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Lauricella's F_C with finite irreducible monodromy group

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Abstract. This paper presents our study of the conditions under which the monodromy group for Lauricella's hypergeometric function $F_C(a,b,c;x)$ is finite irreducible. We provide these conditions in terms of the parameters a, b, c. In addition, we discuss the structure of the finite irreducible monodromy group.

1. Introduction

Lauricella's hypergeometric series F_C of n variables $x_1, \dots, x_n$ with complex parameters a, b, $c_1, \dots, c_n$ is defined by

$$F_C(a,b,c;x) = \sum_{m_1,\dots,m_n=0}^{\infty} \frac{(a, m_1+\cdots+m_n)(b, m_1+\cdots+m_n)}{(c_1,m_1)\cdots(c_n,m_n)m_1!\cdots m_n!} x_1^{m_1}\cdots x_n^{m_n},$$

where $x = (x_1, \dots, x_n)$, $c = (c_1, \dots, c_n)$, $c_1, \dots, c_n \notin \{0, -1, -2, \dots\}$, and $(c_1, m_1) = \Gamma(c_1+m_1)/\Gamma(c_1)$. This series converges in the domain

$$D_C = \left\{ (x_1, \dots, x_n) \in \mathbb{C}^n \;\middle|\; \sum_{k=1}^{n} \sqrt{|x_k|} < 1 \right\}.$$

The hypergeometric system $E_C(a,b,c)$ of differential equations (see (3)) satisfied by $F_C(a,b,c;x)$ was shown [8] to be a holonomic system of rank 2^n with the singular locus

$$S = \Big(\prod_{k=1}^{n} x_k \cdot R(x) = 0 \Big) \subset \mathbb{C}^n, \quad R(x_1, \dots, x_n) = \prod_{\varepsilon_1,\dots,\varepsilon_n=\pm1} \Big(1 + \sum_{k=1}^{n} \varepsilon_k \sqrt{x_k} \Big),$$

and that the system $E_C(a,b,c)$ is irreducible if and only if the parameters a, b, c satisfy

$$(1) \qquad a - \sum_{k=1}^{n} i_k c_k, \quad b - \sum_{k=1}^{n} i_k c_k \notin \mathbb{Z}, \qquad \forall I = (i_1, \dots, i_n) \in \{0,1\}^n.$$

In [7], we constructed a fundamental system $\{\widetilde{F}_I\}$ of solutions to $E_C(a,b,c)$ in a simply connected domain in $D_C - S$ under the condition (1); for details, see Fact 2.2.

Let X be the complement of the singular locus S. The fundamental group of X is generated by $n+1$ loops $\rho_0, \rho_1, \dots, \rho_n$ (see Subsection 2.2). In [4], we expressed the circuit transformations $\mathcal{M}_i$ along ρ_i ($i = 0, \dots, n$) by the intersection form on twisted homology groups associated with the Euler-type integrals of solutions to $E_C(a,b,c)$. In [7], we also obtained their representation matrices M_i ($i = 0, \dots, n$) with respect to the basis $\{\widetilde{F}_I\}$. These matrices are of simple forms.

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In this paper, we present our study of the monodromy group **Mon**, which is a subgroup of $\mathbf{GL}_{2^n}(\mathbb{C})$ generated by these representation matrices. When $n = 2$, Lauricella's F_C is also known as Appell's F_4. Several studies have been conducted on the monodromy group. For example, the finite monodromy group was studied in [9] and [10], and the Zariski closure of **Mon**, which is the Picard–Vessiot group, was studied in [12].

In [6], we previously investigated the Zariski closure of **Mon** for general n. In this study, as a generalization of [9], we provide the conditions under which **Mon** is finite irreducible. As was mentioned in [6, Proposition 2.14], the (in)finiteness conditions are important to classify the Zariski closure of **Mon**. The conditions for the finite irreducible monodromy group are given as follows (another formulation is given in Theorem 2.6).

THEOREM 1.1. *We assume $n \geq 3$. The monodromy group* **Mon** *is finite irreducible if and only if the following two conditions hold:*

(A) for each $k = 1, \dots, n$, the monodromy group for Gauss' hypergeometric differential equation ${}_2E_1(a, b, c_k)$ is finite irreducible;

(B) at least n of $c_1, \dots, c_n$, $b - a$, $c_1 + \cdots + c_n - a - b - (n-1)/2$ are equivalent to $1/2$ modulo $\mathbb{Z}$.

We prove this theorem by focusing on the reflection subgroup **Ref**, which is a normal subgroup generated by the reflection M_0 (see Section 3). Certain concepts in our proofs are based on those of [9]. However, we note that the condition (B) in Theorem 1.1 is not a direct generalization of [9] (see Remark 2.8 (ii)).

The finiteness condition is also known as the algebraicity condition. Namely, the monodromy group is finite if and only if the solutions to $E_C(a, b, c)$ are algebraic functions over $\mathbb{C}(x_1, \dots, x_n)$. In [3], algebraicity conditions of $E_C(a, b, c)$ were determined by using results of [2], which were obtained by studying the algebraicity conditions for A-hypergeometric systems. Their approaches are quite different from ours. The monodromy group was not treated directly in [2] and [3], whereas in our study, we investigate it in detail. Further, using our results for the reflection subgroup, we can also determine the structure of the finite irreducible monodromy group.

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2. Preliminaries

In this section, we present certain pertinent facts about Lauricella's F_C mentioned in previous studies ([7], [8], and [11]). We set

$$\alpha = \exp(2\pi\sqrt{-1}a), \quad \beta = \exp(2\pi\sqrt{-1}b), \quad \gamma_k = \exp(2\pi\sqrt{-1}c_k) \ (k = 1, \dots, n).$$

Under these notations, the condition (1) is equivalent to

$$\alpha - \prod_{k=1}^{n} \gamma_k^{i_k}, \quad \beta - \prod_{k=1}^{n} \gamma_k^{i_k} \neq 0, \qquad \forall I = (i_1, \dots, i_n) \in \{0, 1\}^n. \tag{2}$$

2.1. System of differential equations

For $k = 1, \dots, n$, let ∂_k be the partial differential operator with respect to x_k. We set $\theta_k = x_k \partial_k$ and $\theta = \sum_{k=1}^n \theta_k$. Lauricella's $F_C(a, b, c; x)$ satisfies the system of differential equations

$$[\theta_k(\theta_k + c_k - 1) - x_k(\theta + a)(\theta + b)]\, f(x) = 0 \quad (k = 1, \dots, n). \tag{3}$$

The system (3) is known as Lauricella's hypergeometric system $E_C(a, b, c)$ of differential equations. By [8], the left ideal generated by the differential operators (3) is a holonomic ideal of rank 2^n with the singular locus S, and the system is irreducible if and only if the parameters $a, b, c_1, \dots, c_n$ satisfy (1) (equivalently, $\alpha, \beta, \gamma_1, \dots, \gamma_n$ satisfy (2)).

Set $\dot{x} = \left(\frac{1}{2n^2}, \dots, \frac{1}{2n^2}\right) \in X$, and let $Sol_{\dot{x}}$ be the local solution space to $E_C(a, b, c)$ around $\dot{x}$. For $I = (i_1, \dots, i_n) \in \{0, 1\}^n$, we set

$$F_I(x) = \frac{\prod_{k=1}^n \Gamma((-1)^{i_k}(1 - c_k))}{\Gamma(1 - a^I)\Gamma(1 - b^I)} \cdot \prod_{k=1}^n x_k^{i_k(1-c_k)} \cdot F_C(a^I, b^I, c^I; x),$$

where

$$a^I = a + \sum_{k=1}^n i_k(1 - c_k), \quad b^I = b + \sum_{k=1}^n i_k(1 - c_k),$$
$$c^I = (c_1 + 2i_1(1 - c_1), \dots, c_n + 2i_n(1 - c_n)).$$

It is known that the functions $\{F_I\}_{I \in \{0,1\}^n}$ form the basis of $Sol_{\dot{x}}$ under the conditions (1) and $c_1, \dots, c_n \notin \mathbb{Z}$.

2.2. Monodromy representation and fundamental group

For $\rho \in \pi_1(X, \dot{x})$ and $g \in Sol_{\dot{x}}$, let $\rho_* g$ be the analytic continuation of g along ρ. Since $\rho_* g$ is also a solution to $E_C(a, b, c)$, the map $\rho_* : Sol_{\dot{x}} \to Sol_{\dot{x}};\ g \mapsto \rho_* g$ defines a linear automorphism. Thus, we obtain the *monodromy representation*

$$\mathcal{M} : \pi_1(X, \dot{x}) \to GL(Sol_{\dot{x}});\ \rho \mapsto \rho_*$$

of $E_C(a, b, c)$, where $GL(V)$ is the general linear group on a vector space V.

Next, we introduce the generators of the fundamental group $\pi_1(X, \dot{x})$. Let $\rho_0, \rho_1, \dots, \rho_n$ be loops in X such that

- ρ_0 turns around the hypersurface $(R(x) = 0)$ near the point $\left(\frac{1}{n^2}, \dots, \frac{1}{n^2}\right) \in (R(x) = 0)$, positively;
- ρ_k $(k = 1, \dots, n)$ turns around the hyperplane $(x_k = 0)$, positively.

The precise definitions can be found in [4].

Fact 2.1 ([4], [5], [14]). *The loops $\rho_0, \rho_1, \dots, \rho_n$ generate the fundamental group $\pi_1(X, \dot{x})$. If $n \geq 2$, they satisfy*

$$\rho_i \rho_j = \rho_j \rho_i \quad (i, j = 1, \dots, n), \quad (\rho_0 \rho_k)^2 = (\rho_k \rho_0)^2 \quad (k = 1, \dots, n).$$

In addition, if $n \geq 3$, they also satisfy

$$
\begin{aligned}
&(\rho_{i_1}\cdots\rho_{i_p})\rho_0(\rho_{i_1}\cdots\rho_{i_p})^{-1}\cdot(\rho_{j_1}\cdots\rho_{j_q})\rho_0(\rho_{j_1}\cdots\rho_{j_q})^{-1}\\
&=(\rho_{j_1}\cdots\rho_{j_q})\rho_0(\rho_{j_1}\cdots\rho_{j_q})^{-1}\cdot(\rho_{i_1}\cdots\rho_{i_p})\rho_0(\rho_{i_1}\cdots\rho_{i_p})^{-1},
\end{aligned}
$$

for $I=\{i_1,\dots,i_p\}$, $J=\{j_1,\dots,j_q\}\subset\{1,\dots,n\}$ with $p,q\geq 1$, $p+q\leq n-1$ and $I\cap J=\varnothing$. Further, these relations generate all relations in $\pi_1(X,\dot{x})$.

In [4], $n+1$ linear maps $\mathcal{M}_i=\mathcal{M}(\rho_i)$ $(i=0,\dots,n)$ were investigated in terms of twisted homology groups and the intersection form.

2.3. Representation matrices of monodromy

As in [6] and [7], we define the tensor product $A\otimes B$ of matrices A and $B=(b_{ij})_{1\leq i\leq r,1\leq j\leq s}$ as

$$
A\otimes B=\begin{pmatrix} A\,b_{11} & A\,b_{12} & \cdots & A\,b_{1s}\\ A\,b_{21} & A\,b_{22} & \cdots & A\,b_{2s}\\ \vdots & \vdots & \ddots & \vdots\\ A\,b_{r1} & A\,b_{r2} & \cdots & A\,b_{rs}\end{pmatrix}.
$$

We regard $\mathbb{C}^{2^n}$ as $\mathbb{C}^2\otimes\cdots\otimes\mathbb{C}^2$, and take as basis

$$
\mathbf{e}_I=\mathbf{e}_{i_1,\dots,i_n}=\mathbf{e}_{i_1}\otimes\cdots\otimes\mathbf{e}_{i_n},\qquad \mathbf{e}_0=\begin{pmatrix}1\\0\end{pmatrix},\ \mathbf{e}_1=\begin{pmatrix}0\\1\end{pmatrix},\quad I=(i_1,\dots,i_n)\in\{0,1\}^n.
$$

We align this basis in the order $\prec$ of indices $I=(i_1,\dots,i_n)\in\{0,1\}^n$, which is given by

$$
\begin{aligned}
&(i_1,\dots,i_n)\prec(i'_1,\dots,i'_n)\\
&\Longleftrightarrow \exists r\ \text{ s.t. }\ i_j=i'_j\ (j=r+1,\dots,n),\ i_r=0,\ i'_r=1.
\end{aligned}
$$

For example, we align the basis of $\mathbb{C}^8=\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2$ $(n=3)$ as follows:

$$
(\mathbf{e}_{0,0,0},\mathbf{e}_{1,0,0},\mathbf{e}_{0,1,0},\mathbf{e}_{1,1,0},\mathbf{e}_{0,0,1},\mathbf{e}_{1,0,1},\mathbf{e}_{0,1,1},\mathbf{e}_{1,1,1}).
$$

Fact 2.2 ([7, Theorem 3.3]). *We define $\{\widetilde{F}_I\}_{I\in\{0,1\}^n}$ to be*

$$
(\dots,\widetilde{F}_I(x),\dots)=(\dots,F_I(x),\dots)\cdot\left(\begin{pmatrix}1-\gamma_1 & 1\\ 0 & 1\end{pmatrix}\otimes\cdots\otimes\begin{pmatrix}1-\gamma_n & 1\\ 0 & 1\end{pmatrix}\right).
$$

Then, $\{\widetilde{F}_I\}_{I\in\{0,1\}^n}$ form the basis of $Sol_{\dot{x}}$ under the condition (1) only.

