

HAMILTONIAN STRUCTURES OF ISOMONODROMIC DEFORMATIONS ON MODULI SPACES OF PARABOLIC CONNECTIONS

ARATA KOMYO

ABSTRACT. In this paper, we treat moduli spaces of parabolic connections. We take an affine open covering of the moduli spaces, and we construct a Hamiltonian structure of an algebraic vector field determined by the isomonodromic deformation for each affine open subset.

1. INTRODUCTION

Let (C, \mathbf{t}) ($\mathbf{t} = (t_1, \dots, t_n)$) be an n -pointed smooth projective curve of genus g over \mathbb{C} , where t_1, \dots, t_n are distinct points. We take a positive integer r and an element $\boldsymbol{\nu} = (\nu_j^{(i)})_{\substack{1 \leq i \leq n \\ 0 \leq j \leq r-1}} \in \mathbb{C}^{nr}$ such that $\sum_{i,j} \nu_j^{(i)} = -d \in \mathbb{Z}$. We say $(E, \nabla, \{l_*^{(i)}\}_{1 \leq i \leq n})$ is a $(\mathbf{t}, \boldsymbol{\nu})$ -parabolic connection of rank r if

- (1) E is a rank r algebraic vector bundle of degree d on C ,
- (2) $\nabla: E \rightarrow E \otimes \Omega_C^1(t_1 + \dots + t_n)$ is a connection, and
- (3) for each t_i , $l_*^{(i)}$ is a filtration $E|_{t_i} = l_0^{(i)} \supset l_1^{(i)} \supset \dots \supset l_r^{(i)} = 0$ such that $\dim(l_j^{(i)}/l_{j+1}^{(i)}) = 1$ and $(\text{res}_{t_i}(\nabla) - \nu_j^{(i)} \text{id}_{E|_{t_i}})(l_j^{(i)}) \subset l_{j+1}^{(i)}$ for $j = 0, \dots, r-1$.

Inaba–Iwasaki–Saito [11] (for the general case see Inaba [10]) introduces the $\boldsymbol{\alpha}$ -stability for $(\mathbf{t}, \boldsymbol{\nu})$ -parabolic connections, and constructs the moduli scheme of $\boldsymbol{\alpha}$ -stable $(\mathbf{t}, \boldsymbol{\nu})$ -parabolic connections of rank r , denoted by $M_C^\boldsymbol{\alpha}(\mathbf{t}, \boldsymbol{\nu})$. Moreover, let T be a smooth algebraic scheme which is an étale covering of the moduli stack of n -pointed smooth projective curves of genus g over \mathbb{C} and take a universal family $(\mathcal{C}, \tilde{t}_1, \dots, \tilde{t}_n)$ over T . Let $N_r^{(n)}(d)$ be the set of $\boldsymbol{\nu} = (\nu_j^{(i)}) \in \mathbb{C}^{nr}$ such that $\sum_{i,j} \nu_j^{(i)} = -d \in \mathbb{Z}$. Then we can construct a relative fine moduli scheme $M_{\mathcal{C}/T}^\boldsymbol{\alpha}(\tilde{\mathbf{t}}, r, d) \rightarrow T \times N_r^{(n)}(d)$ of $\boldsymbol{\alpha}$ -stable parabolic connections of rank r and of degree d , which is smooth and quasi-projective [10, Theorem 2.1]. The moduli space $M_C^\boldsymbol{\alpha}(\mathbf{t}, \boldsymbol{\nu})$ is a fiber of $M_{\mathcal{C}/T}^\boldsymbol{\alpha}(\tilde{\mathbf{t}}, r, d) \rightarrow T \times N_r^{(n)}(d)$ and is equipped with a natural symplectic structure.

The moduli space $M_{\mathcal{C}/T}^\boldsymbol{\alpha}(\tilde{\mathbf{t}}, r, d)$ gives a geometric description of the differential equation determined by the isomonodromic deformation. We fix $\boldsymbol{\nu}$. We can regard $M_{\mathcal{C}/T}^\boldsymbol{\alpha}(\tilde{\mathbf{t}}, r, d)_\boldsymbol{\nu} \rightarrow T$ as a phase space of the differential equation determined by the isomonodromic deformation, and T as a space of time variables. A fiber of $M_{\mathcal{C}/T}^\boldsymbol{\alpha}(\tilde{\mathbf{t}}, r, d)_\boldsymbol{\nu} \rightarrow T$ becomes a space of initial conditions. In fact, for the case of $C = \mathbb{P}^1$, $r = 2$ and $n = 4$, these fibers coincide with the spaces of initial conditions for the Painlevé VI equation constructed by Okamoto [18] (see [12]). Inaba–Iwasaki–Saito [11] (for rank 2 and \mathbb{P}^1 cases) and Inaba [10] (for general cases) show that the Riemann–Hilbert correspondence induces a proper surjective bimeromorphic morphism between the moduli space of $\boldsymbol{\alpha}$ -stable parabolic connections and the moduli space of certain equivalence classes of representations of the fundamental group $\pi_1(C \setminus \{t_1, \dots, t_n\}, *)$. By this property of the Riemann–Hilbert correspondence, they show that the differential equation determined by the isomonodromic deformation satisfies the geometric Painlevé property (see [11] and [10]). Note that geometric descriptions of the isomonodromic deformation are also given by Hitchin [8] and Boalch [4], [5] et al. from symplectic points of view.

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One of the important properties of the Painlevé equations is that they can be written as (*non-autonomous*) *Hamiltonian systems* ([19], [20], [15]). The purpose of this paper is to give Hamiltonian descriptions of the vector fields determined by the isomonodromic deformations on the moduli space $M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)_{\nu}$, which is a phase space. Hamiltonian descriptions of the vector fields determined by the isomonodromic deformations on moduli spaces of certain connections were essentially considered by Krichever [16] and Hurtubise [9]. (Wong [23] generalizes those results in the case of principal G -bundles). We apply their ideas to the moduli space $M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)_{\nu}$ of parabolic connections. Accordingly, we give an affine open covering $\{M_i\}_i$ of $M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)_{\nu}$ and a Hamiltonian structure on each M_i . Namely, we construct a 2-form ω on $M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)_{\nu}$ such that the kernel $\text{Ker}(\omega)$ induces the vector fields determined by the isomonodromic deformations and ω is the symplectic form fiberwise (Proposition 4.10). The 2-form ω is considered in [13] and [14]. We define Hamiltonian functions H_i for $i = 1, 2, \dots, \dim T$ on each M_i (Definition 4.13). If we may take good coordinates on $M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)_{\nu}$, we obtain a Hamiltonian description of the vector field on M_i induced by the isomonodromic deformation (Corollary 4.17).

In the argument of the Hamiltonian description of the isomonodromic deformations, it is important to describe the vector fields determined by the isomonodromic deformations on $M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)_{\nu}$ in terms of the hypercohomology of some complex. Indeed, we give such a description of the isomonodromic deformations on the moduli space $M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)_{\nu}$. Biswas–Heu–Hurtubise [3] have described the vector fields determined by the isomonodromic deformations in terms of the hypercohomology of some complex. Essentially, our description is same as their description. However, their description of the isomonodromic deformations are only for certain generic connections. On the other hand, our moduli space $M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)_{\nu}$ is the *full* phase space of the isomonodromic deformations, since the isomonodromic deformations on this moduli space have geometric Painlevé property. So we give a description of the isomonodromic deformations for the full phase space.

The organization of this paper is as follows. In Section 2, we recall the description of the tangent space of $M_{\mathcal{C}}^{\alpha}(\mathbf{t}, \nu)$ and of the natural symplectic structure on $M_{\mathcal{C}}^{\alpha}(\mathbf{t}, \nu)$ in terms of the hypercohomology of a certain complex. In Section 3.1, we recall the *Atiyah algebra*. In Section 3.2, we discuss a description of the tangent space of $M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)_{\nu}$ in terms of the hypercohomology of a certain complex. We use the Atiyah algebra in a definition of this complex. In Section 3.3, we describe the vector field determined by the isomonodromic deformation in terms of the hypercohomology. In Section 4, we give Hamiltonian descriptions of the vector fields determined by the isomonodromic deformations. In Section 4.1, we take an affine open covering $\{M_i\}_i$ of $M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)_{\nu}$ and construct a relative initial connection ∇_0 on each M_i . Note that ∇_0 is defined Zariski locally. In this paper, we will use only the Zariski topology. In Section 4.2, we define vector fields on each M_i associated to the relative initial connection ∇_0 . We use these vector fields instead of vector fields associated to time variables. In Section 4.4, we give the 2-form ω on $M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)_{\nu}$ and define Hamiltonian functions on each M_i . The Hamiltonian functions depend on the choice of the relative initial connection ∇_0 . Finally, we obtain a Hamiltonian description of the vector field on each M_i induced by the isomonodromic deformation.

2. PRELIMINARIES

2.1. Moduli space of stable parabolic connections. Let C be a smooth projective curve of genus g . We put

$$T_n := \{(t_1, \dots, t_n) \in C \times \dots \times C \mid t_i \neq t_j \text{ for } i \neq j\}$$

for a positive integer n . For integers d, r with $r > 0$, we put

$$N_r^{(n)}(d) := \left\{ (\nu_j^{(i)})_{\substack{1 \leq i \leq n \\ 0 \leq j \leq r-1}} \in \mathbb{C}^{nr} \mid d + \sum_{i,j} \nu_j^{(i)} = 0 \right\}.$$

Take members $\mathbf{t} = (t_1, \dots, t_n) \in T_n$ and $\nu = (\nu_j^{(i)})_{1 \leq i \leq n, 0 \leq j \leq r-1} \in N_r^{(n)}(d)$.

Definition 2.1. We say $(E, \nabla, \{l_*^{(i)}\}_{1 \leq i \leq n})$ is a (\mathbf{t}, ν) -parabolic connection of rank r and of degree d over C if

- (1) E is a rank r algebraic vector bundle on C ,
- (2) $\nabla: E \rightarrow E \otimes \Omega_C^1(t_1 + \cdots + t_n)$ is a connection, that is, ∇ is a homomorphism of sheaves satisfying $\nabla(fa) = a \otimes df + f\nabla(a)$ for $f \in \mathcal{O}_C$ and $a \in E$, and
- (3) for each t_i , $l_*^{(i)}$ is a filtration $E|_{t_i} = l_0^{(i)} \supset l_1^{(i)} \supset \cdots \supset l_r^{(i)} = 0$ such that $\dim(l_j^{(i)}/l_{j+1}^{(i)}) = 1$ and $(\text{res}_{t_i}(\nabla) - \nu_j^{(i)} \text{id}_{E|_{t_i}})(l_j^{(i)}) \subset l_{j+1}^{(i)}$ for $j = 0, \dots, r-1$.

The filtration $l_*^{(i)}$ ($1 \leq i \leq n$) is said to be a *parabolic structure* on the vector bundle E .

Remark 2.2. We have

$$\deg E = \deg(\det(E)) = - \sum_{i=1}^n \text{res}_{t_i}(\nabla_{\det(E)}) = - \sum_{i=1}^n \sum_{j=0}^{r-1} \nu_j^{(i)} = d.$$

Take rational numbers

$$0 < \alpha_1^{(i)} < \alpha_2^{(i)} < \cdots < \alpha_r^{(i)} < 1$$

for $i = 1, \dots, n$ satisfying $\alpha_j^{(i)} \neq \alpha_{j'}^{(i')}$ for $(i, j) \neq (i', j')$. We choose a sufficiently generic $\alpha = (\alpha_j^{(i)})$.

Definition 2.3. A parabolic connection $(E, \nabla, \{l_*^{(i)}\}_{1 \leq i \leq n})$ is α -stable (resp. α -semistable) if for any proper nonzero subbundle $F \subset E$ satisfying $\nabla(F) \subset F \otimes \Omega_C^1(t_1 + \cdots + t_n)$, the inequality

$$\frac{\deg F + \sum_{i=1}^n \sum_{j=1}^r \alpha_j^{(i)} \dim((F|_{t_i} \cap l_{j-1}^{(i)})/(F|_{t_i} \cap l_j^{(i)}))}{\text{rank } F} \underset{(\text{resp. } \leq)}{<} \frac{\deg E + \sum_{i=1}^n \sum_{j=1}^r \alpha_j^{(i)} \dim(l_{j-1}^{(i)}/l_j^{(i)})}{\text{rank } E}$$

holds.

Let T be a smooth algebraic scheme which is an étale covering of the moduli stack $\mathcal{M}_{g,n}$ of n -pointed smooth projective curves of genus g over \mathbb{C} and take a universal family $(\mathcal{C}, \tilde{t}_1, \dots, \tilde{t}_n)$ over T .

Definition 2.4. We denote the pull-back of \mathcal{C} and $\tilde{\mathbf{t}}$ by the morphism $T \times N_r^{(n)}(d) \rightarrow T$ by the same characters \mathcal{C} and $\tilde{\mathbf{t}} = \{\tilde{t}_1, \dots, \tilde{t}_n\}$. Then $D(\tilde{\mathbf{t}}) := \tilde{t}_1 + \cdots + \tilde{t}_n$ becomes an effective Cartier divisor on \mathcal{C} flat over $T \times N_r^{(n)}(d)$. We also denote by $\tilde{\nu}$ the pull-back of the universal family on $N_r^{(n)}(d)$ by the morphism $T \times N_r^{(n)}(d) \rightarrow N_r^{(n)}(d)$. We define a functor $\mathcal{M}_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)$ from the category of locally noetherian schemes over $T \times N_r^{(n)}(d)$ to the category of sets by

$$\mathcal{M}_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)(S) := \left\{ (E, \nabla, \{l_j^{(i)}\}) \right\} / \sim$$

for a locally noetherian scheme S over $T \times N_r^{(n)}(d)$, where

- (1) E is a rank r algebraic vector bundle on \mathcal{C}_S ,
- (2) $\nabla: E \rightarrow E \otimes \Omega_{\mathcal{C}_S/S}^1(D(\tilde{\mathbf{t}})_S)$ is a relative connection,
- (3) for each $(\tilde{t}_i)_S$, $l_*^{(i)}$ is a filtration by subbundles $E|_{(\tilde{t}_i)_S} = l_0^{(i)} \supset l_1^{(i)} \supset \cdots \supset l_r^{(i)} = 0$ such that $(\text{res}_{(\tilde{t}_i)_S}(\nabla) - (\tilde{\nu}_j^{(i)})_S \text{id}_{E|_{(\tilde{t}_i)_S}})(l_j^{(i)}) \subset l_{j+1}^{(i)}$ for $j = 0, \dots, r-1$, and
- (4) for any point $s \in S$, $\dim(l_j^{(i)}/l_{j+1}^{(i)}) \otimes k(s) = 1$ for any i, j and $(E, \nabla, \{l_j^{(i)}\}) \otimes k(s)$ is α -stable.

Here $(E, \nabla, \{l_j^{(i)}\}) \sim (E', \nabla', \{l_j'^{(i)}\})$ if there exist a line bundle \mathcal{L} on S and an isomorphism $\sigma: E \xrightarrow{\sim} E' \otimes \mathcal{L}$ such that $\sigma|_{\tilde{t}_i} (l_j^{(i)}) = l_j'^{(i)} \otimes \mathcal{L}$ for any i, j and the diagram

$$\begin{array}{ccc} E & \xrightarrow{\nabla} & E \otimes \Omega_{\mathcal{C}/T}^1(D(\tilde{\mathbf{t}})) \\ \downarrow \sigma & & \downarrow \sigma \otimes \text{id} \\ E' \otimes \mathcal{L} & \xrightarrow{\nabla' \otimes \mathcal{L}} & E' \otimes \Omega_{\mathcal{C}/T}^1(D(\tilde{\mathbf{t}})) \otimes \mathcal{L} \end{array}$$

commutes.

Theorem 2.5 ([10, Theorem 2.1]). *For the moduli functor $\mathcal{M}_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)$, there exists a fine moduli scheme*

$$\varpi: M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d) \longrightarrow T \times N_r^{(n)}(d)$$

of α -stable parabolic connections of rank r and degree d , which is smooth and quasi-projective. The fiber $M_{\mathcal{C}_x}^\alpha(\tilde{\mathbf{t}}_x, \nu)$ over $(x, \nu) \in T \times N_r^{(n)}(d)$ is the moduli space of α -stable $(\tilde{\mathbf{t}}_x, \nu)$ -parabolic connections whose dimension is

$$2r^2(g-1) + nr(r-1) + 2$$

if it is non-empty.

2.2. Description of the relative tangent sheaf of the phase space over the space of time variables. We recall the description of the relative tangent sheaf $\Theta_{M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)/T \times N_r^{(n)}(d)}$ in terms of the hypercohomology of a certain complex ([10, the proof of Theorem 2.1]). Let $(\tilde{E}, \tilde{\nabla}, \{\tilde{l}_j^{(i)}\})$ be a universal family on $\mathcal{C} \times_T M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)$. First, we define a complex \mathcal{F}^\bullet by

$$\begin{aligned} \mathcal{F}^0 &:= \left\{ s \in \mathcal{E}nd(\tilde{E}) \mid s|_{\tilde{t}_i \times M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)}(\tilde{l}_j^{(i)}) \subset \tilde{l}_j^{(i)} \text{ for any } i, j \right\} \\ \mathcal{F}^1 &:= \left\{ s \in \mathcal{E}nd(\tilde{E}) \otimes \Omega_{\mathcal{C}/T}^1(D(\tilde{\mathbf{t}})) \mid \text{res}_{\tilde{t}_i \times M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)}(s)(\tilde{l}_j^{(i)}) \subset \tilde{l}_{j+1}^{(i)} \text{ for any } i, j \right\} \\ \nabla_{\mathcal{F}^\bullet}: \mathcal{F}^0 &\longrightarrow \mathcal{F}^1; \quad \nabla_{\mathcal{F}^\bullet}(s) = \tilde{\nabla} \circ s - s \circ \tilde{\nabla}. \end{aligned}$$

Second, we take an affine open set $M \subset M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)$ and an affine open covering $\mathcal{C}_M = \bigcup_\alpha U_\alpha$ such that $\tilde{E}|_{U_\alpha} \cong \mathcal{O}_{U_\alpha}^{\oplus r}$ for any α , $\#\{i \mid \tilde{t}_i|_{\mathcal{C}_M} \cap U_\alpha \neq \emptyset\} \leq 1$ for any α and $\#\{\alpha \mid \tilde{t}_i|_{\mathcal{C}_M} \cap U_\alpha \neq \emptyset\} \leq 1$ for any i . Take a relative tangent vector field $v \in \Theta_{M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)/T \times N_r^{(n)}(d)}(M)$. The field v corresponds to a member $(E_\epsilon, \nabla_\epsilon, \{(l_\epsilon)_j^{(i)}\}) \in M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)(\text{Spec } \mathcal{O}_M[\epsilon])$ such that $(E_\epsilon, \nabla_\epsilon, \{(l_\epsilon)_j^{(i)}\}) \otimes \mathcal{O}_M[\epsilon]/(\epsilon) \cong (\tilde{E}, \tilde{\nabla}, \{\tilde{l}_j^{(i)}\})|_{\mathcal{C}_M}$, where $\mathcal{O}_M[\epsilon] = \mathcal{O}_M[t]/(t^2)$. There is an isomorphism

$$\varphi_\alpha: E_\epsilon|_{U_\alpha \times \text{Spec } \mathcal{O}_M[\epsilon]} \xrightarrow{\sim} \mathcal{O}_{U_\alpha \times \text{Spec } \mathcal{O}_M[\epsilon]}^{\oplus r} \xrightarrow{\sim} \tilde{E}|_{U_\alpha} \otimes \mathcal{O}_M[\epsilon]$$

such that $\varphi_\alpha \otimes \mathcal{O}_M[\epsilon]/(\epsilon): E_\epsilon \otimes \mathcal{O}_M[\epsilon]/(\epsilon)|_{U_\alpha} \xrightarrow{\sim} \tilde{E}|_{U_\alpha} \otimes \mathcal{O}_M[\epsilon]/(\epsilon) = \tilde{E}|_{U_\alpha}$ is the given isomorphism and that $\varphi_\alpha|_{\tilde{t}_i \otimes \mathcal{O}_M[\epsilon]}((l_\epsilon)_j^{(i)}) = \tilde{l}_j^{(i)}|_{U_\alpha \times \text{Spec } \mathcal{O}_M[\epsilon]}$ if $\tilde{t}_i|_{\mathcal{C}_M} \cap U_\alpha \neq \emptyset$. We put

$$\begin{aligned} u_{\alpha\beta} &:= \varphi_\alpha \circ \varphi_\beta^{-1} - \text{id}_{\tilde{E}|_{U_{\alpha\beta} \times \text{Spec } \mathcal{O}_M[\epsilon]}}, \\ v_\alpha &:= (\varphi_\alpha \otimes \text{id}) \circ \nabla_\epsilon|_{U_\alpha \times \text{Spec } \mathcal{O}_M[\epsilon]} \circ \varphi_\alpha^{-1} - \tilde{\nabla}|_{U_\alpha \times \text{Spec } \mathcal{O}_M[\epsilon]}. \end{aligned}$$

Then $\{u_{\alpha\beta}\} \in C^1((\epsilon) \otimes \mathcal{F}_M^0)$, $\{v_\alpha\} \in C^0((\epsilon) \otimes \mathcal{F}_M^1)$ and

$$d\{u_{\alpha\beta}\} = \{u_{\beta\gamma} - u_{\alpha\gamma} + u_{\alpha\beta}\} = 0; \quad \nabla_{\mathcal{F}^\bullet}\{u_{\alpha\beta}\} = \{v_\beta - v_\alpha\} = d\{v_\alpha\}.$$

So $[(\{u_{\alpha\beta}\}, \{v_\alpha\})]$ determines an element $\sigma_M(v)$ of $\mathbf{H}^1(\mathcal{F}_M^\bullet)$. We can check that $v \mapsto \sigma_M(v)$ determines an isomorphism

$$\sigma_M: \Theta_{M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)/T \times N_r^{(n)}(d)}(M) \xrightarrow{\sim} \mathbf{H}^1(\mathcal{F}_M^\bullet); \quad v \longmapsto \sigma_M(v).$$

The isomorphism σ_M induces a canonical isomorphism

$$\sigma: \Theta_{M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)/T \times N_r^{(n)}(d)} \xrightarrow{\sim} \mathbf{R}^1(\pi_{M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)})_*(\mathcal{F}^\bullet),$$

where $\pi_{M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)}: \mathcal{C}_{M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)} \rightarrow M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)$ is the natural morphism.

2.3. Symplectic structure. For each affine open subset $U \subset M_{\mathcal{C}/T}^\alpha(\tilde{\mathfrak{t}}, r, d)$, we define a pairing

$$(1) \quad \mathbf{H}^1(\mathcal{C} \times_T U, \mathcal{F}_U^\bullet) \otimes \mathbf{H}^1(\mathcal{C} \times_T U, \mathcal{F}_U^\bullet) \longrightarrow \mathbf{H}^2(\mathcal{C} \times_T U, \Omega_{\mathcal{C} \times_T U/U}^\bullet \cong H^0(\mathcal{O}_U)) \\ [[\{u_{\alpha\beta}\}, \{v_\alpha\}]] \otimes [[\{u'_{\alpha\beta}\}, \{v'_\alpha\}]] \longmapsto [[\{\text{Tr}(u_{\alpha\beta} \circ u'_{\beta\gamma})\}, -\{\text{Tr}(u_{\alpha\beta} \circ v'_\beta) - \text{Tr}(v_\alpha \circ u'_{\alpha\beta})\}]],$$

considered in Čech cohomology with respect to an affine open covering $\{U_\alpha\}$ of $\mathcal{C} \times_T U$, $\{u_{\alpha\beta}\} \in C^1(\mathcal{F}^0)$, $\{v_\alpha\} \in C^0(\mathcal{F}^1)$ and so on. This pairing determines a pairing

$$\omega: \mathbf{R}^1(\pi_{M_{\mathcal{C}/T}^\alpha(\tilde{\mathfrak{t}}, r, d)})_*(\mathcal{F}^\bullet) \otimes \mathbf{R}^1(\pi_{M_{\mathcal{C}/T}^\alpha(\tilde{\mathfrak{t}}, r, d)})_*(\mathcal{F}^\bullet) \longrightarrow \mathcal{O}_{M_{\mathcal{C}/T}^\alpha(\tilde{\mathfrak{t}}, r, d)}.$$

This pairing is a nondegenerate relative 2-form. This fact follows from the Serre duality and the five lemma:

$$\begin{array}{ccccccccc} H^0(\mathcal{F}_x^0) & \longrightarrow & H^0(\mathcal{F}_x^1) & \longrightarrow & \mathbf{H}^1(\mathcal{F}_x^\bullet) & \longrightarrow & H^1(\mathcal{F}_x^0) & \longrightarrow & H^1(\mathcal{F}_x^1) \\ \downarrow \sim & & \downarrow \sim & & \downarrow \sim & & \downarrow \sim & & \downarrow \sim \\ H^1(\mathcal{F}_x^1)^\vee & \longrightarrow & H^1(\mathcal{F}_x^0)^\vee & \longrightarrow & \mathbf{H}^1(\mathcal{F}_x^\bullet)^\vee & \longrightarrow & H^0(\mathcal{F}_x^1)^\vee & \longrightarrow & H^0(\mathcal{F}_x^0)^\vee \end{array}$$

for any point $x \in M_{\mathcal{C}/T}^\alpha(\tilde{\mathfrak{t}}, r, d)$. Moreover we have $d\omega = 0$ (see [10, Proposition 7.3]).

3. ISOMONODROMIC DEFORMATION

In this section, we consider a description of the vector field determined by the isomonodromic deformation in terms of a Čech cohomology. In Section 3.1, we recall the Atiyah algebra. By the Atiyah algebra, we obtain descriptions of first-order deformations of pairs of pointed curves and vector bundles. In Section 3.2, we show that the relative tangent sheaf $\Theta_{M_{\mathcal{C}/T}^\alpha(\tilde{\mathfrak{t}}, r, d)/N_r^{(n)}(d)}$ is isomorphic to the hypercohomology of a certain complex using the Atiyah algebra. In Section 3.3, we consider the integrable deformations of parabolic connections when the n -pointed curves vary. The integrable deformations of parabolic connections mean the isomonodromic deformations of the corresponding relative parabolic connections. We describe the vector field determined by the isomonodromic deformation in terms of a Čech cohomology.

3.1. Atiyah algebra. We recall the *Atiyah algebra*. (For details, for example see [2]). Let C be a smooth projective curve and Θ_C be the tangent sheaf. Let E be a vector bundle of rank r on C . We define a sheaf of differential operators on E as follows. (For detail, see [1, Section 1]). We take an affine open subset $U \subset C$. For the $\mathcal{O}(U)$ -bimodule $\text{End}_{\mathbb{C}}(E(U), E(U))$, we define $\text{End}_{\mathbb{C}}(E(U), E(U))_i^\vee$ (for $i \geq -1$) by induction as follows:

$$\text{End}_{\mathbb{C}}(E(U), E(U))_{-1}^\vee := 0 \\ \text{End}_{\mathbb{C}}(E(U), E(U))_i^\vee := \left\{ s \in \text{End}_{\mathbb{C}}(E(U), E(U)) \mid \begin{array}{l} r \cdot s - s \cdot r \in \text{End}_{\mathbb{C}}(E(U), E(U))_{i-1}^\vee \\ \text{for any } r \in \mathcal{O}(U) \end{array} \right\}.$$

Definition 3.1 (see [1, 1.1.6]). We say a \mathbb{C} -linear morphism $s: E \rightarrow E$ is a *differential operator of degree i* if for any affine subset $U \subset C$ the corresponding morphism $s_U: E(U) \rightarrow E(U)$ lies in $\text{End}_{\mathbb{C}}(E(U), E(U))_i^\vee$ and is not contained in $\text{End}_{\mathbb{C}}(E(U), E(U))_{i-1}^\vee$.

Put $\mathcal{D}_E = \text{Diff}(E, E) = \bigcup_i \mathcal{D}_i$, where \mathcal{D}_i is the sheaf of differential operators of degree $\leq i$ on C . We have $\mathcal{D}_i/\mathcal{D}_{i-1} = \mathcal{E}ndE \otimes S^i(\Theta_C)$, where $S^i(\Theta_C)$ is the i -th symmetric product of Θ_C . We denote by symb_1 the composition

$$\mathcal{D}_1 \longrightarrow \mathcal{D}_1/\mathcal{D}_0 \xrightarrow{=} \mathcal{E}ndE \otimes \Theta_C.$$

For the differential operator v of degree 1, the image $\text{symb}_1(v)$ is called the *symbol* of v .

