex of the Dynkin diagram in Figure 1.

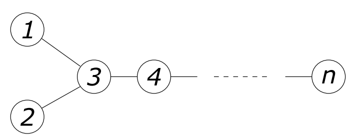

Figure 1: The Dynkin diagram of type $\mathbf{D}_n$

We first assume that $E_1 \cap E_2$ contains a 2-dimensional component P of F. Then P is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, and E_3 is a unique divisor that intersects with P. Also, E_3 contains a $\mathbb{P}^1$-bundle Σ over a diagonal of P (see **Case 1**). E_4 is a unique divisor that intersects with Σ and contains $\mathbb{P}^1$-bundle Σ' over a curve C of Σ with $C^2 = 2$. After all, we will obtain a chain of one $\mathbb{P}^1 \times \mathbb{P}^1$ and $n-2$ copies of Σ_2. This corresponds to $\bar{\mathbf{D}}_{n,\mathrm{I}}$.

Next we consider the case when $E_1 \cap E_2$ does not contain a 2-dimensional component of F. When $n = 4$, we can reduce to the previous case by symmetry. We assume $n \geq 5$. In this case one can see that every pair of non-adjacent divisors intersect along a 2-dimensional component of F. Indeed, for any pair of such 2-dimensional components $P_{i,j} \subset E_i \cap E_j$, there is a chain of $P_{i,j}$'s that connects the pair. For example, $P_{1,4}$ and $P_{2,4}$ are connected by the chain $P_{1,4}$, $P_{1,5}$, $P_{3,5}$, $P_{2,5}$, $P_{2,4}$. Finally we will find that F corresponds to $\bar{\mathbf{D}}_{n,\mathrm{II}}$.

To finish the classification, let us consider the case when F is contained in more than one bunches of exceptional divisors. In this case there are two divisors, say E_1 and E_2, that are not in the same bunch and $E_1 \cap E_2$ contains a 2-dimensional component P of F.

P must be isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ as discussed. Note that the number of bunches cannot be more than two since otherwise another exceptional divisor in a third bunch would contain P, which is a contradiction as explained before **Case 1**. We can use the same method as above to produce the other components of F. In this case all components of F are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ since no divisors are adjacent to both E_1 and E_2 and thus **Case 1** will not happen. It is easy to see that the resulting F is isomorphic to the product of two Dynkin trees of $\mathbb{P}^1$. This completes the proof. □

Proof of Theorem 3.1

By Proposition 3.5, we have shown that (Z, z) and $(\mathrm{Hilb}^2(S), q)$ have isomorphic 2-dimensional fibers in their unique symplectic resolutions for some q. When the fiber is of one of the types in the list (2.2), this implies that (Z, z) and $(\mathrm{Hilb}^2(S), q)$ have the same singularity types [Y, Theorem 3.4].

In the product cases, we will prove the claim using the results in [Y, §3]. Every irreducible component of $F \subset Y$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and thus its formal neighborhood is always isomorphic to the completion of the cotangent bundle along the zero section. We will show that the isomorphism class of the formal neighborhood of F is also determined uniquely.

As we have seen, the fiber $F \subset Y$ is contained in two bunches of exceptional divisors. Let $B_1 = E_1 \cup \cdots \cup E_k$ and $B_2 = E'_1 \cup \cdots \cup E'_l$ be the two bunches. There are contractions π_{B_1} and π_{B_2} of B_1 and B_2 respectively. Then the images $S_1 := \pi_{B_1}(B_1)$ and $S_2 := \pi_{B_2}$ are isomorphic to the minimal resolutions of ADE-singularities. Let T_l (resp. T_k) be the ADE-type of S_1 (resp. S_2). The unique symplectic resolution of the product of two ADE-singularities of types T_l and T_k also admits two bunches of divisors and corresponding surfaces S'_1 and S'_2 which are isomorphic as symplectic varieties to S_1 and S_2 respectively in the formal neighborhoods of the exceptional curves.