In [7], the representation matrices of $\mathcal{M}_i$'s with respect to the basis $\{\widetilde{F}_I\}_I$ were obtained; they are simple matrices. Let E_m be a unit matrix of size m.

Fact 2.3 ([7, Corollary 3.5]). *Let M_i be the representation matrix of $\mathcal{M}_i$ $(i=$*

$0, \dots, n)$ *with respect to the basis* $\{\widetilde{F}_I\}_I$*. For* $k = 1, \dots, n$*, we have*

$$M_k = E_2 \otimes \cdots \otimes E_2 \otimes \underset{kth}{G(\gamma_k)} \otimes E_2 \otimes \cdots \otimes E_2, \quad G(\gamma_k) = \begin{pmatrix} 1 & -\gamma_k^{-1} \\ 0 & \gamma_k^{-1} \end{pmatrix}.$$

The matrix M_0 *is written as*

$$M_0 = E_{2^n} - {}^t(\mathbf{0}, \dots, \mathbf{0}, \boldsymbol{v}),$$

where $\boldsymbol{v} \in \mathbb{C}^{2^n}$ *is a column vector of which the* I*-th entry is*

$$\begin{cases} (-1)^n \frac{(\alpha-1)(\beta-1)\prod_{k=1}^n \gamma_k}{\alpha\beta} & (I = (0, \dots, 0)), \\ (-1)^{n+|I|} \frac{(\alpha\beta + (-1)^{|I|}\prod_{k=1}^n \gamma_k^{i_k})\prod_{k=1}^n \gamma_k^{1-i_k}}{\alpha\beta} & (I \neq (0, \dots, 0)). \end{cases}$$

The aforementioned expressions indicate that these representation matrices depend on the parameters α, β, $\gamma_1, \dots, \gamma_n$. In other words, they are determined by the parameters a, b, $c_1, \dots, c_n$ modulo $\mathbb{Z}$. Thus, we often write $M_i = M_i^{(n)}(\alpha, \beta, \gamma)$ and $\boldsymbol{v} = \boldsymbol{v}^{(n)}(\alpha, \beta, \gamma) = {}^t(\dots, v_I^{(n)}(\alpha, \beta, \gamma), \dots)$.

REMARK 2.4. $\mathbf{e}_{1,\dots,1} = \mathbf{e}_1 \otimes \cdots \otimes \mathbf{e}_1$ is an eigenvector of $M_0 = M_0^{(n)}(\alpha, \beta, \gamma)$, that is,

$$M_0 \mathbf{e}_{1,\dots,1} = \delta_0^{(n)}(\alpha, \beta, \gamma)\mathbf{e}_{1,\dots,1}, \quad \delta_0^{(n)}(\alpha, \beta, \gamma) = (-1)^{n+1}\frac{\gamma_1 \cdots \gamma_n}{\alpha\beta}.$$

The eigenspace of M_0 with eigenvalue 1 is $2^n - 1$ dimensional. The matrix $M_0^{(n)}(\alpha, \beta, \gamma)$ is a "reflection" (see Section 3) with the special eigenvalue $\delta_0^{(n)}(\alpha, \beta, \gamma)$.

EXAMPLE 2.5. In the case $n = 2$, the representation matrices are as follows.

$$M_1 = \begin{pmatrix} 1 & -\gamma_1^{-1} \\ 0 & \gamma_1^{-1} \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -\gamma_1^{-1} & 0 & 0 \\ 0 & \gamma_1^{-1} & 0 & 0 \\ 0 & 0 & 1 & -\gamma_1^{-1} \\ 0 & 0 & 0 & \gamma_1^{-1} \end{pmatrix},$$

$$M_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & -\gamma_2^{-1} \\ 0 & \gamma_2^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & -\gamma_2^{-1} & 0 \\ 0 & 1 & 0 & -\gamma_2^{-1} \\ 0 & 0 & \gamma_2^{-1} & 0 \\ 0 & 0 & 0 & \gamma_2^{-1} \end{pmatrix},$$

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{(\alpha-1)(\beta-1)\gamma_1\gamma_2}{\alpha\beta} & \frac{(\alpha\beta-\gamma_1)\gamma_2}{\alpha\beta} & \frac{(\alpha\beta-\gamma_2)\gamma_1}{\alpha\beta} & -\frac{\gamma_1\gamma_2}{\alpha\beta} \end{pmatrix}.$$

2.4. Monodromy group

Using the basis $\{\widetilde{F}_I\}_I$, we can identify $Sol_{\dot{x}}$ and $\mathbb{C}^{2^n}$. Thus, we regard the monodromy representation as a group homomorphism $\mathcal{M} : \pi_1(X, \dot{x}) \to \mathbf{GL}_{2^n}(\mathbb{C})$. The *monodromy* *group* $\mathbf{Mon} = \mathbf{Mon}^{(n)}(\alpha,\beta,\gamma)$ is defined by

$$\mathbf{Mon}^{(n)}(\alpha,\beta,\gamma) = \mathcal{M}(\pi_1(X,\dot{x})) = \langle M_0, M_1, \dots, M_n\rangle.$$

Recall that the matrices $M_0, M_1, \dots, M_n$ depend on α, β, $\gamma_1, \dots, \gamma_n$. We restate Theorem 1.1 in terms of α, β, and γ.

THEOREM 2.6. *We assume $n \geq 3$. The monodromy group $\mathbf{Mon}^{(n)}(\alpha,\beta,\gamma)$ is finite irreducible if and only if the following two conditions hold:*

(A) each $\mathbf{Mon}^{(1)}(\alpha,\beta,\gamma_k)$ $(k=1,\dots,n)$ is finite irreducible;

(B) at least n of $\gamma_1,\dots,\gamma_n$, $\beta\alpha^{-1}$, $\delta_0^{(n)}(\alpha,\beta,\gamma)$ are -1.

On the other hand, for $n=2$ (Appell's F_4), the finite irreducible condition was given by Kato [9].

FACT 2.7 ([9, Theorem 1]). *The monodromy group $\mathbf{Mon}^{(2)}(\alpha,\beta,(\gamma_1,\gamma_2))$ is finite irreducible if and only if the following two conditions hold:*

(A) $\mathbf{Mon}^{(1)}(\alpha,\beta,\gamma_1)$ and $\mathbf{Mon}^{(1)}(\alpha,\beta,\gamma_2)$ are finite irreducible;

(B') $\delta_0^{(2)}(\alpha,\beta,(\gamma_1,\gamma_2)) = -1$, or at least two of γ_1, γ_2, $\beta\alpha^{-1}$ are -1.

REMARK 2.8.

(i) The monodromy group $\mathbf{Mon}^{(1)}(\alpha,\beta,\gamma_k)$ is nothing but that for Gauss' hypergeometric function ${}_2F_1(a,b,c_k;x)$. Its irreducibility condition is known to be

$$\alpha-1,\ \alpha-\gamma_k,\ \beta-1,\ \beta-\gamma_k \neq 0$$

(see also (2)); the finiteness conditions (the so-called "Schwarz list") are provided in [13]. Interested readers are referred to [3], in which an accessible list is available.

(ii) If $n=2$, then (A) and $\delta_0^{(2)}(\alpha,\beta,(\gamma_1,\gamma_2)) = -1$ imply the finiteness of the monodromy group $\mathbf{Mon}^{(2)}(\alpha,\beta,(\gamma_1,\gamma_2))$. However, if $n \geq 3$, (A) and $\delta_0^{(n)}(\alpha,\beta,\gamma) = -1$ are not sufficient for the finiteness. Thus, Theorem 2.6 is not a direct generalization of Fact 2.7.

3. Reflection subgroup

In this section, we assume that the irreducibility condition (2) holds.

As in [1], we refer to a matrix $g \in \mathbf{GL}_n(\mathbb{C})$ as a *reflection* if $g - E_n$ has rank one. We refer to the determinant of a reflection g as the special eigenvalue of g. As mentioned in Remark 2.4, $M_0^{(n)}(\alpha,\beta,\gamma)$ is a reflection with the special eigenvalue $\delta_0^{(n)}(\alpha,\beta,\gamma)$.

Let $\mathbf{Ref} = \mathbf{Ref}^{(n)}(\alpha,\beta,\gamma) \subset \mathbf{Mon}$ be the smallest normal subgroup containing M_0, that is, a subgroup generated by reflections gM_0g^{-1} $(g \in \mathbf{Mon})$. The reflection subgroup was introduced in [1] for the generalized hypergeometric function ${}_nF_{n-1}$ and subsequently considered in [9] for Appell's F_4. Then, we introduced the *reflection subgroup* $\mathbf{Ref}$ for the study of F_C in [6].

FACT 3.1 ([6, Proposition 2.15]). *The monodromy group $\mathbf{Mon}^{(n)}(\alpha,\beta,\gamma)$ is finite if and only if $\mathbf{Ref}^{(n)}(\alpha,\beta,\gamma)$ is finite.*

To discuss the finiteness of **Mon**, it suffices to consider that of **Ref**. We use the following two lemmas. Although the reducibility was shown in [6, Lemmas 2.20 and 2.21], we need more precise statements about direct product decompositions.

LEMMA 3.2. *If at least two of $\gamma_1,\dots,\gamma_n$ are -1, then the action of* **Ref** *is reducible. For example, if $\gamma_{n-1}=\gamma_n=-1$, then we have the decomposition*

$$\mathbf{Ref}^{(n)}(\alpha,\beta,(\gamma_1,\dots,\gamma_{n-2},-1,-1)) \simeq \Bigl(\mathbf{Ref}^{(n-1)}(\alpha,\beta,(\gamma_1,\dots,\gamma_{n-2},-1))\Bigr)^2.$$

LEMMA 3.3. *If at least one of $\gamma_1,\dots,\gamma_n$ is -1 and $\alpha\beta^{-1}$ is -1, then the action of* **Ref** *is reducible. For example, if $\gamma_n=\beta\alpha^{-1}=-1$, then we have the decomposition*

$$\mathbf{Ref}^{(n)}(\alpha,\beta,(\gamma_1,\gamma_2,\dots,\gamma_{n-1},-1)) \simeq \Bigl(\mathbf{Ref}^{(n-1)}(\alpha,\beta,(\gamma_1,\gamma_2,\dots,\gamma_{n-1}))\Bigr)^2.$$

To prove these lemmas, we use the same decompositions of $\mathbb{C}^{2^n}$ into **Ref**-invariant subspaces as [6]. Recall that for $i,j=1,\dots,n$, we have $M_iM_j=M_jM_i$ by the relation in Fact 2.1.

Proof of Lemma 3.2. Without loss of generality, we may assume $\gamma_{n-1}=\gamma_n=-1$. This implies $M_{n-1}^2=M_n^2=E_{2^n}$. For each $(i_1,\dots,i_{n-2})\in\{0,1\}^{n-2}$, we set

$$\begin{aligned}
g_{i_1,\dots,i_{n-2},0,0} &= \mathbf{e}_{i_1}\otimes\cdots\otimes\mathbf{e}_{i_{n-2}}\otimes\mathbf{e}_0\otimes\mathbf{e}_0,\\
g_{i_1,\dots,i_{n-2},1,0} &= \mathbf{e}_{i_1}\otimes\cdots\otimes\mathbf{e}_{i_{n-2}}\otimes(2\mathbf{e}_1-\mathbf{e}_0)\otimes\mathbf{e}_0,\\
g_{i_1,\dots,i_{n-2},0,1} &= \mathbf{e}_{i_1}\otimes\cdots\otimes\mathbf{e}_{i_{n-2}}\otimes\mathbf{e}_0\otimes(2\mathbf{e}_1-\mathbf{e}_0),\\
g_{i_1,\dots,i_{n-2},1,1} &= \mathbf{e}_{i_1}\otimes\cdots\otimes\mathbf{e}_{i_{n-2}}\otimes(2\mathbf{e}_1-\mathbf{e}_0)\otimes(2\mathbf{e}_1-\mathbf{e}_0),\\
f^{\pm}_{14;i_1,\dots,i_{n-2}} &= g_{i_1,\dots,i_{n-2},0,0}\pm g_{i_1,\dots,i_{n-2},1,1},\\
f^{\pm}_{23;i_1,\dots,i_{n-2}} &= g_{i_1,\dots,i_{n-2},1,0}\pm g_{i_1,\dots,i_{n-2},0,1}.
\end{aligned}$$

We consider a direct sum decomposition of $\mathbb{C}^{2^n}$:

$$\mathbb{C}^{2^n}=W^+\oplus W^-;\quad W^{\pm}=\bigoplus_{(i_1,\dots,i_{n-2})}\mathbb{C}f^{\pm}_{14;i_1,\dots,i_{n-2}}\oplus\bigoplus_{(i_1,\dots,i_{n-2})}\mathbb{C}f^{\pm}_{23;i_1,\dots,i_{n-2}}.$$

The dimension of each factor is $2^{n-2}+2^{n-2}=2^{n-1}$. Note that

$$\mathbf{e}_{1,\dots,1}=\frac{1}{4}(f^{+}_{14;1,\dots,1}+f^{+}_{23;1,\dots,1})\in W^+.$$