Definition 3.2. We define the *Atiyah algebra* of E as

$$\mathcal{A}_E = \{\partial \in \mathcal{D}_1 \mid \text{symb}_1(\partial) \in \text{id}_E \otimes \Theta_C \subset \mathcal{E}nd(E) \otimes \Theta_C\}.$$

We have inclusions $\mathcal{D}_0 = \mathcal{E}ndE \subset \mathcal{A}_E \subset \mathcal{D}_1$ and a short exact sequence

$$0 \longrightarrow \mathcal{E}nd(E) \longrightarrow \mathcal{A}_E \xrightarrow{\text{symb}_1} \Theta_C \longrightarrow 0.$$

Fix a positive integer n . Let $D = t_1 + \cdots + t_n$ be an effective divisor of C , where t_1, \dots, t_n are distinct points of C . We put $\mathcal{A}_E(D) := \text{symb}_1^{-1}(\Theta_C(-D))$. Then we have the following exact sequence

$$0 \longrightarrow \mathcal{E}nd(E) \longrightarrow \mathcal{A}_E(D) \xrightarrow{\text{symb}_1} \Theta_C(-D) \longrightarrow 0.$$

By this exact sequence, we have the following exact sequence

$$0 \longrightarrow H^1(C, \mathcal{E}nd(E)) \longrightarrow H^1(C, \mathcal{A}_E(D)) \xrightarrow{\text{symb}_1} H^1(C, \Theta_C(-D)) \longrightarrow 0$$

when $2g - 2 + n > 0$. Then $H^1(C, \mathcal{A}_E(D))$ means the set of infinitesimal deformations of the pair $((C, t_1, \dots, t_n), E)$ of the n -pointed curve (C, t_1, \dots, t_n) and the vector bundle E on C .

For a connection $\nabla: E \rightarrow E \otimes \Omega_C^1(D)$, we define a splitting

$$(2) \quad \iota(\nabla): \Theta_C(-D) \longrightarrow \mathcal{A}_E(D)$$

by

$$(\iota(\nabla)(v))(u) = \langle v, \nabla(u) \rangle \in E$$

for $v \in \Theta_C(-D)$ and $u \in E$. The splitting $\iota(\nabla)$ is locally described as follows. Let U be an affine open subset of C where we have a trivialization $E|_U \cong \mathcal{O}_U^{\oplus r}$. We denote by $Af^{-1}df$ a connection matrix of ∇ on U , where f is a local defining equation of t_i and $A \in M_r(\mathcal{O}_U)$. For an element $g \frac{\partial}{\partial f} \in \Theta_C(-D)(U)$, we have an equality $\iota(\nabla)(g \frac{\partial}{\partial f}) = g \left(\frac{\partial}{\partial f} + Af^{-1} \right) \in \mathcal{A}_E(D)(U)$. Conversely, a splitting of $\text{symb}_1: \mathcal{A}_E(D) \rightarrow \Theta_C(-D)$ gives a connection as follows. A splitting of symb_1 gives a splitting of $\mathcal{A}_E(D) \otimes \Omega_C(D) \rightarrow \mathcal{O}_C$. Let $(f \frac{\partial}{\partial f} + A) \otimes \frac{df}{f}$ be a local description of the image of $1 \in \mathcal{O}_C$ by this splitting. Remark that the image of $(f \frac{\partial}{\partial f} + A) \otimes \frac{df}{f}$ under the morphism $\mathcal{A}_E(D) \otimes \Omega_C(D) \rightarrow \mathcal{O}_C$ is 1. The image of 1 gives a morphism

$$\begin{aligned} E &\longrightarrow E \otimes \Omega_C^1(D) \\ a &\longmapsto \left(\frac{\partial}{\partial f}(a) + \frac{A}{f}(a) \right) \otimes df. \end{aligned}$$

This is a connection $E \rightarrow E \otimes \Omega_C(D)$.

3.2. Description of the total tangent sheaf of the phase space. We discuss a description of the relative tangent sheaf $\Theta_{M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)/N_r^{(n)}(d)}$ in terms of the hypercohomology of a certain complex. Let

$(\tilde{E}, \tilde{\nabla}, \{\tilde{l}_j^{(i)}\})$ be a universal family on $\mathcal{C} \times_T M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)$. We put

$$\begin{aligned} \mathcal{G}^0 &:= \left\{ s \in \mathcal{A}_{\tilde{E}}(D(\tilde{\mathbf{t}})) \mid s|_{\tilde{l}_i \times_T M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)}(\tilde{l}_j^{(i)}) \subset \tilde{l}_j^{(i)} \text{ for any } i, j \right\} \\ \mathcal{G}^1 &:= \left\{ s \in \mathcal{E}nd(\tilde{E}) \otimes \Omega_{\mathcal{C}/T}^1(D(\tilde{\mathbf{t}})) \mid \text{res}_{\tilde{l}_i \times_T M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)}(s)(\tilde{l}_j^{(i)}) \subset \tilde{l}_{j+1}^{(i)} \text{ for any } i, j \right\}, \end{aligned}$$

where $\mathcal{A}_{\tilde{E}}(D(\tilde{\mathbf{t}}))$ is the relative Atiyah algebra which is the extension

$$0 \longrightarrow \mathcal{E}nd(\tilde{E}) \longrightarrow \mathcal{A}_{\tilde{E}}(D(\tilde{\mathbf{t}})) \longrightarrow \Theta_{\mathcal{C} \times_T M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)/M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)}(-D(\tilde{\mathbf{t}})) \longrightarrow 0.$$

Here note that $s|_{\tilde{l}_i \times_T M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)}(\tilde{l}_j^{(i)})$ is well-defined for $s \in \mathcal{A}_{\tilde{E}}(D(\tilde{\mathbf{t}}))$. Indeed, we take an affine open subset U of $\mathcal{C} \times_T M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)$ such that $(\tilde{l}_i \times_T M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)) \cap U \neq \emptyset$, and we take a trivialization $\bar{\phi}: \tilde{E}|_U \rightarrow \mathcal{O}_U^{\oplus r}$. If we take a local description $s = s_0(\partial/\partial f) + s_1$ (where $s_0(\partial/\partial f) \in \Theta_{U/M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)}(-D(\tilde{\mathbf{t}})|_U)$ and $s_1 \in \mathcal{E}nd(\mathcal{O}_U^{\oplus r})$), then we define

$$s|_{(\tilde{l}_i \times_T M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)) \cap U}(\tilde{l}_j^{(i)}) := \bar{\phi}^{-1}((s_1 \circ \bar{\phi})|_{(\tilde{l}_i \times_T M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)) \cap U}(\tilde{l}_j^{(i)})).$$

This definition is independent of the choice of a trivialization of $\tilde{E}|_U$, since

$$g^{-1} \circ \left(s_0 \frac{\partial}{\partial f} + s_1 \right) \circ g = s_0 \frac{\partial}{\partial f} + s_0 g^{-1} \frac{\partial g}{\partial f} + g^{-1} s_1 g$$

for $g \in \text{Aut}(\mathcal{O}_U^{\oplus r})$ and $s_0 g^{-1} \frac{\partial g}{\partial f}$ vanishes at $(\tilde{t}_i \times_T M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)) \cap U$. We define a complex \mathcal{G}^\bullet by

$$\begin{aligned} \nabla_{\mathcal{G}^\bullet}: \mathcal{G}^0 &\longrightarrow \mathcal{G}^1; \\ s &\longmapsto \tilde{\nabla} \circ \left(s - \iota(\tilde{\nabla}) \circ \text{symb}_1(s) \right) - \left(\left(s - \iota(\tilde{\nabla}) \circ \text{symb}_1(s) \right) \otimes 1 \right) \circ \tilde{\nabla}. \end{aligned}$$

In fact, we use the description (4) below for calculation of the complex \mathcal{G}^\bullet .

Let U be an affine open subset of $\mathcal{C} \times_T M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)$. we take a trivialization of $\tilde{E}|_U$. Let d be the relative exterior derivative on $U \rightarrow M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)$. We take $s \in \mathcal{G}^0(U)$, which has a local description $s_0(\partial/\partial f) + s_1$. Here $s_1 \in \mathcal{E}nd(\mathcal{O}_U^{\oplus r})$ and $\text{symb}_1(s) = s_0(\partial/\partial f)$.

Definition 3.3. For the section $s = s_0(\partial/\partial f) + s_1 \in \mathcal{G}^0(U)$, we define a homomorphism of sheaves $s: \mathcal{O}_U^{\oplus r} \otimes \Omega_{\mathcal{C}/T}^1(D(\tilde{\mathbf{t}})) \rightarrow \mathcal{O}_U^{\oplus r} \otimes \Omega_{\mathcal{C}/T}^1(D(\tilde{\mathbf{t}}))$ by

$$(3) \quad \left(s_0 \frac{\partial}{\partial f} + s_1 \right) (a) := d \left\langle 1 \otimes s_0 \frac{\partial}{\partial f}, a \right\rangle + (s_1 \otimes 1)a$$

for any $a \in \mathcal{O}_U^{\oplus r} \otimes \Omega_{\mathcal{C}/T}^1(D(\tilde{\mathbf{t}}))$. Here, $d\langle 1 \otimes s_0 \partial/\partial f, \cdot \rangle$ means the composition

$$\mathcal{O}_U^{\oplus r} \otimes \Omega_{\mathcal{C}/T}^1(D(\tilde{\mathbf{t}})) \xrightarrow{\langle 1 \otimes s_0 \partial/\partial f, \cdot \rangle} \mathcal{O}_U^{\oplus r} \xrightarrow{d} \mathcal{O}_U^{\oplus r} \otimes \Omega_{\mathcal{C}/T}^1 \longrightarrow \mathcal{O}_U^{\oplus r} \otimes \Omega_{\mathcal{C}/T}^1(D(\tilde{\mathbf{t}})).$$

Since $\mathcal{C} \times_T M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d) \rightarrow M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)$ is a family of smooth projective curves, the definition (3) is independent of the choice of the parameter f . Indeed, let $a_0 df$ be a element of $\mathcal{O}_U^{\oplus r} \otimes \Omega_{\mathcal{C}/T}^1(D(\tilde{\mathbf{t}}))$. Then $\left(s_0 \frac{\partial}{\partial f} + s_1 \right) (a_0 df) = d(s_0 a_0) + (s_1 \otimes 1)a_0 df$. We have the following equalities:

$$\begin{aligned} \left(s_0 \frac{\partial g}{\partial f} \frac{\partial}{\partial g} + s_1 \right) (a_0 df) &= d \left\langle 1 \otimes s_0 \frac{\partial g}{\partial f} \frac{\partial}{\partial g}, a_0 \frac{\partial f}{\partial g} dg \right\rangle + (s_1 \otimes 1)a_0 df \\ &= d(s_0 a_0) + (s_1 \otimes 1)a_0 df. \end{aligned}$$

These equality mean that the definition (3) is independent of the choice of the parameter f . The definition (3) comes from the local description of the Lie derivative.

Lemma 3.4. *We take a trivialization of \tilde{E} on U . Let $Af^{-1}df$ be the connection matrix of $\tilde{\nabla}$ on U with respect to the trivialization. Then we have the following equality:*

$$\begin{aligned} &\left(d + A \frac{df}{f} \right) \left(\left(s_1 - s_0 \frac{A}{f} \right) (a) \right) - \left(s_1 - s_0 \frac{A}{f} \right) \left(\left(d + A \frac{df}{f} \right) (a) \right) \\ &= \left(d + A \frac{df}{f} \right) \left(\left(s_0 \frac{\partial}{\partial f} + s_1 \right) (a) \right) - \left(s_0 \frac{\partial}{\partial f} + s_1 \right) \left(\left(d + A \frac{df}{f} \right) (a) \right) \end{aligned}$$

for any $a \in \mathcal{O}_U^{\oplus r} \otimes \Omega_{\mathcal{C}/T}^1(D(\tilde{\mathbf{t}}))$. That is, we have

$$(4) \quad \nabla_{\mathcal{G}^\bullet}(s)(a) = \tilde{\nabla} \circ s(a) - s \circ \tilde{\nabla}(a)$$

for any $a \in \tilde{E}|_U \otimes \Omega_{\mathcal{C}/T}^1(D(\tilde{\mathbf{t}}))$.

Proof. From the right hand side, we compute as follows:

$$\begin{aligned}
& \left(d + A \frac{df}{f} \right) \left(\left(s_0 \frac{\partial}{\partial f} + s_1 \right) (a) \right) - \left(s_0 \frac{\partial}{\partial f} + s_1 \right) \left(\left(d + A \frac{df}{f} \right) (a) \right) \\
&= \left(d + A \frac{df}{f} \right) \left(s_0 \frac{\partial a}{\partial f} \right) + \left(d + A \frac{df}{f} \right) (s_1(a)) - s_0 \frac{\partial}{\partial f} \left(d(a) + Aa \frac{df}{f} \right) - s_1 \left(d(a) + Aa \frac{df}{f} \right) \\
&= \left(d + A \frac{df}{f} \right) \left(s_0 \frac{\partial a}{\partial f} \right) + \left(d + A \frac{df}{f} \right) (s_1(a)) \\
&\quad - d \left\langle s_0 \frac{\partial}{\partial f}, d(a) \right\rangle - d \left\langle s_0 \frac{\partial}{\partial f}, Aa \frac{df}{f} \right\rangle - s_1 \left(d(a) + Aa \frac{df}{f} \right) \\
&= s_0 \frac{A}{f} \frac{\partial a}{\partial f} df + \left(d + A \frac{df}{f} \right) (s_1(a)) - d \left(s_0 \frac{A}{f} (a) \right) - s_1 \left(d(a) + Aa \frac{df}{f} \right).
\end{aligned}$$

On the other hand, we have the following equalities:

$$\begin{aligned}
& \left(d + A \frac{df}{f} \right) \left(\left(s_1 - s_0 \frac{A}{f} \right) (a) \right) - \left(s_1 - s_0 \frac{A}{f} \right) \left(\left(d + A \frac{df}{f} \right) (a) \right) \\
&= \left(d + A \frac{df}{f} \right) (s_1(a)) - \left(d + A \frac{df}{f} \right) \left(s_0 \frac{A}{f} (a) \right) - \left(s_1 - s_0 \frac{A}{f} \right) \left(\left(d(a) + Aa \frac{df}{f} \right) \right) \\
&= \left(d + A \frac{df}{f} \right) (s_1(a)) - d \left(s_0 \frac{A}{f} (a) \right) - A \frac{df}{f} \left(s_0 \frac{A}{f} (a) \right) \\
&\quad - s_1 \left(d(a) + Aa \frac{df}{f} \right) + s_0 \frac{A}{f} \left(d(a) + Aa \frac{df}{f} \right) \\
&= s_0 \frac{A}{f} \frac{\partial a}{\partial f} df + \left(d + A \frac{df}{f} \right) (s_1(a)) - d \left(s_0 \frac{A}{f} (a) \right) - s_1 \left(d(a) + Aa \frac{df}{f} \right),
\end{aligned}$$

since s_0 is a scalar. By the calculation above, we obtain the equalities in the assertion of this lemma. \square

Proposition 3.5. *There exists a canonical isomorphism*

$$\varsigma: \Theta_{M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)/N_r^{(n)}(d)} \xrightarrow{\sim} \mathbf{R}^1(\pi_{M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)})_*(\mathcal{G}^{\bullet}),$$

where $\pi_{M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)}: \mathcal{C}_{M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)} \rightarrow M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)$ is the natural morphism.

Proof. We take an affine open set $M \subset M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)$. We also denote by $(\tilde{E}, \tilde{\nabla}, \{\tilde{l}_j^{(i)}\})$ the family on $\mathcal{C}_M = \mathcal{C} \times_{T \times N_r^{(n)}(d)} M$ induced by the universal family. Let $D(\tilde{\mathbf{t}})_M$ be the pull-back of $D(\tilde{\mathbf{t}})$ by the morphism $\mathcal{C}_M \rightarrow \mathcal{C}$. We take an affine open covering $\mathcal{C}_M = \bigcup_{\alpha} U_{\alpha}$ such that we have $\bar{\varphi}_{\alpha}: \tilde{E}|_{U_{\alpha}} \xrightarrow{\sim} \mathcal{O}_{U_{\alpha}}^{\oplus r}$ for any α , $\sharp\{i \mid \tilde{l}_i|_{\mathcal{C}_M} \cap U_{\alpha} \neq \emptyset\} \leq 1$ for any α and $\sharp\{\alpha \mid \tilde{l}_i|_{\mathcal{C}_M} \cap U_{\alpha} \neq \emptyset\} \leq 1$ for any i .

Take a relative tangent vector field $v \in \Theta_{M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)/N_r^{(n)}(d)}(M)$. Put $M[\epsilon] = \text{Spec } \mathcal{O}_M[\epsilon]$, where $\mathcal{O}_M[\epsilon] = \mathcal{O}_M[t]/(t^2)$. The field v corresponds to a morphism $M[\epsilon] \rightarrow M$ over $N_r^{(n)}(d)$. Let $(E_{\epsilon}, \nabla_{\epsilon}, \{(l_{\epsilon})_j^{(i)}\})$ be the flat family of parabolic connections on $\mathcal{C} \times_{T \times N_r^{(n)}(d)} M[\epsilon]$ over $M[\epsilon]$ induced by this morphism $M[\epsilon] \rightarrow M$ and the flat family $(\tilde{E}, \tilde{\nabla}, \{\tilde{l}_j^{(i)}\})$ on $\mathcal{C} \times_{T \times N_r^{(n)}(d)} M$ over M . Set

- $\mathcal{C}_{\epsilon} := \mathcal{C} \times_{T \times N_r^{(n)}(d)} M[\epsilon]$,
- $(\tilde{t}_i)_{\epsilon} := \tilde{t}_i \times_{T \times N_r^{(n)}(d)} M[\epsilon]$ ($i = 1, \dots, n$), and
- $D(\tilde{\mathbf{t}})_{\epsilon} := (\tilde{t}_1)_{\epsilon} + \dots + (\tilde{t}_n)_{\epsilon}$.

The tuple $(\mathcal{C}_{\epsilon}, (\tilde{t}_1)_{\epsilon}, \dots, (\tilde{t}_n)_{\epsilon})$ is the family of n -pointed curves over $M[\epsilon]$ induced by the composition $M[\epsilon] \rightarrow M \rightarrow T \times N_r^{(n)}(d)$ over $N_r^{(n)}(d)$ and the family $(\mathcal{C}, \tilde{t}_1, \dots, \tilde{t}_n)$ over $T \times N_r^{(n)}(d)$. Remark that, since the morphism $T \rightarrow \mathcal{M}_{g,n}$ is étale, giving a morphism $M[\epsilon] \rightarrow T \times N_r^{(n)}(d)$ over $N_r^{(n)}(d)$ is equivalent to giving a flat family $(\mathcal{C}_{\epsilon}, (\tilde{t}_1)_{\epsilon}, \dots, (\tilde{t}_n)_{\epsilon})$ of n -pointed curves satisfying $(\mathcal{C}_{\epsilon}, (\tilde{t}_1)_{\epsilon}, \dots, (\tilde{t}_n)_{\epsilon}) \otimes \mathcal{O}_M[\epsilon]/(\epsilon) = (\mathcal{C}, \tilde{t}_1, \dots, \tilde{t}_n)$. Let $\mathcal{C}_{\epsilon} = \bigcup_{\alpha} U_{\alpha}^{\epsilon}$ be the open covering corresponding to the affine open covering of \mathcal{C}_M .

There is an $M[\epsilon]$ -morphism $\sigma_\alpha: U_\alpha^\epsilon \rightarrow U_\alpha \times \text{Spec } \mathbb{C}[\epsilon]$ which is a lift of id_{U_α} preserving the divisor $D(\tilde{\mathbf{t}})_\epsilon \cap U_\alpha^\epsilon$ and $(D(\tilde{\mathbf{t}}) \cap U_\alpha) \times \text{Spec } \mathbb{C}[\epsilon]$. If we put $d_{\alpha\beta} := (\sigma_\alpha^*)^{-1} \circ \sigma_\beta^* - 1: \mathcal{O}_{U_{\alpha\beta}} \rightarrow \epsilon \otimes \mathcal{O}_{U_{\alpha\beta}}$, then $d_{\alpha\beta}$ becomes a derivation and $[\{d_{\alpha\beta}\}] \in H^1(\{U_\alpha\}, \Theta_{\mathcal{C}_M/M}(-D(\tilde{\mathbf{t}})_M))$ gives the Kodaira–Spencer class induced by $M[\epsilon] \xrightarrow{v} M \rightarrow T \rightarrow \mathcal{M}_{g,n}$. If we take a frame $\phi_\alpha: E_\epsilon|_{U_\alpha^\epsilon} \rightarrow \mathcal{O}_{U_\alpha^\epsilon}^{\oplus r}$ such that $\bar{\phi}_\alpha = \phi_\alpha \pmod{\epsilon}$, then there is a composition of isomorphisms

$$(5) \quad \varphi_\alpha: \tilde{E}|_{U_\alpha} \otimes_{\mathbb{C}} \mathbb{C}[\epsilon] \xrightarrow{\bar{\phi}_\alpha \otimes 1} \mathcal{O}_{U_\alpha}^{\oplus r} \otimes_{\mathbb{C}} \mathbb{C}[\epsilon] = \mathcal{O}_{U_\alpha}^{\oplus r} \otimes_{\mathcal{O}_{U_\alpha}} (\mathcal{O}_{U_\alpha} \otimes_{\mathbb{C}} \mathbb{C}[\epsilon]) \xrightarrow{1 \otimes \sigma_\alpha^*} \mathcal{O}_{U_\alpha}^{\oplus r} \otimes_{\mathcal{O}_{U_\alpha}} \mathcal{O}_{U_\alpha^\epsilon} \xrightarrow{\phi_\alpha^{-1}} E_\epsilon|_{U_\alpha^\epsilon}.$$

Since $\#\{\alpha \mid \tilde{t}_i|_{\mathcal{C}_M} \cap U_\alpha \neq \emptyset\} \leq 1$, if $\tilde{t}_i|_{\mathcal{C}_M} \cap U_\alpha \neq \emptyset$, then $\tilde{t}_i|_{\mathcal{C}_M} \subset U_\alpha$. By a change of the frame of $E_\epsilon|_{U_\alpha^\epsilon}$ for each α such that $\tilde{t}_i|_{\mathcal{C}_M} \cap U_\alpha \neq \emptyset$, we can assume that $\varphi_\alpha|_{\tilde{t}_i \times \text{Spec } \mathbb{C}[\epsilon]}(\tilde{l}_j^{(i)} \otimes \mathbb{C}[\epsilon]) = (l_\epsilon)_j^{(i)}$ if $\tilde{t}_i|_{\mathcal{C}_M} \cap U_\alpha \neq \emptyset$. We put

$$u_{\alpha\beta} := \varphi_\alpha^{-1} \circ \varphi_\beta - \text{id}: \tilde{E}|_{U_{\alpha\beta}} \longrightarrow \epsilon \otimes \tilde{E}|_{U_{\alpha\beta}}.$$

Put $\bar{g}_{\alpha\beta} := (\bar{\phi}_\alpha \otimes 1)(\bar{\phi}_\beta \otimes 1)^{-1}$ and $g_{\alpha\beta} := \phi_\alpha \phi_\beta^{-1}$. We define $g'_{\alpha\beta}$ and $b_{\alpha\beta}$ by

$$\bar{g}_{\alpha\beta}(\text{id} + \epsilon g'_{\alpha\beta}) = (1 \otimes \sigma_\alpha^*)^{-1} g_{\alpha\beta} (1 \otimes \sigma_\alpha^*) \quad \text{and} \quad \bar{g}_{\alpha\beta}(\text{id} + \epsilon b_{\alpha\beta}) = (1 \otimes \sigma_\alpha^*)^{-1} g_{\alpha\beta} (1 \otimes \sigma_\beta^*).$$

Here $g'_{\alpha\beta} \in \text{End}(\mathcal{O}_{U_{\alpha\beta}}^{\oplus r}, \mathcal{O}_{U_{\alpha\beta}}^{\oplus r})$ and $b_{\alpha\beta}: \mathcal{O}_{U_{\alpha\beta}}^{\oplus r} \rightarrow \mathcal{O}_{U_{\alpha\beta}}^{\oplus r}$ is a differential operator of degree ≤ 1 satisfying

$$(6) \quad \epsilon b_{\alpha\beta}(fa) = d_{\alpha\beta}(f)a + \epsilon f b_{\alpha\beta}(a)$$

for $f \in \mathcal{O}_{U_{\alpha\beta}}$ and $a \in \mathcal{O}_{U_{\alpha\beta}}^{\oplus r}$. Indeed we have

$$\begin{aligned} (\text{id} + \epsilon b_{\alpha\beta})(fa) &= \left(\bar{g}_{\alpha\beta}^{-1} (1 \otimes \sigma_\alpha^*)^{-1} g_{\alpha\beta} (1 \otimes \sigma_\beta^*) \right) (fa) \\ &= ((\sigma_\alpha^*)^{-1} \sigma_\beta^*(f)) \cdot \left(\bar{g}_{\alpha\beta}^{-1} (1 \otimes \sigma_\alpha^*)^{-1} g_{\alpha\beta} (1 \otimes \sigma_\beta^*) \right) (a) \\ &= (f + d_{\alpha\beta}(f)) \cdot (\text{id} + \epsilon b_{\alpha\beta})(a) \\ &= fa + (d_{\alpha\beta}(f)a + \epsilon f b_{\alpha\beta}(a)). \end{aligned}$$

Since

$$\begin{aligned} \bar{g}_{\alpha\beta}(\text{id} + \epsilon b_{\alpha\beta}) &= (1 \otimes \sigma_\alpha^*)^{-1} g_{\alpha\beta} (1 \otimes \sigma_\beta^*) \\ &= (1 \otimes \sigma_\alpha^*)^{-1} g_{\alpha\beta} (1 \otimes \sigma_\alpha^*) (1 \otimes \sigma_\alpha^*)^{-1} (1 \otimes \sigma_\beta^*) \\ &= \bar{g}_{\alpha\beta}(\text{id} + \epsilon g'_{\alpha\beta}) (1 \otimes (1 + d_{\alpha\beta})) \\ &= \bar{g}_{\alpha\beta}(\text{id} + d_{\alpha\beta} + \epsilon g'_{\alpha\beta}), \end{aligned}$$

we have $\epsilon b_{\alpha\beta} = d_{\alpha\beta} + \epsilon g'_{\alpha\beta}$. Since

$$\begin{aligned} u_{\alpha\beta} &= \varphi_\alpha^{-1} \circ \varphi_\beta - \text{id} \\ &= (\phi_\alpha^{-1} \circ (1 \otimes \sigma_\alpha^*) \circ (\bar{\phi}_\alpha \otimes 1))^{-1} \circ (\phi_\beta^{-1} \circ (1 \otimes \sigma_\beta^*) \circ (\bar{\phi}_\beta \otimes 1)) - \text{id} \\ &= (\bar{\phi}_\alpha \otimes 1)^{-1} \circ (\bar{g}_{\alpha\beta}(\text{id} + \epsilon b_{\alpha\beta})) \circ (\bar{\phi}_\beta \otimes 1) - \text{id} \\ &= (\bar{\phi}_\beta \otimes 1)^{-1} \circ (\epsilon b_{\alpha\beta}) \circ (\bar{\phi}_\beta \otimes 1) \end{aligned}$$

and $\text{symb}_1(u_{\alpha\beta}) = 1 \otimes d_{\alpha\beta} \in \text{id} \otimes \Theta_{\mathcal{C}_M/M}(-D(\tilde{\mathbf{t}})_M)$, we have $u_{\alpha\beta} \in \mathcal{A}_{\tilde{E}}(D(\tilde{\mathbf{t}}))|_{U_{\alpha\beta}}$. Moreover we have

$$v_\alpha := (\varphi_\alpha^{-1} \otimes \text{id}) \circ \nabla_\epsilon|_{U_\alpha^\epsilon} \circ \varphi_\alpha - \tilde{\nabla}|_{U_\alpha \times \text{Spec } \mathbb{C}[\epsilon]}.$$

