Moduli Space of Quasimaps from $\mathbb{P}^1$ with Two Marked Points to $\mathbb{P}(1,1,1,3)$ and j-invariant

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Abstract

In this paper, we construct toric data of moduli space of quasimaps of degree d from $\mathbb{P}^1$ with two marked points to weighted projective space $\mathbb{P}(1,1,1,3)$. With this result, we prove that the moduli space is a compact toric orbifold. We also determine its Chow ring. Moreover, we give a proof of the conjecture proposed by Jinzenji that a series of intersection numbers of the moduli spaces coincides with expansion coefficients of inverse function of $-\log(j(\tau))$.

1 Introduction.

1.1 Overview of Classical Mirror Symmetry

Mirror Symmetry is a symmetry between two topological sigma models, A-model on a Calabi-Yau manifold X and B-model on another Calabi-Yau manifold X^*. Mathematically, A-model has information of world sheet instantons, i.e., holomorphic maps from a Riemann surface Σ to X. On the other hand, B-model has information of deformation of Hodge structure of X^*. Mirror Symmetry surprisingly connects these concepts and has fascinated both physicists and mathematicians.

Classical Mirror Symmetry enables us to compute GW invariants of X by using solutions of Picard-Fuchs equations for Period integral of X^* ([1, 4, 7]). GW invariants are defined as intersection numbers of the moduli spaces of stable maps $\overline{M}_{g,n}(X,\beta)$. But computation of the invariants from this definition is usually done by localization technique and the process is quite complicated. On the other hand, the Picard-Fuchs equations of X^* are linear differential equations and easy to solve in many cases. Furthermore, *mirror map* which connects A-model with B-model is also given by solutions of Picard-Fuchs equations. Hence, the process of computation of GW invariants by using classical mirror symmetry is much simpler than the process by using localization.

In [9], one of the authors (Jinzenji) introduced moduli space $\widetilde{Mp}_{0,2}(N,d)$, which is a compactified moduli space of quasimaps (polynomial maps) from $\mathbb{P}^1$ to $\mathbb{P}^{N-1}$ with two marked points. This moduli space is deeply connected to informations of the B-model in classical mirror symmetry. Namely, generating functions of intersection numbers on $\widetilde{Mp}_{0,2}(N,d)$ directly gives us the solution of Picard-Fuchs equations and the mirror map. The geometrical feature of $\widetilde{Mp}_{0,2}(N,d)$ comes from the fact that its geometrical structure is much simpler than the corresponding moduli space $\overline{M}_{0,2}(\mathbb{P}^{N-1},d)$, which is compactified by using stable maps. In [13], the other author (Saito) constructed a concrete toric data of $\widetilde{Mp}_{0,2}(N,d)$. This toric data gives us a lot of properties of $\widetilde{Mp}_{0,2}(N,d)$. For example, $\widetilde{Mp}_{0,2}(N,d)$ is a compact toric orbifold. We can also determine Chow ring of $\widetilde{Mp}_{0,2}(N,d)$ from the toric data. In [13], Saito also discovered an injective homomorphism from $A^*(\widetilde{Mp}_{0,2}(N,d))$ to $A^*(\overline{M}_{0,2}(\mathbb{P}^{N-1},d))$ for $d=1,2$ cases.[*1] Furthermore, Saito proved a formula that describes genus 0 GW invariants of projective hypersurfaces of $d=1,2$ in terms of Chow ring of $\overline{M}_{0,2}(\mathbb{P}^{N-1},d)$. This formula was found by pursuing analogy of the formula that describes the corresponding intersection numbers of $\widetilde{Mp}_{0,2}(N,d)$ in terms of its Chow ring, that was first presented in [9].

1.2 j-invariant.

The j-invariant is a weight zero modular function of τ which is the coordinate of complex structures of elliptic curves over $\mathbb{C}$:

$$\mathbb{C}/(\mathbb{Z}\oplus\mathbb{Z}\tau),$$

where τ is in the upper half plane. j-invariant is one of the fundamental tools for studying elliptic curves. For example, it determines a group structure of elliptic curves.

Let us introduce $q=e^{2\pi\sqrt{-1}\tau}$, then Fourier expansion of j-invariant is given by,

$$\begin{aligned} j(\tau) &= q^{-1}+744+196884q+21493760q^2+864299970q^3+20245856256q^4+\cdots \\ &=: q^{-1}+\sum_{d=1}^{\infty} j_d\cdot q^{d-1}. \end{aligned} \tag{1.1}$$

These coefficients were found to be related to ranks of irreducible representations of the Monster group (the largest sporadic simple group), which is known as monstrous moonshine.

[*1] In the $d=2$ case, Saito did not construct an explicit birational map from $\overline{M}_{0,2}(\mathbb{P}^{N-1},d)$ to $\widetilde{Mp}_{0,2}(N,d)$, and it is not clear whether the injective map is induced from the expected birational map.

In this paper, we deal with the expansion coefficients of *inverse* function of $-\log(j(\tau))$,

$$2\pi\sqrt{-1}\tau = -\log(j) + 744j^{-1} + 473652j^{-2} + 451734080j^{-3} + 510531007770j^{-4} + \cdots$$
$$=: -\log(j) + \sum_{d=1}^{\infty} w_d \cdot j^{-d}. \qquad (1.2)$$

This function appeared as the mirror map of the K3 surface in $\mathbb{P}(1,1,1,3)$. The expansion coefficient j_d is reconstructed by the expansion coefficient w_d via the following relation[*2]:

$$j_d = \sum_{\sigma_d \in P_d} (-(d-1))^{l(\sigma_d)-1} \frac{1}{\prod_{j=1}^{d}(\mathrm{mul}(\sigma,j))!} \prod_{j=1}^{l(\sigma_d)} w_{d_j}, \qquad (1.3)$$

where P_d is set of partitions of positive integer d,

$$P_d := \{\sigma_d = (d_1, d_2, \cdots, d_{l(\sigma_d)}) \mid \sum_{j=1}^{l(\sigma_d)} d_j = d, \;\; d_1 \geq d_2 \geq \cdots \geq d_{l(\sigma_d)} \geq 1\ \},$$

$l(\sigma_d)$ is length of a partition $\sigma_d \in P_d$ and $\mathrm{mul}(\sigma, j)$ is multiplicity of j's that appear in σ. In [9] (arXiv version), Jinzenji proposed the following:

Conjecture 1.1 *The coefficient w_d is written as an intersection number of moduli space of quasimaps from $\mathbb{P}^1$ to $\mathbb{P}(1,1,1,3)$ of degree d with two marked points.*

The aim of our paper is to prove this conjecture.

1.3 Picard-Fuchs equation for j-invariant.

It is well-known that the Picard-Fuchs equation for 1-parameter deformation of algebraic K3 surfaces can be solved in terms of the j-invariant ([11]). Let us demonstrate it in the case of 1-parameter deformation of algebraic K3 surface in $\mathbb{P}(1,1,1,3)$. We mainly refer to section 5.4. of [11].

The 1-parameter family of K3 surface which we use is given by the following defining equation:

$$x_1^6 + x_2^6 + x_3^6 + x_4^2 + z^{-1/6} x_1 x_2 x_3 x_4 = 0,$$

whics is embedded in $\mathbb{P}(1,1,1,3) = \{(x_1, x_2, x_3, x_4)\}/(\mathbb{C}^* - \text{action})$. The Picard-Fuchs equation for period integrals of the above family is given by,

$$(\Theta^3 - 8z(6\Theta+1)(6\Theta+3)(6\Theta+5))f(z) = 0, \qquad (1.4)$$

where Θ is a differential operator $\Theta = z\frac{d}{dz}$. It is obtained from standard techniques, for example, Griffiths-Dwork method, or A-Hypergeometric equations (see [4], etc.).

[*2] (1.3) follows from the equation (A.10) that appears in Appendix A of [8].

In the following, we briefly review the process of solving the above equation. Let us assume that a solution $f(z)$ is expanded as follows:

$$f(z)=\sum_{n=0}^{\infty}a_n(\epsilon)z^{n+\epsilon},$$

where ϵ is a parameter and $a_0(\epsilon)=1$. By substituting it to (1.4), we obtain,

$$\sum_{n=0}^{\infty}(a_n(\epsilon)(n+\epsilon)^3-8(6n-5+6\epsilon)(6n-3+6\epsilon)(6n-1+6\epsilon)a_{n-1}(\epsilon))z^{n+\epsilon}=0,$$

where $a_{-1}(\epsilon)=0$. Since it holds for any z, we obtain,

$$\begin{aligned}a_n(\epsilon)&=\frac{8(6n-5+6\epsilon)(6n-3+6\epsilon)(6n-1+6\epsilon)}{(n+\epsilon)^3}a_{n-1}(\epsilon)\\&=\frac{\Gamma(6n+6\epsilon+1)}{\Gamma(n+\epsilon+1)^3\Gamma(3n+3\epsilon+1)},\end{aligned}$$

where $\Gamma(x)$ is the Gamma function. Therfore, we can obtain the solutions of (1.4) from the following expression:

$$f(z,\epsilon):=\sum_{n=0}^{\infty}\frac{\Gamma(6n+6\epsilon+1)}{\Gamma(n+\epsilon+1)^3\Gamma(3n+3\epsilon+1)}z^{n+\epsilon}.$$

By setting $\epsilon=0$, it gives a solution which is holomorphic at $z=0$:

$$f_0(z):=\sum_{d=0}^{\infty}\frac{2^{3d}(6d-1)!!}{(d!)^3}z^d. \tag{1.5}$$

If we differentiate $f(z,\epsilon)$ by ϵ and set $\epsilon=0$, we obtain the solution which has a log-singularity at $z=0$:

$$f_1(z):=f_0(z)(\log(z))+\sum_{d=0}^{\infty}(\sum_{j=1}^{3d}\frac{6}{2j-1}-\sum_{j=1}^{d}\frac{3}{j})\frac{2^{3d}(6d-1)!!}{(d!)^3}z^d. \tag{1.6}$$

The mirror map for the K3 surface is given by,

$$f_1(z)/f_0(z).$$

It gives us the inverse function of $-\log(j(\tau))$ ([11]).