As was shown in [6], we obtain the following equalities:

$$\begin{aligned}
&M_k\cdot f^{\pm}_{*;i_1,\dots,i_k,\dots,i_{n-2}}=\begin{cases} f^{\pm}_{*;i_1,\dots,0,\dots,i_{n-2}} & (i_k=0)\\ -\gamma_k^{-1}f^{\pm}_{*;i_1,\dots,0,\dots,i_{n-2}}+\gamma_k^{-1}f^{\pm}_{*;i_1,\dots,1,\dots,i_{n-2}} & (i_k=1),\end{cases}\\
&M_{n-1}\cdot f^{\pm}_{14;i_1,\dots,i_{n-2}}=f^{\mp}_{14;i_1,\dots,i_{n-2}},\quad M_{n-1}\cdot f^{\pm}_{23;i_1,\dots,i_{n-2}}=-f^{\mp}_{23;i_1,\dots,i_{n-2}},\\
&M_n\cdot f^{\pm}_{*;i_1,\dots,i_{n-2}}=f^{\mp}_{*;i_1,\dots,i_{n-2}},
\end{aligned}$$

$$\begin{aligned}
&M_0 \cdot f^-_{*;i_1,\dots,i_{n-2}} = f^-_{*;i_1,\dots,i_{n-2}} \in W^-,\\
&M_0 \cdot f^+_{14;i_1,\dots,i_{n-2}} = f^+_{14;i_1,\dots,i_{n-2}} - 2\lambda_{0;i_1,\dots,i_{n-2}}\mathbf{e}_{1,\dots,1} \in W^+,\\
&M_0 \cdot f^+_{23;i_1,\dots,i_{n-2}} = f^+_{23;i_1,\dots,i_{n-2}} - 2(2\lambda_{1;i_1,\dots,i_{n-2}} - \lambda_{0;i_1,\dots,i_{n-2}})\mathbf{e}_{1,\dots,1} \in W^+,
\end{aligned}$$

where $* = 14$ or 23, $1 \le k \le n-2$, and

$$\begin{aligned}
\lambda_{1;i_1,\dots,i_{n-2}} &= (-1)^{n+i_1+\cdots+i_{n-2}} \frac{(\alpha\beta + (-1)^{i_1+\cdots+i_{n-2}} \prod_{k=1}^{n-2}\gamma_k^{i_k}) \prod_{k=1}^{n-2}\gamma_k^{1-i_k}}{\alpha\beta},\\
\lambda_{0;i_1,\dots,i_{n-2}} &= \begin{cases} (-1)^n \frac{(\alpha-1)(\beta-1)\prod_{k=1}^{n-2}\gamma_k}{\alpha\beta} & ((i_1,\dots,i_{n-2}) = (0,\dots,0)) \\ \lambda_{1;i_1,\dots,i_{n-2}} & ((i_1,\dots,i_{n-2}) \ne (0,\dots,0)). \end{cases}
\end{aligned}$$

These equalities imply that $W^\pm$ are non-trivial **Ref**-subspaces (see [6]). Because

- M_0 and $(M_{n-1}M_n)M_0(M_{n-1}M_n)^{-1}$ act trivially on W^-,
- $M_nM_0M_n^{-1}$ and $M_{n-1}M_0M_{n-1}^{-1}$ act trivially on W^+,

we have the direct product decomposition

(4)

$$\begin{aligned}
&\mathbf{Ref}^{(n)}(\alpha,\beta,(\gamma_1,\dots,\gamma_{n-2},-1,-1))\\
&= \left\langle \begin{matrix} gM_0g^{-1}, gM_{n-1}M_0M_{n-1}^{-1}g^{-1}, \\ gM_nM_0M_n^{-1}g^{-1}, g(M_{n-1}M_n)M_0(M_{n-1}M_n)^{-1}g^{-1} \end{matrix} \,\middle|\, g = M_1^{j_1}\cdots M_{n-2}^{j_{n-2}},\ j_k \in \mathbb{Z} \right\rangle\\
&= \left\langle gM_0g^{-1}, g(M_{n-1}M_n)M_0(M_{n-1}M_n)^{-1}g^{-1} \mid g = M_1^{j_1}\cdots M_{n-2}^{j_{n-2}},\ j_k \in \mathbb{Z} \right\rangle\\
&\quad \times \left\langle gM_{n-1}M_0M_{n-1}^{-1}g^{-1}, gM_nM_0M_n^{-1}g^{-1} \mid g = M_1^{j_1}\cdots M_{n-2}^{j_{n-2}},\ j_k \in \mathbb{Z} \right\rangle\\
&= R^+ \times R^-.
\end{aligned}$$

Here, R^+ (resp. R^-) acts trivially on W^- (resp. W^+). We retake the bases of $W^\pm$ as

$$\widetilde{f}^\pm_{i_1,\dots,i_{n-2},i_{n-1}} = \begin{cases} f^\pm_{14;i_1,\dots,i_{n-2}} & (i_{n-1} = 0) \\ \frac{1}{2}(f^\pm_{14;i_1,\dots,i_{n-2}} + f^\pm_{23;i_1,\dots,i_{n-2}}) & (i_{n-1} = 1), \end{cases}$$

where $(i_1,\dots,i_{n-2},i_{n-1}) \in \{0,1\}^{n-1}$. Note that $\widetilde{f}^+_{1,\dots,1,1} = 2\mathbf{e}_{1,\dots,1}$. We consider the representation matrices of the actions by M_k $(1 \le k \le n-2)$ and $M_{n-1}M_n$ on $W^\pm$, M_0 on W^+, and $M_nM_0M_n^{-1}$ on W^- with respect to these bases.

- For $1 \le k \le n-2$, the representation matrix of the action by M_k is

$$E_2 \otimes \cdots \otimes E_2 \otimes \underset{k\text{th}}{G(\gamma_k)} \otimes E_2 \otimes \cdots \otimes \underset{(n-1)\text{th}}{E_2} = M_k^{(n-1)}(\alpha,\beta,(\gamma_1,\dots,\gamma_{n-2},-1)).$$

- Since we have

$$M_{n-1}M_n \cdot \widetilde{f}^\pm_{i_1,\dots,i_{n-2},i_{n-1}} = \begin{cases} \widetilde{f}^\pm_{i_1,\dots,i_{n-2},0} & (i_{n-1} = 0) \\ \widetilde{f}^\pm_{i_1,\dots,i_{n-2},0} - \widetilde{f}^\pm_{i_1,\dots,i_{n-2},1} & (i_{n-1} = 1), \end{cases}$$

the representation matrix of the action by $M_{n-1}M_n$ is

$$\begin{aligned}\underbrace{E_2 \otimes \cdots \otimes E_2}_{n-2} \otimes \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} &= \underbrace{E_2 \otimes \cdots \otimes E_2}_{n-2} \otimes G(-1) \\ &= M_{n-1}^{(n-1)}(\alpha, \beta, (\gamma_1, \dots, \gamma_{n-2}, -1)).\end{aligned}$$

- Since we have

$$M_0 \cdot \widetilde{f}^+_{i_1,\dots,i_{n-2},i_{n-1}} = \begin{cases} \widetilde{f}^+_{i_1,\dots,i_{n-2},0} - \lambda_{0;i_1,\dots,i_{n-2}} \widetilde{f}^+_{1,\dots,1,1} & (i_{n-1} = 0) \\ \widetilde{f}^+_{i_1,\dots,i_{n-2},1} - \lambda_{1;i_1,\dots,i_{n-2}} \widetilde{f}^+_{1,\dots,1,1} & (i_{n-1} = 1), \end{cases}$$

  the representation matrix of the action by M_0 is

$$E_{2^{n-1}} - {}^t(\mathbf{0}, \dots, \mathbf{0}, \boldsymbol{v}'), \tag{5}$$

  where the $I = (i_1, \dots, i_{n-2}, i_{n-1})$-th entries of $\boldsymbol{v}'$ are

$$\begin{cases} \lambda_{0;i_1,\dots,i_{n-2}} = (-1)^n \frac{(\alpha-1)(\beta-1)\prod_{k=1}^{n-2}\gamma_k}{\alpha\beta} = (-1)^{n-1} \frac{(\alpha-1)(\beta-1)\prod_{k=1}^{n-2}\gamma_k \cdot (-1)}{\alpha\beta} & (i_1 = \cdots = i_{n-1} = 0) \\ \lambda_{0;i_1,\dots,i_{n-2}} = (-1)^{n+i_1+\cdots+i_{n-2}} \frac{(\alpha\beta + (-1)^{i_1+\cdots+i_{n-2}} \prod_{k=1}^{n-2}\gamma_k^{i_k})\prod_{k=1}^{n-2}\gamma_k^{1-i_k}}{\alpha\beta} & \\ \qquad = (-1)^{n-1+|I|} \frac{(\alpha\beta + (-1)^{|I|} \prod_{k=1}^{n-2}\gamma_k^{i_k})\prod_{k=1}^{n-2}\gamma_k^{1-i_k} \cdot (-1)}{\alpha\beta} & (i_{n-1} = 0, \exists i_k \neq 0) \\ \lambda_{1;i_1,\dots,i_{n-2}} = (-1)^{n+i_1+\cdots+i_{n-2}} \frac{(\alpha\beta + (-1)^{i_1+\cdots+i_{n-2}} \prod_{k=1}^{n-2}\gamma_k^{i_k})\prod_{k=1}^{n-2}\gamma_k^{1-i_k}}{\alpha\beta} & \\ \qquad = (-1)^{n-1+|I|} \frac{(\alpha\beta + (-1)^{|I|} \prod_{k=1}^{n-2}\gamma_k^{i_k} \cdot (-1))\prod_{k=1}^{n-2}\gamma_k^{1-i_k}}{\alpha\beta} & (i_{n-1} = 1). \end{cases}$$

  Each of these entries coincides with $v_I^{(n-1)}(\alpha, \beta, (\gamma_1, \dots, \gamma_{n-2}, -1))$. Thus, the representation matrix (5) is nothing but $M_0^{(n-1)}(\alpha, \beta, (\gamma_1, \dots, \gamma_{n-2}, -1))$.

- We consider the action by $M_n M_0 M_n^{-1}$ on W^-. Using

$$M_n \cdot \widetilde{f}^{\pm}_{i_1,\dots,i_{n-2},i_{n-1}} = \widetilde{f}^{\mp}_{i_1,\dots,i_{n-2},i_{n-1}},$$

  we have

$$M_n M_0 M_n^{-1} \cdot \widetilde{f}^-_{i_1,\dots,i_{n-2},i_{n-1}} = \begin{cases} \widetilde{f}^-_{i_1,\dots,i_{n-2},0} - \lambda_{0;i_1,\dots,i_{n-2}} \widetilde{f}^-_{1,\dots,1,1} & (i_{n-1} = 0) \\ \widetilde{f}^-_{i_1,\dots,i_{n-2},1} - \lambda_{1;i_1,\dots,i_{n-2}} \widetilde{f}^-_{1,\dots,1,1} & (i_{n-1} = 1). \end{cases}$$

  Similarly to the aforementioned discussion, we can show that the representation matrix of the action by $M_n M_0 M_n^{-1}$ coincides with $M_0^{(n-1)}(\alpha, \beta, (\gamma_1, \dots, \gamma_{n-2}, -1))$.

Therefore, the subgroups $R^{\pm}$ are isomorphic to the smallest normal subgroup of $\mathbf{Mon}^{(n-1)}(\alpha, \beta, (\gamma_1, \dots, \gamma_{n-2}, -1))$, which contains $M_0^{(n-1)}(\alpha, \beta, (\gamma_1, \dots, \gamma_{n-2}, -1))$. This is nothing but $\mathbf{Ref}^{(n-1)}(\alpha, \beta, (\gamma_1, \dots, \gamma_{n-2}, -1))$. Thus, the decomposition (4) implies the lemma. □

Proof of Lemma 3.3. Without loss of generality, we may assume $\gamma_n = \beta\alpha^{-1} = -1$. Note that $M_n^2 = E_{2^n}$. For each $(i_1, \dots, i_{n-1}) \in \{0,1\}^{n-1}$, we set

$$h_{i_1,\dots,i_{n-1},0} = \mathbf{e}_{i_1} \otimes \mathbf{e}_{i_2} \otimes \cdots \otimes \mathbf{e}_{i_{n-1}} \otimes \mathbf{e}_0,$$

$$h_{i_1,\dots,i_{n-1},1} = \mathbf{e}_{i_1} \otimes \mathbf{e}_{i_2} \otimes \cdots \otimes \mathbf{e}_{i_{n-1}} \otimes (2\mathbf{e}_1 - \mathbf{e}_0),$$
$$f^{\pm}_{12;i_1,\dots,i_{n-1}} = h_{i_1,\dots,i_{n-1},0} \pm h_{i_1,\dots,i_{n-1},1},$$

and consider a direct sum decomposition of $\mathbb{C}^{2^n}$:

$$\mathbb{C}^{2^n} = W^+ \oplus W^-; \quad W^{\pm} = \bigoplus_{(i_1,\dots,i_{n-1})} \mathbb{C} f^{\pm}_{12;i_1,\dots,i_{n-1}}.$$

The dimension of each factor is 2^{n-1}. Note that

$$\mathbf{e}_{1,\dots,1} = \frac{1}{2}(h_{1,\dots,1,0} + h_{1,\dots,1,1}) = \frac{1}{2} f^{+}_{12;1,\dots,1} \in W^+.$$

As was shown in [6], we obtain the following equalities:

$$\begin{aligned}
M_k \cdot f^{\pm}_{12;i_1,\dots,i_k,\dots,i_{n-1}} &= \begin{cases} f^{\pm}_{12;i_1,\dots,0,\dots,i_{n-1}} & (i_k = 0) \\ -\gamma_k^{-1} f^{\pm}_{12;i_1,\dots,0,\dots,i_{n-1}} + \gamma_k^{-1} f^{\pm}_{12;i_1,\dots,1,\dots,i_{n-1}} & (i_k = 1), \end{cases} \\
M_n \cdot f^{\pm}_{12;i_1,\dots,i_{n-1}} &= f^{\mp}_{12;i_1,\dots,i_{n-1}}, \\
M_0 \cdot f^{-}_{12;i_1,\dots,i_{n-1}} &= f^{-}_{12;i_1,\dots,i_{n-1}} \in W^-, \\
M_0 \cdot f^{+}_{12;i_1,\dots,i_{n-1}} &= f^{+}_{12;i_1,\dots,i_{n-1}} + 2\lambda_{i_1,\dots,i_{n-1}} \mathbf{e}_{1,\dots,1} \in W^+,
\end{aligned}$$

where $1 \le k \le n-1$ and

$$\lambda_{i_1,\dots,i_{n-1}} = (-1)^{n+i_1+\cdots+i_{n-1}} \frac{(\alpha\beta + (-1)^{i_1+\cdots+i_{n-1}} \prod_{k=1}^{n-1} \gamma_k^{i_k}) \prod_{k=1}^{n-1} \gamma_k^{1-i_k}}{\alpha\beta}.$$