We can check $v_\alpha \in \epsilon \otimes \mathcal{E}nd(\tilde{E}) \otimes \Omega_{\mathcal{C}/T}^1(D(\tilde{\mathbf{t}}))$ as follows. We have $v_\alpha(a) = 0 \pmod{\epsilon}$ for $a \in \tilde{E}|_{U_\alpha}$, that is, $v_\alpha(a) \in \epsilon \otimes (\tilde{E} \otimes \Omega_{\mathcal{C}/T}^1(D(\tilde{\mathbf{t}})))|_{U_\alpha}$. Since $d(\sigma_\alpha^*(f)) = \sigma_\alpha^*(df)$ for $f \in \mathcal{O}_{U_\alpha}$, we have

$$\begin{aligned} v_\alpha(fa) &= ((\varphi_\alpha^{-1} \otimes \text{id}) \circ \nabla_\epsilon|_{U_\alpha^\epsilon} \circ \varphi_\alpha)(fa) - \tilde{\nabla}|_{U_\alpha \times \text{Spec } \mathbb{C}[\epsilon]}(fa) \\ &= ((\varphi_\alpha^{-1} \otimes \text{id}) \circ \nabla_\epsilon|_{U_\alpha^\epsilon})(\sigma_\alpha^*(f)\varphi_\alpha(a)) - a \otimes df - f\tilde{\nabla}|_{U_\alpha \times \text{Spec } \mathbb{C}[\epsilon]}(a) \\ &= (\varphi_\alpha^{-1} \otimes \text{id}) \circ (\varphi_\alpha(a) \otimes d(\sigma_\alpha^*(f)) + \sigma_\alpha^*(f)\nabla_\epsilon|_{U_\alpha^\epsilon}\varphi_\alpha(a)) - a \otimes df - f\tilde{\nabla}|_{U_\alpha \times \text{Spec } \mathbb{C}[\epsilon]}(a) \\ &= f(\varphi_\alpha^{-1} \otimes \text{id}) \circ \nabla_\epsilon|_{U_\alpha^\epsilon} \circ \varphi_\alpha(a) - f\tilde{\nabla}|_{U_\alpha \times \text{Spec } \mathbb{C}[\epsilon]}(a) = fv_\alpha(a). \end{aligned}$$

Then we have $v_\alpha \in \epsilon \otimes \mathcal{E}nd(\tilde{E}) \otimes \Omega_{\mathbb{C}/T}^1(D(\tilde{\mathbf{t}}))$. Since $\varphi_\alpha|_{\tilde{t}_i \times \text{Spec } \mathbb{C}[\epsilon]}(\tilde{l}_j^{(i)} \otimes_{\mathbb{C}} \mathbb{C}[\epsilon]) = (l_\epsilon)_j^{(i)}$ if $\tilde{t}_i|_{\mathcal{C}_M} \cap U_\alpha \neq \emptyset$, we have $v_\alpha \in \epsilon \otimes \mathcal{G}_M^1$. We can check the equalities

$$(7) \quad u_{\beta\gamma} - u_{\alpha\gamma} + u_{\alpha\beta} = 0 \quad \text{and} \quad \tilde{\nabla} \circ u_{\alpha\beta} - u_{\alpha\beta} \circ \tilde{\nabla} = v_\beta - v_\alpha.$$

In fact, the first equality follows from the equality $g_{\alpha\beta}g_{\beta\gamma} = g_{\alpha\gamma}$. Next, by the equality

$$(8) \quad (\varphi_\beta^{-1} \circ \varphi_\alpha) \circ \varphi_\alpha^{-1} \circ \nabla_\epsilon|_{U_{\alpha\beta}^\epsilon} \circ \varphi_\alpha \circ (\varphi_\alpha^{-1} \circ \varphi_\beta) = \varphi_\beta^{-1} \circ \nabla_\epsilon|_{U_{\alpha\beta}^\epsilon} \circ \varphi_\beta$$

and the definition of $u_{\alpha\beta}$ and v_α , we have the equality

$$(9) \quad (\text{id} - u_{\alpha\beta}) \circ (\tilde{\nabla}|_{U_\alpha \times \text{Spec } \mathbb{C}[\epsilon]} + v_\alpha) \circ (\text{id} + u_{\alpha\beta}) = \tilde{\nabla}|_{U_{\alpha\beta} \times \text{Spec } \mathbb{C}[\epsilon]} + v_\beta \\ \tilde{\nabla}|_{U_{\alpha\beta} \times \text{Spec } \mathbb{C}[\epsilon]} + \left(v_\alpha + \tilde{\nabla}|_{U_{\alpha\beta} \times \text{Spec } \mathbb{C}[\epsilon]} \circ u_{\alpha\beta} - u_{\alpha\beta} \circ \tilde{\nabla}|_{U_{\alpha\beta} \times \text{Spec } \mathbb{C}[\epsilon]} \right) = \tilde{\nabla}|_{U_{\alpha\beta} \times \text{Spec } \mathbb{C}[\epsilon]} + v_\beta$$

Then we have the second equality. So $[(\{u_{\alpha\beta}\}, \{v_\alpha\})]$ determines an element $\varsigma_M(v)$ of $\mathbf{H}^1(\mathcal{G}_M^\bullet)$.

Conversely, by an element $[(\{u_{\alpha\beta}\}, \{v_\alpha\})] \in \mathbf{H}^1(\mathcal{G}_M^\bullet)$ we have a tangent vector field of M as follows. The class $[\{\text{symb}_1(u_{\alpha\beta})\}] \in H^1(\Theta_{\mathcal{C}_M/M}(-D(\tilde{\mathbf{t}}))_M)$ determines a first-order deformation $(\mathcal{C}_\epsilon, D(\tilde{\mathbf{t}})_\epsilon)$ of $(\mathcal{C}_M, D(\tilde{\mathbf{t}})_M)$. We can take an $M[\epsilon]$ -morphism $\sigma_\alpha: U_\alpha^\epsilon \xrightarrow{\sim} U_\alpha \times \text{Spec } \mathbb{C}[\epsilon]$ satisfying $\text{symb}_1(u_{\alpha\beta}) = (\sigma_\alpha^*)^{-1} \circ \sigma_\beta^* - 1$. We put $\tilde{E}_\alpha := (1 \otimes \sigma_\alpha^*)(\tilde{E}|_{U_\alpha} \otimes_{\mathbb{C}} \mathbb{C}[\epsilon])$. Let ∇_α be the homomorphism of sheaves defined by the following composition

$$\nabla_\alpha: \tilde{E}_\alpha \xrightarrow{(1 \otimes \sigma_\alpha^*)^{-1}} \tilde{E}|_{U_\alpha} \otimes_{\mathbb{C}} \mathbb{C}[\epsilon] \xrightarrow{\tilde{\nabla} + v_\alpha} (\tilde{E}|_{U_\alpha} \otimes \Omega_{\mathbb{C}/T}^1(D(\tilde{\mathbf{t}}))) \otimes_{\mathbb{C}} \mathbb{C}[\epsilon] \xrightarrow{1 \otimes \sigma_\alpha^*} \tilde{E}_\alpha \otimes \Omega_{\mathbb{C}_\epsilon/\text{Spec } \mathcal{O}_M[\epsilon]}^1(D(\tilde{\mathbf{t}})_\epsilon).$$

Since for $f \in \mathcal{O}_{U_\alpha^\epsilon}$ and $a \in \tilde{E}_\alpha$,

$$\begin{aligned} \nabla_\alpha(fa) &= (1 \otimes \sigma_\alpha^*) \circ \left(\tilde{\nabla}((\sigma_\alpha^*)^{-1}(f)(1 \otimes \sigma_\alpha^*)^{-1}(a)) + v_\alpha((\sigma_\alpha^*)^{-1}(f)(1 \otimes \sigma_\alpha^*)^{-1}(a)) \right) \\ &= (1 \otimes \sigma_\alpha^*) \circ \left((1 \otimes \sigma_\alpha^*)^{-1}(a) \otimes d((\sigma_\alpha^*)^{-1}(f)) + (\sigma_\alpha^*)^{-1}(f) \tilde{\nabla}((1 \otimes \sigma_\alpha^*)^{-1}(a)) \right. \\ &\quad \left. + (\sigma_\alpha^*)^{-1}(f)v_\alpha((1 \otimes \sigma_\alpha^*)^{-1}(a)) \right) \\ &= (1 \otimes \sigma_\alpha^*) \circ \left((1 \otimes \sigma_\alpha^*)^{-1}(a) \otimes ((\sigma_\alpha^*)^{-1}(df)) + f(1 \otimes \sigma_\alpha^*) \circ (\tilde{\nabla} + v_\alpha)((1 \otimes \sigma_\alpha^*)^{-1}(a)) \right) \\ &= a \otimes df + f \nabla_\alpha(a), \end{aligned}$$

the homomorphism ∇_α becomes a connection. Let $g_{\alpha\beta}$ be the composition

$$(10) \quad g_{\alpha\beta}: E_\beta|_{U_{\alpha\beta}^\epsilon} \xrightarrow{(1 \otimes \sigma_\beta^*)^{-1}|_{U_{\alpha\beta}^\epsilon}} \tilde{E}|_{U_{\alpha\beta}} \otimes_{\mathbb{C}} \mathbb{C}[\epsilon] \xrightarrow{\text{id} + u_{\alpha\beta}} \tilde{E}|_{U_{\alpha\beta}} \otimes_{\mathbb{C}} \mathbb{C}[\epsilon] \xrightarrow{1 \otimes \sigma_\alpha^*|_{U_{\alpha\beta} \times \text{Spec } \mathbb{C}[\epsilon]}} E_\alpha|_{U_{\alpha\beta}^\epsilon}.$$

By the composition $g_{\alpha\beta}$ and the condition (7), we can glue $(\tilde{E}_\alpha, \tilde{\nabla}_\alpha)$. For the subbundle $\tilde{l}_j^{(i)} \subset \tilde{E}|_{(\tilde{t}_i)_S}$, we put $(l_\epsilon)_j^{(i)} := (1 \otimes \sigma_\alpha^*)|_{(\tilde{t}_i)_S \times \text{Spec } \mathbb{C}[\epsilon]}(\tilde{l}_j^{(i)} \otimes_{\mathbb{C}} \mathbb{C}[\epsilon])$, which gives a parabolic structure. Then we obtain a flat family of parabolic connections on $(\mathcal{C}_\epsilon, D(\tilde{\mathbf{t}})_\epsilon)$ over $\text{Spec } \mathcal{O}_M[\epsilon]$ up to isomorphism. This flat family gives a tangent vector field on M . So $v \mapsto \varsigma_M(v)$ determines an isomorphism

$$\varsigma_M: \Theta_{M_{\mathbb{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)/N_r^{(n)}(d)}(M) \xrightarrow{\sim} \mathbf{H}^1(\mathcal{G}_M^\bullet); \quad v \mapsto \varsigma_M(v).$$

The isomorphism ς_M induces a canonical isomorphism

$$\varsigma: \Theta_{M_{\mathbb{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)/N_r^{(n)}(d)} \xrightarrow{\sim} \mathbf{R}^1(\pi_{M_{\mathbb{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)})_*(\mathcal{G}^\bullet).$$

□

3.3. Isomonodromic deformation. Let $p_1: T \times N_r^{(n)}(d) \rightarrow T$ be the projection. There exists an algebraic splitting

$$(11) \quad D: (p_1 \circ \varpi)^*(\Theta_T) \longrightarrow \Theta_{M_{\mathbb{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)/N_r^{(n)}(d)}$$

of the tangent map $\Theta_{M_{\mathbb{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)/N_r^{(n)}(d)} \rightarrow (p_1 \circ \varpi)^*(\Theta_T)$. Here an image of (11) means an algebraic vector field determined by the isomonodromic deformation. (See [10, Proposition 8.1]). We will define

the algebraic splitting (11) rigorously below (see (12)). We describe this algebraic splitting in terms of the description of $\Theta_{M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)/N_r^{(n)}(d)}$ in Proposition 3.5.

Take any affine open set $U \subset T$ and a vector field $v \in H^0(U, \Theta_T)$. Then v corresponds to a morphism $v^v: \text{Spec } \mathcal{O}_U[\epsilon] \rightarrow T$ with $\epsilon^2 = 0$ such that the composite $U \hookrightarrow \text{Spec } \mathcal{O}_U[\epsilon] \rightarrow T$ is just the inclusion $U \hookrightarrow T$. We denote the restriction of the universal family to $\mathcal{C} \times_T (p_1 \circ \varpi)^{-1}(U)$ simply by $(\tilde{E}, \tilde{\nabla}, \{\tilde{l}_j^{(i)}\})$. Consider the fiber product $\mathcal{C} \times_T \text{Spec } \mathcal{O}_{(p_1 \circ \varpi)^{-1}(U)}[\epsilon]$ with respect to the canonical projection $\mathcal{C} \rightarrow T$ and the composite $\text{Spec } \mathcal{O}_{(p_1 \circ \varpi)^{-1}(U)}[\epsilon] \rightarrow \text{Spec } \mathcal{O}_U[\epsilon] \xrightarrow{v^v} T$. We denote the pull-back of $D(\tilde{\mathbf{t}})$ by the morphism $\mathcal{C} \times_T \text{Spec } \mathcal{O}_{(p_1 \circ \varpi)^{-1}(U)}[\epsilon] \rightarrow \mathcal{C}$ simply by $D(\tilde{\mathbf{t}})_{\mathcal{O}_{(p_1 \circ \varpi)^{-1}(U)}[\epsilon]}$. In this section, let d be the relative exterior derivative on $\mathcal{C} \times_T \text{Spec } \mathcal{O}_{(p_1 \circ \varpi)^{-1}(U)}[\epsilon] \rightarrow (p_1 \circ \varpi)^{-1}(U)$. We set

$$\begin{aligned} \hat{\Omega}^1 &:= \Omega_{\mathcal{C} \times_T \text{Spec } \mathcal{O}_{(p_1 \circ \varpi)^{-1}(U)}[\epsilon]/(p_1 \circ \varpi)^{-1}(U)}^1 \quad \text{and} \\ \hat{\Omega}^2 &:= \Omega_{\mathcal{C} \times_T \text{Spec } \mathcal{O}_{(p_1 \circ \varpi)^{-1}(U)}[\epsilon]/(p_1 \circ \varpi)^{-1}(U)}^2. \end{aligned}$$

Here $\hat{\Omega}^1$ is the sheaf of relative differential with respect to the composition

$$\mathcal{C} \times_T \text{Spec } \mathcal{O}_{(p_1 \circ \varpi)^{-1}(U)}[\epsilon] \longrightarrow \text{Spec } \mathcal{O}_{(p_1 \circ \varpi)^{-1}(U)}[\epsilon] \longrightarrow (p_1 \circ \varpi)^{-1}(U)$$

of the trivial projection.

Definition 3.6. We call $(\mathcal{E}, \nabla^{\mathcal{E}}, \{(l_{\mathcal{E}})_j^{(i)}\})$ a *horizontal lift* of $(\tilde{E}, \tilde{\nabla}, \{\tilde{l}_j^{(i)}\})$ if

- (1) \mathcal{E} is a vector bundle on $\mathcal{C} \times_T \text{Spec } \mathcal{O}_{(p_1 \circ \varpi)^{-1}(U)}[\epsilon]$,
- (2) $\mathcal{E}|_{\tilde{t}_i \times \mathcal{O}_{(p_1 \circ \varpi)^{-1}(U)}[\epsilon]} = (l_{\mathcal{E}})_0^{(i)} \supset \dots \supset (l_{\mathcal{E}})_r^{(i)} = 0$ is a filtration by subbundles for $i = 1, \dots, n$,
- (3) $\nabla^{\mathcal{E}}: \mathcal{E} \rightarrow \mathcal{E} \otimes \hat{\Omega}^1 \left(\log \left(D(\tilde{\mathbf{t}})_{\mathcal{O}_{(p_1 \circ \varpi)^{-1}(U)}[\epsilon]} \right) \right)$ is a connection satisfying
 - (a) $\nabla^{\mathcal{E}}(F_j^{(i)}(\mathcal{E})) \subset F_j^{(i)}(\mathcal{E}) \otimes \hat{\Omega}^1 \left(\log \left(D(\tilde{\mathbf{t}})_{\mathcal{O}_{(p_1 \circ \varpi)^{-1}(U)}[\epsilon]} \right) \right)$, where $F_j^{(i)}(\mathcal{E})$ is given by $F_j^{(i)}(\mathcal{E}) := \text{Ker} \left(\mathcal{E} \rightarrow \mathcal{E}|_{\tilde{t}_i \times \mathcal{O}_{(p_1 \circ \varpi)^{-1}(U)}[\epsilon]/(p_1 \circ \varpi)^{-1}(U)}/(l_{\mathcal{E}})_{j+1}^{(i)} \right)$,
 - (b) the curvature $\nabla^{\mathcal{E}} \circ \nabla^{\mathcal{E}}: \mathcal{E} \rightarrow \mathcal{E} \otimes \hat{\Omega}^2 \left(\log \left(D(\tilde{\mathbf{t}})_{\mathcal{O}_{(p_1 \circ \varpi)^{-1}(U)}[\epsilon]} \right) \right)$ is zero,
 - (c) $(\text{res}_{\tilde{t}_i \times \mathcal{O}_{(p_1 \circ \varpi)^{-1}(U)}[\epsilon]}(\tilde{\nabla}^{\mathcal{E}}) - \tilde{\nu}_j^{(i)})((l_{\mathcal{E}})_j^{(i)}) \subset (l_{\mathcal{E}})_{j+1}^{(i)}$ for any i, j , where $\tilde{\nabla}^{\mathcal{E}}$ is the relative connection over $\text{Spec } \mathcal{O}_{(p_1 \circ \varpi)^{-1}(U)}[\epsilon]$ induced by $\nabla^{\mathcal{E}}$ and
 - (d) $(\mathcal{E}, \tilde{\nabla}^{\mathcal{E}}, \{(l_{\mathcal{E}})_j^{(i)}\}) \otimes \mathcal{O}_{(p_1 \circ \varpi)^{-1}(U)}[\epsilon]/(\epsilon) \cong (\tilde{E}, \tilde{\nabla}, \{\tilde{l}_j^{(i)}\})$.

Here, we define the sheaf $\hat{\Omega}^1 \left(\log \left(D(\tilde{\mathbf{t}})_{\mathcal{O}_{(p_1 \circ \varpi)^{-1}(U)}[\epsilon]} \right) \right)$ as the coherent subsheaf of $\hat{\Omega}^1 \left(D(\tilde{\mathbf{t}})_{\mathcal{O}_{(p_1 \circ \varpi)^{-1}(U)}[\epsilon]} \right)$ locally generated by $\tilde{g}^{-1}d\tilde{g}$ and $d\epsilon$ for a local defining equation \tilde{g} of $D(\tilde{\mathbf{t}})_{\mathcal{O}_{(p_1 \circ \varpi)^{-1}(U)}[\epsilon]}$ and the sheaf $\hat{\Omega}^2 \left(\log \left(D(\tilde{\mathbf{t}})_{\mathcal{O}_{(p_1 \circ \varpi)^{-1}(U)}[\epsilon]} \right) \right)$ as the coherent subsheaf of $\hat{\Omega}^2 \left(D(\tilde{\mathbf{t}})_{\mathcal{O}_{(p_1 \circ \varpi)^{-1}(U)}[\epsilon]} \right)$ locally generated by $\tilde{g}^{-1}d\tilde{g} \wedge d\epsilon$.

Let $\mathcal{C} \times_T (p_1 \circ \varpi)^{-1}(U) = \bigcup_{\alpha} U_{\alpha}$ be an affine open covering such that we have $\bar{\phi}_{\alpha}: \tilde{E}|_{U_{\alpha}} \xrightarrow{\sim} \mathcal{O}_{U_{\alpha}}^{\oplus r}$ for any α , $\#\{i \mid \tilde{t}_i|_{\mathcal{C} \times_T (p_1 \circ \varpi)^{-1}(U)} \cap U_{\alpha} \neq \emptyset\} \leq 1$ for any α and $\#\{\alpha \mid \tilde{t}_i|_{\mathcal{C} \times_T (p_1 \circ \varpi)^{-1}(U)} \cap U_{\alpha} \neq \emptyset\} \leq 1$ for any i . Assume that the parabolic connection $(\tilde{E}, \tilde{\nabla}, \{\tilde{l}_j^{(i)}\})$ is locally given in the affine subset U_{α} by a connection matrix $A_{\alpha} f_{\alpha}^{-1} df_{\alpha}$, where

- f_{α} is a local defining equation of $(\tilde{t}_i \times (p_1 \circ \varpi)^{-1}(U)) \cap U_{\alpha}$,
- $A_{\alpha} \in M_r(\mathcal{O}_{U_{\alpha}})$,
- $A_{\alpha}((\tilde{t}_i \times_T (p_1 \circ \varpi)^{-1}(U)) \cap U_{\alpha})$ is an upper triangular matrix, and
- the parabolic structure $\{\tilde{l}_j^{(i)}\}_{U_{\alpha}}$ is given by $(\tilde{l}_j^{(i)})_{U_{\alpha}} = (*, *, \dots, *, 0, \dots, 0)$.

Put $U_{\alpha}^{\epsilon} := U_{\alpha} \times_T \text{Spec } \mathcal{O}_{(p_1 \circ \varpi)^{-1}(U)}[\epsilon]$. We take an $M[\epsilon]$ -morphism $\sigma_{\alpha}: U_{\alpha}^{\epsilon} \rightarrow U_{\alpha} \times \text{Spec } \mathbb{C}[\epsilon]$ which is a lift of $\text{id}_{U_{\alpha}}$ preserving the divisor $D(\tilde{\mathbf{t}})_{\mathcal{O}_{(p_1 \circ \varpi)^{-1}(U)}[\epsilon]} \cap U_{\alpha}^{\epsilon}$ and $(D(\tilde{\mathbf{t}}) \cap U_{\alpha}) \times \text{Spec } \mathbb{C}[\epsilon]$. Put $f_{\alpha}^{\epsilon} := \sigma_{\alpha}^* f_{\alpha}$,

$\tilde{E}_\alpha := (1 \otimes \sigma_\alpha^*)(\tilde{E}|_{U_\alpha} \otimes_{\mathbb{C}} \mathbb{C}[\epsilon])$, and $A_\alpha^\epsilon := (1 \otimes \sigma_\alpha^*)(A_\alpha) \in M_r(\mathcal{O}_{U_\alpha^\epsilon})$. If we denote the composite

$$\mathcal{O}_{U_\alpha^\epsilon} \xrightarrow{d} \Omega_{U_\alpha^\epsilon / (p_1 \circ \varpi)^{-1}(U)}^1 = \mathcal{O}_{U_\alpha^\epsilon} df_\alpha^\epsilon \oplus \mathcal{O}_{U_\alpha^\epsilon} d\epsilon \longrightarrow \mathcal{O}_{U_\alpha^\epsilon} d\epsilon$$

by d_ϵ , then we have $d_\epsilon(A_\alpha^\epsilon) = 0$ (which means integrable). Then \tilde{E}_α and the connection matrix $A_\alpha^\epsilon (f_\alpha^\epsilon)^{-1} df_\alpha^\epsilon$ give a *local* horizontal lift of $(\tilde{E}, \tilde{\nabla}, \{\tilde{l}_j^{(i)}\})|_{U_\alpha}$. By [10, Proposition 8.1], the obstruction for the patching the local horizontal lifts vanishes and the global horizontal lift of $(\tilde{E}, \tilde{\nabla}, \{\tilde{l}_j^{(i)}\})$ to $\mathcal{C} \times_T (p_1 \circ \varpi)^{-1}(U)$ is unique for a vector field $v \in H^0(U, \Theta_T)$. Let M be an affine open subset of $M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)$. If we have $v \in H^0(M, (p_1 \circ \varpi)^* \Theta_T)$, then we have the relative connection $(E^{\text{hor}}, \tilde{\nabla}^{\text{hor}}, \{(\tilde{l}^{\text{hor}})_j^{(i)}\})$ on $\mathcal{C} \times_{T \times N_r^{(n)}(d)} M[\epsilon]$ over $M[\epsilon]$ induced by the global horizontal lift $(\mathcal{E}, \nabla^\mathcal{E}, \{(l_\mathcal{E})_j^{(i)}\})$ on $\mathcal{C} \times_{T \times N_r^{(n)}(d)} M[\epsilon]$ with respect to v . We denote by $D(v) \in \Theta_{M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d) / N_r^{(n)}(d)}(M)$ the vector field on M corresponding to the relative connection $(E^{\text{hor}}, \tilde{\nabla}^{\text{hor}}, \{(\tilde{l}^{\text{hor}})_j^{(i)}\})$. We define a morphism D as

$$(12) \quad \begin{aligned} D: (p_1 \circ \varpi)^* \Theta_T &\longrightarrow \Theta_{M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d) / N_r^{(n)}(d)} \\ v &\longmapsto D(v), \end{aligned}$$

which is the algebraic splitting (11).

Remark 3.7. The connection $\nabla^\mathcal{E}$ on \mathcal{E} on $\mathcal{C} \times_T \text{Spec } \mathcal{O}_{(p_1 \circ \varpi)^{-1}(U)}[\epsilon]$ satisfies the *integrability condition* (Definition 3.6 (3) (b)). The integrability means that the relative connection associated to $\nabla^\mathcal{E}$ is an *isomonodromic family*. (See for example [22, 0.16.6]).

Let

$$\mu: (p_1 \circ \varpi)^* \Theta_T \longrightarrow \mathbf{R}^1(\pi_{M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)})^*(\Theta_{\mathcal{C} \times_T M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d) / M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)}(-D(\tilde{\mathbf{t}})))$$

be the Kodaira–Spencer map, where $\pi_{M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)}: \mathcal{C}_{M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)} \rightarrow M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)$ is the natural morphism. We obtain the desired algebraic splitting D as follows. (The statement of the following proposition is essentially pointed out in [3, Section 4]).

Proposition 3.8. *Let M be an affine open subset of $M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)$. The following morphism*

$$(13) \quad \begin{aligned} (p_1 \circ \varpi)^* \Theta_T(M) &\longrightarrow \mathbf{H}^1(\mathcal{G}_M^\bullet) \cong \Theta_{M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d) / N_r^{(n)}(d)}(M) \\ v &\longmapsto [\{\iota(\tilde{\nabla})(\mu_M(v))\}, \{0\}] \end{aligned}$$

coincides with the algebraic splitting (12). Here

$$\iota(\tilde{\nabla}): H^1(\Theta_{\mathcal{C} \times_T M / M}(-D(\tilde{\mathbf{t}})_M)) \longrightarrow H^1(\mathcal{G}_M^0)$$

is induced by the splitting associated to the universal family $\tilde{\nabla}$ defined in Section 3.1.

Proof. Take an affine open set $U \subset T$ and a vector field $v \in H^0(U, \Theta_T)$. We denote the restriction of the universal family to $\mathcal{C} \times_T (p_1 \circ \varpi)^{-1}(U)$ simply by $(\tilde{E}, \tilde{\nabla}, \{\tilde{l}_j^{(i)}\})$. We take a horizontal lift $(\mathcal{E}, \nabla^\mathcal{E}, \{(l_\mathcal{E})_j^{(i)}\})$ of $(\tilde{E}, \tilde{\nabla}, \{\tilde{l}_j^{(i)}\})$ corresponding to v .