Theorem 1.1 (B. Lian, S. T. Yau, 1996)

$$\frac{f_1(j^{-1})}{f_0(j^{-1})}=2\pi\sqrt{-1}\tau=-\log(j)+\sum_{d=1}^{\infty}w_d\cdot j^{-d}.$$

1.4 The Goal of this paper.

In order to prove the conjecture, we explicitly construct the compactified moduli space of degree d quasimaps from $\mathbb{P}^1$ to $\mathbb{P}(1,1,1,3)$ with two marked points, which we denote by $\widetilde{Mp}_{0,2}(\mathbb{P}(1,1,1,3),d)$. In section 2, we provide a toric data of $\widetilde{Mp}_{0,2}(\mathbb{P}(1,1,1,3),d)$ and prove the following theorem.

Theorem 1.2 *$\widetilde{Mp}_{0,2}(\mathbb{P}(1,1,1,3),d)$ is a compact toric orbifold.*

Furthermore, we show that the Chow ring of $\widetilde{Mp}_{0,2}(\mathbb{P}(1,1,1,3),d)$ is given by,

Proposition 1.1 *$A^*(\widetilde{Mp}_{0,2}(\mathbb{P}(1,1,1,3),d)) \cong \mathbb{C}[H_0, H_1, \ldots, H_d]/\mathcal{I}$, where $\mathcal{I} = (H_0^4(2H_0+H_1), H_1^4(H_0+2H_1)(2H_1+H_2)(-H_0+2H_1-H_2), \ldots, H_{d-1}^4(H_{d-2}+2H_{d-1})(2H_{d-1}+H_d)(-H_{d-2}+2H_{d-1}-H_d), H_d^4(H_{d-1}+2H_d))$.*

With these preparations, we define the intersection number that corresponds to w_d by,

Definition 1.1

$$w(\mathcal{O}_{z^a}\mathcal{O}_{z^b})_{0,d} := \int_{\widetilde{Mp}_{0,2}(\mathbb{P}(1,1,1,3),d)} H_0^a H_d^b \cdot \frac{\prod_{i=1}^{d} e^6(H_{i-1}, H_i)}{\prod_{i=1}^{d-1} 6H_i},$$

where

$$e^6(x,y) := \prod_{j=0}^{6}((6-j)x+y).$$

In this definition, $H_0, H_1, \ldots, H_d$ are generators of Chow rings of $\widetilde{Mp}_{0,2}(\mathbb{P}(1,1,1,3),d)$.

In section 3, we prove our main result.

Theorem 1.3 (Main Theorem)

$$w_d = \frac{1}{2} w(\mathcal{O}_{z^1}\mathcal{O}_{z^0})_{0,d}. \tag{1.7}$$

Of course, our main result also follows from general theory of wall crossing formula of quasimap theory by Cheong, Ciocan-Fontanine and Kim [4, 3]. But our moduli space $\widetilde{Mp}_{0,2}(\mathbb{P}(1,1,1,3),d)$ seems to be a little bit different from their moduli space because we omit some boundary divisors in our construction. In compensation for this simplification, we gave up the condition that images of two marked points of $\mathbb{P}^1$ are well defined in $\mathbb{P}(1,1,1,3)$. At first sight, this point seems to cause some errors in computing the intersection number $w(\mathcal{O}_{z^a}\mathcal{O}_{z^b})_{0,d}$, but in fact, the extra spurious factors that appear in denominator of the residue integral representation of $w(\mathcal{O}_{z^a}\mathcal{O}_{z^b})_{0,d}$ are killed by the factors that appear in $e^6(H_{i-1}, H_i)$. Therefore, it causes no problem. Moreover, we can determine Chow ring of our moduli space of quasimaps from $\mathbb{P}^1$ with two marked points to $\mathbb{P}(1,1,1,3)$. Our approach in this paper is quite direct and provides explicit computation of intersection numbers that correspond to expansion coefficients of the mirror map. Therefore, we expect that our approach also contributes to deeper understanding of machinery of mirror computation.

2 The moduli space $\widetilde{Mp}_{0,2}(\mathbb{P}(1,1,1,3),d)$.

In this section, we provide definition of the quasimap moduli space $\widetilde{Mp}_{0,2}(\mathbb{P}(1,1,1,3),d)$ and prove Theorem 1.2.

Furthermore, we compute its Chow ring:

$$A^*(\widetilde{Mp}_{0,2}(\mathbb{P}(1,1,1,3),d)) \cong \mathbb{C}[H_0, H_1, \dots, H_d]/\mathcal{I},$$

where $\mathcal{I} = (H_0^4(2H_0+H_1), H_1^4(H_0+2H_1)(2H_1+H_2)(-H_0+2H_1-H_2), \cdots, H_j^4(H_{j-1}+2H_j)(2H_j+H_{j+1})(-H_{j-1}+2H_j-H_{j+1}), \cdots, H_{d-1}^4(H_{d-2}+2H_{d-1})(2H_{d-1}+H_d)(-H_{d-2}+2H_{d-1}-H_d), H_d^4(H_{d-1}+2H_d))$.

In addition, we define an intersection numbers $w(\mathcal{O}_{z^a}\mathcal{O}_{z^b})_{0,d}$ of the quasimap moduli space $\widetilde{Mp}_{0,2}(\mathbb{P}(1,1,1,3),d)$.

2.1 The fan of $\widetilde{Mp}_{0,2}(\mathbb{P}(1,1,1,3),d)$.

The generic quasimap from $\mathbb{P}^1$ to $\mathbb{P}(1,1,1,3)$ is given by

$$\begin{aligned}
&\mathbb{P}^1 \to \mathbb{P}(1,1,1,3)\\
&[s:t] \mapsto [f_0(s.t) : f_1(s,t) : f_2(s,t) : f_3(s,t)],
\end{aligned}$$

where

$$\begin{aligned}
f_i(s,t) &:= \sum_{j=0}^{d} a_{i,j}s^{d-j}t^j, && (0 \le i \le 2),\\
f_3(s,t) &:= \sum_{j=0}^{3d} a_{3,j}s^{3d-j}t^j.
\end{aligned}$$

The following $(\mathbb{C}^*)^2$-action on $(a_{i,j})$ is induced from projective equivalence of $\mathbb{P}(1,1,1,3)$ and automorphism group of $\mathbb{P}^1$ which keeps $(0:1), (1:0) \in \mathbb{P}^1$ fixed.

$$\begin{aligned}
&(\mu,\nu)\cdot(\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_d, a_{3,0}, a_{3,1}, \dots, a_{3,3d})\\
=&(\mu\mathbf{a}_0, \mu\nu\mathbf{a}_1, \mu\nu^2\mathbf{a}_2, \dots, \mu\nu^d\mathbf{a}_d, \mu^3 a_{3,0}, \mu^3\nu a_{3,1}, \mu^3\nu^2 a_{3,2}, \dots, \mu^3\nu^{3d}a_{3,3d}).
\end{aligned} \tag{2.1}$$

Here $\mathbf{a}_j \in \mathbf{C}^3$ represents a vector $(a_{0,j}, a_{1,j}, a_{2,j})$. This is equivalent to the following action:

$$\begin{aligned}
&(\mu',\nu')\cdot(\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_d, a_{3,0}, a_{3,1}, \dots, a_{3,3d})\\
=&((\mu')^d\mathbf{a}_0, (\mu')^{d-1}(\nu')\mathbf{a}_1, (\mu')^{d-2}(\nu')^2\mathbf{a}_2, \dots, (\nu')^d\mathbf{a}_d,\\
&(\mu')^{3d}a_{3,0}, (\mu')^{3d-1}(\nu')a_{3,1}, (\mu')^{3d-2}(\nu')^2 a_{3,2}, \dots, (\nu')^{3d}a_{3,3d}).
\end{aligned} \tag{2.2}$$

The set

$$\begin{aligned}
&Mp_{0,2}(\mathbb{P}(1,1,1,3),d)\\
:=&\{(\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_d, a_{3,0}, a_{3,1}, \dots, a_{3,3d}) \in \mathbb{C}^{3(d+1)+3d+1}\}/(\mathbb{C}^*)^2
\end{aligned}$$

is not compact. In order to see it, let us use the $(\mathbb{C}^*)^2$-action in (2.1) to turn $(\mathbf{a}_0, a_{3,0})$ and $(\mathbf{a}_d, a_{3,3d})$ into points in $\mathbb{P}(1,1,1,3)$, $[\mathbf{a}_0, a_{3,0}]$ and $[\mathbf{a}_d, a_{3,3d}]$. Then, we obtain

$$\begin{aligned}
&Mp_{0,2}(\mathbb{P}(1,1,1,3), d)\\
\cong&\{([\mathbf{a}_0, a_{3,0}], \mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{d-1}, [\mathbf{a}_d, a_{3,3d}]\\
&\quad, a_{3,1}, a_{3,2}, \dots, a_{3,3d-1}) \in \mathbb{P}(1,1,1,3) \times \mathbb{C}^{3(d-1)+3d-1} \times \mathbb{P}(1,1,1,3)\}/\mathbb{Z}_d.
\end{aligned}$$

Here the $\mathbb{Z}_d$-action is given by

$$([\mathbf{a}_0, a_{3,0}], \zeta_d\mathbf{a}_1, \zeta_d^2\mathbf{a}_2, \dots, \zeta_d^{d-1}\mathbf{a}_{d-1}, [\mathbf{a}_d, a_{3,3d}], \zeta_d a_{3,1}, \zeta_d^2 a_{3,2}, \dots, \zeta_d^{3d-1} a_{3,3d-1}),$$

where ζ_d is the d-th primitive root of unity.