These equalities imply that $W^{\pm}$ are non-trivial **Ref**-subspaces (see [6]). Because

- M_0 acts trivially on W^-,
- $M_n M_0 M_n^{-1}$ acts trivially on W^+,

we have the direct product decomposition

(6)
$$\begin{aligned}
&\mathbf{Ref}^{(n)}(\alpha, \beta, (\gamma_1, \dots, \gamma_{n-1}, -1)) \\
&= \left\langle g M_0 g^{-1}, g M_n M_0 M_n^{-1} g^{-1} \mid g = M_1^{j_1} \cdots M_{n-1}^{j_{n-1}},\ j_k \in \mathbb{Z} \right\rangle \\
&= \left\langle g M_0 g^{-1} \mid g = M_1^{j_1} \cdots M_{n-1}^{j_{n-1}},\ j_k \in \mathbb{Z} \right\rangle \times \left\langle g M_n M_0 M_n^{-1} g^{-1} \mid g = M_1^{j_1} \cdots M_{n-1}^{j_{n-1}},\ j_k \in \mathbb{Z} \right\rangle \\
&= R^+ \times R^-.
\end{aligned}$$

Here, R^+ (resp. R^-) acts trivially on W^- (resp. W^+). We consider the representation matrices of the actions by M_k $(1 \le k \le n-1)$ on $W^{\pm}$, M_0 on W^+, and $M_n M_0 M_n^{-1}$ on W^- with respect to the bases $\{f^{\pm}_{12;i_1,\dots,i_{n-1}}\}$.

- Similarly to the proof of Lemma 3.2, the representation matrix of the action by M_k $(1 \le k \le n-1)$ is $M_k^{(n-1)}(\alpha, \beta, (\gamma_1, \dots, \gamma_{n-2}, \gamma_{n-1}))$.

- We consider the action by M_0 on W^+. Since
$$M_0 \cdot f^+_{12;i_1,\dots,i_{n-1}} = f^+_{12;i_1,\dots,i_{n-1}} + \lambda_{i_1,\dots,i_{n-1}} f^+_{12;1,\dots,1},$$
the representation matrix is
$$E_{2^{n-1}} - {}^t(\mathbf{0},\dots,\mathbf{0},\boldsymbol{v}'), \tag{7}$$
where the $I=(i_1,\dots,i_{n-2},i_{n-1})$-th entry of $\boldsymbol{v}'$ is
$$-\lambda_{i_1,\dots,i_{n-1}} = (-1)^{n-1+i_1+\cdots+i_{n-1}} \frac{(\alpha\beta + (-1)^{i_1+\cdots+i_{n-1}} \prod_{k=1}^{n-1} \gamma_k^{i_k}) \prod_{k=1}^{n-1} \gamma_k^{1-i_k}}{\alpha\beta}.$$
This entry is nothing but $v_I^{(n-1)}(\alpha,\beta,(\gamma_1,\dots,\gamma_{n-2},\gamma_{n-1}))$ if $I \neq (0,\dots,0)$. Because we have $\alpha+\beta=0$ by the assumption of the lemma, the $(0,\dots,0)$-th entry is written as
$$\begin{aligned} -\lambda_{0,\dots,0} &= -(-1)^n \frac{(\alpha\beta+1)\prod_{k=1}^{n-1}\gamma_k}{\alpha\beta} = (-1)^{n-1} \frac{(\alpha-1)(\beta-1)\prod_{k=1}^{n-1}\gamma_k}{\alpha\beta} \\ &= v^{(n-1)}_{0,\dots,0}(\alpha,\beta,(\gamma_1,\dots,\gamma_{n-2},\gamma_{n-1})). \end{aligned}$$
This implies that the representation matrix (7) coincides with the reflection $M_0^{(n-1)}(\alpha,\beta,(\gamma_1,\dots,\gamma_{n-2},\gamma_{n-1}))$.
- Similarly to the proof of Lemma 3.2, we can show that the representation matrix of the action by $M_n M_0 M_n^{-1}$ on W^- is also $M_0^{(n-1)}(\alpha,\beta,(\gamma_1,\dots,\gamma_{n-2},\gamma_{n-1}))$.

Therefore, the subgroups $R^\pm$ are isomorphic to the reflection subgroup $\mathbf{Ref}^{(n-1)}(\alpha,\beta,(\gamma_1,\dots,\gamma_{n-2},\gamma_{n-1}))$; the decomposition (6) implies the lemma. □

From the proofs of Lemmas 3.2 and 3.3, $\mathbf{Mon}^{(n)}(\alpha,\beta,\gamma)$ is imprimitive (see [1, Definition 5.1]) if at least two of $\gamma_1,\dots,\gamma_n,\alpha\beta^{-1}$ are -1. For $n \geq 3$, if $\mathbf{Mon}^{(n)}(\alpha,\beta,\gamma)$ is finite irreducible, then the condition (B) in Theorem 2.6 implies that at least two of $\gamma_1,\dots,\gamma_n,\alpha\beta^{-1}$ are -1. Thus, we obtain the following as a corollary of Theorem 2.6.

Corollary 3.4. *We assume $n \geq 3$. If the monodromy group $\mathbf{Mon}^{(n)}(\alpha,\beta,\gamma)$ is finite irreducible, then it is imprimitive.*

4. Proof of Theorem 2.6

4.1. Proof of “if” part

We assume the conditions (A) and (B) in Theorem 2.6. When we assume the condition (B), it suffices to consider the following four cases without loss of generality:

(B-a) $\gamma_1 = \cdots = \gamma_n = -1$,

(B-b) $\gamma_2 = \cdots = \gamma_n = \beta\alpha^{-1} = -1$,

(B-c) $\gamma_2 = \cdots = \gamma_n = \delta_0^{(n)}(\alpha,\beta,\gamma) = -1$,

(B-d) $\gamma_3 = \cdots = \gamma_n = \beta\alpha^{-1} = \delta_0^{(n)}(\alpha,\beta,\gamma) = -1$.

Lemma 4.1. *If the conditions (A) and (B) hold, then the irreducibility condition (2) holds, and hence the monodromy group* $\mathbf{Mon}^{(n)}(\alpha,\beta,\gamma)$ *is irreducible.*

Proof. Since $\mathbf{Mon}^{(1)}(\alpha,\beta,\gamma_k)$ is irreducible by the condition (A), we have

$$\alpha-1,\ \alpha-\gamma_k,\ \beta-1,\ \beta-\gamma_k\neq 0 \tag{8}$$

for $k=1,\ldots,n$. We consider the four cases (B-a)–(B-d).

(B-a) In this case, the condition (2) is reduced to $\alpha\pm 1\neq 0$ and $\beta\pm 1\neq 0$. These are nothing but (8) for $k=1$.

(B-b) The non-trivial conditions in (2) are $\alpha+\gamma_1\neq 0$ and $\beta+\gamma_1\neq 0$. By $\beta\alpha^{-1}=-1$, these are equivalent to $-\beta+\gamma_1\neq 0$ and $-\alpha+\gamma_1\neq 0$, respectively. The last conditions follow from (8).

(B-c) Similarly to the case (B-b), the non-trivial conditions in (2) are $\alpha+\gamma_1\neq 0$ and $\beta+\gamma_1\neq 0$. Because of the identity

$$-1=\delta_0^{(n)}(\alpha,\beta,\gamma)=(-1)^{n-1}\gamma_1\cdots\gamma_n\alpha^{-1}\beta^{-1}=\gamma_1\alpha^{-1}\beta^{-1},$$

we have $\gamma_1=-\alpha\beta$, hence, we obtain $\alpha+\gamma_1=-\alpha(\beta-1)\neq 0$ and $\beta+\gamma_1=-\beta(\alpha-1)\neq 0$.

(B-d) In this case, the non-trivial conditions in (2) are

$$\alpha+\gamma_1\neq 0,\quad \alpha+\gamma_2\neq 0,\quad \alpha-\gamma_1\gamma_2\neq 0,\quad \alpha+\gamma_1\gamma_2\neq 0,$$

and those obtained by replacing α with β. Because of the identity

$$-1=\delta_0^{(n)}(\alpha,\beta,\gamma)=(-1)^{n-1}\gamma_1\cdots\gamma_n\alpha^{-1}\beta^{-1}=-\gamma_1\gamma_2\alpha^{-1}\beta^{-1}$$

and $\beta\alpha^{-1}=-1$, we obtain

$$\begin{aligned}\alpha+\gamma_1&=\alpha+\alpha\beta\gamma_2^{-1}=\alpha\gamma_2^{-1}(\gamma_2+\beta)=-\alpha\gamma_2^{-1}(\alpha-\gamma_2)\neq 0,\\ \alpha-\gamma_1\gamma_2&=\alpha-\alpha\beta=\alpha(1-\beta)=-\alpha(\beta-1)\neq 0,\\ \alpha+\gamma_1\gamma_2&=\alpha+\alpha\beta=\alpha(1+\beta)=\alpha(1-\alpha)=-\alpha(\alpha-1)\neq 0.\end{aligned}$$

□

Proposition 4.2. *If the conditions (A) and (B) hold, then the reflection subgroup* $\mathbf{Ref}^{(n)}(\alpha,\beta,\gamma)$ *is finite.*

Proof. By Lemma 4.1, we may assume that the irreducibility condition (2) holds. Thus, we can apply Lemmas 3.2 and 3.3. Let us consider the four cases (B-a)–(B-d).

(B-a) Using Lemma 3.2 repeatedly, we have

$$\begin{aligned}\mathbf{Ref}^{(n)}(\alpha,\beta,(-1,-1,-1,\ldots,-1))&\simeq\Big(\mathbf{Ref}^{(n-1)}(\alpha,\beta,(-1,\ldots,-1,-1))\Big)^2\\ &\simeq\Big(\mathbf{Ref}^{(n-2)}(\alpha,\beta,(-1,\ldots,-1))\Big)^4\end{aligned}$$

$$\simeq \cdots \simeq \left(\mathbf{Ref}^{(1)}(\alpha,\beta,-1)\right)^{2^{n-1}}.$$

The finiteness follows from the condition (A).

(B-b) Using Lemma 3.3 repeatedly, we have

$$\mathbf{Ref}^{(n)}(\alpha,\beta,(\gamma_1,-1,\ldots,-1)) \simeq \left(\mathbf{Ref}^{(1)}(\alpha,\beta,\gamma_1)\right)^{2^{n-1}}.$$

The finiteness follows from the condition (A).

(B-c) Using Lemma 3.2 repeatedly, we obtain

$$\mathbf{Ref}^{(n)}(\alpha,\beta,(\gamma_1,-1,\ldots,-1)) \simeq \left(\mathbf{Ref}^{(2)}(\alpha,\beta,(\gamma_1,-1))\right)^{2^{n-2}}.$$

Since

$$\begin{aligned}&\delta_0^{(2)}(\alpha,\beta,(\gamma_1,-1)) = -\gamma_1\cdot(-1)\cdot\alpha^{-1}\beta^{-1}\\ &= (-1)^{n-1}\gamma_1\cdots\gamma_n\alpha^{-1}\beta^{-1} = \delta_0^{(n)}(\alpha,\beta,\gamma) = -1,\end{aligned}$$

$\mathbf{Ref}^{(2)}(\alpha,\beta,(\gamma_1,-1))$ is finite by Fact 2.7, and hence, $\mathbf{Ref}^{(n)}(\alpha,\beta,(\gamma_1,-1,\ldots,-1))$ is also finite.

(B-d) Using Lemma 3.3 repeatedly, we obtain

$$\mathbf{Ref}^{(n)}(\alpha,\beta,(\gamma_1,\gamma_2,-1,\ldots,-1)) \simeq \left(\mathbf{Ref}^{(2)}(\alpha,\beta,(\gamma_1,\gamma_2))\right)^{2^{n-2}}.$$

Because of the identity

$$\begin{aligned}&\delta_0^{(2)}(\alpha,\beta,(\gamma_1,\gamma_2)) = -\gamma_1\gamma_2\alpha^{-1}\beta^{-1}\\ &= (-1)^{n-1}\gamma_1\cdots\gamma_n\alpha^{-1}\beta^{-1} = \delta_0^{(n)}(\alpha,\beta,\gamma) = -1\end{aligned}$$

and Fact 2.7, $\mathbf{Ref}^{(2)}(\alpha,\beta,(\gamma_1,\gamma_2))$ is finite. Thus, we can conclude that $\mathbf{Ref}^{(n)}(\alpha,\beta,(\gamma_1,\gamma_2,-1,\ldots,-1))$ is also finite.

□

Using Fact 3.1, Lemma 4.1, and Proposition 4.2, we complete the proof of the "if" part of Theorem 2.6.

4.2. Proof of "only if" part

We may assume that the irreducibility condition (2) holds. First, we consider the condition (A).