We take an affine open set $M \subset (p_1 \circ \varpi)^{-1}(U)$ and put $M[\epsilon] = \text{Spec } \mathcal{O}_M[\epsilon]$. We denote the restriction of the horizontal lift $(\mathcal{E}, \nabla^\mathcal{E}, \{(l_\mathcal{E})_j^{(i)}\})$ by $(\mathcal{E}_M, \nabla_M^\mathcal{E}, \{((l_\mathcal{E})_M)_j^{(i)}\})$. As in the proof of Proposition 3.5, take an affine open covering $\{U_\alpha\}$ of $\mathcal{C} \times_T M$. Let $\{U_\alpha^\epsilon\}$ be the affine open covering of $\mathcal{C} \times_T M[\epsilon]$ corresponding to $\{U_\alpha\}$ and let $\sigma_\alpha: U_\alpha^\epsilon \rightarrow U_\alpha \times \text{Spec } \mathbb{C}[\epsilon]$ be an isomorphism as in the proof of Proposition 3.5. Put $d_{\alpha\beta} := (\sigma_\alpha^*)^{-1} \circ \sigma_\beta^* - 1: \mathcal{O}_{U_{\alpha\beta}^\epsilon} \rightarrow \epsilon \otimes \mathcal{O}_{U_{\alpha\beta}}$. Then $\{[d_{\alpha\beta}]\}$ is the Kodaira–Spencer class corresponding to $\mathcal{C} \times_T M[\epsilon] \rightarrow M[\epsilon]$. If we take a frame $\phi_\alpha: \mathcal{E}_M|_{U_\alpha^\epsilon} \rightarrow \mathcal{O}_{U_\alpha^\epsilon}^{\oplus r}$ and put $\bar{\phi}_\alpha := \phi_\alpha \pmod{\epsilon}$, there is a

commutative diagram

$$\begin{array}{ccccccc}
\tilde{E}|_{U_{\alpha\beta}} \otimes_{\mathbb{C}} \mathbb{C}[\epsilon] & \xrightarrow{\bar{\phi}_\beta \otimes 1} & \mathcal{O}_{U_{\alpha\beta}}^{\oplus r} \otimes_{\mathcal{O}_{U_{\alpha\beta}}} \mathcal{O}_{U_{\alpha\beta}}[\epsilon] & \xrightarrow{1 \otimes \sigma_\beta^*} & \mathcal{O}_{U_{\alpha\beta}}^{\oplus r} \otimes_{\mathcal{O}_{U_{\alpha\beta}}} \mathcal{O}_{U_{\alpha\beta}} & \xrightarrow{\phi_\beta^{-1}} & \mathcal{E}_M|_{U_{\alpha\beta}^\epsilon} \\
\downarrow \text{id} + u_{\alpha\beta} & & \downarrow (\bar{\phi}_\alpha \otimes 1)(\bar{\phi}_\beta \otimes 1)^{-1}(\text{id} + \epsilon b_{\alpha\beta}) & & \downarrow \phi_\alpha \phi_\beta^{-1} & & \downarrow \text{id} \\
\tilde{E}|_{U_{\alpha\beta}} \otimes_{\mathbb{C}} \mathbb{C}[\epsilon] & \xrightarrow{\bar{\phi}_\alpha \otimes 1} & \mathcal{O}_{U_{\alpha\beta}}^{\oplus r} \otimes_{\mathcal{O}_{U_{\alpha\beta}}} \mathcal{O}_{U_{\alpha\beta}}[\epsilon] & \xrightarrow{1 \otimes \sigma_\alpha^*} & \mathcal{O}_{U_{\alpha\beta}}^{\oplus r} \otimes_{\mathcal{O}_{U_{\alpha\beta}}} \mathcal{O}_{U_{\alpha\beta}} & \xrightarrow{\phi_\alpha^{-1}} & \mathcal{E}_M|_{U_{\alpha\beta}^\epsilon}.
\end{array}$$

Here $b_{\alpha\beta}: \mathcal{O}_{U_{\alpha\beta}}^{\oplus r} \rightarrow \mathcal{O}_{U_{\alpha\beta}}^{\oplus r}$ is a differential operator of degree ≤ 1 satisfying $\epsilon b_{\alpha\beta}(fa) = d_{\alpha\beta}(f)a + \epsilon f b_{\alpha\beta}(a)$ for $f \in \mathcal{O}_{U_{\alpha\beta}}$ and $a \in \mathcal{O}_{U_{\alpha\beta}}^{\oplus r}$. If we denote the connection matrix of $\nabla_M^\epsilon|_{U_{\alpha\beta}^\epsilon}$ via the frame ϕ_α by $A_\alpha^\epsilon(f_\alpha^\epsilon)^{-1}df_\alpha^\epsilon$, then the connection matrix of $\tilde{\nabla}|_{U_{\alpha\beta}} \otimes_{\mathbb{C}} \mathbb{C}[\epsilon]$ becomes $(1 \otimes \sigma_\alpha^*)^{-1}(A_\alpha^\epsilon(f_\alpha^\epsilon)^{-1}df_\alpha^\epsilon)(1 \otimes \sigma_\alpha^*)$. The patching condition for the connections $\tilde{\nabla}|_{U_{\alpha\beta}} \otimes_{\mathbb{C}} \mathbb{C}[\epsilon]$ becomes

$$\begin{aligned}
& \bar{\phi}_\beta \bar{\phi}_\alpha^{-1} d(\bar{\phi}_\alpha \bar{\phi}_\beta^{-1}) \bar{\phi}_\beta \bar{\phi}_\alpha^{-1} + \bar{\phi}_\beta \bar{\phi}_\alpha^{-1} (1 \otimes \sigma_\alpha^*)^{-1} \left(A_\alpha^\epsilon \frac{df_\alpha^\epsilon}{f_\alpha^\epsilon} \right) (1 \otimes \sigma_\alpha^*) \\
(14) \quad & = (1 \otimes \sigma_\beta^*)^{-1} \left(A_\beta^\epsilon \frac{df_\beta^\epsilon}{f_\beta^\epsilon} \right) (1 \otimes \sigma_\beta^*) \bar{\phi}_\beta \bar{\phi}_\alpha^{-1}.
\end{aligned}$$

Here we denote $\bar{\phi}_\alpha \otimes 1$ simply by $\bar{\phi}_\alpha$. Moreover the patching condition for $\nabla_M^\epsilon|_{U_{\alpha\beta}^\epsilon}$ is the equality

$$(15) \quad (\phi_\alpha \phi_\beta^{-1}) A_\beta^\epsilon \frac{df_\beta^\epsilon}{f_\beta^\epsilon} = d(\phi_\alpha \phi_\beta^{-1}) + A_\alpha^\epsilon \frac{df_\alpha^\epsilon}{f_\alpha^\epsilon} (\phi_\alpha \phi_\beta^{-1}).$$

For $a \in \mathcal{O}_{U_{\alpha\beta}}^{\oplus r} \otimes_{\mathcal{O}_{U_{\alpha\beta}}} \mathcal{O}_{U_{\alpha\beta}}$, we have

$$\begin{aligned}
& d(\phi_\alpha \phi_\beta^{-1}(a)) \\
& = d\left((1 \otimes \sigma_\alpha^*) \bar{\phi}_\alpha \bar{\phi}_\beta^{-1} (1 + \epsilon b_{\alpha\beta}) (1 \otimes \sigma_\beta^*)^{-1}(a)\right) \\
& = (1 \otimes \sigma_\alpha^*) d\left(\bar{\phi}_\alpha \bar{\phi}_\beta^{-1}\right) (1 + \epsilon b_{\alpha\beta}) (1 \otimes \sigma_\beta^*)^{-1}(a) + (1 \otimes \sigma_\alpha^*) \bar{\phi}_\alpha \bar{\phi}_\beta^{-1} d\left((1 + \epsilon b_{\alpha\beta}) (1 \otimes \sigma_\beta^*)^{-1}(a)\right) \\
& = (1 \otimes \sigma_\alpha^*) d\left(\bar{\phi}_\alpha \bar{\phi}_\beta^{-1}\right) \bar{\phi}_\beta \bar{\phi}_\alpha^{-1} (1 \otimes \sigma_\alpha^*)^{-1} \phi_\alpha \phi_\beta^{-1}(a) + (1 \otimes \sigma_\alpha^*) \bar{\phi}_\alpha \bar{\phi}_\beta^{-1} d\left((1 + \epsilon b_{\alpha\beta}) (1 \otimes \sigma_\beta^*)^{-1}(a)\right).
\end{aligned}$$

Set $a' := (1 \otimes \sigma_\beta^*)^{-1}(a)$. Rephrasing the above equality and substituting (14), (15), we have

$$\begin{aligned}
(16) \quad & d((1 + \epsilon b_{\alpha\beta})(a')) \\
& = \bar{\phi}_\beta \bar{\phi}_\alpha^{-1} (1 \otimes \sigma_\alpha^*)^{-1} d\left((\phi_\alpha \phi_\beta^{-1})(1 \otimes \sigma_\beta^*)(a')\right) - \bar{\phi}_\beta \bar{\phi}_\alpha^{-1} d\left(\bar{\phi}_\alpha \bar{\phi}_\beta^{-1}\right) \bar{\phi}_\beta \bar{\phi}_\alpha^{-1} (1 \otimes \sigma_\alpha^*)^{-1} \phi_\alpha \phi_\beta^{-1} (1 \otimes \sigma_\beta^*)(a') \\
& = \bar{\phi}_\beta \bar{\phi}_\alpha^{-1} (1 \otimes \sigma_\alpha^*)^{-1} \phi_\alpha \phi_\beta^{-1} A_\beta^\epsilon \frac{df_\beta^\epsilon}{f_\beta^\epsilon} (1 \otimes \sigma_\beta^*)(a') - \bar{\phi}_\beta \bar{\phi}_\alpha^{-1} (1 \otimes \sigma_\alpha^*)^{-1} A_\alpha^\epsilon \frac{df_\alpha^\epsilon}{f_\alpha^\epsilon} (\phi_\alpha \phi_\beta^{-1})(1 \otimes \sigma_\beta^*)(a') \\
& \quad + \bar{\phi}_\beta \bar{\phi}_\alpha^{-1} (1 \otimes \sigma_\alpha^*)^{-1} (\phi_\alpha \phi_\beta^{-1}) d\left((1 \otimes \sigma_\beta^*)(a')\right) - \bar{\phi}_\beta \bar{\phi}_\alpha^{-1} d\left(\bar{\phi}_\alpha \bar{\phi}_\beta^{-1}\right) \bar{\phi}_\beta \bar{\phi}_\alpha^{-1} (1 \otimes \sigma_\alpha^*)^{-1} \phi_\alpha \phi_\beta^{-1} (1 \otimes \sigma_\beta^*)(a') \\
& = \bar{\phi}_\beta \bar{\phi}_\alpha^{-1} (1 \otimes \sigma_\alpha^*)^{-1} \phi_\alpha \phi_\beta^{-1} A_\beta^\epsilon \frac{df_\beta^\epsilon}{f_\beta^\epsilon} (1 \otimes \sigma_\beta^*)(a') - (1 \otimes \sigma_\beta^*)^{-1} \left(A_\beta^\epsilon \frac{df_\beta^\epsilon}{f_\beta^\epsilon} \right) (1 \otimes \sigma_\beta^*) \bar{\phi}_\beta \bar{\phi}_\alpha^{-1} (1 \otimes \sigma_\alpha^*)^{-1} \phi_\alpha \phi_\beta^{-1} (1 \otimes \sigma_\beta^*)(a') \\
& \quad + \bar{\phi}_\beta \bar{\phi}_\alpha^{-1} (1 \otimes \sigma_\alpha^*)^{-1} (\phi_\alpha \phi_\beta^{-1}) (1 \otimes \sigma_\beta^*)(da') \\
& = (1 + \epsilon b_{\alpha\beta}) \left((1 \otimes \sigma_\beta^*)^{-1} A_\beta^\epsilon \frac{df_\beta^\epsilon}{f_\beta^\epsilon} (1 \otimes \sigma_\beta^*) \right) (a') - \left((1 \otimes \sigma_\beta^*)^{-1} A_\beta^\epsilon \frac{df_\beta^\epsilon}{f_\beta^\epsilon} (1 \otimes \sigma_\beta^*) \right) (1 + \epsilon b_{\alpha\beta})(a') + (1 + \epsilon b_{\alpha\beta})(da').
\end{aligned}$$

In the last line, $b_{\alpha\beta}$ acts on the elements of $(\mathcal{E}nd(\mathcal{O}_{U_{\alpha\beta}}^{\oplus r}) df_\beta/f_\beta) \oplus \mathcal{E}nd(\mathcal{O}_{U_{\alpha\beta}}^{\oplus r}) d\epsilon$. This action is defined as follows. Let $\tilde{d}_{\alpha\beta}: \mathcal{O}_{U_{\alpha\beta}} \rightarrow \mathcal{O}_{U_{\alpha\beta}}$ be the homomorphism of sheaves such that the composition $\epsilon \circ \tilde{d}_{\alpha\beta}: \mathcal{O}_{U_{\alpha\beta}} \rightarrow \mathcal{O}_{U_{\alpha\beta}} \rightarrow \epsilon \circ \mathcal{O}_{U_{\alpha\beta}}$ coincides with $d_{\alpha\beta}$. By the equality (6), we have $b_{\alpha\beta} = \tilde{d}_{\alpha\beta} + g'_{\alpha\beta}$. Let $X_0 df_\beta/f_\beta + X_1 d\epsilon$ be an element of $(\mathcal{E}nd(\mathcal{O}_{U_{\alpha\beta}}^{\oplus r}) df_\beta/f_\beta) \oplus \mathcal{E}nd(\mathcal{O}_{U_{\alpha\beta}}^{\oplus r}) d\epsilon$. Let d_{f_β} be the relative exterior

derivative $\mathcal{E}nd(\mathcal{O}_{U_\beta}^{\oplus r}) \rightarrow \mathcal{E}nd(\mathcal{O}_{U_\beta}^{\oplus r})df_\beta$. The operator $b_{\alpha\beta}$ acts on $X_0df_\beta/f_\beta + X_1d\epsilon$ as follows:

$$\begin{aligned} b_{\alpha\beta}(X_0\frac{df_\beta}{f_\beta} + X_1d\epsilon) &= (\tilde{d}_{\alpha\beta} + g'_{\alpha\beta})(X_0\frac{df_\beta}{f_\beta} + X_1d\epsilon) \\ &= d_{f_\beta} \left\langle \tilde{d}_{\alpha\beta}, X_0\frac{df_\beta}{f_\beta} \right\rangle + (g'_{\alpha\beta} \otimes 1)X_0\frac{df_\beta}{f_\beta} + \left(X_0 \left\langle \tilde{d}_{\alpha\beta}, \frac{df_\beta}{f_\beta} \right\rangle + g'_{\alpha\beta}X_1 \right) d\epsilon. \end{aligned}$$

This action is natural since $d_{\alpha\beta} = (\sigma_\alpha^*)^{-1}\sigma_\beta^* - 1$ and

$$\begin{aligned} (\sigma_\alpha^*)^{-1}\sigma_\beta^* \left((\sigma_\beta^*)^{-1}\frac{df_\beta^\epsilon}{f_\beta^\epsilon} \right) &= (\sigma_\alpha^*)^{-1}\sigma_\beta^* \left(\frac{df_\beta}{f_\beta} \right) = \frac{d(f_\beta + d_{\alpha\beta}(f_\beta))}{f_\beta + d_{\alpha\beta}(f_\beta)} \\ (17) \quad &= \frac{df_\beta}{f_\beta} + \frac{(\tilde{d}_{\alpha\beta}(f_\beta))}{f_\beta}d\epsilon + \epsilon \frac{d(\tilde{d}_{\alpha\beta}(f_\beta))}{f_\beta} - \epsilon \frac{\tilde{d}_{\alpha\beta}(f_\beta)df_\beta}{(f_\beta)^2} \\ &= \frac{df_\beta}{f_\beta} + \left\langle \tilde{d}_{\alpha\beta}, \frac{df_\beta}{f_\beta} \right\rangle d\epsilon + \epsilon d_{f_\beta} \left\langle \tilde{d}_{\alpha\beta}, \frac{df_\beta}{f_\beta} \right\rangle, \end{aligned}$$

where d_{f_β} is the relative exterior derivative $\mathcal{O}_{U_\beta} \rightarrow \mathcal{O}_{U_\beta}df_\beta$. Here the last line of (17) follows from the equalities $\epsilon^2 = 0$ and $2\epsilon d\epsilon = d(\epsilon^2) = 0$.

We compute the $d\epsilon$ terms in the first line and the last line of the equations (16). By the definition of f_β^ϵ , we have $(\sigma_\beta^*)^{-1}(f_\beta^\epsilon) = f_\beta$. We define $a'_0 \in \mathcal{O}_{U_{\alpha\beta}}^{\oplus r}$ and $a'_1 \in \mathcal{O}_{U_{\alpha\beta}}^{\oplus r}$ by $a' = a'_0 + \epsilon a'_1$. The $d\epsilon$ term in $d((1 + \epsilon b_{\alpha\beta})(a'))$ is $(b_{\alpha\beta}(a'_0) + a'_1)d\epsilon$. The $d\epsilon$ term in $(1 + \epsilon b_{\alpha\beta})(da')$ is $(\tilde{d}_{\alpha\beta}(a'_0) + a'_1)d\epsilon$ by direct computation as in the equalities (17). By comparing the $d\epsilon$ term in the first line of the equations (16) with the $d\epsilon$ term in the last line of the equations (16), we have

$$b_{\alpha\beta}(a'_0)d\epsilon + a'_1d\epsilon = \left\langle \tilde{d}_{\alpha\beta}, A_\beta \frac{df_\beta}{f_\beta}(a'_0) \right\rangle d\epsilon + \tilde{d}_{\alpha\beta}(a'_0)d\epsilon + a'_1d\epsilon,$$

where $A_\beta f_\beta^{-1}df_\beta$ is the connection matrix of $\tilde{\nabla}$ via the frame $\bar{\phi}_\beta$. Then we obtain

$$(18) \quad u_{\alpha\beta} = \bar{\phi}_\beta^{-1} \circ \left(d_{\alpha\beta} + \left\langle d_{\alpha\beta}, A_\beta \frac{df_\beta}{f_\beta} \right\rangle \right) \circ \bar{\phi}_\beta.$$

On the other hand, we take the relative connection $\bar{\nabla}_M^\mathcal{E}$ on \mathcal{E}_M associated to $\nabla_M^\mathcal{E}$. Since $d_\epsilon(A_\alpha^\epsilon) = 0$, we have

$$(19) \quad v_\alpha = (\varphi_\alpha^{-1} \otimes \text{id}) \circ \bar{\nabla}_M^\mathcal{E}|_{U_\alpha^\epsilon} \circ \varphi_\alpha - \tilde{\nabla}|_{U_\alpha \times \text{Spec } \mathbb{C}[\epsilon]} = 0.$$

So for $v \in H^0(U, \Theta_T)$, we obtain the element of $\mathbf{H}^1(\mathcal{G}_M^\bullet)$ given by (18) and (19). This correspondence from v to this element of $\mathbf{H}^1(\mathcal{G}_M^\bullet)$ is just the morphism D defined in (12). On the other hand, by the local description of $\iota(\tilde{\nabla})$ in Section 3.1, we have the following equality:

$$\iota(\tilde{\nabla})(\{d_{\alpha\beta}\}) = \bar{\phi}_\beta^{-1} \circ \left(d_{\alpha\beta} + \left\langle d_{\alpha\beta}, A_\beta \frac{df_\beta}{f_\beta} \right\rangle \right) \circ \bar{\phi}_\beta.$$

This equality means that the morphism (13) coincides with the morphism $D: (p_1 \circ \varpi)^*(\Theta_T) \rightarrow \Theta_{M_{\mathcal{C}/T}^\alpha(\bar{\mathbf{t}}, r, d)/N_r^{(n)}(d)}$ defined in (12). \square

4. HAMILTONIAN DESCRIPTION

In this section, we give a Hamiltonian description of the vector field determined by the isomonodromic deformation. We fix $\nu \in N_r^{(n)}(d)$. In Section 4.1, we take an affine open covering $\{M\}$ of $M_{\mathcal{C}/T}^\alpha(\bar{\mathbf{t}}, r, d)_\nu$. For each M , we construct an initial connection ∇_0 by the following idea. First, for the underlying vector bundle on $\mathcal{C} \times_T M$ induced by a universal family on $\mathcal{C} \times_T M_{\mathcal{C}/T}^\alpha(\bar{\mathbf{t}}, r, d)$, we give an injective morphism (of locally free sheaves) from some fixed vector bundle having same rank. (This construction is not canonical). Such a vector bundle with an injective morphism is treated in [9, Section 2]. Second, we construct a connection on the fixed vector bundle. By the injective morphism and the connection on the fixed vector bundle, we have a connection on the underlying vector bundle on M induced by a universal

family on $M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)$. This connection is an initial connection ∇_0 . In Section 4.2, we construct vector fields on each M associated to an initial connection ∇_0 . Note that an initial connection ∇_0 has poles along divisors on $\mathcal{C}_M \setminus D(\tilde{\mathbf{t}})_M$. Then, for construction of the vector fields, we need some condition of deformations of n -pointed curves on neighborhoods of the poles along divisors on $\mathcal{C}_M \setminus D(\tilde{\mathbf{t}})_M$. We use these vector fields instead of vector fields associated to time variables. In Section 4.4, we describe the main theorem. First, we give a 2-form ω on $M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)_\nu$ such that the kernel $\text{Ker}(\omega)$ induces the vector fields determined by the isomonodromic deformations and ω is the symplectic form fiberwise. Second, we define Hamiltonian functions on each M . Finally, if we take good coordinates on each M , we obtain a Hamiltonian description of the vector field on each M induced by the isomonodromic deformation.

4.1. Construction of initial connections $\tilde{\nabla}_0^{\sigma_M}$. Let T be a smooth algebraic scheme which is an étale covering of the moduli stack $\mathcal{M}_{g,n}$ of n -pointed smooth projective curves of genus g over \mathbb{C} and take a universal family $(\mathcal{C}, \tilde{t}_1, \dots, \tilde{t}_n)$ over T . We take a T -ample line bundle $\mathcal{O}_{\mathcal{C}}(1)$ on \mathcal{C} . Let m be an integer sufficiently large. We take an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathcal{C}}(-m) \longrightarrow \mathcal{O}_{\mathcal{C}} \longrightarrow \mathcal{O}_{\mathcal{C}}/\mathcal{O}_{\mathcal{C}}(-m) \longrightarrow 0.$$

From this section, we fix $\nu \in N_r^{(n)}(d)$ and we denote by

$$\varpi_\nu: M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)_\nu \longrightarrow T \times \{\nu\}$$

the fiber product of $M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d) \rightarrow T \times N_r^{(n)}(d)$ and $T \times \{\nu\} \rightarrow T \times N_r^{(n)}(d)$. Let \tilde{E} be the underlying vector bundle of the universal family of $M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)_\nu$. Let M be an affine open subset of $M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)_\nu$. We put $\mathcal{C}_M := \mathcal{C} \times_T M$. Let $D(m)$ be the Cartier divisor of \mathcal{C}_M such that for any $x \in M$,

$$D(m)_{\mathcal{O}_{c_x}} = \sum_{p \in \mathcal{C}_x} \text{length} \left(\text{Coker} \left(\mathcal{O}_{\mathcal{C}_M}(-m) \rightarrow \mathcal{O}_{\mathcal{C}_M}_p \right) [p] \right).$$

We assume that $D(m)_{\mathcal{O}_{c_x}}$ consists of distinct points for any $x \in M$. Let $\pi_{M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)_\nu}: \mathcal{C} \times_T M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)_\nu \rightarrow M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)_\nu$ and $\pi_M: \mathcal{C}_M \rightarrow M$ be the projections, respectively.

Proposition 4.1. *There exists an affine open covering $\{M\}$ of $M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)_\nu$ such that for each M ,*

- (1) $(\pi_{M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)_\nu})_*(\tilde{E}(m))|_M$ is a free sheaf on M , and
 - (2) there is a subbundle inclusion $\mathcal{O}_M^{\oplus r} \hookrightarrow (\pi_{M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)_\nu})_*(\tilde{E}(m))|_M$ such that for each $x \in M$, the composition
- $$(20) \quad \mathcal{O}_M^{\oplus r} \otimes \mathcal{O}_{\mathcal{C}_M} \otimes k(x) \xrightarrow{\subset} (\pi_{M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)_\nu})_*(\tilde{E}(m))|_M \otimes \mathcal{O}_{\mathcal{C}_M} \otimes k(x) \longrightarrow \tilde{E}(m) \otimes k(x)$$
- is an injection and the cokernel of this injection gives a reduced divisor whose support is disjoint from the supports of $D(\tilde{\mathbf{t}})_{\mathcal{O}_{c_x}}$ and $D(m)_{\mathcal{O}_{c_x}}$.

We denote by $\sigma_M: \mathcal{O}_{\mathcal{C}_M}^{\oplus r} \rightarrow \tilde{E}(m)|_{\mathcal{C}_M}$ the composition

$$\mathcal{O}_{\mathcal{C}_M}^{\oplus r} \cong \mathcal{O}_M^{\oplus r} \otimes \mathcal{O}_{\mathcal{C}_M} \xrightarrow{\subset} (\pi_{M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)_\nu})_*(\tilde{E}(m))|_M \otimes \mathcal{O}_{\mathcal{C}_M} \longrightarrow \tilde{E}(m)|_{\mathcal{C}_M}.$$

Proof. Since m is a sufficiently large integer, $(\pi_{M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)_\nu})_*(\tilde{E}(m))$ is a locally free sheaf on $M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)_\nu$. So we can take an affine open covering $\{M\}$ of $M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)_\nu$ such that $(\pi_{M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)_\nu})_*(\tilde{E}(m))|_M$ is a free sheaf for each M . We take a point $x \in M$. We denote by E_x the restriction $\tilde{E}|_{\mathcal{C}_x}$ of the vector bundle \tilde{E} . Let y be a point on \mathcal{C}_x . Set

$$Y_y := \left\{ (\bar{s}_1, \dots, \bar{s}_r) \in (E_x|_{2y})^{\oplus r} \mid \bar{s}_1 \wedge \dots \wedge \bar{s}_r \text{ does not generate } \bigwedge^r E_x|_{2y} \right\} \text{ and}$$

$$Z_y := \left\{ (\bar{s}_1, \dots, \bar{s}_r) \in Y_y \mid \bar{s}_1 \wedge \dots \wedge \bar{s}_r = 0 \text{ in } \bigwedge^r E_x|_{2y} \right\}.$$

Then we have $Z_y \subsetneq Y_y \subsetneq (E_x|_{2y})^{\oplus r}$ and $\dim Z_y \leq \dim(E_x|_{2y})^{\oplus r} - 2$. If we choose an integer m sufficiently large, we have an exact sequence

$$0 \longrightarrow H^0(E_x(-2y)(m)) \longrightarrow H^0(E_x(m)) \longrightarrow E|_{2y} \longrightarrow 0$$

for any $y \in \mathcal{C}_x$. So the restriction map

$$\begin{aligned} \varphi_y: H^0(E_x(m))^{\oplus r} &\longrightarrow (E|_{2y})^{\oplus r} \\ (s_1, \dots, s_r) &\longmapsto (\bar{s}_1, \dots, \bar{s}_r) \end{aligned}$$

is surjective. Fibers of φ_y are isomorphic to $H^0(E_x(-2y)(m))^{\oplus r}$. Then we get the inequality $\dim \varphi_y^{-1}(Z_y) \leq \dim H^0(E_x(m))^{\oplus r} - 2$. This inequality implies that $H^0(E_x(m))^{\oplus r} \setminus \bigcup_{y \in \mathcal{C}_x} \varphi_y^{-1}(Z_y) \neq \emptyset$. We take a basis $\{e_1, \dots, e_r\}$ of the free sheaf $(\pi_{M_{\mathcal{C}_T}^\alpha(\tilde{\mathbf{t}}, r, d)})_*(\tilde{E}(m))|_M$ such that

$$\{(e_1)_x, \dots, (e_r)_x\} \in H^0(E_x(m))^{\oplus r} \setminus \bigcup_{y \in \mathcal{C}_x} \varphi_y^{-1}(Z_y)$$

for the point $x \in M$. Then we may assume that the subbundle inclusion $\mathcal{O}_M^{\oplus r} \hookrightarrow (\pi_{M_{\mathcal{C}_T}^\alpha(\tilde{\mathbf{t}}, r, d)_\nu})_*(\tilde{E}(m))|_M$ defined by $\{e_1, \dots, e_r\}$ satisfies the condition (2) in the statement of this proposition by taking a refined affine open covering $\{M\}$. \square

For each family $\pi_M: \mathcal{C}_M \rightarrow M$ of curves, we assume that $\text{Supp}(D(m))$ is disjoint from $\text{Supp}(D(\tilde{\mathbf{t}}))$, since $\mathcal{O}_{\mathcal{C}}(1)$ is a T -ample line bundle. Let $D(\sigma_M)$ be the Cartier divisor of \mathcal{C}_M such that for any $x \in M$,

$$D(\sigma_M)_{\mathcal{O}_{\mathcal{C}_x}} = \sum_{p \in \mathcal{C}_x} \text{length} \left(\text{Coker}(\sigma_M)_p \right) [p].$$

Let d_m be the relative connection induced by $0 \rightarrow \mathcal{O}_{\mathcal{C}_M}^{\oplus r}(-m) \rightarrow \mathcal{O}_{\mathcal{C}_M}^{\oplus r}$ and the relative exterior derivative $d_{\mathcal{C}_M/M}$ on $\mathcal{O}_{\mathcal{C}_M}^{\oplus r}$, that is, the diagram

$$\begin{array}{ccc} \mathcal{O}_{\mathcal{C}_M}^{\oplus r}(-m) & \xrightarrow{d_m} & \mathcal{O}_{\mathcal{C}_M}^{\oplus r}(-m) \otimes \Omega_{\mathcal{C}_M/M}^1(D(m)) \\ \downarrow & & \downarrow \\ \mathcal{O}_{\mathcal{C}_M}^{\oplus r} & \xrightarrow{d_{\mathcal{C}_M/M}} & \mathcal{O}_{\mathcal{C}_M}^{\oplus r} \otimes \Omega_{\mathcal{C}_M/M}^1(D(m)) \end{array}$$

is commutative.