We fix two symplectic (or equivalently, Poisson) isomorphisms $S_1 \cong S'_1$ and $S_2 \cong S'_2$. Then, by applying [Y, Lemma 3.5 and Remark 3.4] repeatedly, we obtain a (non-canonical) isomorphism of the formal neighborhoods of central fibers of Y and the product resolution which induces the given isomorphisms $S_i \cong S'_i$. This implies that the germ (Z, z) is isomorphic to the product of two ADE-singularities. As we already see in the proof of Proposition 3.5, such a germ is also presented as $(\mathrm{Hilb}^2(S), q)$ for some S and q. □

Remark 3.2. As we will see below, Theorem 3.1 can apply to $F(X)$. The proof of this theorem shows that three or more irreducible components of the singular locus $\mathrm{Sing}(Z)$ of Z do not intersect at one point. This was discussed for $F(X)$ in Lemma 2.3. Moreover, from the proof we also know that all irreducible components of $\mathrm{Sing}(Z)$ are birational to each other. This is also a generalization of the result for $F(X)$ in [O'G, Proposition 5.8(c)]. □

Proof of Theorems 2.4 and 2.5

As mentioned in the previous section, we have a birational morphism $\mathrm{Hilb}^2(S_p) \to F(X)$. Since S_p is a surface with only ADE-singularities, $\mathrm{Hilb}^2(S_p)$ has a symplectic resolution Y. By [GHJ, Theorem 27.8], we see that Y is of $K3^{[2]}$-type.

To show that $Y \to F(X)$ is a unique symplectic resolution, it suffices to show that two-dimensional fibers do not contain $\mathbb{P}^2$ since two different symplectic resolutions are related by a sequence of Mukai flops [WW, Theorem 1.2]. We already know that the fibers of $Y \to \mathrm{Hilb}^2(S_p)$ do not contain $\mathbb{P}^2$. One can check that the fiber of $\mathrm{Hilb}^2(S_p) \to F(X)$ of a point $\ell \in S_p \subset F(X)$ can be identified with the set of lines passing through p that are contained in the quadric $\mathcal{Q}$. Thus it is 1-dimensional or isomorphic to a non-degenerate or corank-1 quadric in $\mathbb{P}^3$. Therefore, no irreducible components which is isomorphic to $\mathbb{P}^2$ cannot appear in the fiber.

Now we can apply Theorem 3.1 to $Y \to F(X)$. Let $\ell \in F(X)$ be a 0-dimensional symplectic leaf. If $\mathrm{Sing}(F(X))$ is reducible at ℓ, then we fall into the product case. The types of S_i's are determined by the types of ℓ in S_{p_i}'s, which are T_{n_i}'s. This proves Theorem 2.4.

If $\mathrm{Sing}(F(X))$ is irreducible at ℓ, then we fall into one of the types in the list (2.2). In this case, however, the types in (2.2) are not made distinct just by looking at the type of ℓ in S_p. For example, $\bar{\mathbf{A}}_7$ and $\bar{\mathbf{E}}_6$ both have $\mathbf{A}_5$ as the types in the 2-dimensional symplectic leaves. By taking the number of the irreducible exceptional divisors of $Y \to F(X)$ also into account, we can uniquely determine the type of $\ell \in F(X)$. Note that the resolution $Y \to F(X)$ factors through $\mathrm{Hilb}^2(S_p)$, and thus we see that the number of the irreducible exceptional divisors is equal to n for type T_n. This proves Theorem 2.5. □

Remark 3.3. 1. The correspondence of the ADE-types of singularities of X and $F(X)$ can also be explained in terms of local monodromy groups associated to deformation spaces of X and $F(X)$. This will be discussed in Subsection 4.2.