LEMMA 4.3. *Let $\mathbf{H}$ be the subgroup of $\mathbf{Mon}^{(n)}(\alpha,\beta,\gamma)$ generated by M_1, $M_2,\ldots,M_{n-1}$ and $M_0M_nM_0$. Then, there exists a surjective group homomorphism $\mathbf{H}\to\mathbf{Mon}^{(n-1)}(\alpha,\beta,(\gamma_1,\ldots,\gamma_{n-1}))$.*

Proof. We consider a subspace

$$W = \bigoplus_{(i_1,\dots,i_{n-1})} \mathbb{C}\mathbf{e}_{i_1,\dots,i_{n-1},0}$$

of $\mathbb{C}^{2^n}$ of which the dimension is 2^{n-1}. We prove the following two claims.

(i) For $k = 1,\dots,n-1$, M_k acts on W and its representation matrix coincides with $M_k^{(n-1)}(\alpha,\beta,(\gamma_1,\dots,\gamma_{n-1}))$.

(ii) $M_0M_nM_0$ acts on W and its representation matrix coincides with $M_0^{(n-1)}(\alpha,\beta,(\gamma_1,\dots,\gamma_{n-1}))$.

Proving these claims would confirm that each element of $\mathbf{H}$ is of the form

$$\begin{pmatrix} H & H' \\ O & H'' \end{pmatrix}, \quad \begin{array}{l} H \in \mathbf{Mon}^{(n-1)}(\alpha,\beta,(\gamma_1,\dots,\gamma_{n-1})), \\ H' \text{ and } H'' \text{ are square matrices of size } 2^{n-1}. \end{array}$$

Thus, we can define the group homomorphism

$$\mathbf{H} \longrightarrow \mathbf{Mon}^{(n-1)}(\alpha,\beta,(\gamma_1,\dots,\gamma_{n-1})); \qquad \begin{pmatrix} H & H' \\ O & H'' \end{pmatrix} \mapsto H,$$

and it is clearly surjective. First, we present the proof of the claim (i). Since the nth factor of

$$M_k = E_2 \otimes \cdots \otimes E_2 \otimes G(\gamma_k) \otimes E_2 \otimes \cdots \otimes \underline{E_2}, \tag{9}$$

is E_2 (underlined), that of $\mathbf{e}_{i_1,\dots,i_{n-1},0} = \mathbf{e}_{i_1} \otimes \cdots \otimes \mathbf{e}_{i_{n-1}} \otimes \underline{\mathbf{e}_0}$ is not changed. This means that M_k acts on W. Its representation matrix is obtained by removing the nth factor E_2 from (9). This is nothing but $M_k^{(n-1)}(\alpha,\beta,(\gamma_1,\dots,\gamma_{n-1}))$, and the claim (i) is proved. Next, we provide the proof of the claim (ii). We have

$$\begin{aligned}
&M_0M_nM_0\mathbf{e}_{i_1,\dots,i_{n-1},0} = M_0M_n(\mathbf{e}_{i_1,\dots,i_{n-1},0} - v^{(n)}_{i_1,\dots,i_{n-1},0}\mathbf{e}_{1,\dots,1}) \\
&= M_0(\mathbf{e}_{i_1,\dots,i_{n-1},0} - v^{(n)}_{i_1,\dots,i_{n-1},0}(-\gamma_n^{-1}\mathbf{e}_{1,\dots,1,0} + \gamma_n^{-1}\mathbf{e}_{1,\dots,1})) \\
&= \mathbf{e}_{i_1,\dots,i_{n-1},0} - v^{(n)}_{i_1,\dots,i_{n-1},0}\mathbf{e}_{1,\dots,1} \\
&\qquad + v^{(n)}_{i_1,\dots,i_{n-1},0}\gamma_n^{-1}(\mathbf{e}_{1,\dots,1,0} - v^{(n)}_{1,\dots,1,0}\mathbf{e}_{1,\dots,1}) - v^{(n)}_{i_1,\dots,i_{n-1},0}\gamma_n^{-1}(\mathbf{e}_{1,\dots,1} - v^{(n)}_{1,\dots,1}\mathbf{e}_{1,\dots,1}) \\
&= \mathbf{e}_{i_1,\dots,i_{n-1},0} + v^{(n)}_{i_1,\dots,i_{n-1},0}\gamma_n^{-1}\mathbf{e}_{1,\dots,1,0} - v^{(n)}_{i_1,\dots,i_{n-1},0}(1 + \gamma_n^{-1}v^{(n)}_{1,\dots,1,0} + \gamma_n^{-1}(1 - v^{(n)}_{1,\dots,1}))\mathbf{e}_{1,\dots,1}.
\end{aligned}$$

Because of the identity

$$\begin{aligned}
&1 + \gamma_n^{-1}v^{(n)}_{1,\dots,1,0} + \gamma_n^{-1}(1 - v^{(n)}_{1,\dots,1}) = 1 + \gamma_n^{-1}v^{(n)}_{1,\dots,1,0} + \gamma_n^{-1}\delta_0^{(n)}(\alpha,\beta,\gamma) \\
&= 1 + \gamma_n^{-1} \cdot (-1) \cdot \frac{(\alpha\beta + (-1)^{n-1}\prod_{k=1}^{n-1}\gamma_k)\gamma_n}{\alpha\beta} + \gamma_n^{-1} \cdot (-1)^{n-1}\frac{\prod_{k=1}^{n}\gamma_k}{\alpha\beta} = 0,
\end{aligned}$$

we obtain

$$M_0M_nM_0\mathbf{e}_{i_1,\dots,i_{n-1},0} = \mathbf{e}_{i_1,\dots,i_{n-1},0} + v^{(n)}_{i_1,\dots,i_{n-1},0}\gamma_n^{-1}\mathbf{e}_{1,\dots,1,0} \in W,$$

and hence, $M_0 M_n M_0$ acts on W. The representation matrix is equal to

$$E_{2^{n-1}} - {}^t(\mathbf{0}, \dots, \mathbf{0}, \boldsymbol{v}'), \tag{10}$$

where the $(i_1, \dots, i_{n-1})$-th entry $v'_{i_1,\dots,i_{n-1}}$ of $\boldsymbol{v}'$ is $v'_{i_1,\dots,i_{n-1}} = -v^{(n)}_{i_1,\dots,i_{n-1},0} \cdot \gamma_n^{-1}$. If $(i_1, \dots, i_{n-1}) = (0, \dots, 0)$, then we have

$$\begin{aligned} v'_{0,\dots,0} &= -v^{(n)}_{0,\dots,0} \cdot \gamma_n^{-1} = -(-1)^n \frac{(\alpha-1)(\beta-1)\prod_{k=1}^n \gamma_k}{\alpha\beta} \cdot \gamma_n^{-1} \\ &= (-1)^{n-1} \frac{(\alpha-1)(\beta-1)\prod_{k=1}^{n-1} \gamma_k}{\alpha\beta} = v^{(n-1)}_{0,\dots,0}(\alpha, \beta, (\gamma_1, \dots, \gamma_{n-1})). \end{aligned}$$

Otherwise, we have

$$\begin{aligned} v'_{i_1,\dots,i_{n-1}} &= -v^{(n)}_{i_1,\dots,i_{n-1},0} \cdot \gamma_n^{-1} \\ &= -(-1)^{n+|I|} \frac{(\alpha\beta + (-1)^{|I|}\prod_{k=1}^{n-1} \gamma_k^{i_k}) \prod_{k=1}^{n-1} \gamma_k^{1-i_k} \gamma_n}{\alpha\beta} \cdot \gamma_n^{-1} \\ &= (-1)^{n-1+|I|} \frac{(\alpha\beta + (-1)^{|I|}\prod_{k=1}^{n-1} \gamma_k^{i_k}) \prod_{k=1}^{n-1} \gamma_k^{1-i_k}}{\alpha\beta} \\ &= v^{(n-1)}_{i_1,\dots,i_{n-1}}(\alpha, \beta, (\gamma_1, \dots, \gamma_{n-1})). \end{aligned}$$

Therefore, the representation matrix (10) coincides with $M_0^{(n-1)}(\alpha, \beta, (\gamma_1, \dots, \gamma_{n-1}))$, and the proof is completed. □

Using this lemma, we obtain the following corollary.

Corollary 4.4. *For $1 \le j_1 < j_2 < \cdots < j_k \le n$ ($k = 1, \dots, n$), $\mathbf{Mon}^{(k)}(\alpha, \beta, (\gamma_{j_1}, \dots, \gamma_{j_k}))$ is isomorphic to a subquotient group of $\mathbf{Mon}^{(n)}(\alpha, \beta, \gamma)$.*

Let us show that the condition (A) holds.

Proposition 4.5. *Suppose that $\mathbf{Mon}^{(n)}(\alpha, \beta, \gamma)$ is finite irreducible. Then, for $1 \le j_1 < j_2 < \cdots < j_k \le n$ ($k = 1, \dots, n$), $\mathbf{Mon}^{(k)}(\alpha, \beta, (\gamma_{j_1}, \dots, \gamma_{j_k}))$ is also finite irreducible. Especially, the condition (A) holds.*

Proof. The irreducibility of $\mathbf{Mon}^{(k)}(\alpha, \beta, (\gamma_{j_1}, \dots, \gamma_{j_k}))$ immediately follows from that of $\mathbf{Mon}^{(n)}(\alpha, \beta, \gamma)$ (recall that the irreducibility condition is given by (2)). The finiteness follows from Corollary 4.4. □

Next, we consider the condition (B).

Lemma 4.6. *Let $n \ge 3$ and $\mathbf{Mon}^{(n)}(\alpha, \beta, \gamma)$ be finite irreducible. For distinct $i, j, k \in \{1, \dots, n\}$, $\mathbf{Mon}^{(2)}(\alpha, \alpha\gamma_k^{-1}, (\gamma_i, \gamma_j))$ and $\mathbf{Mon}^{(2)}(\beta, \beta\gamma_k^{-1}, (\gamma_i, \gamma_j))$ are also finite irreducible.*

Proof. For simplicity, we prove the claim only for $\mathbf{Mon}^{(2)}(\alpha, \alpha\gamma_n^{-1}, (\gamma_{n-2}, \gamma_{n-1}))$. As was mentioned in [4, Remark 5.10], $x_n^{-a} f(\frac{x_1}{x_n}, \dots, \frac{x_{n-1}}{x_n}, \frac{1}{x_n})$ is a solution to $E_C(a, b, c)$ if and only if $f(\xi_1, \dots, \xi_n)$ is a solution to $E_C(a, a - c_n + 1, (c_1, \dots, c_{n-1}, a - b + 1))$

with variables $\xi_1, \ldots, \xi_n$. Then, the finiteness of $\mathbf{Mon}^{(n)}(\alpha, \beta, \gamma)$ implies that of $\mathbf{Mon}^{(n)}(\alpha, \alpha\gamma_n^{-1}, (\gamma_1, \ldots, \gamma_{n-1}, \alpha\beta^{-1}))$. Using Proposition 4.5 with $(j_1, j_2) = (n-2, n-1)$, we conclude that $\mathbf{Mon}^{(2)}(\alpha, \alpha\gamma_n^{-1}, (\gamma_{n-2}, \gamma_{n-1}))$ is finite irreducible. □

Lemma 4.7. *For $n \geq 3$, if $\mathbf{Mon}^{(n)}(\alpha, \beta, \gamma)$ is finite irreducible, then at least $n-2$ of $\gamma_1, \ldots, \gamma_n$ are -1.*

Proof. When the number of k such that $\gamma_k \neq -1$ is at most one, the claim holds. We assume that $\gamma_1 \neq -1$, $\gamma_2 \neq -1$, and show that $\gamma_k = -1$ $(k = 3, \ldots, n)$. By Lemma 4.6, $\mathbf{Mon}^{(2)}(\alpha, \alpha\gamma_k^{-1}, (\gamma_1, \gamma_2))$ and $\mathbf{Mon}^{(2)}(\beta, \beta\gamma_k^{-1}, (\gamma_1, \gamma_2))$ are finite irreducible. By Fact 2.7 (B') for $\mathbf{Mon}^{(2)}(\alpha, \alpha\gamma_k^{-1}, (\gamma_1, \gamma_2))$, we have two possibilities:

(i) $\delta_0^{(2)}(\alpha, \alpha\gamma_k^{-1}, (\gamma_1, \gamma_2)) = -1$;

(ii) at least two of $\gamma_1, \gamma_2, \alpha\gamma_k^{-1} \cdot \alpha^{-1} = \gamma_k^{-1}$ are -1.

By the assumption, (ii) does not occur and we obtain

$$1 = -\delta_0^{(2)}(\alpha, \alpha\gamma_k^{-1}, (\gamma_1, \gamma_2)) = -(-\gamma_1\gamma_2\alpha^{-1}(\alpha\gamma_k^{-1})^{-1}) = \gamma_1\gamma_2\gamma_k\alpha^{-2}.$$

Similarly, we obtain $\gamma_1\gamma_2\gamma_k\beta^{-2} = 1$ from the finiteness of $\mathbf{Mon}^{(2)}(\beta, \beta\gamma_k^{-1}, (\gamma_1, \gamma_2))$. Thus, we have

$$1 = (\gamma_1\gamma_2\gamma_k\alpha^{-2})(\gamma_1\gamma_2\gamma_k\beta^{-2}) = \left(\delta_0^{(2)}(\alpha, \beta, (\gamma_1, \gamma_2))\right)^2 \gamma_k^2.$$

On the other hand, Proposition 4.5 implies that $\mathbf{Mon}^{(2)}(\alpha, \beta, (\gamma_1, \gamma_2))$ is also finite irreducible. By Fact 2.7 (B') and the assumption, we have $\delta_0^{(2)}(\alpha, \beta, (\gamma_1, \gamma_2)) = -1$. Therefore, we obtain $\gamma_k^2 = 1$, that is, $\gamma_k = 1$ or $\gamma_k = -1$. Because the matrix $G(1) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ has infinite order, the matrix M_k also has infinite order if $\gamma_k = 1$. This is a contradiction. Therefore, we conclude that $\gamma_k = -1$. □

By the following proposition, we complete the proof of Theorem 2.6.