Definition 4.2. For each affine open subset $M \subset M_{\mathcal{C}_T}^\alpha(\tilde{\mathbf{t}}, r, d)_\nu$ of Proposition 4.1, we fix $\sigma_M: \mathcal{O}_{\mathcal{C}_M}^{\oplus r} \rightarrow \tilde{E}(m)|_{\mathcal{C}_M}$ as in this proposition. We define a *relative initial connection*

$$\tilde{\nabla}_0^{\sigma_M}: \tilde{E}_{\mathcal{C}_M} \longrightarrow \tilde{E}_{\mathcal{C}_M} \otimes \Omega_{\mathcal{C}_M/M}^1(D(m) + D(\sigma_M))$$

by the relative connection induced by $0 \rightarrow \mathcal{O}_{\mathcal{C}_M}^{\oplus r}(-m) \xrightarrow{\sigma_M} \tilde{E}|_{\mathcal{C}_M}$ and the relative connection d_m on $\mathcal{O}_{\mathcal{C}_M}^{\oplus r}(-m)$.

We will construct a parabolic structure of $(\tilde{E}_{\mathcal{C}_M}, \tilde{\nabla}_0^{\sigma_M})$ as follows. Let $\{\tilde{\mathbf{p}}_1, \dots, \tilde{\mathbf{p}}_{N_1}\}$ be the support of the divisor $D(m)$ and $\{\tilde{\mathbf{p}}'_1, \dots, \tilde{\mathbf{p}}'_{N_2}\}$ be the support of the divisor $D(\sigma_M)$. Here we put $N_1 := \deg D(m)|_{\mathcal{C}_x}$ and $N_2 := \deg D(\sigma_M)|_{\mathcal{C}_x}$ for each $x \in M$. Let $\{(U_\alpha, g_\alpha)\}_\alpha$ and $\{(U_\alpha, h_\alpha)\}_\alpha$ be the Cartier divisors corresponding to $D(m)$ and $D(\sigma_M)$, respectively. Here $\{U_\alpha\}_\alpha$ is an affine open covering of \mathcal{C}_M . We take a basis $e_1 \otimes g_\alpha, \dots, e_r \otimes g_\alpha$ of $\mathcal{O}_{\mathcal{C}_M}^{\oplus r}(-m)|_{U_\alpha}$. We put $s_{\alpha, j} := \sigma_M|_{U_\alpha}(e_j \otimes g_\alpha)$ for $j = 1, \dots, r$. First, we consider a filtration on $\tilde{\mathbf{p}}_{i'}$ ($i' = 1, 2, \dots, N_1$). Since $\{\tilde{\mathbf{p}}_1, \dots, \tilde{\mathbf{p}}_{N_1}\}$ and $\{\tilde{\mathbf{p}}'_1, \dots, \tilde{\mathbf{p}}'_{N_2}\}$ are disjoint, for each $\tilde{\mathbf{p}}_{i'}$, we can define a filtration $(l_*^{\sigma_M})^{(i')}$ by subbundles $\tilde{E}|_{\tilde{\mathbf{p}}_{i'}} = (l_0^{\sigma_M})^{(i')} \supset (l_1^{\sigma_M})^{(i')} \supset \dots \supset (l_r^{\sigma_M})^{(i')} = 0$ by using the basis $\{s_{\alpha, 1}|_{\tilde{\mathbf{p}}_{i'}}, \dots, s_{\alpha, r}|_{\tilde{\mathbf{p}}_{i'}}\}$ where $\tilde{\mathbf{p}}_{i'} \in U_\alpha$. By the definition of $\tilde{\nabla}_0^{\sigma_M}$, the residue matrix

of $\nabla_0^{\sigma_M}$ along $\tilde{\mathbf{p}}_i$, associated to the trivialization given by $\{\mathbf{s}_{\alpha,j}\}_{1 \leq j \leq r}$ is

$$(21) \quad \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Then we have that the filtration $(l_*^{\sigma_M})^{(i')}$ is compatible with $\tilde{\nabla}_0^{\sigma_M}$. Second, we consider a filtration on $\tilde{\mathbf{p}}'_{i''}$ ($i'' = 1, 2, \dots, N_2$). For each i'' , we take α such that $\tilde{\mathbf{p}}'_{i''} \in U_\alpha$. We change the order of $(\mathbf{s}_{\alpha,1}, \mathbf{s}_{\alpha,2}, \dots, \mathbf{s}_{\alpha,r})$ as follows. For each i'' , we set $\mathbf{s}_{\alpha,i'',k} := \mathbf{s}_{\alpha,j(i'',k)}$ (where $k = 1, 2, \dots, r$ and $\{j(i'',1), \dots, j(i'',r)\} = \{1, 2, \dots, r\}$) such that $(\mathbf{s}_{\alpha,i'',2}|_{\tilde{\mathbf{p}}'_{i''}}, \dots, (\mathbf{s}_{\alpha,i'',r}|_{\tilde{\mathbf{p}}'_{i''}})$ are linearly independent. Let $a_{\alpha,i'',2}, \dots, a_{\alpha,i'',r}$ be elements of $\pi_M^*(\mathcal{O}_M)|_{U_\alpha}$ such that $\mathbf{s}_{\alpha,i'',1}|_{\tilde{\mathbf{p}}'_{i''}} = a_{\alpha,i'',2} \cdot \mathbf{s}_{\alpha,i'',2}|_{\tilde{\mathbf{p}}'_{i''}} + \cdots + a_{\alpha,i'',r} \cdot \mathbf{s}_{\alpha,i'',r}|_{\tilde{\mathbf{p}}'_{i''}}$. We define $\hat{\mathbf{s}}_{\alpha,i'',1}$ as

$$\mathbf{s}_{\alpha,i'',1} = h_\alpha \cdot \hat{\mathbf{s}}_{\alpha,i'',1} + a_{\alpha,i'',2} \cdot \mathbf{s}_{\alpha,i'',2} + \cdots + a_{\alpha,i'',r} \cdot \mathbf{s}_{\alpha,i'',r}.$$

For each $\tilde{\mathbf{p}}'_{i''}$, we can define a filtration $(l_*^{\sigma_M})^{(i'')}$ by subbundles $\tilde{E}|_{\tilde{\mathbf{p}}'_{i''}} = (l_0^{\sigma_M})^{(i'')} \supset (l_1^{\sigma_M})^{(i'')} \supset \cdots \supset (l_r^{\sigma_M})^{(i'')} = 0$ by using the basis $\{\hat{\mathbf{s}}_{\alpha,i'',1}|_{\tilde{\mathbf{p}}'_{i''}}, \mathbf{s}_{\alpha,i'',2}|_{\tilde{\mathbf{p}}'_{i''}}, \dots, \mathbf{s}_{\alpha,i'',r}|_{\tilde{\mathbf{p}}'_{i''}}\}$. Next we check the filtration $(l_*^{\sigma_M})^{(i'')}$ is compatible with $\tilde{\nabla}_0^{\sigma_M}$. We set

$$(22) \quad T_{\alpha,i''} := \begin{pmatrix} h_\alpha & 0 & 0 & \cdots & 0 \\ a_{\alpha,i'',2} & 1 & 0 & \cdots & 0 \\ a_{\alpha,i'',3} & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{\alpha,i'',r} & 0 & 0 & \cdots & 1 \end{pmatrix},$$

which gives a commutative diagram

$$(23) \quad \begin{array}{ccc} \mathcal{O}_{\mathcal{C}_M}^{\oplus r}(-m)|_{U_\alpha} & \xleftarrow{g_\alpha} & \mathcal{O}_{U_\alpha}^{\oplus r} \\ \downarrow \sigma_M|_{U_\alpha} & & \downarrow T_{\alpha,i''} \\ \tilde{E}|_{U_\alpha} & \xrightarrow{\cong} & \mathcal{O}_{U_\alpha}^{\oplus r} \end{array}$$

where the morphism $\tilde{E}|_{U_\alpha} \rightarrow \mathcal{O}_{U_\alpha}^{\oplus r}$ is defined by $\hat{\mathbf{s}}_{\alpha,i''} := \{\hat{\mathbf{s}}_{\alpha,i'',1}, \mathbf{s}_{\alpha,i'',2}, \dots, \mathbf{s}_{\alpha,i'',r}\}$. Here U_α shrinks so that $\tilde{\mathbf{p}}'_{i''} \in U_\alpha$ and the morphism from $\tilde{E}|_{U_\alpha}$ to $\mathcal{O}_{U_\alpha}^{\oplus r}$ in the diagram (23) is an isomorphism. This isomorphism gives a trivialization of $\tilde{E}|_{U_\alpha}$. The residue matrix of $\nabla_0^{\sigma_M}$ along $\tilde{\mathbf{p}}'_{i''}$ associated to this trivialization is

$$(24) \quad \begin{pmatrix} -\operatorname{res}_{h_\alpha=0}(h_\alpha^{-1}dh_\alpha) & 0 & \cdots & 0 \\ -\operatorname{res}_{h_\alpha=0}(h_\alpha^{-1}da_{\alpha,i'',2}) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\operatorname{res}_{h_\alpha=0}(h_\alpha^{-1}da_{\alpha,i'',r}) & 0 & \cdots & 0 \end{pmatrix}.$$

Then we have that the filtration $(l_*^{\sigma_M})^{(i'')}$ is compatible with $\tilde{\nabla}_0^{\sigma_M}$.

Definition 4.3. For each affine open subset $M \subset M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)_\nu$ of Proposition 4.1, we fix $\sigma_M: \mathcal{O}_{\mathcal{C}_M}^{\oplus r} \rightarrow \tilde{E}(m)|_{\mathcal{C}_M}$ as in this proposition. Let $(\tilde{E}_{\mathcal{C}_M}, \tilde{\nabla}_0^{\sigma_M})$ be the relative initial connection defined in Definition 4.2. Let $\mathbf{l}^{\sigma_M} := \{(l_*^{\sigma_M})^{(i')}, (l_*^{\sigma_M})^{(i'')}\}_{1 \leq i' \leq N_1, 1 \leq i'' \leq N_2}$ be the filtrations defined as above. We call $(\tilde{E}_{\mathcal{C}_M}, \tilde{\nabla}_0^{\sigma_M}, \mathbf{l}^{\sigma_M})$ a *relative initial parabolic connection* on $\pi_M: \mathcal{C}_M \rightarrow M$.

Definition 4.4. For α such that $\tilde{\mathbf{p}}_i \in U_\alpha$ and $\tilde{\mathbf{p}}'_{i''} \notin U_\alpha$, the sections $\{\mathbf{s}_{\alpha,i'',k}\}_{1 \leq k \leq r}$ give a trivialization of $E|_{U_\alpha}$. For α such that $\tilde{\mathbf{p}}'_{i''} \in U_\alpha$, the sections $\{\hat{\mathbf{s}}_{\alpha,i'',1}, \mathbf{s}_{\alpha,i'',k}\}_{2 \leq k \leq r}$ give a trivialization of $E|_{U_\alpha}$. We say trivializations $(U_\alpha, \bar{\phi}_\alpha: \tilde{E}|_{U_\alpha} \rightarrow \mathcal{O}_{U_\alpha}^{\oplus r})$ are *compatible with σ_M* if the trivialization $\bar{\phi}_\alpha$ (for α such that $\tilde{\mathbf{p}}_i \in U_\alpha$ or $\tilde{\mathbf{p}}'_{i''} \in U_\alpha$) coincides with the trivialization given by the sections $\{\mathbf{s}_{\alpha,i'',k}\}_{1 \leq k \leq r}$ or the trivialization given by the sections $\{\hat{\mathbf{s}}_{\alpha,i'',1}, \mathbf{s}_{\alpha,i'',k}\}_{2 \leq k \leq r}$.

Now we have two families of parabolic connections parametrized by $M \subset M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)_{\nu}$:

$$(\tilde{E}_{\mathcal{C}_M}, \tilde{\nabla}_{\mathcal{C}_M}, \{l_*^{(i)}\}_{1 \leq i \leq n}) \quad \text{and} \quad (\tilde{E}_{\mathcal{C}_M}, \tilde{\nabla}_0^{\sigma_M}, \mathbf{l}^{\sigma_M}).$$

The former is induced by a universal family of the moduli space $M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)_{\nu}$. The latter is defined in Definition 4.3. By these two families, we have two morphisms from M to some moduli spaces. The first morphism is by taking difference of the connections: $\tilde{\nabla}_{\mathcal{C}_M} - \tilde{\nabla}_0^{\sigma_M}$. Note that the difference of connections is a *Higgs field* on the vector bundle. We call a vector bundle with parabolic structures a *parabolic bundle*. We call a parabolic bundle with a Higgs field (which is compatible with the parabolic structures) a *parabolic Higgs bundle*. Then we have a morphism to a moduli space of parabolic Higgs bundles, roughly speaking. The second morphism is by taking $\tilde{\nabla}_0^{\sigma_M}$. Then we have a morphism to another moduli space of parabolic connections, roughly speaking. We will see a rough sketch of construction of these morphisms. We set $N := N_1 + N_2$. Let $\mathcal{M}_{g, n+N}$ be the moduli stack of smooth projective curves of genus g with (ordered) points of degree $n+N$. First for each $(C, \text{Supp}(D(t) + D(p))) \in \mathcal{M}_{g, n+N}$, we consider parabolic Higgs bundles of rank r and of degree d on C with poles on $D(t) + D(p)$. Let $\varpi'_H: M'_H \rightarrow \mathcal{M}_{g, n+N}$ be the moduli space of such parabolic Higgs bundles. We define a morphism from M to M'_H as follows:

$$(25) \quad \begin{aligned} h_{\nabla - \nabla_0}: M &\longrightarrow M'_H \\ x &\longmapsto ((\mathcal{C}_x, \text{Supp}(D(\tilde{\mathbf{t}})_{\mathcal{C}_x} + D(\tilde{\mathbf{p}})_{\mathcal{C}_x})), (\tilde{E}_{\mathcal{C}_x}, (\tilde{\nabla} - \tilde{\nabla}_0^{\sigma_M})|_{\mathcal{C}_x}, \mathbf{l}^{\sigma_M}|_{\mathcal{C}_x} \cup \{l_*^{(i)}|_{\mathcal{C}_x}\}_i)). \end{aligned}$$

Second, for each $(C, \text{Supp}(D(t) + D(p))) \in \mathcal{M}_{g, n+N}$, we consider parabolic connections of rank r and of degree d on C with poles on $D(t) + D(p)$. Let $\varpi': M' \rightarrow \mathcal{M}_{g, n+N}$ be the moduli space of such parabolic connections. We define a morphism from M to M' as follows:

$$(26) \quad \begin{aligned} h_{\nabla_0}: M &\longrightarrow M' \\ x &\longmapsto ((\mathcal{C}_x, \text{Supp}(D(\tilde{\mathbf{t}})_{\mathcal{C}_x} + D(\tilde{\mathbf{p}})_{\mathcal{C}_x})), (\tilde{E}_{\mathcal{C}_x}, \tilde{\nabla}_0^{\sigma_M}|_{\mathcal{C}_x}, \mathbf{l}^{\sigma_M}|_{\mathcal{C}_x} \cup \{l_*^{(i)}|_{\mathcal{C}_x}\}_i)). \end{aligned}$$

4.2. Algebraic vector fields associated to $\tilde{\nabla}_0^{\sigma_M}$. We take an affine open covering $\{M\}$ of $M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)_{\nu}$ as in Proposition 4.1. We denote by the same notations $D(\tilde{\mathbf{t}})$ and $(\tilde{E}, \tilde{\nabla}, \{\tilde{l}_j^{(i)}\})$ the pull-backs of the divisor $D(\tilde{\mathbf{t}})$ and a universal family $(\tilde{E}, \tilde{\nabla}, \{\tilde{l}_j^{(i)}\})$ under the morphism $\mathcal{C}_M \rightarrow \mathcal{C} \times_T M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)_{\nu}$, respectively. We fix an injection $\sigma_M: \mathcal{O}_{\mathcal{C}_M}^{\oplus r} \rightarrow \tilde{E}(m)$ for each M as in Proposition 4.1. We put $D(\tilde{\mathbf{p}}) := D(m) + D(\sigma_M)$. In this section, we show the following.

Proposition 4.5. *Let $\tilde{\nabla}_0^{\sigma_M}$ be the relative initial connection of Definition 4.2. For $\mu \in H^1(\mathcal{C}_M, \Theta_{\mathcal{C}_M/M}(-D(\tilde{\mathbf{t}})))$, we take a lift $\hat{\mu} \in H^1(\mathcal{C}_M, \Theta_{\mathcal{C}_M/M}(-D(\tilde{\mathbf{t}}) - D(\tilde{\mathbf{p}})))$ as (27) below. Then we can construct an algebraic vector field associated to $\tilde{\nabla}_0^{\sigma_M}$ on M (Lemma 4.7 below). This vector field is described by $[\{u_{\alpha\beta}^{\hat{\mu}\nabla_0}\}, \{-v_{\alpha}^{\hat{\mu}\nabla_0}\}] \in \mathbf{H}^1(\mathcal{G}_M^{\bullet})$ in Lemma 4.7.*

Let $\{U_{\alpha}\}_{\alpha}$ be an affine open covering of \mathcal{C}_M such that $\#\{i \mid \hat{t}_i|_{\mathcal{C}_M} \cap U_{\alpha} \neq \emptyset\} \leq 1$ for any α and $\#\{\alpha \mid \hat{t}_i|_{\mathcal{C}_M} \cap U_{\alpha} \neq \emptyset\} \leq 1$ for any i . Here $\{\hat{t}_i\}$ is the set of the supports of the Cartier divisor $D(\tilde{\mathbf{t}}) + D(\tilde{\mathbf{p}})$. Set

$$I_{D(\tilde{\mathbf{p}})} := \{\alpha \mid U_{\alpha} \cap \text{Supp}(D(\tilde{\mathbf{p}})) \neq \emptyset\}.$$

We denote by $D(\tilde{\mathbf{t}}) = \{(U_{\alpha}, f_{\alpha})\}_{\alpha}$, $D(m) = \{(U_{\alpha}, g_{\alpha})\}_{\alpha}$ and $D(\sigma_M) = \{(U_{\alpha}, h_{\alpha})\}_{\alpha}$ the Cartier divisors. For any α , we assume that there exists a trivialization $\bar{\phi}_{\alpha}: \tilde{E}|_{U_{\alpha}} \xrightarrow{\sim} \mathcal{O}_{U_{\alpha}}^{\oplus r}$ of \tilde{E} over U_{α} .

We take $\mu \in H^1(\mathcal{C}_M, \Theta_{\mathcal{C}_M/M}(-D(\tilde{\mathbf{t}})))$. Since

$$H^1(\mathcal{C}_M, \Theta_{\mathcal{C}_M/M}(-D(\tilde{\mathbf{t}}) - D(\tilde{\mathbf{p}}))) \longrightarrow H^1(\mathcal{C}_M, \Theta_{\mathcal{C}_M/M}(-D(\tilde{\mathbf{t}}))) \longrightarrow 0,$$

we choose a lift

$$(27) \quad \hat{\mu} = \left[\{\hat{d}_{\alpha\beta}\} \right] \in H^1(\mathcal{C}_M, \Theta_{\mathcal{C}_M/M}(-D(\tilde{\mathbf{t}}) - D(\tilde{\mathbf{p}})))$$

of μ as follows. First we take the isomonodromic lift $M[\epsilon] \rightarrow M$ associated to μ as in Section 3.3. Here we set $M[\epsilon] := M \times \text{Spec } \mathbb{C}[\epsilon]$. We denote by $\text{im}_{M'}^{\mu}: M[\epsilon] \rightarrow M$ this isomonodromic lift. Let \mathcal{C}_{ϵ} be the

fiber product $\mathcal{C}_M \times_M M[\epsilon]$ with respect to the projection $\mathcal{C}_M \rightarrow M$ and the morphism $\text{im}_M^\mu: M[\epsilon] \rightarrow M$. The lift im_M^μ induces a morphism

$$(28) \quad \text{IM}_M^\mu: \mathcal{C}_\epsilon \longrightarrow \mathcal{C}_M.$$

Second we take the pull-back $(\text{IM}_M^\mu)^*(D(\tilde{\mathbf{t}}) + D(\tilde{\mathbf{p}}))$, which is a Cartier divisor on \mathcal{C}_ϵ . We consider the pair $(\mathcal{C}_\epsilon, \text{Supp}((\text{IM}_M^\mu)^*(D(\tilde{\mathbf{t}}) + D(\tilde{\mathbf{p}}))))$, which is a flat family of curves with (ordered) points. This flat family gives a morphism $M[\epsilon] \rightarrow \mathcal{M}_{g,n+N}$ to the moduli stack $\mathcal{M}_{g,n+N}$ of smooth projective curves of genus g with points of degree $n+N$. The morphism $M[\epsilon] \rightarrow \mathcal{M}_{g,n+N}$ determines a class of $H^1(\mathcal{C}_M, \Theta_{\mathcal{C}_M/M}(-D(\tilde{\mathbf{t}}) - D(\tilde{\mathbf{p}})))$. We denote by $\hat{\mu}$ this class, which is a lift of μ .

Now we define an algebraic vector field on M by the relative initial connection $(\tilde{E}, \tilde{\nabla}_0^{\sigma_M})$ and the lift $\hat{\mu} = [\{\hat{d}_{\alpha\beta}\}]$ of μ . Let \mathcal{G}_M^0 and \mathcal{G}_M^1 be the pull-backs of \mathcal{G}^0 and \mathcal{G}^1 by the morphism $\mathcal{C}_M \rightarrow \mathcal{C} \times_T M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)_\nu$, respectively. We have the morphism $\nabla_{\mathcal{G}_M^\bullet}: \mathcal{G}_M^0 \rightarrow \mathcal{G}_M^1$ induced by $\nabla_{\mathcal{G}^\bullet}: \mathcal{G}^0 \rightarrow \mathcal{G}^1$. Let $\{(U_\alpha, z_\alpha)\}_\alpha$ be the Cartier divisor $D(\tilde{\mathbf{t}}) + D(\tilde{\mathbf{p}})$. Let $\tilde{A}_\alpha^0 z_\alpha^{-1} dz_\alpha$ and $\tilde{A}_\alpha f_\alpha^{-1} df_\alpha$ be connection matrices of $\tilde{\nabla}_0^{\sigma_M}$ and $\tilde{\nabla}$ on U_α associated to the trivialization $\bar{\phi}_\alpha$, respectively. First, we set

$$(29) \quad u_{\alpha\beta}^{\hat{\mu}\nabla_0} := \bar{\phi}_\beta^{-1} \circ \left(\hat{d}_{\alpha\beta} + \left\langle \hat{d}_{\alpha\beta}, \tilde{A}_\beta^0 \frac{dz_\beta}{z_\beta} \right\rangle \right) \circ \bar{\phi}_\beta \in \mathcal{G}_M^0(U_{\alpha\beta}),$$

which satisfy the equality $u_{\beta\gamma}^{\hat{\mu}\nabla_0} - u_{\alpha\gamma}^{\hat{\mu}\nabla_0} + u_{\alpha\beta}^{\hat{\mu}\nabla_0} = 0$. The idea of the definition of $u_{\alpha\beta}^{\hat{\mu}\nabla_0}$ is ‘‘isomonodromic deformation’’ of $\tilde{\nabla}_0^{\sigma_M}$ associated to $\hat{\mu}$.

Now we construct a vector field on M such that its image of the natural morphism $\mathbf{H}^1(\mathcal{G}_M^\bullet) \rightarrow H^1(\mathcal{G}_M^0)$ is $[\{u_{\alpha\beta}^{\hat{\mu}\nabla_0}\}]$. We will define $v_{\alpha\beta}^{\hat{\mu}\nabla_0}$ later at (32) such that $[\{u_{\alpha\beta}^{\hat{\mu}\nabla_0}\}, \{-v_{\alpha\beta}^{\hat{\mu}\nabla_0}\}] \in \mathbf{H}^1(\mathcal{G}_M^\bullet)$. For $\{\hat{d}_{\alpha\beta}\}_{\alpha\beta}$, we put

$$(30) \quad \hat{u}_{\alpha\beta}^{\text{IMD}} := \bar{\phi}_\beta^{-1} \circ \left(\hat{d}_{\alpha\beta} + \left\langle \hat{d}_{\alpha\beta}, \tilde{A}_\beta \frac{df_\beta}{f_\beta} \right\rangle \right) \circ \bar{\phi}_\beta, \quad \hat{v}_\alpha^{\text{IMD}} := 0.$$

By Proposition 3.8, the class $[\{\hat{u}_{\alpha\beta}^{\text{IMD}}\}, \{\hat{v}_\alpha^{\text{IMD}}\}] \in \mathbf{H}^1(\mathcal{G}_M^\bullet)$ means the vector field determined by the isomonodromic deformations of $\tilde{\nabla}$ associated to μ . Hence $[\{\hat{u}_{\alpha\beta}^{\text{IMD}}\}, \{\hat{v}_\alpha^{\text{IMD}}\}]$ gives the isomonodromic lift $\text{im}_M^\mu: M[\epsilon] \rightarrow M$. Let $(E_\epsilon, (\text{IM}_M^\mu)^*\tilde{\nabla}, \{(l_\epsilon)_j^{(i)}\})$ be the pull-back of $(\tilde{E}, \tilde{\nabla}, \{\tilde{l}_j^{(i)}\})$ under the morphism $\text{IM}_M^\mu: \mathcal{C}_\epsilon \rightarrow \mathcal{C}_M$, which is induced by im_M^μ . Let $D_\epsilon(\tilde{\mathbf{t}}) = \{(U_\alpha^\epsilon, f_\alpha^\epsilon)\}_\alpha$ and $D_\epsilon(\tilde{\mathbf{p}}) = \{(U_\alpha^\epsilon, z_\alpha^\epsilon)\}_\alpha$ be the pull-backs of the Cartier divisors $D(\tilde{\mathbf{t}}) = \{(U_\alpha, f_\alpha)\}_\alpha$ and $D(\tilde{\mathbf{p}}) = \{(U_\alpha, z_\alpha)\}_\alpha$ by the morphism $\text{IM}_M^\mu: \mathcal{C}_\epsilon \rightarrow \mathcal{C}_M$, respectively.

Definition 4.6. Let $(\tilde{E}, \tilde{\nabla}_0^{\sigma_M})$ be the relative initial connection defined in Definition 4.2. We define an infinitesimal deformation

$$\nabla_{0,\epsilon}^{\sigma_M, \text{IMD}}: E_\epsilon \longrightarrow E_\epsilon \otimes \Omega_{\mathcal{C}_\epsilon/M[\epsilon]}^1(D_\epsilon(\tilde{\mathbf{p}}))$$

of the relative initial connection $(\tilde{E}, \tilde{\nabla}_0^{\sigma_M})$ by taking the pull-back of $(\tilde{E}, \tilde{\nabla}_0^{\sigma_M})$ by $\text{IM}_M^\mu: \mathcal{C}_\epsilon \rightarrow \mathcal{C}_M$.