In order to compactify $Mp(\mathbb{P}(1,1,1,3), d)$, we add variables $u_1, u_2, \dots, u_{d-1}$. They should be added as in the case of $\widetilde{Mp}_{0,2}(N,d)$ (see [9] or [13]). Hence, we want to obtain a toric data of

$$\begin{aligned}
&\widetilde{Mp}_{0,2}(\mathbb{P}(1,1,1,3), d)\\
:=&\{(\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_d, a_{3,0}, a_{3,1}, \dots, a_{3,3d}, u_1, u_2, \dots, u_{d-1}) \in U\}/(\mathbb{C}^*)^{d+1},
\end{aligned}$$

where U is a dense open subset of $\mathbb{C}^{3(d+1)+(3d+1)+(d-1)}$, and the $(\mathbb{C}^*)^{d+1}$-action is given by

$$\begin{aligned}
&(\lambda_0, \dots, \lambda_d) \cdot (\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_d, a_{3,0}, a_{3,1}, \dots, a_{3,3d}, u_1, u_2, \dots, u_{d-1})\\
=&(\lambda_0\mathbf{a}_0, \lambda_1\mathbf{a}_1, \dots, \lambda_d\mathbf{a}_d, \lambda_0^3 a_{3,0}, \lambda_0^2\lambda_1 a_{3,1}, \lambda_0\lambda_1^2 a_{3,2}, \lambda_1^3 a_{3,3}, \lambda_1^2\lambda_2 a_{3,4}, \dots, \lambda_d^3 a_{3,3d},\\
&\lambda_0^{-1}\lambda_1^2\lambda_2^{-1}u_1, \lambda_1^{-1}\lambda_2^2\lambda_3^{-1}u_2, \dots, \lambda_{d-2}^{-1}\lambda_{d-1}^2\lambda_d^{-1}u_{d-1}).
\end{aligned}$$

The structure of U will be given by constructing toric data corresponding to $\widetilde{Mp}_{0,2}(\mathbb{P}(1,1,1,3), d)$.

In the following, we construct a fan which is complete and simplicial, and realizes this $(\mathbb{C}^*)^{d+1}$-action.

Let $p_0, p_1, p_2 \in \mathbb{Z}^2$ be integer vectors given by,

$$(p_0, p_1, p_2) = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

Next, we introduce $(d+1)$ column vectors

$$v'_0, v'_1, \dots, v'_d \in \mathbb{Z}^{d-1},$$

defined by,

$$(v'_0, v'_1, \dots, v'_{d-1}, v'_d) = \begin{pmatrix}
-1 & 2 & -1 & 0 & \cdots & 0 & 0 \\
0 & -1 & 2 & -1 & \cdots & 0 & 0 \\
0 & 0 & -1 & 2 & \cdots & 0 & 0 \\
0 & 0 & 0 & -1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & 0 \\
0 & 0 & 0 & 0 & \cdots & -1 & 0 \\
0 & 0 & 0 & 0 & \cdots & 2 & -1
\end{pmatrix} \in M_{d-1,d+1}(\mathbb{Z}).$$

in the same way as the case of $\widetilde{Mp}_{0,2}(N,d)$ in [13].

In addtion, we have to introduce the following vectors:

$$(w_0, w_1, w_2, \ldots, w_d) := \begin{pmatrix} 3 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 2 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 3 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 3 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 2 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 3 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 2 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 2 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 3 \end{pmatrix} \in M_{3d+1,d+1}(\mathbb{Z})$$

Finally, we define column vectors,

$$\begin{aligned} & v_{i,j} \ \ (0 \leq i \leq 2, 0 \leq j \leq d), \\ & v_{3,j} \ \ (0 \leq j \leq 3d), \\ & u_k \ \ (1 \leq k \leq d-1) \end{aligned}$$

as follows:
for $i \neq 0$,

$$v_{i,j} = \begin{pmatrix} \mathbf{0}_2 \\ \vdots \\ p_i \\ \vdots \\ \mathbf{0}_2 \\ \mathbf{0}_{3d+1} \\ \mathbf{0}_{d-1} \end{pmatrix} \begin{matrix} \\ \\ \leftarrow j \\ \\ \\ \\ \\ \end{matrix} \in \mathbb{Z}^{2(d+1)+(3d+1)+(d-1)},$$

for $i = 0$,

$$v_{0,j} = \begin{pmatrix} \mathbf{0}_2 \\ \vdots \\ p_0 \\ \vdots \\ \mathbf{0}_2 \\ -w_j \\ v'_j \end{pmatrix} \begin{matrix} \\ \\ \leftarrow j \\ \\ \\ \\ \\ \end{matrix} \in \mathbb{Z}^{2(d+1)+(3d+1)+(d-1)},$$

for $0 \leq j \leq 3d$,

$$v_{3,j} = \begin{pmatrix} \mathbf{0}_{2(d+1)} \\ e_j^{3d+1} \\ \mathbf{0}_{d-1} \end{pmatrix} \quad \in \mathbb{Z}^{2(d+1)+(3d+1)+(d-1)},$$

and for $k = 1, \ldots, d-1$,

$$u_k = \begin{pmatrix} \mathbf{0}_{2(d+1)} \\ \mathbf{0}_{3d+1} \\ -e_k^{d-1} \end{pmatrix} \quad \in \mathbb{Z}^{2(d+1)+(3d+1)+(d-1)},$$

where $\mathbf{0}_\alpha$ is the zero vector in $\mathbb{Z}^\alpha$ and e_k^β is the k-th standard basis of $\mathbb{Z}^\beta$.

Definition 2.1 *Let*

$$\begin{aligned} &P_0 := \{v_{0,0}, v_{1,0}, v_{2,0}, v_{3,0}, v_{3,1}\}, \\ &P_d := \{v_{0,d}, v_{1,d}, v_{2,d}, v_{3,3d-1}, v_{3,3d}\}, \\ &P_i := \{v_{0,i}, v_{1,i}, v_{2,i}, v_{3,3i-1}, v_{3,3i}, v_{3,3i+1}, u_i\} \ (1 \leq i \leq d-1). \end{aligned}$$

Then, we define

$$\Sigma_d$$

as a set of cones generated by the union of proper subsets (involving empty set) of $P_0, P_1, \ldots, P_d$. (A cone corresponding to empty set is $\{0\}$).

We have to show that Σ_d is a fan.

Theorem 2.1 *For arbitrary positive integer d, Σ_d is a simplicial complete fan.*

In order to prove that Σ_d is a simplicial complete fan, we should check the following claim:

Lemma 2.1 *For all $v \in \mathbb{R}^{6d+2}$, there uniquely exist $a_{i,j} \in \mathbb{R}$ and $b_k \in \mathbb{R}$ that satisfy the following conditions.*

$$\begin{aligned} &(a) \qquad v = \sum_{i=0}^{2}\sum_{j=0}^{d} a_{i,j} v_{i,j} + \sum_{j=0}^{3d} a_{3,j} v_{3,j} + \sum_{k=1}^{d-1} b_k u_k, \\ &(b) \qquad \min(\{a_{0,0}, a_{1,0}, a_{2,0}, a_{3,0}, a_{3,1}\}) = 0, \\ &\qquad\quad\ \ \min(\{a_{0,d}, a_{1,d}, a_{2,d}, a_{3,3d-1}, a_{3,3d}\}) = 0, \\ &(c) \qquad \min(\{a_{0,i}, a_{1,i}, a_{2,i}, a_{3,3i-1}, a_{3,3i}, a_{3,3i+1}, b_i\}) = 0 \ \ (i = 1, 2, \cdots, d-1). \end{aligned}$$

Proof. The following relations for $\{v_{i,j}\}, \{u_k\}$ hold:

$$v_{0,0} + v_{1,0} + v_{2,0} + 3v_{3,0} + 2v_{3,1} + v_{3,2} - u_1 = 0, \tag{2.3}$$

$$\begin{aligned} &v_{0,i} + v_{1,i} + v_{2,i} + v_{3,3i-2} + 2v_{3,3i-1} + 3v_{3,3i} + 2v_{3,3i+1} + v_{3,3i+2} \\ &\qquad - u_{i-1} + 2u_i - u_{i+1} = 0, \quad (1 \leq i \leq d-1) \end{aligned} \tag{2.4}$$

$$v_{0,d} + v_{1,d} + v_{2,d} + v_{3,3d-2} + 2v_{3,3d-1} + 3v_{3,3d} - u_d = 0. \tag{2.5}$$

We can easily show them by definition of Σ_d.