2. The assumption that X does not contain a plane is used only for the uniqueness of a symplectic resolution of $F(X)$. Recall that the uniqueness of the resolution enables us to determine the structure of the exceptional locus of the resolution (cf. Lemma 3.4, Proposition 3.5) and that this structure characterizes the singularity of $F(X)$. The above proof of the main results shows the uniqueness of the resolution by analyzing the structure of the birational morphism $\pi : \mathrm{Hilb}^2(S_p) \to F(X)$. If X contains a plane passing through a singular point $p \in X$, however, π is just a birational map which is undefined on that plane, and it seems much more difficult to study the structure of fibers of a symplectic resolution of $F(X)$ in this case. Note that the existence of a symplectic resolution of $F(X)$ is guaranteed (see e.g. [KLSV, Remark 5.4]).

3. It is still expected that $F(X)$ has the same singularities as the Hilbert square of ADE singularities even if X contains a plane since $F(X)$ is always a symplectic variety of local complete intersection (in the Grassmannian $\mathrm{Grass}(1, \mathbb{P}^5)$): As far as the author knows, all known symplectic singularities of complete intersection are the nilpotent cones in semisimple Lie algebras and their Slodowy slices (see e.g. [LNS] for definitions). The singularities of the Hilbert square of ADE singularities are exactly those of the 4-dimensional Slodowy slices for the Lie algebras of simply-laced types [Y, §4]. Note that being of simply-laced types corresponds to the triviality of the graph automorphism of the Dynkin diagram of type ADE associated to a symplectic resolution (see the proof of Proposition 3.5). □

4 Discussion: $F(X)$ as a degeneration of $K3^{[2]}$-type manifolds

Cubic fourfolds with at worst simple singularities are considered as mild degenerations of smooth cubic fourfolds. Indeed, such cubic fourfolds are stable in the sense of geometric invariant theory (GIT) [La1] and thus they behave as smooth ones. Such degenerations are also characterized as the ones having finite monodromy groups [La2] and usually called Type I.

Accordingly, the Fano varieties of lines on such cubic fourfolds are also degenerations of holomorphic symplectic manifolds of $K3^{[2]}$-type since the assignment $X \mapsto F(X)$ can work in a family of cubic fourfolds. Such a degeneration is treated in §6 of [KLSV], and the authors show that, in general, a degeneration of Type I (i.e. with finite monodromy) of projective hyper-Kähler manifolds admits a smooth filling (after base change and birational modifications). A smooth filling in the case of the degeneration of $F(X)$ is given by a symplectic resolution (which is unique under our assumption) of singular $F(X)$. For this reason, note that taking resolutions of a degeneration of $F(X)$ will never produce IHS manifolds of new deformation types.

In this section we discuss mild degenerations of cubic fourfolds and their Fano varieties, and try to understand the main results (Theorems 2.4, 2.5) from the viewpoint of such degenerations in the period domain.

4.1 Finite monodromy degenerations

First let us recall the basic facts briefly about the moduli space and the period domain of cubic fourfolds (cf. e.g. [Ha]). Let $V = \mathbb{C}^6$. The moduli space $\mathcal{C}^0$ of smooth cubic fourfolds $X \subset \mathbb{P}(V)$ is obtained by taking the (GIT-)quotient of an open set of the vector space $\mathrm{Sym}^3(V^*)$ of cubic polynomials by the action of $PGL(V)$. For each $X \in \mathcal{C}^0$, the 4th cohomology group $H^4(X, \mathbb{Z})$ becomes a lattice of rank 23 with respect to the intersection form. Let L_0 be a lattice which is isomorphic to the orthogonal complement $(h^2)^\perp \subset H^4(X, \mathbb{Z})$ of the square h^2 of the class of the hyperplane of X. Then we can define the local period domain $\mathcal{D} \subset \mathbb{P}(L_0 \otimes \mathbb{C})$ as a connected open subset of a quadric hypersurface. Let Γ be the orthogonal group of L_0 which preserves the connected component $\mathcal{D}$. Then the global period domain $\mathcal{D}/\Gamma$ is a 20-dimensional quasi-projective variety. The Hodge structure of X gives a 1-dimensional subspace $H^{3,1}(X) \subset (h^2)^\perp \otimes \mathbb{C} \cong L_0 \otimes \mathbb{C}$, and we can define the well-defined period map $p : \mathcal{C}^0 \to \mathcal{D}/\Gamma$, which assigns a cubic fourfold X to its period $H^{3,1}(X) \in \mathbb{P}(L_0 \otimes \mathbb{C})$. Voisin showed that this map is an open immersion [V], and in particular the global Torelli theorem holds like the case of $K3$ surfaces.