Let $\mathcal{C}_\epsilon = \bigcup_\alpha U_\alpha^\epsilon$ be the open covering corresponding to the affine open covering $\{U_\alpha\}_\alpha$ of \mathcal{C}_M . Here, the affine open covering $\{U_\alpha\}_\alpha$ is defined after Proposition 4.5. We can take trivializations $\{\phi_\alpha\}_\alpha$ of E_ϵ on U_α^ϵ such that $\bar{\phi}_\alpha = \phi_\alpha \pmod{\epsilon}$ and

$$(31) \quad \varphi_\alpha^{-1} \circ \varphi_\beta - \text{id} = \hat{u}_{\alpha\beta}^{\text{IMD}},$$

where φ_α is defined as in the proof of Proposition 3.5. We set

$$(32) \quad v_\alpha^{\hat{\mu}\nabla_0} := (\varphi_\alpha^{-1} \otimes \text{id}) \circ (\nabla_{0,\epsilon}^{\sigma_M, \text{IMD}}) \circ \varphi_\alpha - \tilde{\nabla}_0^{\sigma_M}$$

for any α .

Lemma 4.7. For $u_{\alpha\beta}^{\hat{\mu}\nabla_0}$ defined as (30) and $v_\alpha^{\hat{\mu}\nabla_0}$ defined as (32), we have the following equality

$$(33) \quad \tilde{\nabla} \circ u_{\alpha\beta}^{\hat{\mu}\nabla_0} - u_{\alpha\beta}^{\hat{\mu}\nabla_0} \circ \tilde{\nabla} = -(v_\beta^{\hat{\mu}\nabla_0} - v_\alpha^{\hat{\mu}\nabla_0}).$$

Moreover $v_\alpha^{\hat{\mu}\nabla_0}$ is an element of $\epsilon \otimes \mathcal{G}_M^1$. As a result, the pair $\left[\left(\left\{u_{\alpha\beta}^{\hat{\mu}\nabla_0}\right\}, \left\{-v_\alpha^{\hat{\mu}\nabla_0}\right\}\right)\right]$ is a class of $\mathbf{H}^1(\mathcal{G}_M^\bullet)$. This class is independent of the choice of a representative of the lift $\hat{\mu} = [\{\hat{d}_{\alpha\beta}\}] \in H^1(\mathcal{C}_M, \Theta_{\mathcal{C}_M/M}(-D(\tilde{\mathbf{t}}) - D(\tilde{\mathbf{p}})))$ of μ .

Proof. First, we show the equality (33). We can check the equalities

$$\begin{aligned} & (\tilde{\nabla} \circ u_{\alpha\beta}^{\hat{\mu}\nabla_0} - u_{\alpha\beta}^{\hat{\mu}\nabla_0} \circ \tilde{\nabla})(a) \\ &= (\bar{\phi}_\beta^{-1} \otimes \text{id}) \circ \left(d + \tilde{A}_\beta \frac{df_\beta}{f_\beta}\right) \circ \left(\hat{d}_{\alpha\beta} + \left\langle \hat{d}_{\alpha\beta}, \tilde{A}_\beta^0 \frac{dz_\beta}{z_\beta} \right\rangle\right) (\bar{\phi}_\beta(a)) \\ &\quad - (\bar{\phi}_\beta^{-1} \otimes \text{id}) \circ \left(\hat{d}_{\alpha\beta} + \left\langle \hat{d}_{\alpha\beta}, \tilde{A}_\beta^0 \frac{dz_\beta}{z_\beta} \right\rangle\right) \circ \left(d + \tilde{A}_\beta \frac{df_\beta}{f_\beta}\right) (\bar{\phi}_\beta(a)) \\ &= (\bar{\phi}_\beta^{-1} \otimes \text{id}) \circ \left(d \left(\left\langle \hat{d}_{\alpha\beta}, \tilde{A}_\beta^0 \frac{dz_\beta}{z_\beta} - \tilde{A}_\beta \frac{df_\beta}{f_\beta} \right\rangle\right) (\bar{\phi}_\beta(a)) + \left[\tilde{A}_\beta \frac{df_\beta}{f_\beta}, \left\langle \hat{d}_{\alpha\beta}, \tilde{A}_\beta^0 \frac{dz_\beta}{z_\beta} \right\rangle\right] (\bar{\phi}_\beta(a))\right) \end{aligned}$$

and

$$\begin{aligned} & (\tilde{\nabla}_0^{\sigma_M} \circ \hat{u}_{\alpha\beta}^{\text{IMD}} - \hat{u}_{\alpha\beta}^{\text{IMD}} \circ \tilde{\nabla}_0^{\sigma_M})(a) \\ &= (\bar{\phi}_\beta^{-1} \otimes \text{id}) \circ \left(d + \tilde{A}_\beta^0 \frac{dz_\beta}{z_\beta}\right) \circ \left(\hat{d}_{\alpha\beta} + \left\langle \hat{d}_{\alpha\beta}, \tilde{A}_\beta \frac{df_\beta}{f_\beta} \right\rangle\right) (\bar{\phi}_\beta(a)) \\ &\quad - (\bar{\phi}_\beta^{-1} \otimes \text{id}) \circ \left(\hat{d}_{\alpha\beta} + \left\langle \hat{d}_{\alpha\beta}, \tilde{A}_\beta \frac{df_\beta}{f_\beta} \right\rangle\right) \circ \left(d + \tilde{A}_\beta^0 \frac{dz_\beta}{z_\beta}\right) (\bar{\phi}_\beta(a)) \\ &= (\bar{\phi}_\beta^{-1} \otimes \text{id}) \circ \left(d \left(\left\langle \hat{d}_{\alpha\beta}, \tilde{A}_\beta \frac{df_\beta}{f_\beta} - \tilde{A}_\beta^0 \frac{dz_\beta}{z_\beta} \right\rangle\right) (\bar{\phi}_\beta(a)) + \left[\tilde{A}_\beta^0 \frac{dz_\beta}{z_\beta}, \left\langle \hat{d}_{\alpha\beta}, \tilde{A}_\beta \frac{df_\beta}{f_\beta} \right\rangle\right] (\bar{\phi}_\beta(a))\right) \end{aligned}$$

for $a \in \tilde{E}|_{U_{\alpha\beta}}$. Since $\mathcal{C}_M \rightarrow M$ is a family of smooth projective curves, we have $\langle \hat{d}_{\alpha\beta}, df_\beta \rangle dz_\beta = \langle \hat{d}_{\alpha\beta}, dz_\beta \rangle df_\beta$. By the equalities above, we have

$$\tilde{\nabla} \circ u_{\alpha\beta}^{\hat{\mu}\nabla_0} - u_{\alpha\beta}^{\hat{\mu}\nabla_0} \circ \tilde{\nabla} = -(\tilde{\nabla}_0^{\sigma_M} \circ \hat{u}_{\alpha\beta}^{\text{IMD}} - \hat{u}_{\alpha\beta}^{\text{IMD}} \circ \tilde{\nabla}_0^{\sigma_M}).$$

On the other hand, since $v_\alpha^{\hat{\mu}\nabla_0}$ is defined by the infinitesimal deformation of the relative initial connection $\tilde{\nabla}_0^{\sigma_M}$ associated to $(\{\hat{u}_{\alpha\beta}^{\text{IMD}}\}, \{\hat{v}_\alpha^{\text{IMD}}\})$, we can check the equality

$$(34) \quad \tilde{\nabla}_0^{\sigma_M} \circ \hat{u}_{\alpha\beta}^{\text{IMD}} - \hat{u}_{\alpha\beta}^{\text{IMD}} \circ \tilde{\nabla}_0^{\sigma_M} = v_\beta^{\hat{\mu}\nabla_0} - v_\alpha^{\hat{\mu}\nabla_0}.$$

Then we obtain the equality

$$\tilde{\nabla} \circ u_{\alpha\beta}^{\hat{\mu}\nabla_0} - u_{\alpha\beta}^{\hat{\mu}\nabla_0} \circ \tilde{\nabla} = -(v_\beta^{\hat{\mu}\nabla_0} - v_\alpha^{\hat{\mu}\nabla_0}).$$

Next, we show that $v_\alpha^{\hat{\mu}\nabla_0} \in \epsilon \otimes \mathcal{E}nd(\tilde{E}) \otimes \Omega_{\mathcal{C}/T}^1(D(\tilde{\mathbf{t}}))$. For this purpose, we show that $\{v_\alpha^{\hat{\mu}\nabla_0}\}$ has no pole on the supports of $D(m)$ and $D(\sigma_M)$. Let $\sigma_\alpha: U_\alpha^\epsilon \rightarrow U_\alpha \times \text{Spec } \mathbb{C}[\epsilon]$ be an isomorphism such that the isomorphism σ_α preserves the divisor $D_\epsilon(\tilde{\mathbf{p}}) \cap U_\alpha^\epsilon$ and $(D(\tilde{\mathbf{p}}) \cap U_\alpha) \times \text{Spec } \mathbb{C}[\epsilon]$ and the isomorphisms $\{\sigma_\alpha\}_\alpha$ give a representative of $\hat{\mu}$ by $\hat{d}_{\alpha\beta} = (\sigma_\alpha^*)^{-1} \sigma_\beta^* - \text{id}: \mathcal{O}_{U_{\alpha\beta}} \rightarrow \epsilon \otimes \mathcal{O}_{U_{\alpha\beta}}$. We set $U_{I_D(\tilde{\mathbf{p}})} := \bigcup_{\alpha \in I_D(\tilde{\mathbf{p}})} U_\alpha$. We consider a subset $(\text{IM}_M^\mu)^{-1}(U_{I_D(\tilde{\mathbf{p}})}) \subset \mathcal{C}_\epsilon$ as the tuple of open subsets $\{U_\alpha \times \text{Spec } \mathbb{C}[\epsilon]\}_{\alpha \in I_D(\tilde{\mathbf{p}})}$ and gluing data $\{(\sigma_\alpha^*)^{-1} \sigma_\beta^* = \text{id} + \hat{d}_{\alpha\beta}\}_{\alpha, \beta \in I_D(\tilde{\mathbf{p}})}$:

$$(\text{IM}_M^\mu)^{-1}(U_{I_D(\tilde{\mathbf{p}})}) = (\{U_\alpha \times \text{Spec } \mathbb{C}[\epsilon]\}_{\alpha \in I_D(\tilde{\mathbf{p}})}, \{\text{id} + \hat{d}_{\alpha\beta}\}_{\alpha, \beta \in I_D(\tilde{\mathbf{p}})}).$$

By the definition of the lift $\hat{\mu}$, we have an isomorphism

$$(\text{IM}_M^\mu)^{-1}(U_{I_D(\tilde{\mathbf{p}})}) \cong (\{U_\alpha \times \text{Spec } \mathbb{C}[\epsilon]\}_{\alpha \in I_D(\tilde{\mathbf{p}})}, \{\text{id}\}_{\alpha, \beta \in I_D(\tilde{\mathbf{p}})})$$

such that this isomorphism preserves the divisor $D_\epsilon(\tilde{\mathbf{p}}) \cap (\text{IM}_M^\mu)^{-1}(U_{I_D(\tilde{\mathbf{p}})})$ and $(D(\tilde{\mathbf{p}}) \cap U_{I_D(\tilde{\mathbf{p}})}) \times \text{Spec } \mathbb{C}[\epsilon]$. That is, there exists a set $\{\hat{d}_\alpha\}_{\alpha \in I_D(\tilde{\mathbf{p}})}$ such that $\hat{d}_\alpha \in \Theta_{\mathcal{C}_M/M}(-D(\tilde{\mathbf{t}}) - D(\tilde{\mathbf{p}}))(U_\alpha)$ and $\hat{d}_{\alpha\beta} = \hat{d}_\beta - \hat{d}_\alpha$ for $\alpha, \beta \in I_D(\tilde{\mathbf{p}})$. We consider a vector bundle $E_\epsilon|_{(\text{IM}_M^\mu)^{-1}(U_{I_D(\tilde{\mathbf{p}})})}$ as a tuple of an open covering $\{U_\alpha \times \text{Spec } \mathbb{C}[\epsilon]\}_{\alpha \in I_D(\tilde{\mathbf{p}})}$ and gluing data $\{g_{\alpha\beta}\}_{\alpha, \beta \in I_D(\tilde{\mathbf{p}})}$, where

$$g_{\alpha\beta} := (\bar{\phi}_\alpha \otimes 1) \circ (\text{id} + \hat{u}_{\alpha\beta}^{\text{IMD}}) \circ (\bar{\phi}_\beta \otimes 1)^{-1}.$$

That is, $E_\epsilon|_{(\mathrm{IM}_M^\mu)^{-1}(U_{I_D(\tilde{\mathbf{p}})})} = (\{U_\alpha \times \mathrm{Spec} \mathbb{C}[\epsilon]\}_{\alpha \in I_D(\tilde{\mathbf{p}})}, \{g_{\alpha\beta}\}_{\alpha, \beta \in I_D(\tilde{\mathbf{p}})})$. Set

$$u_\alpha^{\hat{d}_\alpha \nabla} := \bar{\phi}_\alpha^{-1} \circ \left(\hat{d}_\alpha + \left\langle \hat{d}_\alpha, \tilde{A}_\alpha \frac{df_\alpha}{f_\alpha} \right\rangle \right) \circ \bar{\phi}_\alpha$$

for $\alpha \in I_D(\tilde{\mathbf{p}})$. Since $\hat{u}_{\alpha\beta}^{\mathrm{IMD}} = u_\beta^{\hat{d}_\beta \nabla} - u_\alpha^{\hat{d}_\alpha \nabla}$, we have an isomorphism

$$\phi_{I_D(\tilde{\mathbf{p}})} : E_\epsilon|_{(\mathrm{IM}_M^\mu)^{-1}(U_{I_D(\tilde{\mathbf{p}})})} \xrightarrow{\cong} (\{U_\alpha \times \mathrm{Spec} \mathbb{C}[\epsilon]\}_{\alpha \in I_D(\tilde{\mathbf{p}})}, \{(\bar{\phi}_\alpha \otimes 1) \circ (\bar{\phi}_\beta \otimes 1)^{-1}\}_{\alpha, \beta \in I_D(\tilde{\mathbf{p}})}).$$

That is, $\phi_{I_D(\tilde{\mathbf{p}})}$ is an isomorphism from $E_\epsilon|_{(\mathrm{IM}_M^\mu)^{-1}(U_{I_D(\tilde{\mathbf{p}})})}$ to the trivial deformation of $\tilde{E}|_{U_{I_D(\tilde{\mathbf{p}})}}$. Now we consider a connection

$$(35) \quad (\phi_{I_D(\tilde{\mathbf{p}})}^{-1})^* (\nabla_{0, \epsilon}^{\sigma_M, \mathrm{IMD}}|_{(\mathrm{IM}_M^\mu)^{-1}(U_{I_D(\tilde{\mathbf{p}})})}).$$

Let $\tilde{A}_\alpha^0 \frac{dz_\alpha}{z_\alpha}$ be the connection matrix of $\tilde{\nabla}_0^{\sigma_M}$ associated to the trivialization $\bar{\phi}_\alpha$. The connection $\nabla_{0, \epsilon}^{\sigma_M, \mathrm{IMD}}$ is defined by the injection $\sigma_M : \mathcal{O}_{\mathcal{C}_M}(-m) \hookrightarrow \tilde{E}_{\mathcal{C}_M}$ and the injection $\mathcal{O}_{\mathcal{C}_M}(-m) \hookrightarrow \mathcal{O}_{\mathcal{C}_M}$, which are defined in Section 4.1. We restrict the injections $\sigma_M : \mathcal{O}_{\mathcal{C}_M}^{\oplus r}(-m) \hookrightarrow \tilde{E}_{\mathcal{C}_M}$ and $\mathcal{O}_{\mathcal{C}_M}(-m) \hookrightarrow \mathcal{O}_{\mathcal{C}_M}$ to the open set $U_{I_D(\tilde{\mathbf{p}})}$. The morphism $\mathrm{IM}_M^\mu : \mathcal{C}_\epsilon \rightarrow \mathcal{C}_M$ and the restricted injections induce injections over $(\mathrm{IM}_M^\mu)^{-1}(U_{I_D(\tilde{\mathbf{p}})})$. The images of these induced injections under the isomorphism $\phi_{I_D(\tilde{\mathbf{p}})}$ are independent of ϵ . Then we have that the connection matrix of the connection (35) on $U_\alpha \times \mathrm{Spec} \mathbb{C}[\epsilon]$ by the trivialization $(\phi_{I_D(\tilde{\mathbf{p}})}^{-1})^*(\phi_\alpha)$ is just $\tilde{A}_\alpha^0 \frac{dz_\alpha}{z_\alpha}$. That is, the ϵ -term of this connection matrix of the connection (35) vanishes. Since this ϵ -term vanishes, we have

$$v_\alpha^{\hat{u}_\alpha \nabla_0} = \tilde{\nabla}_0^{\sigma_M} \circ u_\alpha^{\hat{d}_\alpha \nabla} - u_\alpha^{\hat{d}_\alpha \nabla} \circ \tilde{\nabla}_0^{\sigma_M}.$$

We may check this equality as in (8) and (9). Since \hat{d}_α vanishes at the support of $D(\tilde{\mathbf{p}})$, $u_\alpha^{\hat{d}_\alpha \nabla}$ also vanishes at the support of $D(\tilde{\mathbf{p}})$. This fact means that $\{v_\alpha^{\hat{u}_\alpha \nabla_0}\}$ has no pole on the supports of $D(m)$ and $D(\sigma_M)$. Since $v_\alpha^{\hat{u}_\alpha \nabla_0}$ has no poles along $D(\tilde{\mathbf{t}})$, we have the compatibility of $v_\alpha^{\hat{u}_\alpha \nabla_0}$ with the parabolic structures $\{\tilde{l}_j^{(i)}\}$. Then we have $v_\alpha^{\hat{u}_\alpha \nabla_0} \in \epsilon \otimes \mathcal{G}_M^1$. So $[(\{u_\alpha^{\hat{u}_\alpha \nabla_0}\}, \{-v_\alpha^{\hat{u}_\alpha \nabla_0}\})]$ is an element of $\mathbf{H}^1(\mathcal{G}_M^\bullet)$.

Let d_α be an element of $\Theta_{\mathcal{C}_M/M}(-D(\tilde{\mathbf{t}}) - D(\tilde{\mathbf{p}}))(U_\alpha)$. Set $\hat{d}'_{\alpha\beta} := \hat{d}_{\alpha\beta} + d_\beta - d_\alpha$ and

$$u_\alpha^{d_\alpha \nabla_0} := \bar{\phi}_\alpha^{-1} \circ \left(d_\alpha + \left\langle d_\alpha, \tilde{A}_\alpha^0 \frac{dz_\alpha}{z_\alpha} \right\rangle \right) \circ \bar{\phi}_\alpha, \text{ and } u_\alpha^{d_\alpha \nabla} := \bar{\phi}_\alpha^{-1} \circ \left(d_\alpha + \left\langle d_\alpha, \tilde{A}_\alpha \frac{df_\alpha}{f_\alpha} \right\rangle \right) \circ \bar{\phi}_\alpha.$$

For the representative $\{\hat{d}'_{\alpha\beta}\}$, we define $'u_{\alpha\beta}^{\hat{u}_\alpha \nabla_0}$ and $(\{'\hat{u}_{\alpha\beta}^{\mathrm{IMD}}\}, \{'\hat{v}_\alpha^{\mathrm{IMD}}\})$ as (29) and (30), respectively. We can define the infinitesimal deformation of the relative initial connection $\tilde{\nabla}_0^{\sigma_M}$ along the vector field $(\{'\hat{u}_{\alpha\beta}^{\mathrm{IMD}}\}, \{'\hat{v}_\alpha^{\mathrm{IMD}}\})$ as Definition 4.6. By this infinitesimal deformation, we can define $'v_\alpha^{\hat{u}_\alpha \nabla_0}$ as (32). We can check the following equalities:

$$\begin{aligned} 'u_{\alpha\beta}^{\hat{u}_\alpha \nabla_0} &= u_{\alpha\beta}^{\hat{u}_\alpha \nabla_0} + \left(u_\beta^{d_\beta \nabla_0} - u_\alpha^{d_\alpha \nabla_0} \right) \text{ and} \\ 'v_\alpha^{\hat{u}_\alpha \nabla_0} &= v_\alpha^{\hat{u}_\alpha \nabla_0} + \left(\tilde{\nabla}_0^{\sigma_M} \circ u_\alpha^{d_\alpha \nabla} - u_\alpha^{d_\alpha \nabla} \circ \tilde{\nabla}_0^{\sigma_M} \right) = v_\alpha^{\hat{u}_\alpha \nabla_0} - \left(\tilde{\nabla} \circ u_\alpha^{d_\alpha \nabla_0} - u_\alpha^{d_\alpha \nabla_0} \circ \tilde{\nabla} \right). \end{aligned}$$

Note that $d_\alpha + \left\langle d_\alpha, \tilde{A}_\alpha^0 \frac{dz_\alpha}{z_\alpha} \right\rangle$ has no pole at the support of $D(\tilde{\mathbf{p}})$. Moreover for any $a \in \tilde{E}|_{U_\alpha}$,

$$\begin{aligned} & \left(\tilde{\nabla} \circ u_\alpha^{d_\alpha \nabla_0} - u_\alpha^{d_\alpha \nabla_0} \circ \tilde{\nabla} \right) (a) \\ &= (\bar{\phi}_\alpha^{-1} \otimes \mathrm{id}) \circ \left(d \left(\left\langle \hat{d}_\alpha, \tilde{A}_\alpha^0 \frac{dz_\alpha}{z_\alpha} - \tilde{A}_\alpha \frac{df_\alpha}{f_\alpha} \right\rangle \right) (\bar{\phi}_\alpha(a)) + \left[\tilde{A}_\alpha \frac{df_\alpha}{f_\alpha}, \left\langle \hat{d}_\alpha, \tilde{A}_\alpha^0 \frac{dz_\alpha}{z_\alpha} \right\rangle \right] (\bar{\phi}_\alpha(a)) \right) \end{aligned}$$

has no pole at the support of $D(\tilde{\mathbf{p}})$. Then the class $[(\{'\hat{u}_{\alpha\beta}^{\hat{u}_\alpha \nabla_0}\}, \{-'v_\alpha^{\hat{u}_\alpha \nabla_0}\})]$ coincides with the class $[(\{u_{\alpha\beta}^{\hat{u}_\alpha \nabla_0}\}, \{-v_\alpha^{\hat{u}_\alpha \nabla_0}\})]$ in $\mathbf{H}^1(\mathcal{G}_M^\bullet)$. So the class $[(\{u_{\alpha\beta}^{\hat{u}_\alpha \nabla_0}\}, \{-v_\alpha^{\hat{u}_\alpha \nabla_0}\})]$ is independent of the choice of a representative of the class $[\{\hat{d}_{\alpha\beta}\}] \in H^1(\mathcal{C}_M, \Theta_{\mathcal{C}_M/M}(-D(\tilde{\mathbf{t}}) - D(\tilde{\mathbf{p}})))$. \square

Now we consider meaning of the vector field $[(\{u_{\alpha\beta}^{\hat{u}_\alpha \nabla_0}\}_{\alpha\beta}, \{-v_\alpha^{\hat{u}_\alpha \nabla_0}\}_\alpha)] \in \mathbf{H}^1(\mathcal{G}_M^\bullet)$ by using the morphisms

$$h_{\nabla - \nabla_0} : M \longrightarrow M'_H \quad \text{and} \quad h_{\nabla_0} : M \longrightarrow M'$$

(see (25) and (26)). In some sense, we define tangent sheaves $\Theta_{M'_H}$ and $\Theta_{M'}$ of the moduli spaces M'_H and M' , respectively. Let $v_1 \in h_{\tilde{\nabla}-\nabla_0}^* \Theta_{M'_H}$ be the image of the vector field of isomonodromic deformations of $\tilde{\nabla}$ associated to μ on M under the morphism $\Theta_M \rightarrow h_{\tilde{\nabla}-\nabla_0}^* \Theta_{M'_H}$. First, we consider the infinitesimal deformation of $\tilde{\nabla} - \tilde{\nabla}_0^{\sigma_M}$ on $M[\epsilon]$ corresponding to v_1 . Recall that $\{v_\alpha^{\hat{\mu}\nabla_0}\}_\alpha$ is defined by $\nabla_{0,\epsilon}^{\sigma_M, \text{IMD}}$, which is the pull-back of $\nabla_0^{\sigma_M}$ under the morphism $\text{IM}_M^\mu: \mathcal{C}_\epsilon \rightarrow \mathcal{C}_M$. Then the infinitesimal deformation of $\tilde{\nabla} - \tilde{\nabla}_0^{\sigma_M}$ on $M[\epsilon]$ corresponding to v_1 is $(\text{IM}_M^\mu)^* \tilde{\nabla} - \nabla_{0,\epsilon}^{\sigma_M, \text{IMD}}$. By $\hat{v}_\alpha^{\text{IMD}} = 0$ in (30) and the definition (32) of $v_\alpha^{\hat{\mu}\nabla_0}$, we have the following equality

$$(\varphi_\alpha^{-1} \otimes \text{id}) \circ ((\text{IM}_M^\mu)^* \tilde{\nabla} - \nabla_{0,\epsilon}^{\sigma_M, \text{IMD}})|_{U_\epsilon} \circ \varphi_\alpha - (\tilde{\nabla} - \tilde{\nabla}_0^{\sigma_M}) = -v_\alpha^{\hat{\mu}\nabla_0}.$$

Roughly speaking, the pair $(\{\hat{u}_{\alpha\beta}^{\text{IMD}}\}_{\alpha\beta}, \{-v_\alpha^{\hat{\mu}\nabla_0}\}_\alpha)$ means the infinitesimal deformation of $(\tilde{E}, \tilde{\nabla} - \tilde{\nabla}_0^{\sigma_M}, \mathbf{l}^{\sigma_M} \cup \{\tilde{l}_*^{(i)}\})$ corresponding to the pushforward v_1 . Second, we will define $v_2 \in h_{\tilde{\nabla}-\nabla_0}^* \Theta_{M'_H}$ later at (36) and we consider the difference $v_1 - v_2$ (see (37) below). We have the following equality

$$\hat{u}_{\alpha\beta}^{\text{IMD}} - u_{\alpha\beta}^{\hat{\mu}\nabla_0} = \bar{\phi}_\beta^{-1} \circ \left(\left\langle \hat{d}_{\alpha\beta}, \tilde{A}_\beta \frac{df_\beta}{f_\beta} - \tilde{A}_\beta^0 \frac{dz_\beta}{z_\beta} \right\rangle \right) \circ \bar{\phi}_\beta.$$

Here $\tilde{A}_\beta \frac{df_\beta}{f_\beta}$ and $\tilde{A}_\beta^0 \frac{dz_\beta}{z_\beta}$ are the connection matrices of $\tilde{\nabla}$ and $\tilde{\nabla}_0^{\sigma_M}$ associated to the trivialization $\bar{\phi}_\beta$, respectively. Then we have $[\tilde{\nabla} - \tilde{\nabla}_0^{\sigma_M}, \hat{u}_{\alpha\beta}^{\text{IMD}} - u_{\alpha\beta}^{\hat{\mu}\nabla_0}] = 0$. Here $[\cdot, \cdot]$ is the commutator. This equality means that the pair

$$(36) \quad (\{\hat{u}_{\alpha\beta}^{\text{IMD}} - u_{\alpha\beta}^{\hat{\mu}\nabla_0}\}_{\alpha\beta}, \{0\}_\alpha)$$

gives an infinitesimal deformation of $(\tilde{E}, \tilde{\nabla} - \tilde{\nabla}_0^{\sigma_M}, \mathbf{l}^{\sigma_M} \cup \{\tilde{l}_*^{(i)}\})$ parametrized by $M[\epsilon]$. We denote by v_2 the corresponding vector field. We consider the difference $v_1 - v_2$, which is the image of the vector field of isomonodromic deformations of $\tilde{\nabla}$ associated to μ under the following composition:

$$\begin{array}{ccc} \Theta_M & \longrightarrow & h_{\tilde{\nabla}-\nabla_0}^* \Theta_{M'_H} \\ & \searrow & \downarrow \text{adding } -v_2 \\ & & h_{\tilde{\nabla}-\nabla_0}^* \Theta_{M'_H}. \end{array}$$