For all $v \in \mathbb{R}^{6d+2}$, it is clear that there uniquely exist real numbers $x_{i,j}$, $(i = 1, 2,\ 0 \leq j \leq d)$, $x_{3,j}$, $(0 \leq j \leq 3d)$, y_k, $(1 \leq k \leq d-1)$ such that

$$v = \sum_{i=0}^{d}(x_{1,i}v_{1,i} + x_{2,i}v_{2,i}) + \sum_{j=0}^{3d} x_{3,j}v_{3,j} + \sum_{k=1}^{d-1} y_k u_k.$$

Then, we obtain:

$$\begin{aligned}
&v \\
=&v + \alpha_0(v_{0,0} + v_{1,0} + v_{2,0} + 3v_{3,0} + 2v_{3,1} + v_{3,2} - u_1) \\
&+ \alpha_d(v_{0,d} + v_{1,d} + v_{2,d} + v_{3,3d-2} + 2v_{3,3d-1} + 3v_{3,3d} - u_d) \\
&+ \sum_{i=1}^{d-1} \alpha_i(v_{0,i} + v_{1,i} + v_{2,i} + v_{3,3i-2} + 2v_{3,3i-1} + 3v_{3,3i} + 2v_{3,3i+1} + v_{3,3i+2})
\end{aligned}$$

$$\begin{aligned}
=&(\alpha_0 + x_{0,0})v_{0,0} + (\alpha_0 + x_{1,0})v_{1,0} + (\alpha_0 + x_{2,0})v_{2,0} + (3\alpha_0 + x_{3,0})v_{3,0} \\
&\quad + (2\alpha_0 + \alpha_1 + x_{3,1})v_{3,1} \\
&+ (\alpha_d + x_{0,d})v_{0,d} + (\alpha_d + x_{1,d})v_{1,d} + (\alpha_d + x_{2,d})v_{2,d} + (3\alpha_d + x_{3,3d})v_{3,3d} \\
&\quad + (2\alpha_d + \alpha_{d-1} + x_{3,3d-1})v_{3,3d-1}
\end{aligned}$$

$$\begin{aligned}
&+ \sum_{i=1}^{d-1}(\alpha_i v_{0,i} + (\alpha_i + x_{1,i})v_{1,i} + (\alpha_i + x_{2,i})v_{2,i} \\
&\quad + (\alpha_{i-1} + 2\alpha_i + x_{3,3i-1})v_{3,3i-1} + (3\alpha_i + x_{3,3i})v_{3,3i} \\
&\quad + (2\alpha_i + \alpha_{i+1} + x_{3,3i+1})v_{3,3i+1} + (-\alpha_{i-1} + 2\alpha_i - \alpha_{i+1} + y_i)u_i).
\end{aligned}$$

Hence, we should show the following claim:

Claim 2.1 *A map F_d from $\mathbb{R}^{d+1}$ to $\mathbb{R}^{d+1}$:*

$$\begin{pmatrix}
\min\{\alpha_0 + z_0, 2\alpha_0 + \alpha_1 + x_{3,1}\} \\
\min\{\alpha_1 + z_1, \alpha_0 + 2\alpha_1 + x_{3,2}, 2\alpha_1 + \alpha_2 + x_{3,4}, -\alpha_0 + 2\alpha_1 - \alpha_2 + y_1\} \\
\min\{\alpha_2 + z_2, \alpha_1 + 2\alpha_2 + x_{3,5}, 2\alpha_2 + \alpha_3 + x_{3,7}, -\alpha_1 + 2\alpha_2 - \alpha_3 + y_2\} \\
\vdots \\
\min\{\alpha_{d-1} + z_{d-1}, \alpha_{d-2} + 2\alpha_{d-1} + x_{3,3d-4}, 2\alpha_{d-1} + \alpha_d + x_{3,3d-2}, \\
-\alpha_{d-2} + 2\alpha_{d-1} - \alpha_d + y_{d-1}\} \\
\min\{\alpha_d + z_d, \alpha_{d-1} + 2\alpha_d + x_{3,3d-1}\}
\end{pmatrix}$$

is bijective, where $z_i := \max\{0, x_{0,i}, x_{1,i}, x_{2,i}, x_{3,3i}/3\}$.

The following two lemmas lead us to proof of the above claim:

Lemma 2.2 *$F_d(\alpha)$ is coherently oriented piecewise affine map (i.e. for any component $P \subset \mathbb{R}^{d+1}$ which $F(\alpha)$ is linear, $\det(F|_P)$ is positive).*

Lemma 2.3 *The* recession function *of F_d*

$$F_d^{\infty}(\alpha) := \begin{pmatrix} \min\{\alpha_0, 2\alpha_0 + \alpha_1\} \\ \min\{\alpha_1, \alpha_0 + 2\alpha_1, 2\alpha_1 + \alpha_2, -\alpha_0 + 2\alpha_1 - \alpha_2\} \\ \min\{\alpha_2, \alpha_1 + 2\alpha_2, 2\alpha_2 + \alpha_3, -\alpha_1 + 2\alpha_2 - \alpha_3\} \\ \vdots \\ \min\{\alpha_{d-1}, \alpha_{d-2} + 2\alpha_{d-1}, 2\alpha_{d-1} + \alpha_d, -\alpha_{d-2} + 2\alpha_{d-1} - \alpha_d\} \\ \min\{\alpha_d, \alpha_{d-1} + 2\alpha_d\} \end{pmatrix}$$

is bijective.

If these lemmas are proved, we can use the following theorem:

Theorem 2.2 (Theorem 2.5.1. of [12]) *A coherently oriented piecewise affine function is a homeomorphism if and only if its recession function is a homeomorphism.*

Hence, $F_d(\alpha)$ is also a homeomorphism (i.e., bijective).

2.2 Proof of Lemma 2.2.

The coefficient matrix of F_d is given by the following block representation of a $(d+1) \times (d+1)$ matrix:

$$F_d = \left(\begin{array}{cccccccccc} A_0 & \mathbf{c}_1 & & & & & & & & \\ & 1 & & & & & & & & \\ & \mathbf{d}_1 & A_1 & \mathbf{c}_2 & & & & & & \\ & & & 1 & & & & & & \\ & & & \mathbf{d}_2 & A_3 & \mathbf{c}_3 & & & & \\ & & & & & \ddots & & & & \\ & & & & & & \ddots & & & \\ & & & & & & & & 1 & \\ & & & & & & & & \mathbf{d}_r & A_r \end{array}\right), \tag{2.6}$$

where blank spaces are filled with 0 entries.

At this stage, we introduce the following notations.

$$0 < k_1 < k_2 < \cdots < k_r < d,$$
$$(\epsilon_0, \epsilon_1, \epsilon_1', \ldots, \epsilon_{d-1}, \epsilon_{d-1}', \epsilon_d) \in \{0, \pm 1\}^{2d}.$$

Then A_0 is a $k_1 \times k_1$ matrix given by,

$$A_0 \quad := \quad \begin{pmatrix} 1+\epsilon_0 & \epsilon_0 & 0 & \cdots & 0 & 0 \\ \epsilon_1 & 2 & \epsilon_1' & \cdots & 0 & 0 \\ 0 & \epsilon_2 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & \epsilon_{k_1-2}' \\ 0 & 0 & 0 & \cdots & \epsilon_{k_1-1} & 2 \end{pmatrix},$$

A_i $(i = 1, \cdots, r-1)$ is a $(k_{i+1} - k_i - 1) \times (k_{i+1} - k_i - 1)$ matrix given by,

$$A_i \quad := \quad \begin{pmatrix} 2 & \epsilon'_{k_i+1} & 0 & \cdots & 0 & 0 \\ \epsilon_{k_i+2} & 2 & \epsilon'_{k_i+2} & \cdots & 0 & 0 \\ 0 & \epsilon_{k_i+3} & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & \epsilon'_{k_{i+1}-2} \\ 0 & 0 & 0 & \cdots & \epsilon_{k_{i+1}-1} & 2 \end{pmatrix}, \quad (i = 1, 2, \cdots, r-1),.$$

and A_r is a $(d - k_r) \times (d - k_r)$ matrix given by,

$$A_r \quad := \quad \begin{pmatrix} 2 & \epsilon'_{k_r+1} & 0 & \cdots & 0 & 0 \\ \epsilon_{k_r+2} & 2 & \epsilon'_{k_r+2} & \cdots & 0 & 0 \\ 0 & \epsilon_{k_r+3} & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & \epsilon'_{d-1} \\ 0 & 0 & 0 & \cdots & \epsilon_d & 1 + \epsilon_d \end{pmatrix}.$$

$\mathbf{c}_i$'s and $\mathbf{d}_i$'s are column vectors whose sizes fit to the block representation (2.6).

$$\mathbf{c}_i := \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ \epsilon'_{k_i-1} \end{pmatrix}, \quad \mathbf{d}_i := \begin{pmatrix} \epsilon_{k_i+1} \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}, \quad (i = 1, 2, \cdots, r).$$

In the above matrices, we can assume that for $1 \leq i \leq d-1$,

$$(\epsilon_i, \epsilon'_i) = \begin{cases} (1,0), \ (0,1), \text{ or } (-1,-1), & (i \notin K) \\ (0,0), & (i \in K) \end{cases}$$

and

$$(\epsilon_0, \epsilon_d) \in \{0, 1\}^2,$$

where $K := \{k_1, k_2, \ldots, k_r\}$.

Then what we have to show is that determinant of the matrix is positive. From a formula

$$\det \begin{pmatrix} X & 0 \\ Z & Y \end{pmatrix} = \det(X)\det(Y),$$

and elementary operations of matrix, we can reduce the problem to check positivity of the determinant of the following matrix:

$$B_k := \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix} \in M_{k+1,k+1},$$

where $k > 0$. We can easily compute the determinant:

$$\det B_k = 9k - 6 > 0.$$

□

2.3 Proof of Lemma 2.3.

It is clear that

$$F_d^{\infty}(t\alpha) = tF_d^{\infty}(\alpha),$$

for all $t \in \mathbb{R}_{\geq 0}$ Hence, in order to prove that F_d^{∞} is injective, we only have to show that

$$G := \pi \circ F_d^{\infty}|_{S^d} : S^d \to S^d,$$

is injective. where $\pi : \mathbb{R}^{d+1} - \{0\} \to S^d$ is a projection.