Proposition 4.8. *For $n \geq 3$, if $\mathbf{Mon}^{(n)}(\alpha, \beta, \gamma)$ is finite irreducible, then the condition (B) holds.*

Proof. By Lemma 4.7, we may assume that

$$\gamma_3 = \cdots = \gamma_n = -1 \tag{11}$$

without loss of generality. Proposition 4.5 implies that $\mathbf{Mon}^{(2)}(\alpha, \beta, (\gamma_1, \gamma_2))$ and $\mathbf{Mon}^{(2)}(\alpha, \beta, (\gamma_i, -1))$ $(i = 1, 2)$ are finite irreducible. By Fact 2.7 (B') for $\mathbf{Mon}^{(2)}(\alpha, \beta, (\gamma_1, \gamma_2))$, we have two possibilities:

(i) $\delta_0^{(2)}(\alpha, \beta, (\gamma_1, \gamma_2)) = -1$, that is, $-\gamma_1\gamma_2\alpha^{-1}\beta^{-1} = -1$;

(ii) at least two of γ_1, γ_2, $\beta\alpha^{-1}$ are -1.

In the case (ii), the condition (11) implies (B-a) or (B-b), and the proposition holds. We consider the case when (ii) does not hold. We may assume $\gamma_1 \neq -1$. Since (i) holds, we have

$$\delta_0^{(n)}(\alpha,\beta,\gamma) = (-1)^{n-1}\gamma_1\gamma_2\gamma_3\cdots\gamma_n\alpha^{-1}\beta^{-1} = -\gamma_1\gamma_2\alpha^{-1}\beta^{-1} = -1.$$

Therefore, if $\beta\alpha^{-1} = -1$, then the condition (B-d) holds. We assume $\beta\alpha^{-1} \neq -1$ and show $\gamma_2 = -1$, which implies (B-c). By Fact 2.7 (B') for $\mathbf{Mon}^{(2)}(\alpha,\beta,(\gamma_1,-1))$, we have two possibilities:

(iii) $\delta_0^{(2)}(\alpha,\beta,(\gamma_1,-1)) = -1$, that is, $\gamma_1\alpha^{-1}\beta^{-1} = -1$;

(iv) at least two of $\gamma_1, -1$, $\beta\alpha^{-1}$ are -1.

By the assumption, (iv) does not hold and we obtain $\gamma_1\alpha^{-1}\beta^{-1} = -1$. This and (i) imply

$$\gamma_2 = (\gamma_1\alpha^{-1}\beta^{-1})^{-1} = -1.$$

Therefore, we complete the proof. □

5. Structure of the finite irreducible monodromy group

In this section, we consider the structure of $\mathbf{Mon}^{(n)}$ when it is finite irreducible. Note that $\alpha,\beta,\gamma_1,\dots,\gamma_n$ are roots of unity (equivalently, $a,b,c_1,\dots,c_n \in \mathbb{Q}$) by the condition (A). For $q \in \{3,4,5,\dots\}$, we set $\zeta_q = \exp(2\pi\sqrt{-1}/q)$.

By the definition of the reflection subgroup, we have

$$\mathbf{Mon}^{(n)}(\alpha,\beta,\gamma) = \mathbf{Ref}^{(n)}(\alpha,\beta,\gamma)\cdot\langle M_1,\dots,M_n\rangle.$$

To determine the structure of $\mathbf{Mon}^{(n)}(\alpha,\beta,\gamma)$, we need to examine the intersection $\mathbf{Ref}^{(n)}(\alpha,\beta,\gamma)\cap\langle M_1,\dots,M_n\rangle$. By the proof of Proposition 4.2, the reflection subgroup $\mathbf{Ref}^{(n)}$ can be decomposed into a product of some $\mathbf{Ref}^{(1)}$'s or $\mathbf{Ref}^{(2)}$'s. First, we consider the intersections $\mathbf{Ref}^{(1)}\cap\langle M_1^{(1)}\rangle$ and $\mathbf{Ref}^{(2)}\cap\langle M_1^{(2)}, M_2^{(2)}\rangle$. To examine these intersections, the following lemma is useful.

Lemma 5.1. *Suppose that $Q \in \mathbf{Mon}^{(n)}$ satisfies $Q^q = E_{2^n}$. If $(QM_0^j)^{qr+1} = E_{2^n}$ with $r \in \mathbb{Z}$, then we have $Q \in \mathbf{Ref}^{(n)}$.*

Proof. By $Q^{-(q-1)} = Q$, we have

$$(QM_0^j)^q = QM_0^jQ^{-1}\cdot Q^2M_0^jQ^{-2}\cdot Q^3M_0^jQ^{-3}\cdots Q^{q-1}M_0^jQ^{-(q-1)}\cdot M_0^j \in \mathbf{Ref}^{(n)}.$$

Therefore, we obtain $Q = (QM_0^j)^{-qr}\cdot M_0^{-j} \in \mathbf{Ref}^{(n)}$. □

5.1. A lemma on $\mathbf{Mon}^{(1)}(\alpha,\beta,\gamma_1)$

In this subsection, we set $c = c_1$ and $\gamma = \gamma_1$. For our study, we need to examine the intersection $\mathbf{Ref}^{(1)}(\alpha,\beta,\gamma)\cap\langle M_1\rangle$ when at least one of γ and $\beta\alpha^{-1}$ is -1. Although this intersection was studied in [9], all the cases were not considered. To take all the cases into consideration, we improve the results in [9, Section 6].

Lemma 5.2. *The intersection $\mathbf{Ref}^{(1)}\cap\langle M_1\rangle$ is given as follows.*

(I-1) *If* $\gamma = \beta\alpha^{-1} = -1$*, then* $\mathbf{Ref}^{(1)} \cap \langle M_1 \rangle = \{E_2\}$*.*

(I-2) *Assume* $\gamma = -1$ *and* $\beta\alpha^{-1} \neq -1$*.*

- *(I-2-1)* *If* $\gamma\alpha^{-1}\beta^{-1} = -1$ *and* α *is a primitive* q*th root of unity for an odd number* q*, then* $\mathbf{Ref}^{(1)} \cap \langle M_1 \rangle = \langle M_1 \rangle$*.*
- *(I-2-2)* *Otherwise,* $\mathbf{Ref}^{(1)} \cap \langle M_1 \rangle = \{E_2\}$*.*

(I-3) *If* $\gamma \neq -1$ *and* $\beta\alpha^{-1} = \gamma\alpha^{-1}\beta^{-1} = -1$*, then* $\mathbf{Ref}^{(1)} \cap \langle M_1 \rangle = \{E_2\}$*.*

(I-4) *Assume* $\gamma \neq -1$*,* $\beta\alpha^{-1} = -1$ *and* $\gamma\alpha^{-1}\beta^{-1} \neq -1$*.*

- *(I-4-1)* *If* $\gamma = \gamma\alpha^{-1}\beta^{-1}$ *and it is a primitive 3rd root of unity, then* $\mathbf{Ref}^{(1)} \cap \langle M_1 \rangle = \langle M_1 \rangle$*.*
- *(I-4-2)* *If both* γ *and* $\gamma\alpha^{-1}\beta^{-1}$ *are primitive 5th roots of unity, then* $\mathbf{Ref}^{(1)} \cap \langle M_1 \rangle = \langle M_1 \rangle$*.*
- *(I-4-3)* *Otherwise,* $\mathbf{Ref}^{(1)} \cap \langle M_1 \rangle = \{E_2\}$*.*

Proof. First, the claim (I-1) follows from [9, Lemma 6.1].

Second, we assume $\gamma = -1$ and $\beta\alpha^{-1} \neq -1$. By [9, Lemma 6.2], if $\gamma\alpha^{-1}\beta^{-1} \neq -1$, then we have $\mathbf{Ref}^{(1)} \cap \langle M_1 \rangle = \{E_2\}$. We consider the case when $\gamma\alpha^{-1}\beta^{-1} = -1$ and assume that α is a primitive qth root of unity. In this case, we have $\beta = \alpha^{-1}$ and $M_0^2 = M_1^2 = E_2$. The characteristic polynomial of

$$M_1 M_0 = \begin{pmatrix} \alpha + \alpha^{-1} - 1 & -1 \\ -\alpha - \alpha^{-1} + 2 & 1 \end{pmatrix}$$

is $\phi(x) = x^2 - (\alpha + \alpha^{-1})x + 1$, and its roots are α and α^{-1}. Because of $\phi(x) | (x^q - 1)$, we have $(M_1 M_0)^q = E_2$. It is sufficient to show that $M_1 \in \mathbf{Ref}^{(1)}$ if and only if q is an odd integer. If $q = 2r + 1$ ($r \in \mathbb{Z}_{>0}$), then $(M_1 M_0)^{2r+1} = E_2$ and Lemma 5.1 imply $M_1 \in \mathbf{Ref}^{(1)}$. Conversely, we assume $M_1 \in \mathbf{Ref}^{(1)}$. Since $\mathbf{Ref}^{(1)} = \langle M_0, M_1 M_0 M_1 \rangle$ and each generator is of order 2, M_1 is expressed as one of the following:

$$\begin{aligned} &(M_0 \cdot M_1 M_0 M_1)^r = M_0 (M_1 M_0)^{2r-1} M_1, \quad (M_0 \cdot M_1 M_0 M_1)^r M_0 = M_0 (M_1 M_0)^{2r}, \\ &(M_1 M_0 M_1 \cdot M_0)^r = (M_1 M_0)^{2r}, \quad (M_1 M_0 M_1 \cdot M_0)^r M_1 M_0 M_1 = (M_1 M_0)^{2r+1} M_1, \end{aligned}$$

where r is some integer. Since $\det M_1 = -1$, we have $M_1 = M_0 (M_1 M_0)^{2r}$ or $M_1 = (M_1 M_0)^{2r+1} M_1$. In either case, we obtain $(M_1 M_0)^{2r+1} = E_2$, and hence, $\phi(x)$ divides $x^{2r+1} - 1$. Since α is a root of $\phi(x)$, we have $\alpha^{2r+1} = 1$. Thus, q is an odd number, and the claim (I-2) is proved.

Third, we assume $\gamma \neq -1$ and $\beta\alpha^{-1} = \gamma\alpha^{-1}\beta^{-1} = -1$. Since $\det M_0 = -1$ and $\det M_1 = \gamma^{-1}$, the possibilities of non-trivial intersection $\mathbf{Ref}^{(1)} \cap \langle M_1 \rangle$ are only those cases in which γ is a primitive $2r$-th root of unity ($r \in \mathbb{Z}_{>1}$) and $\mathbf{Ref}^{(1)} \cap \langle M_1 \rangle = \langle M_1^r \rangle$. We prove that these cases do not occur. We assume γ is a primitive $2r$-th root of unity. By the assumption $\beta\alpha^{-1} = \gamma\alpha^{-1}\beta^{-1} = -1$, we have

$$M_0 = \begin{pmatrix} 1 & 0 \\ \gamma - 1 & -1 \end{pmatrix}, \qquad M_1 = \begin{pmatrix} 1 & -\gamma^{-1} \\ 0 & \gamma^{-1} \end{pmatrix}.$$

By a straightforward calculation, we can show that the characteristic polynomial of $M_1^j M_0$ is $x^2 - \gamma^{-j}$ and hence, we have $(M_1^j M_0)^2 = \gamma^{-j} E_2$. We thus obtain

$$\begin{aligned} M_1^j M_0 M_1^{-j} &= M_1^j M_0 M_1^{2r-j} = M_1^j \cdot M_1^j M_1^{2r-j} \cdot M_0 M_1^{2r-j} \cdot M_0 M_0 \\ &= \gamma^{-(2r-j)} M_1^{2j} M_0 = \gamma^j M_1^{2j} M_0. \end{aligned}$$

This expression and the identity $\gamma^r = -1$ imply $M_1^j M_0 M_1^{-j} = -M_1^{j-r} M_0 M_1^{-(j-r)}$. Especially for $j = r$, we have $M_1^r M_0 M_1^{-r} = -M_0$ and hence, we obtain $-E_2 \in \mathbf{Ref}^{(1)}$. Therefore, $\mathbf{Ref}^{(1)}$ is expressed as

$$\begin{aligned} \mathbf{Ref}^{(1)} &= \langle M_0, M_1 M_0 M_1^{-1}, M_1^2 M_0 M_1^{-2}, \dots, M_1^{2r-1} M_0 M_1^{-(2r-1)} \rangle \\ &= \langle M_0, \pm\gamma M_1^2 M_0, \pm\gamma^2 M_1^4 M_0, \dots, \pm\gamma^{r-1} M_1^{2(r-1)} M_0, -M_0 \rangle \\ &= \langle M_0, -E_2, \gamma M_1^2 \rangle. \end{aligned}$$

Because of the identity

$$\begin{aligned} (\gamma M_1^2)^i M_0 (\gamma M_1^2)^j &= \gamma^{i+j} M_1^{2i} M_1^{2j} M_1^{-2j} M_0 M_1^{2j} = \gamma^{i+j} M_1^{2i} M_1^{2j} \cdot \gamma^{-2j} M_1^{-4j} M_0 \\ &= \gamma^{i-j} M_1^{2(i-j)} M_0 = (\gamma M_1^2)^{i-j} M_0, \end{aligned}$$

we obtain the expression

$$\mathbf{Ref}^{(1)} = \{\pm(\gamma M_1^2)^j \mid j = 0, 1, \dots, r-1\} \cup \{\pm(\gamma M_1^2)^j M_0 \mid j = 0, 1, \dots, r-1\}.$$

Comparing the entries of matrices, we can show $M_1^r \notin \mathbf{Ref}^{(1)}$. Thus, the claim (I-3) is proved.