The difference $v_1 - v_2$ is described by

$$(37) \quad (\{u_{\alpha\beta}^{\hat{\mu}\nabla_0}\}_{\alpha\beta}, \{-v_\alpha^{\hat{\mu}\nabla_0}\}_\alpha),$$

which gives an infinitesimal deformation of $(\tilde{E}, \tilde{\nabla} - \tilde{\nabla}_0^{\sigma_M}, \mathbf{l}^{\sigma_M} \cup \{\tilde{l}_*^{(i)}\})$ parametrized by $M[\epsilon]$. Third, we construct an infinitesimal deformation of $\tilde{\nabla} = \tilde{\nabla}_0^{\sigma_M} + (\tilde{\nabla} - \tilde{\nabla}_0^{\sigma_M})$ by the infinitesimal deformation of $\tilde{\nabla}_0^{\sigma_M}$ corresponding to the isomonodromic deformation of $\tilde{\nabla}_0^{\sigma_M}$ and the infinitesimal deformation of $\tilde{\nabla} - \tilde{\nabla}_0^{\sigma_M}$ corresponding to $v_1 - v_2$. For this purpose, we consider the following commutative diagram:

$$(38) \quad \begin{array}{ccc} M' \times_{\text{par-Bun}} M'_H & \xrightarrow{\text{Add}} & M' \\ & \searrow (h_{\nabla_0}, h_{\nabla-\nabla_0}) & \uparrow h_\nabla \\ & & M, \end{array}$$

where par-Bun is the moduli space of parabolic bundles (in some sense),

$$\begin{aligned} (h_{\nabla_0}, h_{\nabla-\nabla_0}): (E, \tilde{\nabla}, \{\tilde{l}_*^{(i)}\}; \tilde{\nabla}_0^{\sigma_M}, \mathbf{l}^{\sigma_M}) &\longmapsto ((\tilde{E}, \mathbf{l}^{\sigma_M} \cup \{\tilde{l}_*^{(i)}\}), \tilde{\nabla}_0^{\sigma_M}, \tilde{\nabla} - \tilde{\nabla}_0^{\sigma_M}), \\ h_\nabla: (E, \tilde{\nabla}, \{\tilde{l}_*^{(i)}\}; \tilde{\nabla}_0^{\sigma_M}, \mathbf{l}^{\sigma_M}) &\longmapsto (E, \tilde{\nabla}, \mathbf{l}^{\sigma_M} \cup \{\tilde{l}_*^{(i)}\}), \end{aligned}$$

and

$$\begin{aligned} \text{Add}((\tilde{E}, \mathbf{l}^{\sigma_M} \cup \{\tilde{l}_*^{(i)}\}), \tilde{\nabla}_0^{\sigma_M}, \tilde{\nabla} - \tilde{\nabla}_0^{\sigma_M}) &= (E, \tilde{\nabla}_0^{\sigma_M} + (\tilde{\nabla} - \tilde{\nabla}_0^{\sigma_M}), \mathbf{l}^{\sigma_M} \cup \{\tilde{l}_*^{(i)}\}) \\ &= (E, \tilde{\nabla}, \mathbf{l}^{\sigma_M} \cup \{\tilde{l}_*^{(i)}\}). \end{aligned}$$

We have a lift $M[\epsilon] \rightarrow M' \times_{\text{par-Bun}} M'_H$ of $(h_{\nabla_0}, h_{\nabla-\nabla_0}): M \rightarrow M' \times_{\text{par-Bun}} M'_H$ by the infinitesimal deformation of $\tilde{\nabla}_0^{\sigma_M}$ corresponding to the isomonodromic deformation of $\tilde{\nabla}_0^{\sigma_M}$ and the infinitesimal deformation of $\tilde{\nabla} - \tilde{\nabla}_0^{\sigma_M}$ corresponding to $v_1 - v_2$. By the diagram (38), we have a lift $M[\epsilon] \rightarrow M'$

of $h_{\nabla}: M \rightarrow M'$. The desired infinitesimal deformation of $\tilde{\nabla} = \tilde{\nabla}_0^{\sigma_M} + (\tilde{\nabla} - \tilde{\nabla}_0^{\sigma_M})$ just corresponds to this lift $M[\epsilon] \rightarrow M'$. We describe this infinitesimal deformation of $\tilde{\nabla} = \tilde{\nabla}_0^{\sigma_M} + (\tilde{\nabla} - \tilde{\nabla}_0^{\sigma_M})$ by the Čech cohomology as follows (The conclusion is (40) below). A description of the infinitesimal deformation of $\tilde{\nabla} - \tilde{\nabla}_0^{\sigma_M}$ corresponding to $v_1 - v_2$ by the Čech cohomology is (37). We consider the infinitesimal deformation of $\tilde{\nabla}_0^{\sigma_M}$ corresponding to the isomonodromic deformation of $\tilde{\nabla}_0^{\sigma_M}$. We will give a description of this infinitesimal deformation of $\tilde{\nabla}_0^{\sigma_M}$ by the Čech cohomology ((39) below). As in Section 3.3, we may define the vector field of the isomonodromic deformation for $\varpi': M' \rightarrow \mathcal{M}_{g,n+N}$. That is, we define a splitting $(\varpi')^* \Theta_{\mathcal{M}_{g,n+N}} \rightarrow \Theta_{M'}$. By h_{∇_0} , we have a morphism $h_{\nabla_0}^* (\varpi')^* \Theta_{\mathcal{M}_{g,n+N}} \rightarrow h_{\nabla_0}^* \Theta_{M'}$. The lift $\hat{\mu}$ of μ gives a section of $h_{\nabla_0}^* (\varpi')^* \Theta_{\mathcal{M}_{g,n+N}}$. We take the image of $\hat{\mu}$ by this morphism $h_{\nabla_0}^* (\varpi')^* \Theta_{\mathcal{M}_{g,n+N}} \rightarrow h_{\nabla_0}^* \Theta_{M'}$. This image gives an infinitesimal deformation of $\tilde{\nabla}_0^{\sigma_M}$ parametrized by $M[\epsilon]$. By this infinitesimal deformation of $\tilde{\nabla}_0^{\sigma_M}$, we have a pair

$$(39) \quad (\{u_{\alpha\beta}^{\hat{\mu}\nabla_0}\}_{\alpha\beta}, \{0\}_{\alpha})$$

as Proposition 3.8. The first components of the pairs (37) and (39) mean infinitesimal deformations of the underlying parabolic bundles. The pairs (37) and (39) have same first components. Then we can add the infinitesimal deformation of $\tilde{\nabla}_0^{\sigma_M}$ corresponding to the pair (39) to the infinitesimal deformation of $\tilde{\nabla} - \tilde{\nabla}_0^{\sigma_M}$ corresponding to the pair (37). By this addition, we have a pair

$$(40) \quad (\{u_{\alpha\beta}^{\hat{\mu}\nabla_0}\}_{\alpha\beta}, \{-v_{\alpha}^{\hat{\mu}\nabla_0}\}_{\alpha}).$$

This pair corresponds to the lift $M[\epsilon] \rightarrow M'$. By this infinitesimal deformation of $\tilde{\nabla}$ and forgetting the parabolic structure \mathbf{l}^{σ_M} , we have a morphism $M[\epsilon] \rightarrow M$. Here this morphism is given by the universal property of the moduli space M . Finally, this morphism $M[\epsilon] \rightarrow M$ corresponds to the vector field

$$[(\{u_{\alpha\beta}^{\hat{\mu}\nabla_0}\}_{\alpha\beta}, \{-v_{\alpha}^{\hat{\mu}\nabla_0}\}_{\alpha})] \in \mathbf{H}^1(\mathcal{G}_M^{\bullet}).$$

4.3. Trivializations of the infinitesimal deformation of \tilde{E} given by the infinitesimal deformation of σ_M . We take any $v \in \Theta_M$. For the vector field v , we have a morphism $f_v: M[\epsilon] \rightarrow M$. Put $\mathcal{C}_{\epsilon}^v := \mathcal{C}_M \times_M M[\epsilon]$ given by the family $\mathcal{C}_M \rightarrow M$ and $f_v: M[\epsilon] \rightarrow M$. We denote by $F_v: \mathcal{C}_{\epsilon}^v \rightarrow \mathcal{C}_M$ the natural morphism from \mathcal{C}_{ϵ}^v to \mathcal{C}_M . For $\text{symb}_1(v) \in H^1(\mathcal{C}_M, \Theta_{\mathcal{C}_M/M}(-D(\tilde{\mathbf{t}})))$, we take the lift of $\text{symb}_1(v)$ in $H^1(\mathcal{C}_M, \Theta_{\mathcal{C}_M/M}(-D(\tilde{\mathbf{t}}) - D(\tilde{\mathbf{p}})))$ defined by the family $(\mathcal{C}_{\epsilon}^v, \text{Supp}(F_v^*(D(\tilde{\mathbf{t}}) + D(\tilde{\mathbf{p}}))))$. Let $\{U_{\alpha}\}_{\alpha}$ be an affine open covering of \mathcal{C}_M . Set $U_{\alpha}^{\epsilon} := F_v^{-1}(U_{\alpha})$. Let $\sigma_{\alpha}: U_{\alpha}^{\epsilon} \rightarrow U_{\alpha} \times \text{Spec } \mathbb{C}[\epsilon]$ be an isomorphism corresponding to the lift. The infinitesimal deformation $(F_v^* \tilde{E}, F_v^* \tilde{\nabla}_0^{\sigma_M})$ is defined by taking the pull-back of $(\tilde{E}, \tilde{\nabla}_0^{\sigma_M})$ under the morphism $F_v: \mathcal{C}_{\epsilon}^v \rightarrow \mathcal{C}_M$. Let $\sigma_M: \mathcal{O}_{\mathcal{C}_M}^{\oplus r} \rightarrow \tilde{E}(m)|_{\mathcal{C}_M}$ be the injection in Proposition 4.1. Let $F_v^* \sigma_M$ be the pull-back of $\sigma_M \in \text{Hom}(\mathcal{O}_{\mathcal{C}_M}^{\oplus r}(-m), \tilde{E}|_{\mathcal{C}_M})$ under the morphism $F_v: \mathcal{C}_{\epsilon}^v \rightarrow \mathcal{C}_M$:

$$(41) \quad F_v^* \sigma_M: \mathcal{O}_{\mathcal{C}_{\epsilon}^v}^{\oplus r}(-m) \longrightarrow F_v^* \tilde{E}.$$

In this section we will define trivializations of $F_v^* \tilde{E}$ which are compatible with $F_v^* \sigma_M$. We will use these trivializations in the proof of Lemma 4.14. Let $D_{\epsilon}(m) = \{(U_{\alpha}^{\epsilon}, g_{\alpha}^{\epsilon})\}_{\alpha}$ and $D_{\epsilon}(\sigma) = \{(U_{\alpha}^{\epsilon}, h_{\alpha}^{\epsilon})\}_{\alpha}$ be the pull-backs of the Cartier divisors $D(m) = \{(U_{\alpha}, g_{\alpha})\}_{\alpha}$ and $D(\sigma) = \{(U_{\alpha}, h_{\alpha})\}_{\alpha}$ by the morphism $F_v: \mathcal{C}_{\epsilon}^v \rightarrow \mathcal{C}_M$, respectively.

We define trivializations $\{(U_{\alpha}^{\epsilon}, \psi_{\alpha})\}_{\alpha}$ of $F_v^* \tilde{E}$ as follows. First, we consider an affine open subset U_{α} such that the intersection of U_{α} and the support of $D(\tilde{\mathbf{p}})$ is empty. On such an affine open subset U_{α}^{ϵ} , the natural injection $\mathcal{O}_{U_{\alpha}^{\epsilon}}^{\oplus r}(-m) \rightarrow \mathcal{O}_{U_{\alpha}^{\epsilon}}^{\oplus r}$ and $F_v^* \sigma_M: \mathcal{O}_{U_{\alpha}^{\epsilon}}^{\oplus r}(-m) \rightarrow F_v^* \tilde{E}|_{U_{\alpha}^{\epsilon}}$ are isomorphisms. Then the morphisms

$$\mathcal{O}_{U_{\alpha}^{\epsilon}}^{\oplus r} \xleftarrow{\cong} \mathcal{O}_{U_{\alpha}^{\epsilon}}^{\oplus r}(-m) \xrightarrow[\cong]{F_v^* \sigma_M} F_v^* \tilde{E}|_{U_{\alpha}^{\epsilon}}$$

give a trivialization of $F_v^* \tilde{E}|_{U_{\alpha}^{\epsilon}}$. We denote by $(U_{\alpha}^{\epsilon}, \psi_{\alpha})$ this trivialization of $F_v^* \tilde{E}|_{U_{\alpha}^{\epsilon}}$. Second, we consider an affine open set U_{α}^{ϵ} such that $U_{\alpha}^{\epsilon} \cap \text{Supp}(D_{\epsilon}(m)) \neq \emptyset$ and $U_{\alpha}^{\epsilon} \cap \text{Supp}(D_{\epsilon}(\sigma_M)) = \emptyset$. On such an affine open subset U_{α}^{ϵ} , the composition

$$\mathcal{O}_{U_{\alpha}^{\epsilon}}^{\oplus r} \xrightarrow[\cong]{g_{\alpha}^{\epsilon}} \mathcal{O}_{U_{\alpha}^{\epsilon}}^{\oplus r}(-m) \xrightarrow[\cong]{F_v^* \sigma_M} F_v^* \tilde{E}|_{U_{\alpha}^{\epsilon}}$$

is an isomorphism. This isomorphism gives a trivialization of $F_v^* \tilde{E}|_{U_\alpha^\epsilon}$. We denote by $(U_\alpha^\epsilon, \psi_\alpha)$ this trivialization of $F_v^* \tilde{E}|_{U_\alpha^\epsilon}$. Third, we consider an affine open set U_α^ϵ where $U_\alpha^\epsilon \cap \text{Supp}(D_\epsilon(m)) = \emptyset$ and $U_\alpha^\epsilon \cap \text{Supp}(D_\epsilon(\sigma_M)) \neq \emptyset$. Set $(\tilde{\mathbf{p}}_{i''})^\epsilon := \tilde{\mathbf{p}}_{i''}^\epsilon \times_M \text{Spec } \mathcal{O}_M[\epsilon]$. Here $\{\tilde{\mathbf{p}}_1, \dots, \tilde{\mathbf{p}}_{N_2}\}$ are the components of the support of the divisor $D(\sigma_M)$, where $N_2 := \deg D(\sigma_M)|_{\mathcal{C}_x}$ for each $x \in M$. We put $\mathbf{s}_{\alpha, j}^\epsilon := (F_v^* \sigma_M)|_{U_\alpha^\epsilon}(e_j \otimes g_\alpha^\epsilon)$ for $j = 1, \dots, r$. We change the order of $(\mathbf{s}_{\alpha, 1}^\epsilon, \mathbf{s}_{\alpha, 2}^\epsilon, \dots, \mathbf{s}_{\alpha, r}^\epsilon)$ as follows. For each i'' , we set $\mathbf{s}_{\alpha, i'', k}^\epsilon := \mathbf{s}_{\alpha, j(i'', k)}^\epsilon$ where $k = 1, 2, \dots, r$ and $\{j(i'', 1), \dots, j(i'', r)\} = \{1, 2, \dots, r\}$. We assume that $\mathbf{s}_{\alpha, i'', 2}^\epsilon|_{(\tilde{\mathbf{p}}_{i''})^\epsilon}, \dots, \mathbf{s}_{\alpha, i'', r}^\epsilon|_{(\tilde{\mathbf{p}}_{i''})^\epsilon}$ are linearly independent. Let $a_{\alpha, i'', 2}^\epsilon, \dots, a_{\alpha, i'', r}^\epsilon$ be elements of $\pi_{M[\epsilon]}^*(\mathcal{O}_{M[\epsilon]})|_{U_\alpha^\epsilon}$ such that

$$\mathbf{s}_{\alpha, i'', 1}^\epsilon|_{(\tilde{\mathbf{p}}_{i''})^\epsilon} = a_{\alpha, i'', 2}^\epsilon \cdot \mathbf{s}_{\alpha, i'', 2}^\epsilon|_{(\tilde{\mathbf{p}}_{i''})^\epsilon} + \dots + a_{\alpha, i'', r}^\epsilon \cdot \mathbf{s}_{\alpha, i'', r}^\epsilon|_{(\tilde{\mathbf{p}}_{i''})^\epsilon}.$$

Here $\pi_{M[\epsilon]}: \mathcal{C}_\epsilon^v \rightarrow M[\epsilon]$ is the projection. We define $\hat{\mathbf{s}}_{\alpha, i'', 1}^\epsilon$ as

$$\mathbf{s}_{\alpha, i'', 1}^\epsilon = h_\alpha^\epsilon \cdot \hat{\mathbf{s}}_{\alpha, i'', 1}^\epsilon + a_{\alpha, i'', 2}^\epsilon \cdot \mathbf{s}_{\alpha, i'', 2}^\epsilon + \dots + a_{\alpha, i'', r}^\epsilon \cdot \mathbf{s}_{\alpha, i'', r}^\epsilon.$$

The tuple $\hat{\mathbf{s}}_{\alpha, i''}^\epsilon := (\hat{\mathbf{s}}_{\alpha, i'', 1}^\epsilon, \mathbf{s}_{\alpha, i'', 2}^\epsilon, \dots, \mathbf{s}_{\alpha, i'', r}^\epsilon)$ gives a trivialization of $F_v^* \tilde{E}|_{U_\alpha^\epsilon}$. We denote by $(U_\alpha^\epsilon, \psi_\alpha)$ this trivialization of $F_v^* \tilde{E}|_{U_\alpha^\epsilon}$.

Definition 4.8. We say trivializations $\{(U_\alpha^\epsilon, \psi_\alpha)\}_\alpha$ of $F_v^* \tilde{E}$ are *compatible with $F_v^* \sigma_M$* if $\{(U_\alpha^\epsilon, \psi_\alpha)\}_\alpha$ are the trivializations constructed above for any α .

We set $\bar{\psi}_\alpha := \psi_\alpha \pmod{\epsilon}$. For any α , we define $\varphi'_\alpha: \tilde{E}|_{U_\alpha} \rightarrow F_v^* \tilde{E}|_{U_\alpha^\epsilon}$ by the composition

$$(42) \quad \varphi'_\alpha: \tilde{E}|_{U_\alpha} \otimes_{\mathbb{C}} \mathbb{C}[\epsilon] \xrightarrow{\bar{\psi}_\alpha \otimes 1} \mathcal{O}_{U_\alpha}^{\oplus r} \otimes_{\mathbb{C}} \mathbb{C}[\epsilon] = \mathcal{O}_{U_\alpha}^{\oplus r} \otimes_{\mathcal{O}_{U_\alpha}} (\mathcal{O}_{U_\alpha} \otimes_{\mathbb{C}} \mathbb{C}[\epsilon]) \xrightarrow{1 \otimes \sigma_\alpha^*} \mathcal{O}_{U_\alpha}^{\oplus r} \otimes_{\mathcal{O}_{U_\alpha}} \mathcal{O}_{U_\alpha^\epsilon} \xrightarrow{\psi_\alpha^{-1}} F_v^* \tilde{E}|_{U_\alpha^\epsilon}$$

as (5).

Lemma 4.9. *For the isomorphisms σ_α ($\alpha \in I_{D(\tilde{\mathbf{p}})}$), we assume that $(\sigma_\alpha^*)^{-1} g_\alpha^\epsilon = g_\alpha$ and $(\sigma_\alpha^*)^{-1} h_\alpha^\epsilon = h_\alpha$. Moreover assume that the trivializations $\{(U_\alpha^\epsilon, \psi_\alpha)\}_\alpha$ of $F_v^* \tilde{E}$ are compatible with $F_v^* \sigma_M$. Then*

$$(43) \quad ((\varphi'_\alpha)^{-1} \otimes \text{id}) \circ (F_v^* \tilde{\nabla}_0^{\sigma_M}) \circ \varphi'_\alpha - \tilde{\nabla}_0^{\sigma_M} = 0$$

for any α .

Proof. First, we consider an affine open subset U_α such that the intersection of U_α and the support of $D(\tilde{\mathbf{p}})$ is empty. On such an affine open subset U_α^ϵ , we have the following commutative diagram

$$\begin{array}{ccccc} \mathcal{O}_{U_\alpha^\epsilon}^{\oplus r} & \xleftarrow[\cong]{\supset} & \mathcal{O}_{U_\alpha^\epsilon}^{\oplus r}(-m) & \xrightarrow[\cong]{F_v^* \sigma_M} & F_v^* \tilde{E}|_{U_\alpha^\epsilon} \\ \downarrow d_{\mathcal{C}_\epsilon^v/M[\epsilon]}|_{U_\alpha^\epsilon} & & \downarrow F_v^* d_m|_{U_\alpha^\epsilon} & & \downarrow F_v^* \tilde{\nabla}_0^{\sigma_M}|_{U_\alpha^\epsilon} \\ \mathcal{O}_{U_\alpha^\epsilon}^{\oplus r} \otimes \Omega_{U_\alpha^\epsilon/M[\epsilon]}^1(D_\epsilon(\tilde{\mathbf{p}})) & \xleftarrow[\cong]{\supset} & \mathcal{O}_{U_\alpha^\epsilon}^{\oplus r}(-m) \otimes \Omega_{U_\alpha^\epsilon/M[\epsilon]}^1(D_\epsilon(\tilde{\mathbf{p}})) & \xrightarrow[\cong]{F_v^* \sigma_M} & F_v^* \tilde{E}|_{U_\alpha^\epsilon} \otimes \Omega_{U_\alpha^\epsilon/M[\epsilon]}^1(D_\epsilon(\tilde{\mathbf{p}})). \end{array}$$

By this commutative diagram and the definition of the trivialization $(U_\alpha^\epsilon, \psi_\alpha)$, we have

$$(\psi_\alpha \otimes \text{id}) \circ (F_v^* \tilde{\nabla}_0^{\sigma_M}|_{U_\alpha^\epsilon}) \circ \psi_\alpha^{-1} = d_{\mathcal{C}_\epsilon^v/M[\epsilon]}|_{U_\alpha^\epsilon}.$$

By this equality, we may check the equality (43).

Second, we consider an affine open set U_α^ϵ such that $U_\alpha^\epsilon \cap \text{Supp}(D_\epsilon(m)) \neq \emptyset$ and $U_\alpha^\epsilon \cap \text{Supp}(D_\epsilon(\sigma_M)) = \emptyset$. By the definition of the trivialization $(U_\alpha^\epsilon, \psi_\alpha)$, we may check that

$$(\psi_\alpha \otimes \text{id}) \circ (F_v^* \sigma_M|_{U_\alpha^\epsilon}) \circ \psi_\alpha^{-1} = d_{\mathcal{C}_\epsilon^v/M[\epsilon]}|_{U_\alpha^\epsilon} + \text{diag}\left(\frac{dg_\alpha^\epsilon}{g_\alpha^\epsilon}, \dots, \frac{dg_\alpha^\epsilon}{g_\alpha^\epsilon}\right).$$

By this equality and $(\sigma_\alpha^*)^{-1} g_\alpha^\epsilon = g_\alpha$, we may check the equality (43).

Third, we consider an affine open set U_α^ϵ where $U_\alpha^\epsilon \cap \text{Supp}(D_\epsilon(m)) = \emptyset$ and $U_\alpha^\epsilon \cap \text{Supp}(D_\epsilon(\sigma_M)) \neq \emptyset$. On such an affine open subset U_α^ϵ , we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{U_\alpha^\epsilon}^{\oplus r}(-m) & \xleftarrow[\cong]{g_\alpha^\epsilon} & \mathcal{O}_{U_\alpha^\epsilon}^{\oplus r} & \xrightarrow{T_{\alpha, i''}^\epsilon} & \mathcal{O}_{U_\alpha^\epsilon}^{\oplus r} \\ \downarrow F_v^* d_m|_{U_\alpha^\epsilon} & & & & \downarrow (\psi_\alpha \otimes \text{id}) \circ (F_v^* \tilde{\nabla}_0^{\sigma_M}) \circ \psi_\alpha^{-1}|_{U_\alpha^\epsilon} \\ \mathcal{O}_{U_\alpha^\epsilon}^{\oplus r}(-m) \otimes \Omega_{U_\alpha^\epsilon/M[\epsilon]}^1(D_\epsilon(\tilde{\mathbf{p}})) & \xleftarrow[\cong]{g_\alpha^\epsilon} & \mathcal{O}_{U_\alpha^\epsilon}^{\oplus r} \otimes \Omega_{U_\alpha^\epsilon/M[\epsilon]}^1(D_\epsilon(\tilde{\mathbf{p}})) & \xrightarrow{T_{\alpha, i''}^\epsilon} & \mathcal{O}_{U_\alpha^\epsilon}^{\oplus r} \otimes \Omega_{U_\alpha^\epsilon/M[\epsilon]}^1(D_\epsilon(\tilde{\mathbf{p}})) \end{array}$$

(see the diagram (23)). Here we set

$$T_{\alpha, i''}^\epsilon := \begin{pmatrix} h_\alpha^\epsilon & 0 & 0 & \cdots & 0 \\ a_{\alpha, i'', 2}^\epsilon & 1 & 0 & \cdots & 0 \\ a_{\alpha, i'', 3}^\epsilon & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{\alpha, i'', r}^\epsilon & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

By the commutative diagram, we have

$$(\psi_\alpha \otimes \text{id}) \circ (F_v^* \tilde{\nabla}_0^{\sigma_M}) \circ \psi_\alpha^{-1} = d_{\mathcal{C}_\epsilon/M[\epsilon]}|_{U_\alpha^\epsilon} + T_{\alpha, i''}^\epsilon d((T_{\alpha, i''}^\epsilon)^{-1}) + \text{diag}\left(\frac{dg_\alpha^\epsilon}{g_\alpha^\epsilon}, \dots, \frac{dg_\alpha^\epsilon}{g_\alpha^\epsilon}\right).$$

By this equality, $(\sigma_\alpha^*)^{-1}g_\alpha^\epsilon = g_\alpha$, and $(\sigma_\alpha^*)^{-1}h_\alpha^\epsilon = h_\alpha$, we may check the equality (43). \square

4.4. Hamiltonians on the moduli spaces. Let M_{aff} be an affine open set of $M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)_\nu$. We define an algebraic splitting $\eta: \mathbf{H}^1(\mathcal{G}_{M_{\text{aff}}}^\bullet) \rightarrow \mathbf{H}^1(\mathcal{F}_{M_{\text{aff}}}^\bullet)$ of the tangent map $\mathbf{H}^1(\mathcal{F}_{M_{\text{aff}}}^\bullet) \rightarrow \mathbf{H}^1(\mathcal{G}_{M_{\text{aff}}}^\bullet)$ by

$$\eta: \mathbf{H}^1(\mathcal{G}_{M_{\text{aff}}}^\bullet) \longrightarrow \mathbf{H}^1(\mathcal{F}_{M_{\text{aff}}}^\bullet); \quad [(\{u_{\alpha\beta}\}, \{v_\alpha\})] \longmapsto [(\{\eta(u_{\alpha\beta})\}, \{v_\alpha\})].$$

Here we set

$$(44) \quad \eta(s) := s - \iota(\tilde{\nabla}) \circ \text{symb}_1(s), \quad s \in \mathcal{G}_{M_{\text{aff}}}^0.$$

First, we define a lift

$$(45) \quad \omega: \mathbf{R}^1(\pi_{M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)_\nu})_*(\mathcal{G}^\bullet) \otimes \mathbf{R}^1(\pi_{M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)_\nu})_*(\mathcal{G}^\bullet) \longrightarrow \mathcal{O}_{M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)}$$

of the symplectic form defined in Section 2.3 as follows. We define a pairing

$$(46) \quad \begin{aligned} \omega_{M_{\text{aff}}}: \mathbf{H}^1(\mathcal{C}_{M_{\text{aff}}}, \mathcal{G}_{M_{\text{aff}}}^\bullet) \otimes \mathbf{H}^1(\mathcal{C}_{M_{\text{aff}}}, \mathcal{G}_{M_{\text{aff}}}^\bullet) &\longrightarrow \mathbf{H}^2(\mathcal{C}_{M_{\text{aff}}}, \Omega_{\mathcal{C}_{M_{\text{aff}}}/M_{\text{aff}}}^\bullet) \cong H^0(\mathcal{O}_{M_{\text{aff}}}) \\ v \otimes v' &\longmapsto \omega_{M_{\text{aff}}}(v, v'), \end{aligned}$$

where we put

$$\omega_{M_{\text{aff}}}(v, v') := [(\{\text{Tr}(\eta(u_{\alpha\beta}) \circ \eta(u'_{\beta\gamma}))\}, -\{\text{Tr}(\eta(u_{\alpha\beta}) \circ v'_\beta) - \text{Tr}(v_\alpha \circ \eta(u'_{\alpha\beta}))\})]$$

for $v = [(\{u_{\alpha\beta}\}, \{v_\alpha\})]$ and $v' = [(\{u'_{\alpha\beta}\}, \{v'_\alpha\})]$. Here we consider in Čech cohomology with respect to an affine open covering $\{U_\alpha\}$ of $\mathcal{C} \times_T M_{\text{aff}}$, $\{u_{\alpha\beta}\} \in C^1(\mathcal{G}_{M_{\text{aff}}}^0)$, $\{v_\alpha\} \in C^0(\mathcal{G}_{M_{\text{aff}}}^1)$. This pairing induces a lift ω .