Obviously G is continuous. Let $\tilde{G}$ be a smoothing of G. Note that we can take volume of smoothing locus of G as small as we like because non-smooth locus of G is measure 0. Then, $G(\alpha) \neq -\alpha$ holds for all $\alpha \in S^d$ since the diagonal elements of F_d^{∞} are all positive. Therefore,

$$H(t, \alpha) := \pi(t\alpha + (1-t)\tilde{G}(\alpha))$$

gives us a homotopy from $\tilde{G}$ to the identity mapping id_{S^d}. Thus, the mapping degree of $\tilde{G}$ is 1. Hence, $\tilde{G}$ is injective since Jacobian of $\tilde{G}$ is always positive. Accordingly, G is also injective. □

2.4 Some Properties of $\widetilde{Mp}_{0,2}(\mathbb{P}(1,1,1,3),d)$.

Since the fan Σ_d which we constructed in the previous section is complete and simplicial, the corresponding toric variety X_{Σ_d} is a compact orbifold.

In this subsection, we will show that the toric variety X_{Σ_d} realizes the action (2.2), and prove that $\widetilde{Mp}_{0,2}(\mathbb{P}(1,1,1,3),d)$ is a compact toric orbifold.

First, we determine the primitive collections of the fan Σ_d.

Lemma 2.4 *The primitive collections of the fan* Σ_d *are*

$$\begin{aligned} P_0 &:= \{v_{0,0}, v_{1,0}, v_{2,0}, v_{3,0}, v_{3,1}\}, \\ P_d &:= \{v_{0,d}, v_{1,d}, v_{2,d}, v_{3,3d-1}, v_{3,3d}\}, \\ P_i &:= \{v_{0,i}, v_{1,i}, v_{2,i}, v_{3,3i-1}, v_{3,3i}, v_{3,3i+1}, u_i\} \ (1 \leq i \leq d-1). \end{aligned}$$

Proof. By Definition 2.1, it is clear that they are primitive collections of Σ_d. If P is a primitive collection of Σ_d, then P does not generate any cone of Σ_d. Hence P has to contain some P_i ($i = 0, 1, \ldots, d$). If P_i is proper subset of P for some $i = 0, 1, \ldots, d$, then P_i does not generate any cone of Σ_d. Therefore, P should not be a primitive collection. Accordingly, $P = P_i$. □

We introduce the following notation,

$$\mathbb{C}^{\Sigma_d(1)} = \{(\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_d, a_{3,0}, a_{3,1}, \ldots, a_{3,3d}, u_1, u_2, \ldots, u_{d-1}) \mid \mathbf{a}_i \in \mathbb{C}^3,\ a_{3,j}, u_i \in \mathbb{C}\}.$$

where $\Sigma_d(1)$ is a collection of 1-dimensional cones of Σ_d. Then, we define a subset $Z(\Sigma_d)$ of $\mathbb{C}^{\Sigma_d(1)}$ as follows [*3].

$$Z(\Sigma_d) = \left\{ x \in \mathbb{C}^{\Sigma_d(1)} \mid \begin{array}{ll} (\mathbf{a}_0, a_{3,0}, a_{3,1}) = 0, & \\ (\mathbf{a}_i, a_{3,3i-1}, a_{3,3i}, a_{3,3i+1}, u_i) = 0, & (1 \leq i \leq d-1) \\ (\mathbf{a}_d, a_{3,3d-1}, a_{3,3d}) = 0, & \end{array} \right\}.$$

[*3] From the description of $Z(\Sigma_d)$, wa can see that both $(\mathbf{a}_0, a_{3,0})$ and $(\mathbf{a}_d, a_{3,3d})$ can be 0. Therefore, we allow images of $(1:0), (0:1) \in \mathbb{P}^1$ to be undefined. This is because we omit introducing boundary divisors that describe $a_{3,1}, a_{3,3d-1} \to \infty$.

Note that the toric variety corresponding to the fan Σ_d is given by quotient space $(\mathbb{C}^{\Sigma_d(1)}\backslash Z(\Sigma_d))/G$, where $G := \mathrm{Hom}_{\mathbb{Z}}(A_{\dim(X_{\Sigma_d})-1}(X_{\Sigma_d}), \mathbb{C}^*)$ (see [5] or Chap. 3 of [4]). The G-action is determined as follows.

Let $[D_{i,j}]$ (resp. $[U_k]$) be a divisor class that corresponds to 1-dimensional cone $v_{i,j}$ (resp. u_k). Furthermore, let

$$n := \dim(X_{\Sigma_d}) = 6d + 2.$$

Recall the following exact sequence:

$$0 \to M \to \mathbb{Z}^{\Sigma_d(1)} \to A_{n-1}(X_{\Sigma_d}) \to 0. \tag{2.7}$$

Here $M = \mathbb{Z}^{6d+2}$. Note that $A_{n-1}(X_{\Sigma_d})$ is generated by $[D_{i,j}]$ and $[U_k]$. $M \to \mathbb{Z}^{\Sigma_d(1)}$ and $\mathbb{Z}^{\Sigma_d(1)} \to A_{n-1}(X_{\Sigma_d})$ are given by

$$\begin{aligned}
&M \to \mathbb{Z}^{\Sigma_d(1)}; m \mapsto (\langle m, v_\rho\rangle)_{\rho\in\Sigma_d(1)}\\
&\mathbb{Z}^{\Sigma_d(1)} \to A_{n-1}(X_{\Sigma_d}); (a_\rho)_{\rho\in\Sigma_d(1)} \mapsto \sum_{\rho\in\Sigma_d(1)} a_\rho[D_\rho].
\end{aligned}$$

Here $v_\rho \in M$ is a generator of 1-dimensional cone $\rho \in \Sigma_d(1)$ and D_ρ is a divisor corresponding to $\rho \in \Sigma_d(1)$.

By the exact sequence (2.7) and definition of $v_{i,j}$ and u_k, we obtain the following relations of Chow group $A_{n-1}(X_{\Sigma_d})$:

$$\begin{cases}
[D_{0,j}] = [D_{1,j}] = [D_{2,j}]\ (0 \le j \le d),\\
[D_{3,3j}] = 3[D_{0,j}],\ (0 \le j \le d)\\
[D_{3,3j+1}] = 2[D_{0,j}] + [D_{0,j+1}],\ (0 \le j \le d-1)\\
[D_{3,3j+2}] = [D_{0,j}] + 2[D_{0,j+1}],\ (0 \le j \le d-1)\\
[U_k] = -[D_{0,k-1}] + 2[D_{0,k}] - [D_{0,k+1}]\ (1 \le k \le d-1).
\end{cases} \tag{2.8}$$

From these relations, it is easily shown that $G = \mathrm{Hom}_{\mathbb{Z}}(A_{n-1}(X_{\Sigma_d}), (\mathbb{C}^*)) \cong (\mathbb{C}^*)^{d+1}$. Let $\lambda_i := g([D_{i,1}])$ $(g \in G = \mathrm{Hom}_{\mathbb{Z}}(A_{n-1}(X_{\Sigma_d}), (\mathbb{C}^*)))$. The above relation tells us that $g([U_k]) = \lambda_{k-1}^{-1}\lambda_k^2\lambda_{k+1}^{-1}$, and the $(\mathbb{C}^*)^{d+1}$-action turns out to be,

$$\begin{aligned}
&(\lambda_0, \ldots, \lambda_d) \cdot (\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_d, a_{3,0}, a_{3,1}, \ldots, a_{3,3d}, u_1, u_2, \ldots, u_{d-1})\\
=&(\lambda_0\mathbf{a}_0, \lambda_1\mathbf{a}_1, \ldots, \lambda_d\mathbf{a}_d, \lambda_0^3 a_{3,0}, \lambda_0^2\lambda_1 a_{3,1}, \lambda_0\lambda_1^2 a_{3,2}, \lambda_1^3 a_{3,3}, \lambda_1^2\lambda_2 a_{3,4}, \ldots, \lambda_d^3 a_{3,3d},\\
&\lambda_0^{-1}\lambda_1^2\lambda_2^{-1}u_1, \lambda_1^{-1}\lambda_2^2\lambda_3^{-1}u_2, \ldots, \lambda_{d-2}^{-1}\lambda_{d-1}^2\lambda_d^{-1}u_{d-1}).
\end{aligned}$$

When $u_k = 1$ for all $k = 1, 2, \ldots, d-1$, by setting $\lambda_i = (\mu')^{d-i}(\nu')^i$, we obtain the action which is similar to (2.2).

Accordingly, we can identify $X_{\Sigma_d} = \widetilde{Mp}_{0,2}(\mathbb{P}(1,1,1,3), d)$. Hence, we obtain Theorem 1.2.

2.5 The Chow Ring of $\widetilde{Mp}_{0,2}(\mathbb{P}(1,1,1,3), d)$ (proof of proposition 1.1).

In this subsection, we will compute the Chow ring of $\widetilde{Mp}_{0,2}(\mathbb{P}(1,1,1,3), d)$. The recipe of computation is the same as the one in [13].

Let us recall the structure of Chow ring of general toric variety is given by

$$A^*(X_\Sigma) \cong \mathbb{C}[x_\rho | \rho \in \Sigma(1)]/(P(\Sigma) + SR(\Sigma)).$$

Here,

$$\begin{aligned} P(\Sigma) &:= \left\langle \sum_{\rho \in \Sigma(1)} \langle m, v_\rho \rangle \, x_\rho | m \in M \right\rangle \\ SR(\Sigma) &:= \langle x_{\rho_1} \cdots x_{\rho_k} | \{\rho_1, \ldots, \rho_k\} \,\text{is a primitive collection of}\, \Sigma \rangle . \end{aligned}$$

(see [6]).