However, p is not surjective. Laza and Looijenga showed that the complement of the image of p consists of two irreducible divisors $\mathscr{H}_\Delta$ and $\mathscr{H}_\infty$ and that $p : \mathcal{C}^0 \to \mathcal{D}/\Gamma$ extends to a morphism $\bar{p} : \mathcal{C} \to \mathcal{D}/\Gamma$ whose image is the complement of $\mathscr{H}_\infty$ where $\mathcal{C} \supset \mathcal{C}^0$ is the moduli space of cubic fourfolds with at worst simple singularities [La2, Theorem 1.1], [Loo, Theorem 4.1]. The following proposition suggests that cubic fourfolds with at worst simple singularities are especially mild degenerations.

Proposition 4.1. *$F(X)$ is not a symplectic variety for $X \in \bar{\mathcal{C}} \setminus \mathcal{C}$ where $\bar{\mathcal{C}}$ is the GIT compactification of $\mathcal{C}$.*

Proof. We use the explicit description of semistable cubic fourfolds in [La1]. In order for $F(X)$ with $X \in \bar{\mathcal{C}}$ to be a symplectic variety, the normalization of each irreducible component of $\operatorname{Sing} F(X) = \bigcup_{p \in \operatorname{Sing} X} S_p$ must be a (symplectic) surface with ADE singularities. For this, singularities of X must be isolated since otherwise $\dim(\operatorname{Sing} F(X)) \geq 3$. Moreover, if X contains an isolated singular point p which is not of type ADE and another singular point q, then a singular point of the same type as $p \in X$ would appear in S_q (cf. Proposition 2.1) and thus $F(X)$ would not be a symplectic variety. Combining these facts with the classification in [La1], we see that X must be in $\mathcal{C}$. □

Since degeneration of a cubic fourfold X (inside $\mathcal{C}$) gives degeneration of a symplectic variety $F(X)$, it is natural consider the period domain of $F(X)$. In fact, $\mathcal{D}/\Gamma$ is identified with the period domain of $F(X)$. Indeed, Beauville and Donagi showed that the Abel-Jacobi map

$$\alpha : H^4(X, \mathbb{Z}) \to H^2(F(X), \mathbb{Z})$$

is a Hodge isomorphism of type $(-1,-1)$ [BD]. α also preserves the intersection form and the Beauville-Bogomorov form on the both sides up to sign, and it maps h^2 to the natural polarization g of $F(X)$ which comes from the Plücker embedding. Similarly to the case of cubic fourfolds, we can construct a moduli space $\mathcal{M}$ of polarized irreducible holomorphic symplectic manifolds with the same numerical invariants as $F(X)$ (cf. [DM]). Also, we have the period map $p' : \mathcal{M} \to \mathcal{D}/\Gamma$, and Verbitsky proved that the global Torelli theorem holds, that is, p' is an open immersion [DM, Theorem 3.2]. The image of p' is the the complement of $\mathscr{H}_\Delta$ [DM, Example 6.4]. The points of $\mathscr{H}_\infty \setminus \mathscr{H}_\Delta$ (as the points of $\mathcal{M}$) correspond to the polarized symplectic manifolds $(\operatorname{Hilb}^2(S), 2\widetilde{H} - \delta)$ where (S, H) is a degree two polarized K3 surface, $\widetilde{H}$ is the nef and big divisor induced from H, and δ is the half of the Hilbert-Chow exceptional divisor.