Finally, we assume $\gamma \neq -1$, $\beta\alpha^{-1} = -1$ and $\gamma\alpha^{-1}\beta^{-1} \neq -1$. When the triple $(\lambda, \mu, \nu) = (1-c, c-a-b, b-a) = (1-c, c-a-b, 1/2)$ is one in the Schwarz list ([3, Table 1], [13]), our claim is already proved in [9]. However, $\mathbf{Mon}^{(1)}$ is also finite irreducible when a triple (λ, μ, ν) is one in the Schwarz list after performing the following operations: permutation of λ, μ, ν; sign change of each of λ, μ, ν; addition of $(l_1, l_2, l_3) \in \mathbb{Z}^3$ with $l_1 + l_2 + l_3 \in 2\mathbb{Z}$. A triple such as this was not considered in [9]. Here, we consider all the cases. As was mentioned in the proof of [9, Lemma 6.3], if the denominators of c and $c - a - b$ are different, then we have $\mathbf{Ref}^{(1)} \cap \langle M_1 \rangle = \{E_2\}$. The remaining cases are

$$\begin{aligned} (\gamma, \gamma\alpha^{-1}\beta^{-1}) = {} & (\zeta_3, \zeta_3),\ (\zeta_3, \zeta_3^2),\ (\zeta_3^2, \zeta_3),\ (\zeta_3^2, \zeta_3^2), \\ & (\zeta_5, \zeta_5^2),\ (\zeta_5, \zeta_5^3),\ (\zeta_5^4, \zeta_5^2),\ (\zeta_5^4, \zeta_5^3),\ (\zeta_5^2, \zeta_5),\ (\zeta_5^3, \zeta_5),\ (\zeta_5^2, \zeta_5^4),\ (\zeta_5^3, \zeta_5^4). \end{aligned}$$

In each case, the matrices M_0 and M_1 are defined over the field $\mathbb{Q}(\zeta_3)$ or $\mathbb{Q}(\zeta_5)$. The relations between M_0 and M_1 are preserved under the actions in the Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta_3)/\mathbb{Q}) \simeq \mathbb{Z}/2\mathbb{Z}$ or $\mathrm{Gal}(\mathbb{Q}(\zeta_5)/\mathbb{Q}) \simeq \mathbb{Z}/4\mathbb{Z}$. Therefore, it suffices to consider the four cases:

$$(\gamma, \gamma\alpha^{-1}\beta^{-1}) = (\zeta_3, \zeta_3),\ (\zeta_3^2, \zeta_3),\ (\zeta_5^4, \zeta_5^2),\ (\zeta_5^4, \zeta_5^3).$$

Two cases $(\gamma, \gamma\alpha^{-1}\beta^{-1}) = (\zeta_3^2, \zeta_3),\ (\zeta_5^4, \zeta_5^2)$ are discussed in the proof of [9, Lemma 6.3], and our claim is true. We discuss the other two cases. If we assume $(\gamma, \gamma\alpha^{-1}\beta^{-1}) =$

(ζ_3, ζ_3) (resp. (ζ_5^4, ζ_5^3)), then we have $(M_1 M_0)^4 = E_2$ (resp. $(M_1 M_0^3)^6 = E_2$) and hence, $M_1 \in \mathbf{Ref}^{(1)}$ by Lemma 5.1. Therefore, the proof of the claim (I-4) is completed. $\square$

5.2. A lemma on $\mathbf{Mon}^{(2)}(\boldsymbol{\alpha}, \boldsymbol{\beta}, (\boldsymbol{\gamma_1}, \boldsymbol{\gamma_2}))$

We consider the intersection $\mathbf{Ref}^{(2)} \cap \langle M_1, M_2 \rangle$. If two of γ_1, γ_2, $\beta\alpha^{-1}$ are -1, then $\mathbf{Ref}^{(2)}$ is decomposed into $\mathbf{Ref}^{(1)} \times \mathbf{Ref}^{(1)}$ by Lemmas 3.2 and 3.3. These cases were already discussed in [9, Section 7]. In this subsection, we assume that $\delta_0^{(2)} = -1$, $\gamma_1 \neq -1$ and either γ_2 or $\beta\alpha^{-1}$ is -1. Certain facts are verified by computer.

LEMMA 5.3. *We assume $\delta_0^{(2)} = -1$ (i.e., $\gamma_1\gamma_2\alpha^{-1}\beta^{-1} = 1$) and $\gamma_1 \neq -1$. The intersection $\mathbf{Ref}^{(2)} \cap \langle M_1, M_2 \rangle$ is given as follows.*

(II-1) If $\gamma_2 = -1$ and $\beta\alpha^{-1} \neq -1$, then $\mathbf{Ref}^{(2)} \cap \langle M_1, M_2 \rangle = \{E_4\}$.

(II-2) Assume $\beta\alpha^{-1} = -1$ and $\gamma_2 \neq -1$. Let γ_k be a primitive q_kth root of unity ($q_k \in \{3, 4, 5\}$).

- *(II-2-1) If $q_1 \neq q_2$, then $\mathbf{Ref}^{(2)} \cap \langle M_1, M_2 \rangle = \{E_4\}$.*
- *(II-2-2) If $q_1 = q_2 = 3$ and $\gamma_2 = \gamma_1$, then $\mathbf{Ref}^{(2)} \cap \langle M_1, M_2 \rangle = \{E_4\}$.*
- *(II-2-3) If $q_1 = q_2 = 3$ and $\gamma_2 = \gamma_1^2$, then $\mathbf{Ref}^{(2)} \cap \langle M_1, M_2 \rangle = \langle M_1 M_2 \rangle \simeq \mathbb{Z}/3\mathbb{Z}$.*
- *(II-2-4) If $q_1 = q_2 = 5$, then $\mathbf{Ref}^{(2)} \cap \langle M_1, M_2 \rangle = \langle M_1 M_2^j \rangle \simeq \mathbb{Z}/5\mathbb{Z}$, where j is an integer such that $\gamma_1 \gamma_2^j = 1$.*

REMARK 5.4. In the case (II-2), when $q_1 = q_2$, we can show that $q_k \neq 4$. Indeed, if we assume $q_1 = q_2 = 4$, the irreducibility condition (2) does not hold.

Proof of Lemma 5.3. Since $\det M_0 = \delta_0^{(2)} = -1$ and $\det(M_1^{j_1} M_2^{j_2}) = \gamma_1^{-2j_1}\gamma_2^{-2j_2}$, the intersection $\mathbf{Ref}^{(2)} \cap \langle M_1, M_2 \rangle$ is trivial except for the following possibilities:

(i) $\gamma_2 = -1$, $\beta\alpha^{-1} \neq -1$;

(ii) $\gamma_2 \neq -1$, $\beta\alpha^{-1} = -1$, $q_1 \neq q_2 = 4$ and $M_2^2 \in \mathbf{Ref}^{(2)} \cap \langle M_1, M_2 \rangle$;

(iii) $\gamma_2 \neq -1$, $\beta\alpha^{-1} = -1$, $q_1 = q_2 = 3$, $\gamma_2 = \gamma_1$ and $M_1 M_2^2 \in \mathbf{Ref}^{(2)} \cap \langle M_1, M_2 \rangle$;

(iv) $\gamma_2 \neq -1$, $\beta\alpha^{-1} = -1$, $q_1 = q_2 = 3$, $\gamma_2 = \gamma_1^2$ and $M_1 M_2 \in \mathbf{Ref}^{(2)} \cap \langle M_1, M_2 \rangle$;

(v) $\gamma_2 \neq -1$, $\beta\alpha^{-1} = -1$, $q_1 = q_2 = 5$, $\gamma_1\gamma_2^j = 1$ and $M_1 M_2^j \in \mathbf{Ref}^{(2)} \cap \langle M_1, M_2 \rangle$.

To prove the lemma, it suffices to show that (i), (ii), (iii) are false and (iv), (v) are true under our assumption. We prove the fact that (i), (ii), (iii) are false by using a computer[1]. In the case (i), we have 18 possibilities of the pair $(\gamma_1, \beta\alpha^{-1})$ up to the complex conjugate. Table 1 lists the cardinalities of $\mathbf{Mon}^{(2)}$, $\mathbf{Ref}^{(2)}$ and the cyclic group $\langle M_1, M_2 \rangle$ for all the pairs. This implies $\mathbf{Ref}^{(2)} \cap \langle M_1, M_2 \rangle = \{E_4\}$. Similarly, by computing the cardinalities (Table 2), we can verify that (ii) and (iii) are also false.

[1] We use the system GAP (`https://www.gap-system.org`). The author does not have an elegant proof. We use the functions

- `Group` and `NormalClosure` (`https://www.gap-system.org/Manuals/doc/ref/chap39.html`) to define $\mathbf{Mon}^{(2)}$ and $\mathbf{Ref}^{(2)}$, respectively;
- `Size` (`https://www.gap-system.org/Manuals/doc/ref/chap30.html`) to compute the cardinalities of the groups $\mathbf{Mon}^{(2)}$ and $\mathbf{Ref}^{(2)}$.

γ_1	$\beta\alpha^{-1}$	$\lvert\mathbf{Mon}^{(2)}\rvert$	$\lvert\mathbf{Ref}^{(2)}\rvert$	$\lvert\mathbf{Mon}^{(2)}\rvert/\lvert\mathbf{Ref}^{(2)}\rvert$	$\lvert\langle M_1, M_2\rangle\rvert$
ζ_3	ζ_3^j	1152	192	6	6
ζ_3	ζ_4^{2j-1}	6912	1152	6	6
ζ_3	ζ_5^k	86400	14400	6	6
ζ_4	ζ_3^j	9216	1152	8	8
ζ_5^i	ζ_3^j	144000	14400	10	10
ζ_5	ζ_5^2	144000	14400	10	10
ζ_5	ζ_5^3	144000	14400	10	10
ζ_5^2	ζ_5	144000	14400	10	10
ζ_5^2	ζ_5^4	144000	14400	10	10

Table 1. (i) $\gamma_2 = \delta_0^{(2)} = -1$ and $\beta\alpha^{-1} \neq -1$ $(i, j \in \{1, 2\},\ k \in \{1, 2, 3, 4\})$

γ_1	γ_2	$\lvert\mathbf{Mon}^{(2)}\rvert$	$\lvert\mathbf{Ref}^{(2)}\rvert$	$\lvert\mathbf{Mon}^{(2)}\rvert/\lvert\mathbf{Ref}^{(2)}\rvert$	$\lvert\langle M_1, M_2\rangle\rvert$
ζ_3	ζ_4	13824	1152	12	12
ζ_3^2	ζ_4	13824	1152	12	12
ζ_3	ζ_3	1728	192	9	9

Table 2. $\beta\alpha^{-1} = \delta_0^{(2)} = -1$ and (ii) $\gamma_2 = \zeta_4$, (iii) $\gamma_1 = \gamma_2 = \zeta_3$

We consider the case (iv). We may assume $\gamma_1 = \zeta_3$, $\gamma_2 = \zeta_3^2$ and $\beta\alpha^{-1} = -1$. By a straightforward calculation, we have $(M_1M_2M_0)^4 = E_4$. Thus, Lemma 5.1 yields $M_1M_2 \in \mathbf{Ref}^{(2)}$.

Finally, we consider the case (v). By $\beta\alpha^{-1} = -1$, $\gamma_1\alpha^{-1}\beta^{-1} = \gamma_2^{-1}$ and the Schwarz list, it is sufficient to discuss two cases $(\gamma_1, \gamma_2) = (\zeta_5, \zeta_5^2), (\zeta_5, \zeta_5^3)$, up to the actions in $\mathrm{Gal}(\mathbb{Q}(\zeta_5)/\mathbb{Q})$. If we assume $(\gamma_1, \gamma_2) = (\zeta_5, \zeta_5^2)$, then we have $(M_1^3M_2M_0)^6 = E_4$. By Lemma 5.1, we obtain $M_1^3M_2 \in \mathbf{Ref}^{(2)}$ and hence, $M_1M_2^2 = (M_1^3M_2)^2 \in \mathbf{Ref}^{(2)}$. If we assume $(\gamma_1, \gamma_2) = (\zeta_5, \zeta_5^3)$, then we have $(M_1M_2^3M_0)^6 = E_4$ which implies $M_1M_2^3 \in \mathbf{Ref}^{(2)}$. Therefore, the proof is completed. □

5.3. Structure of $\mathbf{Mon}^{(n)}(\alpha, \beta, \gamma)$

Now, we provide the structure of $\mathbf{Mon}^{(n)}(\alpha, \beta, \gamma)$. Without loss of generality, we may assume that the condition (B-a), (B-b), (B-c) or (B-d) holds. Let γ_k be a primitive q_kth root of unity $(k \in \{1, 2\},\ q_k \in \{2, 3, 4, 5, \dots\})$. The structure of $\mathbf{Mon}^{(n)}(\alpha, \beta, \gamma)$ is classified into the following four types:

(Type 1) $\mathbf{Mon}^{(n)}(\alpha, \beta, \gamma) = \mathbf{Ref}^{(n)}(\alpha, \beta, \gamma) \cdot \langle M_1, \dots, M_n\rangle$ with

$$\mathbf{Ref}^{(n)} \cap \langle M_1, \dots, M_n\rangle = \{E_{2^n}\}, \quad \mathbf{Ref}^{(n)}(\alpha, \beta, \gamma) \simeq \Big(\mathbf{Ref}^{(1)}(\alpha, \beta, \gamma_1)\Big)^{2^{n-1}},$$
$$\langle M_1, \dots, M_n\rangle \simeq \mathbb{Z}/q_1\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^{n-1}, \quad \mathbf{Mon}^{(1)}(\alpha, \beta, \gamma_1)/\mathbf{Ref}^{(1)}(\alpha, \beta, \gamma_1) \simeq \mathbb{Z}/q_1\mathbb{Z};$$

(Type 2) $\mathbf{Mon}^{(n)}(\alpha, \beta, \gamma) = \mathbf{Ref}^{(n)}(\alpha, \beta, \gamma) \cdot \langle M_2, \dots, M_n\rangle$ with

$$\mathbf{Ref}^{(n)} \cap \langle M_2, \dots, M_n\rangle = \{E_{2^n}\}, \quad \mathbf{Ref}^{(n)}(\alpha, \beta, \gamma) \simeq \Big(\mathbf{Ref}^{(1)}(\alpha, \beta, \gamma_1)\Big)^{2^{n-1}},$$

$$\langle M_2,\dots,M_n\rangle \simeq (\mathbb{Z}/2\mathbb{Z})^{n-1},\quad \mathbf{Mon}^{(1)}(\alpha,\beta,\gamma_1)=\mathbf{Ref}^{(1)}(\alpha,\beta,\gamma_1);$$

(Type 3) $\mathbf{Mon}^{(n)}(\alpha,\beta,\gamma)=\mathbf{Ref}^{(n)}(\alpha,\beta,\gamma)\cdot\langle M_1,\dots,M_n\rangle$ with

$$\mathbf{Ref}^{(n)}\cap\langle M_1,\dots,M_n\rangle=\{E_{2^n}\},\quad \mathbf{Ref}^{(n)}(\alpha,\beta,\gamma)\simeq\Big(\mathbf{Ref}^{(2)}(\alpha,\beta,(\gamma_1,\gamma_2))\Big)^{2^{n-2}},$$
$$\langle M_1,\dots,M_n\rangle\simeq\mathbb{Z}/q_1\mathbb{Z}\times\mathbb{Z}/q_2\mathbb{Z}\times(\mathbb{Z}/2\mathbb{Z})^{n-2},$$
$$\mathbf{Mon}^{(2)}(\alpha,\beta,(\gamma_1,\gamma_2))/\mathbf{Ref}^{(2)}(\alpha,\beta,(\gamma_1,\gamma_2))\simeq\mathbb{Z}/q_1\mathbb{Z}\times\mathbb{Z}/q_2\mathbb{Z};$$

(Type 4) $\mathbf{Mon}^{(n)}(\alpha,\beta,\gamma)=\mathbf{Ref}^{(n)}(\alpha,\beta,\gamma)\cdot\langle M_2,\dots,M_n\rangle$ with

$$\mathbf{Ref}^{(n)}\cap\langle M_2,\dots,M_n\rangle=\{E_{2^n}\},\quad \mathbf{Ref}^{(n)}(\alpha,\beta,\gamma)\simeq\Big(\mathbf{Ref}^{(2)}(\alpha,\beta,(\gamma_1,\gamma_2))\Big)^{2^{n-2}},$$
$$\langle M_2,\dots,M_n\rangle\simeq\mathbb{Z}/q_2\mathbb{Z}\times(\mathbb{Z}/2\mathbb{Z})^{n-2},\quad \mathbf{Mon}^{(2)}(\alpha,\beta,(\gamma_1,\gamma_2))/\mathbf{Ref}^{(2)}(\alpha,\beta,(\gamma_1,\gamma_2))\simeq\mathbb{Z}/q_2\mathbb{Z}.$$

Note that the structures of $\mathbf{Ref}^{(2)}(\alpha,\beta,(\gamma_1,\gamma_2))$ in Type 3 and Type 4 were investigated in [10].

Theorem 5.5. *Let $n\geq 3$ and assume that $\mathbf{Mon}^{(n)}(\alpha,\beta,\gamma)$ is finite irreducible. We may also assume that γ_k is a primitive q_kth root of unity ($k\in\{1,2\}$, $q_k\in\{2,3,4,5,\dots\}$) and $\gamma_3=\cdots=\gamma_n=-1$. The structure of $\mathbf{Mon}^{(n)}(\alpha,\beta,\gamma)$ is given as follows.*

(B-a) Assume $\gamma_1=\gamma_2=-1$ ($q_1=q_2=2$).

- *(B-a-1) If $\beta\alpha^{-1}\neq-1$, $\alpha\beta=1$ and α is a primitive qth root of unity for an odd number q, then $\mathbf{Mon}^{(n)}(\alpha,\beta,\gamma)$ is of Type 2 and $\mathbf{Ref}^{(n)}\cap\langle M_1,\dots,M_n\rangle=\langle M_1\cdots M_n\rangle$.*
- *(B-a-2) Otherwise, $\mathbf{Mon}^{(n)}(\alpha,\beta,\gamma)$ is of Type 1.*

(B-b) Assume $\gamma_2=\beta\alpha^{-1}=-1$ and $\gamma_1\neq-1$.

- *(B-b-1) If $\gamma_1\alpha^{-1}\beta^{-1}\neq-1$, $\gamma_1=\gamma_1\alpha^{-1}\beta^{-1}$ and $q_1=3$, then $\mathbf{Mon}^{(n)}(\alpha,\beta,\gamma)$ is of Type 2 and $\mathbf{Ref}^{(n)}\cap\langle M_1,\dots,M_n\rangle=\langle M_1\rangle$.*
- *(B-b-2) If $\gamma_1\alpha^{-1}\beta^{-1}\neq-1$ and both γ_1 and $\gamma_1\alpha^{-1}\beta^{-1}$ are primitive 5th roots of unity, then $\mathbf{Mon}^{(n)}(\alpha,\beta,\gamma)$ is of Type 2 and $\mathbf{Ref}^{(n)}\cap\langle M_1,\dots,M_n\rangle=\langle M_1\rangle$.*
- *(B-b-3) Otherwise, $\mathbf{Mon}^{(n)}(\alpha,\beta,\gamma)$ is of Type 1.*

(B-c) Assume $\gamma_2=\delta_0^{(n)}(\alpha,\beta,\gamma)=-1$ and $\gamma_1\neq-1$, $\beta\alpha^{-1}\neq-1$. Then $\mathbf{Mon}^{(n)}(\alpha,\beta,\gamma)$ is of Type 3.

(B-d) Assume $\beta\alpha^{-1}=\delta_0^{(n)}(\alpha,\beta,\gamma)=-1$ and $\gamma_1\neq-1$, $\gamma_2\neq-1$.

- *(B-d-1) If $q_1=q_2=3$ and $\gamma_2=\gamma_1^2$, then $\mathbf{Mon}^{(n)}(\alpha,\beta,\gamma)$ is of Type 4 and $\mathbf{Ref}^{(n)}\cap\langle M_1,\dots,M_n\rangle=\langle M_1M_2\rangle$.*
- *(B-d-2) If $q_1=q_2=5$, then $\mathbf{Mon}^{(n)}(\alpha,\beta,\gamma)$ is of Type 4 and $\mathbf{Ref}^{(n)}\cap\langle M_1,\dots,M_n\rangle=\langle M_1M_2^j\rangle$, where j is an integer such that $\gamma_1\gamma_2^j=1$.*
- *(B-d-3) Otherwise, $\mathbf{Mon}^{(n)}(\alpha,\beta,\gamma)$ is of Type 3.*

Proof. We use an approach similar to that of [9, Theorems 7.1 and 7.2] to prove the theorem. We only have to discuss the intersection $\mathbf{Ref}^{(n)} \cap \langle M_1, \ldots, M_n \rangle$.

First, we consider the case (B-a). In the proof of Lemma 3.2, $W^{\pm}$ are invariant under $\mathbf{Ref}^{(n)}$, while M_{n-1}, M_n interchange W^+ and W^-. Because $M_1, \ldots, M_{n-2}$ preserve W^+ and W^-, we have

$$M_1^{i_1} \cdots M_{n-2}^{i_{n-2}} M_{n-1} \notin \mathbf{Ref}^{(n)}, \qquad (i_1, \ldots, i_{n-2}) \in \mathbb{Z}^{n-2}. \tag{12}$$

By rearranging the indices $\{1, \ldots, n\}$, we obtain $M_1^{i_1} \cdots M_n^{i_n} \notin \mathbf{Ref}^{(n)}$, except for E_{2^n} and $M_1 \cdots M_n$. Therefore, we obtain $\mathbf{Ref}^{(n)} \cap \langle M_1, \ldots, M_n \rangle = \mathbf{Ref}^{(n)} \cap \langle M_1 \cdots M_n \rangle$. Repeated applications of the decomposition in the proof of Lemma 3.2 show that $\mathbb{C}^{2^n}$ is decomposed into a direct sum of two-dimensional subspaces. The restriction of the action of $M_1 \cdots M_n$ and $\mathbf{Ref}^{(n)}$ on each of these two-dimensional subspaces are $M_1^{(1)} \in \mathbf{Mon}^{(1)}(\alpha, \beta, -1)$ and $\mathbf{Ref}^{(1)}(\alpha, \beta, -1)$, respectively. The intersection $\mathbf{Ref}^{(n)} \cap \langle M_1, \ldots, M_n \rangle$ coincides with $\langle M_1 \cdots M_n \rangle$ if and only if $\mathbf{Ref}^{(1)}(\alpha, \beta, -1) \cap \langle M_1^{(1)} \rangle = \langle M_1^{(1)} \rangle$. Thus, our claim follows from Lemma 5.2 (I-1) and (I-2).

Next, we consider the case (B-b). In the proof of Lemma 3.3, $W^{\pm}$ are invariant under $\mathbf{Ref}^{(n)}$, whereas M_n interchange W^+ and W^-. Because $M_1, \ldots, M_{n-1}$ preserve W^+ and W^-, we have

$$M_1^{i_1} \cdots M_{n-1}^{i_{n-1}} M_n \notin \mathbf{Ref}^{(n)}, \qquad (i_1, \ldots, i_{n-1}) \in \mathbb{Z}^{n-1}. \tag{13}$$

By rearranging the indices $\{2, \ldots, n\}$, we obtain $M_1^{i_1} \cdots M_n^{i_n} \notin \mathbf{Ref}^{(n)}$, except for $M_1^{i_1}$. Therefore, we obtain $\mathbf{Ref}^{(n)} \cap \langle M_1, \ldots, M_n \rangle = \mathbf{Ref}^{(n)} \cap \langle M_1 \rangle$. Repeated applications of the decomposition in the proof of Lemma 3.3 show that $\mathbb{C}^{2^n}$ is decomposed into a direct sum of two-dimensional subspaces. The restriction of the action of M_1 and $\mathbf{Ref}^{(n)}$ on each of these two-dimensional subspaces are $M_1^{(1)} \in \mathbf{Mon}^{(1)}(\alpha, \beta, \gamma_1)$ and $\mathbf{Ref}^{(1)}(\alpha, \beta, \gamma_1)$, respectively. The intersection $\mathbf{Ref}^{(n)} \cap \langle M_1, \ldots, M_n \rangle$ coincides with $\langle M_1 \rangle$ if and only if $\mathbf{Ref}^{(1)}(\alpha, \beta, \gamma_1) \cap \langle M_1^{(1)} \rangle = \langle M_1^{(1)} \rangle$. Thus, our claim follows from Lemma 5.2 (I-3) and (I-4).

Finally, we consider the cases (B-c) and (B-d).

(B-c) Similarly to the case (B-a), we can show $\mathbf{Ref}^{(n)} \cap \langle M_1, \ldots, M_n \rangle = \mathbf{Ref}^{(n)} \cap \langle M_1, M_2 \cdots M_n \rangle$. Note that in this case, we can rearrange the indices $\{2, \ldots, n\}$ in (12). We also have the decomposition of $\mathbb{C}^{2^n}$ into a direct sum of four-dimensional subspaces, and the restriction of the action of M_1, $M_2 \cdots M_n$ and $\mathbf{Ref}^{(n)}$ on each of these four-dimensional subspaces are $M_1^{(2)}, M_2^{(2)} \in \mathbf{Mon}^{(2)}(\alpha, \beta, (\gamma_1, \gamma_2))$ and $\mathbf{Ref}^{(2)}(\alpha, \beta, (\gamma_1, \gamma_2))$, respectively.

(B-d) Similarly to the case (B-b), we can show $\mathbf{Ref}^{(n)} \cap \langle M_1, \ldots, M_n \rangle = \mathbf{Ref}^{(n)} \cap \langle M_1, M_2 \rangle$. Note that in this case, we can rearrange the indices $\{3, \ldots, n\}$ in (13). We also have the decomposition of $\mathbb{C}^{2^n}$ into a direct sum of four-dimensional subspaces, and the restriction of the action of M_1, M_2 and $\mathbf{Ref}^{(n)}$ on each of these four-dimensional subspaces are $M_1^{(2)}, M_2^{(2)} \in \mathbf{Mon}^{(2)}(\alpha, \beta, (\gamma_1, \gamma_2))$ and $\mathbf{Ref}^{(2)}(\alpha, \beta, (\gamma_1, \gamma_2))$, respectively.

Therefore, the structure of the intersection $\mathbf{Ref}^{(n)} \cap \langle M_1, \ldots, M_n \rangle$ is determined according to that of $\mathbf{Ref}^{(2)} \cap \langle M_1^{(2)}, M_2^{(2)} \rangle$. By Lemma 5.3, our claims are proved. $\square$

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