Proof of proposition 1.1. It is easily see that

$$\mathbb{C}[x_\rho | \rho \in \Sigma_d(1)]/P(\Sigma_d) \cong \mathbb{C}[H_0, H_1, \ldots, H_d]$$

by setting $H_j := [D_{0,j}]$ $(j = 0, 1, \ldots, d)$ and the relations (2.8).

Recall that the primitive sets of Σ_d are

$$\begin{aligned} &P_0 := \{v_{0,0}, v_{1,0}, v_{2,0}, v_{3,0}, v_{3,1}\}, \\ &P_d := \{v_{0,d}, v_{1,d}, v_{2,d}, v_{3,3d-1}, v_{3,3d}\}, \\ &P_i := \{v_{0,i}, v_{1,i}, v_{2,i}, v_{3,3i-1}, v_{3,3i}, v_{3,3i+1}, u_i\} \ (1 \leq i \leq d-1). \end{aligned}$$

Then, the Stanley-Reisner ideal is

$$\begin{aligned} SR(\Sigma_d) =& (H_0^4(2H_0 + H_1), H_1^4(H_0 + 2H_1)(2H_1 + H_2)(-H_0 + 2H_1 - H_2), \\ & \ldots, H_{d-1}^4(H_{d-2} + 2H_{d-1})(2H_{d-1} + H_d)(-H_{d-2} + 2H_{d-1} - H_d), \\ & H_d^4(H_{d-1} + 2H_d)). \end{aligned}$$

□

2.6 The Intersection Numbers $w(\mathcal{O}_{h^a}\mathcal{O}_{h^b})_{0,d}$.

We use

$$\begin{aligned} \mathrm{Vol}_d : &= \left(\prod_{i=0}^{d} [D_{1,i}][D_{2,i}]\right) \cdot \left(\prod_{j=0}^{3d} [D_{3,j}]\right) \cdot \left(\prod_{k=1}^{d-1} [U_k]\right) \\ &= 3^{d+1} \left(\prod_{i=0}^{d} H_i^3\right) \cdot \left(\prod_{i=0}^{d-1} (2H_i + H_{i+1})(H_i + 2H_{i+1})\right) \\ &\quad \times \left(\prod_{k=1}^{d-1} (-H_{k-1} + 2H_k - H_{k+1})\right). \end{aligned}$$

as a volume form of $A^*(\widetilde{Mp}_{0,2}(\mathbb{P}(1,1,1,3), d))$ since it corresponds to Poincaré dual of a smooth point on the toric variety.

Let us explain Definition 1.1 of intersection numbers of $\widetilde{Mp}_{0,2}(\mathbb{P}(1,1,1,3), d)$ again:

Definition 2.2 *Let*

$$e^6(x,y) := \prod_{j=0}^{6}((6-j)x+y).$$

Then, we define the intersection number $w(\mathcal{O}_a\mathcal{O}_b)_{0,d}$ *over* $\widetilde{Mp}_{0,2}(\mathbb{P}(1,1,1,3),d)$ *as*

$$w(\mathcal{O}_{z^a}\mathcal{O}_{z^b})_{0,d} := \int_{\widetilde{Mp}_{0,2}(\mathbb{P}(1,1,1,3),d)} H_0^a H_d^b \cdot \frac{\displaystyle\prod_{i=1}^{d} e^6(H_{i-1},H_i)}{\displaystyle\prod_{i=1}^{d-1} 6H_i}.$$

This intersection number is a quasimap analogue of genus 0 degree d two point GW invariants of K3 surface in $\mathbb{P}(1,1,1,3)$.

Proposition 2.1 *For* $\Omega \in A^*(\widetilde{Mp}_{0,2}(\mathbb{P}(1,1,1,3),d))$*, the following equality holds.*

$$\int_{\widetilde{Mp}_{0,2}(\mathbb{P}(1,1,1,3),d)} \Omega = \prod_{i=0}^{d}\left(\frac{1}{2\pi\sqrt{-1}}\oint_{C_j} dz_j\right)\frac{\tilde{\Omega}}{R}$$

where

$$\begin{aligned} R \;=\; & 3^{d+1}\left(\prod_{i=0}^{d}(z_i)^4\right)(2z_0+z_1)\left(\prod_{i=1}^{d-1}(z_{i-1}+2z_i)(2z_i+z_{i+1})\right)(z_{d-1}+2z_d)\times \\ & \left(\prod_{i=1}^{d-1}(2z_j-z_{j-1}-z_{j+1})\right). \end{aligned}$$

$\frac{1}{2\pi\sqrt{-1}}\oint_{C_j}$ $(j=1,2,\cdots,d-1)$ *means taking residues at* $z_j=0$, $z_j=-\frac{z_{j-1}}{2}$ $z_j=-\frac{z_{j+1}}{2}$, $\frac{z_{j-1}+z_{j+1}}{2}$ *and* $\frac{1}{2\pi\sqrt{-1}}\oint_{C_0}$ *(resp.* $\frac{1}{2\pi\sqrt{-1}}\oint_{C_d}$*) means taking residues at* $z_0=0$, $z_0=-\frac{z_1}{2}$ *(resp.* $z_d=0$, $z_d=-\frac{z_{d-1}}{2}$*).* $\tilde{\Omega}$ *is a polynomial obtained from turning* H_i *into* z_i *in* Ω*.*

Proof. Obviouly, the correspondence $Res : \tilde{\Omega} \to \prod_{i=0}^{d}\left(\frac{1}{2\pi\sqrt{-1}}\oint_{C_j} dz_j\right)\frac{\tilde{\Omega}}{R}$ defines a linear map from $\mathbb{C}[z_0,z_1,\cdots,z_d]$ to $\mathbb{C}$. Let $\mathcal{I}$ be an ideal of $\mathbb{C}[z_0,z_1,\cdots,z_d]$ generated by,

$$\begin{aligned} & r_0 := z_0^4(2z_0+z_1), \\ & r_1 := z_1^4(2z_1+z_0)(2z_1+z_2)(2z_1-z_0-z_2), \\ & \qquad \vdots \\ & r_{d-1} := z_{d-1}^4(2z_{d-1}+z_{d-2})(2z_{d-1}+z_d)(2z_{d-1}-z_{d-2}-z_d), \\ & r_d := z_d^4(2z_d+z_{d-1}). \end{aligned} \tag{2.9}$$

If $\tilde{\Omega}$ takes the form $r_i \cdot f$ $(f \in \mathbb{C}[z_0,z_1,\cdots,z_d])$, $Res(r_i \cdot f) = 0$ since the integrand is holomorphic at the points where we take residues of z_i. Moreover, we can easily see by degree counting that $Res(\tilde{\Omega})$ vanishes if $\tilde{\Omega}$ is a homogeneous polynomial whose degree is not equal to $6d+2$. Therefore, *Res* turns out to be a map from $\big(\mathbb{C}[z_0, z_1, \cdots, z_d]/\mathcal{I}\big)_{6d+2}$ (homogeneous degree $6d+2$ part of $\mathbb{C}[z_0, z_1, \cdots, z_d]/\mathcal{I}$) to $\mathbb{C}$. By investigating Hilbert polynomial of the ring $\mathbb{C}[z_0, z_1, \cdots, z_d]/\mathcal{I}$ (which is isomorphic to $A^*(\widetilde{Mp}_{0,2}(\mathbb{P}(1,1,1,3), d))$), $\big(\mathbb{C}[z_0, z_1, \cdots, z_d]/\mathcal{I}\big)_{6d+2}$ turns out to be 1-dimensional. Hence we only have to check the following equality:

$$
\begin{aligned}
& Res(\widetilde{\mathrm{Vol}}_d) = 1, \\
\Longleftrightarrow \quad & \prod_{i=0}^{d}\left(\frac{1}{2\pi\sqrt{-1}}\oint_{C_j} dz_j\right)\left(\prod_{j=0}^{d}\frac{1}{z_j}\right) = 1.
\end{aligned}
\tag{2.10}
$$

But this is obvious. □.

Remark 2.1 *We can easily see from the proof of Proposition 2.1 that result of the residue integral does not depend on order of integration with respect to subscript j of z_j.*

3 Proof of Main Theorem 1.3

In this section, we prove our main theorem of this paper. Let us restate the theorem here.

Theorem 3.1 *Let w_d be the d-th expansion coefficient of inverse function of $-\log(j(\tau))$. Then*

$$w_d = \frac{1}{2} w(\mathcal{O}_{z^1}\mathcal{O}_{z^0})_{0,d}.$$

In order to prove it, we should show that

$$\frac{f_1(e^x)}{f_0(e^x)} = x + \sum_{d=1}^{\infty} w_d e^{dx} = x + \sum_{d=1}^{\infty} \frac{1}{2} w(\mathcal{O}_{z^1}\mathcal{O}_{z^0})_{0,d} e^{dx}, \tag{3.1}$$

where

$$
\begin{aligned}
f_0(z) &:= \sum_{d=0}^{\infty} \frac{2^{3d}\cdot(6d-1)!!}{(d!)^3} z^d, \\
f_1(z) &:= f_0(z)(\log(z)) + \sum_{d=0}^{\infty}\Big(\sum_{j=1}^{3d}\frac{6}{2j-1} - \sum_{k=1}^{d}\frac{3}{j}\Big)\frac{(6d-1)!!}{(d!)^3} z^d
\end{aligned}
$$

are two solutions of Picard-Fuchs equation (1.4) as subsection 1.3.