As for the points on the divisor $\mathscr{H}_\Delta \subset \mathcal{D}/\Gamma$, they should correspond to singular symplectic varieties $F(X)$ for $X \in \mathcal{C} \setminus \mathcal{C}_0$. Note that $\mathscr{H}_\Delta$ is birational to the global period domain $\mathcal{F}_6$ of degree six $K3$ surfaces [Ha, Theorem 5.2.2]. A typical example of a family of polarized singular symplectic varieties over $\mathcal{F}_6$ is the family of the symmetric squares of (possibly singular) $K3$ surfaces. However, our family over $\mathscr{H}_\Delta$ is different from this one.

Proposition 4.2. *Take a cubic fourfold X corresponding to a general point of $\mathscr{H}_\Delta$ so that the singular locus of $F(X)$ is a smooth $K3$ surface S of degree six with $\rho(S) = 1$. Then $F(X)$ is not isomorphic (but birational) to the symmetric square $\operatorname{Sym}^2(S)$.*

Proof. Recall that $F(X)$ admits a natural symplectic resolution $\pi : \operatorname{Hilb}^2(S) \to F(X)$ (see §2). One can easily check that the exceptional divisor of π is different from the Hilbert-Chow exceptional divisor of $\operatorname{Hilb}^2(S)$. Using the standard generators δ and $\widetilde{H}$ (with $\widetilde{H}^2 = 6$) of $NS(\operatorname{Hilb}^2(S))$, the contraction $\pi : \operatorname{Hilb}^2(S) \to F(X)$ corresponds to the spherical class $\widetilde{H} - 2\delta$ (see §3). See also [BM, §13] for the structure of the movable cone or the nef cone in $NS(\mathrm{Hilb}^2(S))$. If $F(X)$ were isomorphic to $\mathrm{Sym}^2(S)$, then $\mathrm{Hilb}^2(S)$ would admit a nontrivial automorphism since the Hilbert-Chow morphism is the unique symplectic resolution of the symmetric square. However, this cannot happen when S is of degree six [BCNWS]. □

Remark 4.1. In fact we can say that $\mathrm{Sym}^2(S)$ and $F(X)$ belong to different moduli spaces since (smooth polarized deformations of) such symmetric squares have divisibility one while $F(X)$ has divisibility two. Here, the divisibility is an numerical invariant of a polarized $K3^{[2]}$-type manifold (M, H) defined as the number $\gamma \in \{1, 2\}$ such that $H.H^2(M, \mathbb{Z}) = \gamma\mathbb{Z}$. See [DM] for details. □

Since $F(X)$ is a symplectic variety when $X \in \mathcal{C}$, the period domain $\mathcal{D}/\Gamma$ parametrizes possibly singular symplectic varieties at least outside of the codimension two locus $\mathscr{H}_\Delta \cap \mathscr{H}_\infty$. In fact, like the case of $K3$ surfaces, the period domains associated to any polarized symplectic manifolds can be regarded as the moduli spaces of singular symplectic varieties of given numerical types [OO, Theorem 8.3].

The Fano variety $F(X)$ for $X \in \mathcal{C} \setminus \mathcal{C}^0$ that we have discussed so far is a very special case of degeneration of symplectic manifolds. Then it is natural to ask what happens for more general symplectic manifolds:

Question. What symplectic varieties can be realized as degenerations of (polarized) holomorphic symplectic manifolds?

It seems also interesting to study symplectic singularities which are obtained from degeneration of holomorphic symplectic manifolds. For example, the singularities of $F(X)$ with $X \in \mathcal{C} \setminus \mathcal{C}_0$ (namely, the Hilbert squares of ADE singularities) are also studied from different viewpoints such as (geometric) representation theory (cf. [CGGS]).

As a related result, the moduli space of certain sheaves over a $K3$ surface with non-generic polarization has the same symplectic singularities as Nakajima quiver varieties [AS]. A Nakajima quiver variety admits a symplectic resolution by varying the stability parameters and has a smoothing by varying deformation parameters. Note that, in general, a symplectic variety admitting symplectic resolutions has a smoothing [Nam2] and in particular the singular moduli space of sheaves above can be regarded as a degeneration of compact symplectic manifolds. This suggests that such construction can give many analytic types of symplectic singularities on compact varieties other than symmetric products of ADE singularities or its its partial resolutions. See also [KL] for the structure of singularities in a degeneration of the 10-dimensional symplectic manifold constructed by O'Grady.