Let us introduce the following generating functions.

$$
\begin{aligned}
L_0(e^x) &:= 1 + \sum_{d=1}^{\infty} \frac{d}{2} w(\mathcal{O}_{z^2}\mathcal{O}_{z^{-1}})_{0,d} e^{dx}, \\
L_1(e^x) &:= 1 + \sum_{d=1}^{\infty} \frac{d}{2} w(\mathcal{O}_{z^1}\mathcal{O}_{z^0})_{0,d} e^{dx}.
\end{aligned}
$$

First lemma claims that $L_0(e^x)$ gives us the solution $f_0(e^x)$ of Picard-Fuchs equation (1.4).

Lemma 3.1

$$f_0(e^x) = L_0(e^x).$$

Proof. We have to prove that

$$\begin{aligned} \frac{2^{3d}(6d-1)!!}{(d!)^3} &= \frac{d}{2} w(\mathcal{O}_{z^2}\mathcal{O}_{z^{-1}})_{0,d} \\ \Longleftrightarrow \quad \frac{1}{2} w(\mathcal{O}_{z^2}\mathcal{O}_{z^{-1}})_{0,d} &= \frac{1}{d} \cdot \frac{2^{3d}(6d-1)!!}{(d!)^3} \end{aligned} \tag{3.2}$$

for all d. From Proposition 2.1, we have the following equality:

$$\begin{aligned} & \frac{1}{2} w(\mathcal{O}_{z^2}\mathcal{O}_{z^{-1}})_{0,d} \\ = \quad & \frac{1}{2} \left(\frac{1}{2\pi\sqrt{-1}} \oint_{C_j} dz_j \right) z_0^2 \left(\prod_{j=0}^{d-1} e^6(z_j, z_{j+1}) \right) \left(\prod_{j=1}^{d-1} \frac{1}{6z_j} \right) \frac{1}{z_d} \cdot \frac{1}{R} \\ = \quad & \frac{1}{2 \cdot 3^{d+1}} \prod_{j=0}^{d} \left(\frac{1}{2\pi\sqrt{-1}} \oint_{C_j} \frac{dz_j}{z_j^4} \right) z_0^2 \left(\prod_{j=1}^{d} \tilde{e}(z_{j-1}, z_j) \right) \\ & \times \left(\prod_{j=1}^{d-1} \frac{1}{6z_j(2z_j - z_{j-1} - z_{j+1})} \right) \frac{1}{z_d}, \end{aligned} \tag{3.3}$$

where

$$\tilde{e}(x,y) := 2^4 \cdot 3^2 xy \prod_{i=0}^{2} ((2i+1)x + (5-2i)y).$$

Since the integral is holomorphic at $z_0 = -\frac{z_1}{2}$, $z_j = -\frac{z_{j-1}}{2}, -\frac{z_{j+1}}{2}$ $(j = 1, \cdots, d-1)$, $z_d = -\frac{z_{d-1}}{2}$, we only have to take residues at $z_0 = 0, z_j = 0, \frac{z_{j-1}+z_{j+1}}{2}$ $(j = 1, \cdots, d-1), z_d = 0$. Note that the integral does not depend on order of integration. Therefore, we integrate the last line of (3.3) in ascending order of subscript j.

First, we integrate out z_0 variable. By picking up the factors containing z_0, integration is done as follows:

$$\begin{aligned} & \frac{1}{2 \cdot 3^{d+1}} \frac{1}{2\pi\sqrt{-1}} \oint_{C_{(0)}} \frac{dz_0}{z_0} 2^4 \cdot 3^2 z_1 \prod_{i=0}^{2} ((2i+1)z_0 + (5-2i)z_1) \frac{1}{2z_1 - z_0 - z_2} \\ = & \frac{2^3 \cdot 5!!}{3^{d-1}} \frac{z_1^4}{2z_1 - z_2}. \end{aligned}$$

Then we integrate z_1 variable. Since the integrand is holomorphic at $z_1 = 0$, we only have to take residue at $z_1 = z_2/2$.

$$\begin{aligned} & \frac{1}{2 \cdot 3^{d+1}} \prod_{j=0}^{d} \frac{1}{2\pi\sqrt{-1}} \oint_{C_{z_2/2}} dz_1 \cdot 2^3 \cdot 3z_2 \prod_{i=0}^{2} ((2i+1)z_1 + (5-2i)z_2) \frac{1}{2z_2 - z_1 - z_3} \\ = & \frac{1}{2} \cdot \frac{2^6 \cdot 11!!}{3^{d-2} \cdot (2!)^3} \cdot \frac{z_2^4}{\frac{3}{2}z_2 - z_2}. \end{aligned}$$

Here, we use the identity:

$$\prod_{i=0}^{2}((2i+1)\frac{z_2}{2}+(5-2i)z_2)=(z_2)^3\prod_{i=0}^{2}\frac{11-2i}{2}=z_2^3\frac{11!!}{5!!\cdot 2^3}.$$

Integration of z_i $(i=1,2,\ldots,d-1)$ goes in the same way. We only have to take residue at $z_j=\frac{j}{j+1}z_{j+1}$. After finishing integration of z_{d-1}, what remains to do is the following integration.

$$\frac{1}{d}\cdot\frac{2^{3d}\cdot(6d-1)!!}{(d!)^3}\frac{1}{2\pi\sqrt{-1}}\oint_{C_{(0)}}\frac{dz_d}{z_d}.$$

Hence we obtain the equality (3.2). □

The following is the second lemma:

Lemma 3.2

$$\begin{aligned}&\frac{1}{2\cdot 3^{d+1}}\prod_{j=0}^{d}\left(\frac{1}{2\pi\sqrt{-1}}\oint_{C_j}\frac{dz_j}{z_j^4}\right)z_0z_1\left(\prod_{j=1}^{d}\tilde{e}(z_{j-1},z_j)\right)\\&\times\left(\prod_{j=1}^{d-1}\frac{1}{6z_j(2z_j-z_{j-1}-z_{j+1})}\right)\frac{1}{z_d}\\&.=\frac{1}{d}\cdot\frac{2^{3d}(6d-1)!!}{(d!)^3}\left(1-\frac{1}{d}+\sum_{j=1}^{3d}\frac{6}{2j-1}-\sum_{j=1}^{d}\frac{3}{j}\right).\end{aligned}$$

Proof. In the same way as the proof previous lemma, we begin by integrating out z_0 variable.

$$\begin{aligned}&\frac{1}{2\cdot 3^{d+1}}\frac{1}{2\pi\sqrt{-1}}\oint_{C_{(0)}}\frac{dz_0}{z_0^2}2^4\cdot 3^2z_1\prod_{i=0}^{2}((2i+1)z_0+(5-2i)z_1)\frac{1}{2z_1-z_0-z_2}\\&=\frac{2^3\cdot 5!!}{3^{d-1}}\left((a_1)\frac{z_1^3}{2z_1-z_2}+\frac{z_1^4}{(2z_1-z_2)^2}\right),\end{aligned}\tag{3.4}$$

where

$$a_1:=\sum_{i=0}^{2}\frac{2i+1}{5-2i}.$$

In deriving (3.4), we used the following equality:

$$\begin{aligned}&\frac{\partial}{\partial z_0}\left(\prod_{i=0}^{2}((2i+1)z_0+(5-2i)z_1)\frac{1}{2z_1-z_0-z_2}\right)\\&=\left(\prod_{i=0}^{2}((2i+1)z_0+(5-2i)z_1)\right)\\&\qquad\times\frac{1}{2z_1-z_0-z_2}\left(\sum_{i=0}^{2}\frac{2i+1}{(2i+1)z_0+(5-2i)z_1}+\frac{1}{2z_1-z_0-z_2}\right).\end{aligned}$$

Since we have another z_1 factor in the integrand, it becomes holomorphic at $z_1 = 0$ after integration of z_0. Hence integration of z_1 variable is done by taking residue at $z_1 = z_2/2$.

$$
\begin{aligned}
&\frac{3^3\cdot 5!!}{3^{d-1}}\frac{1}{2\pi\sqrt{-1}}\oint_{C_{(z_2/2)}} dz_1 (a_1\frac{1}{2z_1-z_2}+\frac{z_1}{(2z_1-z_2)^2}) \\
&\quad \times 2^4\cdot 3^2 z_2\prod_{i=0}^{2}((2i+1)z_1+(5-2i)z_2)\frac{1}{2z_1-z_0-z_2} \\
=&\frac{1}{2}\cdot\frac{2^6\cdot 11!!}{3^{d-2}(2!)^3}\left(a_2\frac{z_2^4}{\frac{3}{2}z_2-z_3}+\frac{1}{4}\frac{z_2^5}{(\frac{3}{2}z_2-z_3)^2}\right). \qquad (3.5)
\end{aligned}
$$

Here, a_2 is given by

$$
a_2 := a_1 + \frac{1}{2\cdot 1} + \frac{1}{4}\sum_{i=0}^{2}\frac{4i+2}{11-2i}.
$$

In deriving (3.5), we used the following equality:

$$
\begin{aligned}
&\frac{\partial}{\partial z_1}\left(z_1(\prod_{i=0}^{2}((2i+1)z_1+(5-2i)z_2))\frac{1}{2z_2-z_1-z_3}\right) \\
=&z_1\left(\prod_{i=0}^{2}((2i+1)z_1+(5-2i)z_2)\right)\frac{1}{2z_2-z_1-z_3} \\
&\times\left(\frac{1}{z_1}+\sum_{i=0}^{2}\frac{2i+1}{(2i+1)z_1+(5-2i)z_2}+\frac{1}{2z_2-z_1-z_3}\right).
\end{aligned}
$$

Integration of z_j ($j = 2, 3, \ldots, d-2$) goes in the same way. After finishing integration of z_{d-1}, the LHS of this lemma becomes

$$
\frac{1}{d}\cdot\frac{2^3 d\cdot(6d-1)!!}{(d!)^3}a_d\frac{1}{2\pi\sqrt{-1}}\oint_{C_{(0)}}\frac{dz_d}{z_d},
$$

where

$$
\begin{aligned}
a_d &= \sum_{j=2}^{d}\frac{1}{j(j-1)}+\sum_{j=1}^{d}\frac{1}{j^2}\sum_{i=0}^{2}\frac{j(2i+1)}{6j-1-2i} \\
&= \sum_{j=2}^{d}\left(\frac{1}{j-1}-\frac{1}{j}\right)+\sum_{j=1}^{d}\sum_{i=0}^{2}\frac{(2i+1)}{(6j-1-2i)j} \\
&= 1-\frac{1}{d}+\sum_{j=1}^{d}\sum_{i=0}^{2}(\frac{6}{6j-1-2i}-\frac{6}{6j}) \\
&= 1-\frac{1}{d}+\sum_{j=1}^{3d}\frac{6}{2j-1}-\sum_{j=1}^{d}\frac{3}{j}.
\end{aligned}
$$

Integration of z_d immediately leads us to the assertion of the lemma. □

We prove the last lemma:

Lemma 3.3

$$
\begin{aligned}
&\frac{1}{2}w(\mathcal{O}_{z^1}\mathcal{O}_{z^0})_{0,d-f}\cdot w(\mathcal{O}_{z^2}\mathcal{O}_{z^{-1}})_{0,f}\\
=&\frac{1}{2\cdot 3^{d+1}}\prod_{j=0}^{d}\left(\frac{1}{2\pi\sqrt{-1}}\oint_{C_j}\frac{dz_j}{z_j^4}\right)z_0(2z_{d-f}-z_{d-f-1}-z_{d-f+1})\\
&\times\left(\prod_{j=1}^{d}\tilde{e}(z_{j-1},z_j)\right)\left(\prod_{j=1}^{d-1}\frac{1}{6z_j(2z_j-z_{j-1}-z_{j+1})}\right)\frac{1}{z_d} \qquad (3.6)
\end{aligned}
$$

for all $1\leq f\leq d-1$*. Here,* $\frac{1}{2\pi\sqrt{-1}}\oint_{C_j}dz_j$ *means taking residues at* $z_0=0$ $(j=0), z_j=0,\ \frac{z_{j-1}+z_{j+1}}{2}$ $(j=1,\cdots,d-1),\ z_d=0$ $(j=d)$.

Proof. Note that $\frac{1}{2}w(\mathcal{O}_{z^1}\mathcal{O}_{z^0})_{0,d-f}$ is given by,

$$
\begin{aligned}
&\frac{1}{2\cdot 3^{d-f+1}}\prod_{j=0}^{d-f}\left(\frac{1}{2\pi\sqrt{-1}}\oint_{C_j}\frac{dz_j}{z_j^4}\right)z_0\left(\prod_{j=1}^{d-f}\tilde{e}(z_{j-1},z_j)\right)\\
&\times\left(\prod_{j=1}^{d-f-1}\frac{1}{6z_j(2z_j-z_{j-1}-z_{j+1})}\right).
\end{aligned}
$$

Since the integrand in (3.6) is holomorphic at $z_{d-f}=(z_{d-f-1}+z_{d-f+1})/2$, the assertion of the lemma follows from integration of z_j's in ascending order of the subscript i. □

Proof of the Main Theorem 1.3.
Let

$$
\int L_1(e^x)dx = x+\sum_{d=1}^{\infty}\frac{1}{2}w(\mathcal{O}_{z^1}\mathcal{O}_{z^0})_{0,d}e^{dx}
$$

be a primitive function of $L_1(e^x)$. Assertion of the theorem 1.3 is equivalent to the following equality:

$$
f_1(e^x)=f_0(e^x)\int L_1(e^x)dx = L_0(e^x)\int L_1(e^x)dx, \qquad (3.7)
$$

where we used Lemma 3.1. Expanding RHS of (3.7), we obtain,

$$
x\cdot L_0(e^x)+\sum_{d=1}^{\infty}\left(\frac{1}{2}w(\mathcal{O}_{z^1}\mathcal{O}_{z^0})_{0,d}+\sum_{f=1}^{d-1}(\frac{f}{4}w(\mathcal{O}_{z^1}\mathcal{O}_{z^0})_{0,d-f}\cdot w(\mathcal{O}_{z^2}\mathcal{O}_{z^{-1}})_{0,f})\right)e^{dx}.
$$

We can easily derive the equality:$\sum_{f=1}^{d-1}f(2z_{d-f}-z_{d-f-1}-z_{d-f+1})=d(z_1-z_0)+z_0-z_d$. Hence application of Lemma 3.3 and Lemma 3.1 to the RHS of (3.7) results in,

$$
x\cdot f_0(e^x)+\sum_{d=1}^{\infty}R_d e^{dx}, \qquad (3.8)
$$

where

$$\begin{aligned}R_d :=& \frac{1}{2\cdot 3^{d+1}}\prod_{j=0}^{d}\left(\frac{1}{2\pi\sqrt{-1}}\oint_{C_j}\frac{dz_j}{z_j^4}\right) z_0(d(z_1-z_0)+z_0)\\ &\times\left(\prod_{j=1}^{d}\tilde{e}(z_{j-1},z_j)\right)\left(\prod_{j=1}^{d-1}\frac{1}{6z_j(2z_j-z_{j-1}-z_{j+1})}\right)\frac{1}{z_d}.\end{aligned}$$

By combining Lemma 3.1 and 3.2, we can derive

$$R_d = \frac{2^{3d}(6d-1)!!}{(d!)^3}\left(\sum_{j=1}^{3d}\frac{6}{2j-1}-\sum_{j=1}^{d}\frac{3}{j}\right).$$

Therefore, the RHS of (3.7) becomes,

$$x\cdot f_0(e^x)+\sum_{d=1}^{\infty}(\sum_{j=1}^{3d}\frac{6}{2j-1}-\sum_{j=1}^{d}\frac{3}{j})\frac{2^{3d}(6d-1)!!}{(d!)^3}e^{dx}. \qquad (3.9)$$

The formula (1.6) tells us that it is nothing but $f_1(e^x)$. $\Box$

Remark 3.1

In the proof of the main theorem in section 3, we used the technique of residue integral. In this remark, we demonstrate how to translate this analytic argument into ring theoretic one by taking Lemma 3.1 as an example.

We have to compute

$$\int_{\widetilde{Mp}_{0,2}(\mathbb{P}(1,1,1,3),d)}\frac{H_0^2}{H_d}E_d, \qquad (3.10)$$

where

$$\begin{aligned}E_d :=& \frac{\prod_{i=1}^{d}e^6(H_{i-1},H_i)}{\prod_{i=1}^{d-1}6H_i}\\ =& \frac{1}{3^{d+1}}\cdot\frac{\prod_{i=1}^{d}\tilde{e}(H_{i-1},H_i)}{\prod_{i=1}^{d-1}6H_i}\cdot\left(3^{d+1}\prod_{i=0}^{d-1}(2H_i+H_{i+1})(H_i+2H_{i+1})\right).\end{aligned}$$

Since E_d has a factor $3^{d+1}\prod_{i=0}^{d-1}(2H_i+H_{i-1})(H_i+2H_{i+1})$, we can compute (3.10) as

$$\frac{H_0^2}{H_d}\frac{1}{3^{d+1}}\cdot\frac{\prod_{i=1}^{d}\tilde{e}(H_{i-1},H_i)}{\prod_{i=1}^{d-1}6H_i}$$

in $\mathbb{C}[H_0,\dots,H_d]/(H_0^4, H_1^4(-H_0+2H_1-H_2),\dots,H_{d-1}^4(-H_{d-2}+2H_{d-1}-H_d),H_d^4)$.

We can prove that

$$H_0^3H_1^4H_2^4\cdots H_{j-1}^4H_j^5=\frac{j}{j+1}H_0^3H_1^4H_2^4\cdots H_j^4H_{j+1}$$

for all $j = 0, 1, 2, \ldots, d-1$ by induction. Then, we obtain

$$\begin{aligned}
&\frac{H_0^3 H_1^4 H_2^4 \cdots H_{j-2}^4 H_{j-1}^3 \tilde{e}(H_{j-1}, H_j)}{H_j} \\
=&2^4 \cdot 3^2 H_0^3 H_1^4 H_2^4 \cdots H_{j-2}^4 H_{j-1}^4 (5H_{j-1} + H_j)(3H_{j-1} + 3H_j)(H_{j-1} + H_j) \\
=&2^4 \cdot 3^2 H_0^3 H_1^4 H_2^4 \cdots H_{j-2}^4 H_{j-1}^4 (\frac{5(j-1)}{j} + 1)(\frac{3(j-1)}{j} + 3)(\frac{j-1}{j} + 5) H_j^3 \\
=&2^4 \cdot 3^2 \frac{(6j-1)!!}{j^3 \cdot (6j-7)!!} H_0^3 H_1^4 H_2^4 \cdots H_{j-2}^4 H_{j-1}^4 H_j^3 .
\end{aligned}$$

Successive applications of the above equality lead us to the assertion of Lemma 3.1.

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