4.2 Correspondence of local monodromy groups

In the rest of this section we discuss local monodromy groups associated to deformation spaces of a singular cubic fourfold and its Fano variety of lines. The proofs of the main results particularly imply that the Fano variety $F(X)$ of lines on a cubic fourfold X has symplectic singularities whose codimension-two strata have transversal ADE singularities of the same types as the singular points of X when X has at worst simple singularities (even if X contains a plane). This correspondence of the types of singularities of X and $F(X)$ can be viewed as the correspondence of the local monodromy groups of these varieties in their deformation spaces (Proposition 4.3).

Let $X_0 \in \mathcal{C} \setminus \mathcal{C}^0$ be a cubic fourfold with simple singularities. As discussed in the proof of [La2, Proposition 3.2], the moduli space $\mathcal{C}$ gives a simultaneous versal deformation of the singular points of X_0. We refer the reader to [AGZV] for a general theory of versal deformations and monodromy groups of isolated singularities. In a small neighborhood U of $X_0 \in \mathcal{C}$, the discriminant locus $\Sigma = U \setminus \mathcal{C}^0 \subset U$ gives a hypersurface. For $t_1 \in U \setminus \Sigma$, each loop in $U \setminus \Sigma$ passing through t gives an action on the cohomology group and thus we obtain a monodromy representation

$$\rho : \pi_1(U \setminus \Sigma, t_1) \to \mathrm{Aut}(H^4(X_{t_1}, \mathbb{Z}))$$

where X_{t_1} is the smooth cubic fourfold corresponding to t_1. The image W of ρ, which is called the monodromy group of X_0, is the product of the Weyl groups of the same types of the singular points of X_0 acting on the cohomology group by reflections.

Next we recall that there is a deformation theory for symplectic singularities which has similar feature to that for isolated singularities: symplectic singularities admitting symplectic resolutions have their smooth universal Poisson deformation spaces of finite dimension and they admit a simultaneous resolution after a base change by a finite Galois cover whose Galois group is a (product of) Weyl group(s). See [Nam2],[Nam3] for details (and see also [Nam1] for compact symplectic varieties). Note that the usual deformation theory does not work very well since symplectic singularities are usually non-isolated and will have infinite dimensional versal deformation spaces. By considering a deformation of not only singularities but the Poisson structure, which comes from a symplectic structure, we obtain a similar theory to the theory for isolated (in particular ADE) singularities. The universal Poisson deformation space is determined by the complement of a codimension-four subset of a given symplectic variety, and also the Weyl group of the singularity is determined by its codimension-two symplectic leaves and the monodromy on them. In the case of a Fano variety of lines, the monodromy is trivial i.e., the Dynkin diagram of the Weyl group is simply-laced (see the proof of Proposition 3.5).

Proposition 4.3. *Keep the notation as above. Then the image W of the monodromy representation ρ is isomorphic to the Weyl group associated to a symplectic variety $F(X_0)$. In particular, the ADE-types of singular points of X_0 coincide with those of codimension-two symplectic leaves of $F(X_0)$.*

Proof. We regard the moduli space $\mathcal{C}$ also as a deformation space of symplectic varieties via the assignment $X \mapsto F(X)$. Then it gives a monodromy representation $\pi_1(U \setminus \Sigma, t_1) \to \mathrm{Aut}(H^2(F(X_{t_1}), \mathbb{Z}))$ similarly as above. Note that the Abel-Jacobi map can be constructed in a family and then the isomorphism $\alpha_{t_1} : H^4(X_{t_1}, \mathbb{Z}) \to H^2(F(X_{t_1}), \mathbb{Z})$ is clearly equivariant with respect to the action of $\pi_1(U \setminus \Sigma, t_1)$. Therefore, the monodromy group of X_0 also acts on $H^2(F(X_{t_1}), \mathbb{Z})$ by reflections, which should be associated to the classes of the exceptional divisors of the symplectic resolution $Y_0 \to F(X_0)$ (cf. [Ma2, §3]).

We fix symplectic structures ω_t on $F(X_t)$ for $t \in U$ and regard U as the base space of the family of pairs $(F(X_t), \omega_t)$. Then, after a finite base change $\widetilde{U} \to U$, the family admits a simultaneous symplectic resolution $\pi_t : (Y_t, \widetilde{\omega}_t) \to (F(X_t), \omega_t)$. The tangent space $T_0\widetilde{U}$ is identified with a codimension-one subspace of $H^1(Y_0, T_{Y_0})$ and, by using the isomorphism $T_{Y_0} \cong \Omega^1_{Y_0}$ given by the symplectic form $\widetilde{\omega}_0$, this is also regarded as the orthogonal complement $(\pi_0^* g)^\perp \subset H^{1,1}(Y_0)$ where g is the natural polarization of $F(X_0)$. Take a small open neighborhood $V \subset F(X_0)$ of a general point of a 2-dimensional symplectic leaf of $F(X_0)$ and let U' be the base space of the universal Poisson deformation of $(V, \omega_0|_V)$. Let $\widetilde{U'} \to U'$ be the covering map by the Weyl group W' associated to the local singularity V. Then the tangent space $T_0\widetilde{U'}$ is canonically isomorphic to $H^2(\widetilde{V}, \mathbb{C})$ where $\widetilde{V} = \pi_0^{-1}(V) \subset Y_0$ is the symplectic resolution of V. Note that the classes of the exceptional divisors of $\widetilde{V} \to V$ form a basis of $H^2(\widetilde{V}, \mathbb{C})$ (cf. Lemma 3.3).

The universality of U' gives a map $U \to U'$ which lifts to $\lambda : \widetilde{U} \to \widetilde{U'}$. The induced tangent map $d\lambda$ (called the Poisson-Kodaira-Spencer map) is just the composition

$$(\pi_0^* g)^\perp \subset H^{1,1}(Y_0) \subset H^2(Y_0, \mathbb{C}) \to H^2(\widetilde{V}, \mathbb{C})$$

of inclusions and the restriction map. This is surjective since the exceptional divisors in Y_0 are orthogonal to the pullback $\pi_0^* g$. Therefore, λ is a submersion and in particular the monodromy actions on $H^2(F(X_{t_1}, \mathbb{Z}))$ associated to U' come from the ones associated to U. By considering the Poisson deformations of V for points of all 2-dimensional symplectic leaves of $F(X_0)$, we obtain a surjection $W \to W_0'$ where W_0' is the Weyl group associated to the symplectic variety $F(X_0)$, which is the product of the Weyl groups W' over all 2-dimensional symplectic leaves of $F(X_0)$. W is also the product of Weyl groups, and the restriction of $W \to W_0$ to each factor of W is injective since such a factor faithfully acts on $H^4(X_{t_1}, \mathbb{Z})$ and the Abel-Jacobi map $H^4(X_{t_1}, \mathbb{Z}) \cong H^2(F(X_{t_1}), \mathbb{Z})$ is equivariant with respect to $W \to W_0'$. Therefore, the map $W \to W_0'$ gives an isomorphism between the local monodromy groups as desired. □

Remark 4.2. The above argument implies that $\mathcal{C}$ gives universal Poisson deformations of local singularities of $F(X_0)$. Note, however, that $\mathcal{C}$ does not give a universal (Poisson) deformation of global $F(X_0)$ since $F(X_0)$ can be deformed to non-projective symplectic manifolds: $\mathcal{C}$ has dimension 20 while the Kuranishi deformation space $\mathrm{Def}(F(X_0))$ has dimension $h^{1,1}(Y_0) = 21$. □

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