Proposition 4.10. *The kernel $\text{Ker}(\omega_{M_{\text{aff}}})$ of $\omega_{M_{\text{aff}}}: \mathbf{H}^1(\mathcal{G}_{M_{\text{aff}}}^\bullet) \rightarrow \text{Hom}_{\mathcal{O}_{M_{\text{aff}}}}(\mathbf{H}^1(\mathcal{G}_{M_{\text{aff}}}^\bullet), \mathcal{O}_{M_{\text{aff}}})$ induces the vector fields on M_{aff} determined by the isomonodromic deformations.*

Proof. The algebraic splitting (13) for $\mathbf{H}^1(\mathcal{G}_{M_{\text{aff}}}^\bullet) \rightarrow H^1(\Theta_{\mathcal{C} \times_T M_{\text{aff}}/M_{\text{aff}}}(-D(\tilde{\mathbf{t}})_{M_{\text{aff}}}))$ gives an isomorphism

$$\begin{aligned} H^1(\Theta_{\mathcal{C} \times_T M_{\text{aff}}/M_{\text{aff}}}(-D(\tilde{\mathbf{t}})_{M_{\text{aff}}})) \oplus \mathbf{H}^1(\mathcal{F}_{M_{\text{aff}}}^\bullet) &\xrightarrow{\cong} \mathbf{H}^1(\mathcal{G}_{M_{\text{aff}}}^\bullet) \\ ([\{d_{\alpha\beta}\}], [(\{u_{\alpha\beta}\}, \{v_\alpha\})]) &\longmapsto [\{\iota(\tilde{\nabla})(d_{\alpha\beta}) + u_{\alpha\beta}\}, \{v_\alpha\}]. \end{aligned}$$

By this isomorphism, we can define a composition

$$(47) \quad H^1(\Theta_{\mathcal{C} \times_T M_{\text{aff}}/M_{\text{aff}}}(-D(\tilde{\mathbf{t}})_{M_{\text{aff}}})) \oplus \mathbf{H}^1(\mathcal{G}_{M_{\text{aff}}}^\bullet) \longrightarrow \mathbf{H}^1(\mathcal{G}_{M_{\text{aff}}}^\bullet) \otimes \mathbf{H}^1(\mathcal{G}_{M_{\text{aff}}}^\bullet) \xrightarrow{\omega_{M_{\text{aff}}}} \mathbf{H}^2(\Omega_{\mathcal{C}_{M_{\text{aff}}}/M_{\text{aff}}}^\bullet)$$

The image of $[\{d_{\alpha\beta}\}] \in H^1(\Theta_{\mathcal{C} \times_T M_{\text{aff}}/M_{\text{aff}}}(-D(\tilde{\mathbf{t}})_{M_{\text{aff}}}))$ is $[\{\iota(\tilde{\nabla})(d_{\alpha\beta})\}, \{0\}]$. Moreover, we have

$$\begin{aligned} \eta([\{\iota(\tilde{\nabla})(d_{\alpha\beta})\}, \{0\}]) &= [\{\iota(\tilde{\nabla})(d_{\alpha\beta}) - \iota(\tilde{\nabla})(d_{\alpha\beta})\}, \{0\}] \\ &= [\{0\}, \{0\}]. \end{aligned}$$

By (46), the composition (47) is the zero morphism. Then the image of $H^1(\Theta_{\mathcal{C} \times_T M_{\text{aff}}/M_{\text{aff}}}(-D(\tilde{\mathbf{t}})_{M_{\text{aff}}}))$ in $\mathbf{H}^1(\mathcal{G}_{M_{\text{aff}}}^\bullet)$ is contained in $\text{Ker}(\omega_{M_{\text{aff}}})$. On the other hand, the composition

$$\begin{aligned} \mathbf{H}^1(\mathcal{F}_{M_{\text{aff}}}^\bullet) \oplus \mathbf{H}^1(\mathcal{F}_{M_{\text{aff}}}^\bullet) &\longrightarrow \mathbf{H}^1(\mathcal{G}_{M_{\text{aff}}}^\bullet) \otimes \mathbf{H}^1(\mathcal{G}_{M_{\text{aff}}}^\bullet) \xrightarrow{\omega_{M_{\text{aff}}}} \mathbf{H}^2(\Omega_{\mathcal{C}_{M_{\text{aff}}}/M_{\text{aff}}}^\bullet) \\ &\cong H^0(\mathcal{O}_{M_{\text{aff}}}). \end{aligned}$$

is just the symplectic structure (1) in Section 2.3. In particular, this pairing is nondegenerate. Then the image of $H^1(\Theta_{\mathcal{C} \times_T M_{\text{aff}}/M_{\text{aff}}}(-D(\tilde{\mathbf{t}})_{M_{\text{aff}}}))$ in $\mathbf{H}^1(\mathcal{G}_{M_{\text{aff}}}^\bullet)$ coincides with $\text{Ker}(\omega_{M_{\text{aff}}})$. By Proposition 3.8, this image of $H^1(\Theta_{\mathcal{C} \times_T M_{\text{aff}}/M_{\text{aff}}}(-D(\tilde{\mathbf{t}})_{M_{\text{aff}}}))$ means the vector fields on M_{aff} determined by the isomonodromic deformations. \square

Second, we define Hamiltonian functions for each affine open subset $M \subset M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)_\nu$ in Proposition 4.1 as follows. We take an affine open subset $U \subset T$ and we take $\mu_1, \dots, \mu_{3g-3+n} \in \Theta_T(U)$ such that $\mu_1, \dots, \mu_{3g-3+n}$ give a trivialization of $\Theta_T|_U$. By taking a refined covering $\{M\}$, we may define a morphism $\varpi_M: M \rightarrow U$. The sections $\mu_1, \dots, \mu_{3g-3+n}$ induce classes of $H^1(\mathcal{C}_M, \Theta_{\mathcal{C}_M/M}(-D(\tilde{\mathbf{t}})))$ by $\varpi_M: M \rightarrow U$. We also denote by $\mu_1, \dots, \mu_{3g-3+n}$ the corresponding classes in $H^1(\mathcal{C}_M, \Theta_{\mathcal{C}_M/M}(-D(\tilde{\mathbf{t}})))$. We take a representative $\{(d_k)_{\alpha\beta}\}_{\alpha\beta}$ of the class μ_k :

$$(48) \quad \mu_k = [\{(d_k)_{\alpha\beta}\}] \in H^1(\mathcal{C}_M, \Theta_{\mathcal{C}_M/M}(-D(\tilde{\mathbf{t}})))$$

for each k ($k = 1, \dots, 3g-3+n$). Let $\text{im}_M^{\mu_k}: M[\epsilon] \rightarrow M$ be the isomonodromic lift associated to μ_k and $\text{IM}_M^{\mu_k}: \mathcal{C}_\epsilon^k \rightarrow \mathcal{C}_M$ be the induced morphism by $\text{im}_M^{\mu_k}: M[\epsilon] \rightarrow M$. We take a representative $\{(\hat{d}_k)_{\alpha\beta}\}_{\alpha\beta}$ of the lift

$$\hat{\mu}_k = [\{(\hat{d}_k)_{\alpha\beta}\}_{\alpha\beta}] \in H^1(\mathcal{C}_M, \Theta_{\mathcal{C}_M/M}(-D(\tilde{\mathbf{t}}) - D(\tilde{\mathbf{p}})))$$

defined as (27).

Definition 4.11. We describe a representative $\{(\hat{d}_k)_{\alpha\beta}\}_{\alpha\beta}$ as $(\hat{d}_k)_{\alpha\beta} = (d_k)_{\alpha\beta} - (\delta_k)_\alpha + (\delta_k)_\beta$, where

$$(\delta_k)_\alpha = \tilde{\delta}_{k,\alpha} \frac{\partial}{\partial z_\alpha} \in \Theta_{\mathcal{C}_M/M}(-D(\tilde{\mathbf{t}}))(U_\alpha)$$

for any α and $(\delta_k)_\alpha = 0$ for $\alpha \notin I_D(\tilde{\mathbf{p}})$. Here $\tilde{\delta}_{k,\alpha} \in \pi_M^*(\mathcal{O}_M)|_{U_\alpha}$. By changing of coordinates $\{z_\alpha\}$ ($\alpha \in I_D(\tilde{\mathbf{p}})$), we may assume that

$$(49) \quad f_v^*(\tilde{\delta}_{k,\alpha}) - f_0^*(\tilde{\delta}_{k,\alpha}) = 0$$

for any relative vector field $v \in \Theta_{M/T}$. Here $f_v: M[\epsilon] \rightarrow M$ is induced by $v \in \Theta_{M/T}$. Remark that $f_0: M[\epsilon] \rightarrow M$ is the trivial projection. Note that T is the base space of the family $(\mathcal{C}, \tilde{\mathbf{t}}) \rightarrow T$ of n -pointed curves. We say such coordinates $\{z_\alpha\}_\alpha$ are *refined*. Now we assume that $\{(\delta_k)_\alpha\}_{\alpha \in I_D(\tilde{\mathbf{p}})}$ are described by using the refined coordinates.

Remark that we avoid the refined coordinate to describe the representative $(d_k)_{\alpha\beta}$. We describe $(d_k)_{\alpha\beta}$ as $(d_k)_{\alpha\beta} = \tilde{d}_{k,\alpha\beta} \frac{\partial}{\partial \tilde{f}_\beta}$ (where $\tilde{d}_{k,\alpha\beta} \in \mathcal{O}_{\mathcal{C}_M}|_{U_{\alpha\beta}}$) so that

$$(50) \quad F_{U_{\alpha\beta}, v}^*(\tilde{d}_{k,\alpha\beta}) - F_{U_{\alpha\beta}, 0}^*(\tilde{d}_{k,\alpha\beta}) = 0$$

for any relative vector field $v \in \Theta_{M/T}$. Here $F_{U_{\alpha\beta}, v}: U_{\alpha\beta} \times_M M[\epsilon] \rightarrow U_{\alpha\beta}$ is defined by the projection $U_{\alpha\beta} \rightarrow M$ and $f_v: M[\epsilon] \rightarrow M$ for $v \in \Theta_{M/T}$. For the representative $\{(\hat{d}_k)_{\alpha\beta}\}_{\alpha\beta}$ of $\hat{\mu}_k$, we take a lift

$$[\{(\hat{d}_k)_{\alpha\beta}\}_{\alpha\beta}] \in H^1(\mathcal{C}_M, \Theta_{\mathcal{C}_M/M}(-2D(\tilde{\mathbf{t}}) - 2D(\tilde{\mathbf{p}}))).$$

We also denote by $\hat{\mu}_k$ this lift of $\hat{\mu}_k \in H^1(\mathcal{C}_M, \Theta_{\mathcal{C}_M/M}(-D(\tilde{\mathbf{t}}) - D(\tilde{\mathbf{p}})))$.

Remark 4.12. In Definition 4.13 below, we will define Hamiltonians. In this definition, we will couple $(\hat{d}_k)_{\alpha\beta}$ on the square of $\tilde{\nabla} - \tilde{\nabla}_0^{\sigma_M}$ (see (51) below). This square of $\tilde{\nabla} - \tilde{\nabla}_0^{\sigma_M}$ has poles of order 2 at $D(\tilde{\mathbf{t}}) + D(\tilde{\mathbf{p}})$. Then the Hamiltonians will depend on the choice of a lift in $H^1(\mathcal{C}_M, \Theta_{\mathcal{C}_M/M}(-2D(\tilde{\mathbf{t}}) - 2D(\tilde{\mathbf{p}})))$ of $\hat{\mu}_k \in H^1(\mathcal{C}_M, \Theta_{\mathcal{C}_M/M}(-D(\tilde{\mathbf{t}}) - D(\tilde{\mathbf{p}})))$. Now we take a lift of $\hat{\mu}_k \in H^1(\mathcal{C}_M, \Theta_{\mathcal{C}_M/M}(-D(\tilde{\mathbf{t}}) - D(\tilde{\mathbf{p}})))$ so that the equality (69) (below) satisfies.

We take a trivialization $\bar{\phi}_\alpha: \tilde{E}|_{U_\alpha} \rightarrow \mathcal{O}_{U_\alpha}^{\oplus r}$. We assume that the trivializations $(U_\alpha, \bar{\phi}_\alpha)_\alpha$ are compatible with σ_M (see Definition 4.4). Let $\tilde{A}_\alpha^0 z_\alpha^{-1} dz_\alpha$ and $\tilde{A}_\alpha f_\alpha^{-1} df_\alpha$ be connection matrices of $\tilde{\nabla}_0^{\sigma_M}$ and $\tilde{\nabla}$ on U_α via the trivialization $\bar{\phi}_\alpha$, respectively. Set

$$(51) \quad (H_k)_{\alpha\beta} := \bar{\phi}_\beta^{-1} \circ \left\langle (\hat{d}_k)_{\alpha\beta}, \left(\tilde{A}_\beta \frac{df_\beta}{f_\beta} - \tilde{A}_\beta^0 \frac{dz_\beta}{z_\beta} \right)^2 \right\rangle \circ \bar{\phi}_\beta$$

which is an element of $(\mathcal{E}nd(\tilde{E}) \otimes \Omega_{\mathcal{C}_M/M}^1)(U_{\alpha\beta})$ since $\#\{i \mid \hat{t}_i|_{\mathcal{C}_M} \cap U_\alpha \neq \emptyset\} \leq 1$ for any α and $\#\{\alpha \mid \hat{t}_i|_{\mathcal{C}_M} \cap U_\alpha \neq \emptyset\} \leq 1$ for any i . Here $\{\hat{t}_i\}$ is the set of the supports of the Cartier divisor $D(\tilde{\mathbf{t}}) + D(\tilde{\mathbf{p}})$.

Definition 4.13. We define *Hamiltonian functions* H_k ($k = 1, \dots, 3g - 3 + n$) on M as

$$H_k = \frac{1}{2} [\{\text{Tr}((H_k)_{\alpha\beta})\}] \in H^1(\Omega_{\mathcal{C}_M/M}^1) \cong H^0(\mathcal{O}_M)$$

for the lifts $\hat{\mu}_1, \dots, \hat{\mu}_{3g-3+n} \in H^1(\mathcal{C}_M, \Theta_{\mathcal{C}_M/M}(-2D(\tilde{\mathbf{t}}) - 2D(\tilde{\mathbf{p}})))$.

4.5. Calculation of the Hamiltonians. By Proposition 4.5, for the lifts $\hat{\mu}_1, \dots, \hat{\mu}_{3g-3+n}$, we have the vector fields $v_{\hat{\mu}_k}$ on M :

$$v_{\hat{\mu}_k} := \left[\left(\left\{ u_{\alpha\beta}^{\hat{\mu}_k \nabla_0} \right\}, \left\{ -v_{\alpha}^{\hat{\mu}_k \nabla_0} \right\} \right) \right] \in \mathbf{H}^1(\mathcal{G}_M^\bullet).$$

Let $v' = [(\{u'_{\alpha\beta}\}, \{v'_\alpha\})]$ be an element of $\mathbf{H}^1(\mathcal{G}_M^\bullet)$. We put $v'_\eta := [(\{\eta(u'_{\alpha\beta})\}, \{v'_\alpha\})] \in \mathbf{H}^1(\mathcal{F}_M^\bullet)$. Let $f_{v'_\eta}: M[\epsilon] \rightarrow M$ be the morphism induced by $v'_\eta \in \mathbf{H}^1(\mathcal{F}_M^\bullet)$. Remark that $\mathbf{H}^1(\mathcal{F}_M^\bullet) \cong \Theta_{M_{\mathcal{G}_T}(\tilde{\mathbf{t}}, r, d)_\nu/T}(M)$. That is, the family \mathcal{C}_M of curves is constant along the direction of the vector field v'_η . We denote by

$$F_{v'_\eta}: \mathcal{C}_M \times_M M[\epsilon] \longrightarrow \mathcal{C}_M$$

the morphism induced by $f_{v'_\eta}: M[\epsilon] \rightarrow M$. We define an infinitesimal deformation of $\tilde{\nabla}_0^{\sigma_M}$ on $F_{v'_\eta}^* \tilde{E}$ by taking the pull-back of $\tilde{\nabla}_0^{\sigma_M}$ by $F_{v'_\eta}: \mathcal{C}_M \times_M M[\epsilon] \rightarrow \mathcal{C}_M$:

$$F_{v'_\eta}^* \tilde{\nabla}_0^{\sigma_M}: F_{v'_\eta}^* \tilde{E} \longrightarrow \tilde{F}_{v'_\eta}^* \tilde{E} \otimes \Omega_{\mathcal{C}_M \times_M M[\epsilon]/M[\epsilon]}^1 (F_{v'_\eta}^* D(m) + F_{v'_\eta}^* D(\sigma_M)).$$

Let

$$\varphi_\alpha(v'_\eta): F_{v'_\eta}^* \tilde{E}|_{U_\alpha \times \text{Spec } \mathcal{O}_M[\epsilon]} \xrightarrow[\sim]{\phi_\alpha(v'_\eta)} \mathcal{O}_{U_\alpha \times \text{Spec } \mathcal{O}_M[\epsilon]}^{\oplus r} \xrightarrow{\bar{\phi}_\alpha^{-1} \otimes 1} \tilde{E}|_{U_\alpha} \otimes \mathcal{O}_M[\epsilon]$$

be an isomorphism as in Section 2.2. We put

$$(52) \quad (v_\alpha^0)' := (\varphi_\alpha(v'_\eta) \otimes \text{id}) \circ (F_{v'_\eta}^* \tilde{\nabla}_0^{\sigma_M}) \circ (\varphi_\alpha(v'_\eta))^{-1} - \tilde{\nabla}_0^{\sigma_M},$$

which is an element of $\epsilon \otimes \mathcal{E}nd(\tilde{E}) \otimes \Omega_{\mathcal{C}_M/M}^1(D(m) + D(\sigma_M))$. The collection $(\{\eta(u'_{\alpha\beta})\}, \{(v_\alpha^0)'\})$ satisfies

$$(53) \quad \tilde{\nabla}_0^{\sigma_M} \circ (\eta(u'_{\alpha\beta})) - (\eta(u'_{\alpha\beta})) \circ \tilde{\nabla}_0^{\sigma_M} = (v_\beta^0)' - (v_\alpha^0)'$$

as with (34).

Lemma 4.14. We take any $v' = [(\{u'_{\alpha\beta}\}, \{v'_\alpha\})] \in \mathbf{H}^1(\mathcal{G}_M^\bullet)$. For v' , we define $(v_\beta^0)'$ by (52). Then we have the following equality:

$$(54) \quad \omega(v_{\hat{\mu}_k}, [(\{\eta(u'_{\alpha\beta})\}, \{v'_\alpha\})]) = \left[\left(\{0\}_{\alpha\beta\gamma}, \left\{ \text{Tr} \left((-\eta(u_{\alpha\beta}^{\hat{\mu}_k \nabla_0})) \circ (v'_\beta - (v_\beta^0)') \right) \right\}_{\alpha\beta} \right) \right] \in \mathbf{H}^2(\Omega_{\mathcal{C}_M/M}^\bullet).$$

Proof. We consider the 1-form $\omega_M(v_{\hat{\mu}_k}, \cdot) \in \mathcal{H}om_{\mathcal{O}_M}(\mathbf{H}^1(\mathcal{G}_M^\bullet), \mathcal{O}_M)$. For v' , we compute

$$\omega_M(v_{\hat{\mu}_k}, v') = [(\{\mathrm{Tr}(\eta(u_{\alpha\beta}^{\hat{\mu}_k \nabla^0}) \circ \eta(u'_{\beta\gamma}))\}, -\{\mathrm{Tr}(\eta(u_{\alpha\beta}^{\hat{\mu}_k \nabla^0}) \circ v'_\beta) - \mathrm{Tr}(-v_\alpha^{\hat{\mu}_k \nabla^0} \circ \eta(u'_{\alpha\beta}))\})],$$

which is a class of $\mathbf{H}^2(\mathcal{C}_M, \Omega_{\mathcal{C}_M/M}^\bullet)$. We have

$$\begin{aligned} & -\mathrm{Tr}(\eta(u_{\alpha\beta}^{\hat{\mu}_k \nabla^0}) \circ v'_\beta) + \mathrm{Tr}(-v_\alpha^{\hat{\mu}_k \nabla^0} \circ \eta(u'_{\alpha\beta})) \\ (55) \quad & = \mathrm{Tr}(\left(-\eta(u_{\alpha\beta}^{\hat{\mu}_k \nabla^0}) \circ (v'_\beta - (v_\beta^0)')\right) + \left(\mathrm{Tr}(-\eta(u_{\alpha\beta}^{\hat{\mu}_k \nabla^0}) \circ (v_\beta^0)') - \mathrm{Tr}(v_\alpha^{\hat{\mu}_k \nabla^0} \circ \eta(u'_{\alpha\beta}))\right)). \end{aligned}$$

We consider the second term of the right hand side of (55). We set

$$(56) \quad \eta_0(s) := s - \iota(\tilde{\nabla}_0^{\sigma_M}) \circ \mathrm{symb}_1(s).$$

As in the proof of Lemma 3.4, we may check the equality

$$\tilde{\nabla}_0^{\sigma_M} \circ u_{\alpha\beta}^{\hat{\mu}_k \nabla^0} - u_{\alpha\beta}^{\hat{\mu}_k \nabla^0} \circ \tilde{\nabla}_0^{\sigma_M} = \tilde{\nabla}_0^{\sigma_M} \circ \eta_0(u_{\alpha\beta}^{\hat{\mu}_k \nabla^0}) - \eta_0(u_{\alpha\beta}^{\hat{\mu}_k \nabla^0}) \circ \tilde{\nabla}_0^{\sigma_M}.$$

Since $\eta_0(u_{\alpha\beta}^{\hat{\mu}_k \nabla^0}) = 0$, we have $\tilde{\nabla}_0^{\sigma_M} \circ u_{\alpha\beta}^{\hat{\mu}_k \nabla^0} - u_{\alpha\beta}^{\hat{\mu}_k \nabla^0} \circ \tilde{\nabla}_0^{\sigma_M} = 0$. By this equality and the equality (34), we have the following equalities:

$$\begin{aligned} & \tilde{\nabla}_0^{\sigma_M} \circ (-\eta(u_{\alpha\beta}^{\hat{\mu}_k \nabla^0})) - (-\eta(u_{\alpha\beta}^{\hat{\mu}_k \nabla^0})) \circ \tilde{\nabla}_0^{\sigma_M} = \tilde{\nabla}_0^{\sigma_M} \circ (\hat{u}_{\alpha\beta}^{\mathrm{IMD}} - u_{\alpha\beta}^{\hat{\mu}_k \nabla^0}) - (\hat{u}_{\alpha\beta}^{\mathrm{IMD}} - u_{\alpha\beta}^{\hat{\mu}_k \nabla^0}) \circ \tilde{\nabla}_0^{\sigma_M} \\ (57) \quad & = \tilde{\nabla}_0^{\sigma_M} \circ \hat{u}_{\alpha\beta}^{\mathrm{IMD}} - \hat{u}_{\alpha\beta}^{\mathrm{IMD}} \circ \tilde{\nabla}_0^{\sigma_M} \\ & = v_\beta^{\hat{\mu}_k \nabla^0} - v_\alpha^{\hat{\mu}_k \nabla^0}. \end{aligned}$$

Set $\omega_{\alpha\beta}^0 := \mathrm{Tr}(-\eta(u_{\alpha\beta}^{\hat{\mu}_k \nabla^0}) \circ (v_\beta^0)') - \mathrm{Tr}(v_\alpha^{\hat{\mu}_k \nabla^0} \circ \eta(u'_{\alpha\beta}))$. By the cocycle conditions of $\eta(u_{\alpha\beta}^{\hat{\mu}_k \nabla^0})$ and $\eta(u'_{\alpha\beta})$, and by the equalities (53) and (57), we have the following equalities:

$$\begin{aligned} \omega_{\beta\gamma}^0 - \omega_{\alpha\gamma}^0 + \omega_{\alpha\beta}^0 & = \mathrm{Tr}(-\eta(u_{\alpha\beta}^{\hat{\mu}_k \nabla^0}) \circ ((v_\beta^0)' - (v_\gamma^0)')) + \mathrm{Tr}((v_\alpha^{\hat{\mu}_k \nabla^0} - v_\beta^{\hat{\mu}_k \nabla^0}) \circ \eta(u'_{\beta\gamma})) \\ & = -\mathrm{Tr}\left(-\eta(u_{\alpha\beta}^{\hat{\mu}_k \nabla^0}) \circ \left(\tilde{\nabla}_0^{\sigma_M} \circ (\eta(u'_{\beta\gamma})) - (\eta(u'_{\beta\gamma})) \circ \tilde{\nabla}_0^{\sigma_M}\right)\right) \\ & \quad + \mathrm{Tr}\left(\left(\tilde{\nabla}_0^{\sigma_M} \circ (\eta(u_{\alpha\beta}^{\hat{\mu}_k \nabla^0})) - (\eta(u_{\alpha\beta}^{\hat{\mu}_k \nabla^0})) \circ \tilde{\nabla}_0^{\sigma_M}\right) \circ \eta(u'_{\beta\gamma})\right). \end{aligned}$$

Let $\tilde{A}_\beta^0 \frac{dz_\beta}{z_\beta}$ be the connection matrices of $\tilde{\nabla}_0^{\sigma_M}$ associated to the trivialization $\bar{\phi}_\beta$. That is $\tilde{\nabla}_0^{\sigma_M}|_{U_\beta} = (\bar{\phi}_\beta^{-1} \otimes \mathrm{id}) \circ (d + \tilde{A}_\beta^0 \frac{dz_\beta}{z_\beta}) \circ \bar{\phi}_\beta$. By this local description of $\tilde{\nabla}_0^{\sigma_M}$, we have the following equalities:

$$\begin{aligned} \omega_{\beta\gamma}^0 - \omega_{\alpha\gamma}^0 + \omega_{\alpha\beta}^0 & = \mathrm{Tr}\left(\eta(u_{\alpha\beta}^{\hat{\mu}_k \nabla^0}) \circ \left(d(\eta(u'_{\beta\gamma})) + [(\bar{\phi}_\beta^{-1} \otimes \mathrm{id}) \circ \tilde{A}_\beta^0 \frac{dz_\beta}{z_\beta} \circ \bar{\phi}_\beta, \eta(u'_{\beta\gamma})]\right)\right) \\ (58) \quad & \quad + \mathrm{Tr}\left(\left(d(\eta(u_{\alpha\beta}^{\hat{\mu}_k \nabla^0})) + [(\bar{\phi}_\beta^{-1} \otimes \mathrm{id}) \circ \tilde{A}_\beta^0 \frac{dz_\beta}{z_\beta} \circ \bar{\phi}_\beta, \eta(u_{\alpha\beta}^{\hat{\mu}_k \nabla^0})]\right) \circ \eta(u'_{\beta\gamma})\right) \\ & = -d\left(\mathrm{Tr}(\eta(-u_{\alpha\beta}^{\hat{\mu}_k \nabla^0}) \circ \eta(u'_{\beta\gamma}))\right). \end{aligned}$$

Then the pair $(-\{\mathrm{Tr}(\eta(-u_{\alpha\beta}^{\hat{\mu}_k \nabla^0}) \circ \eta(u'_{\beta\gamma}))\}, \{\omega_{\alpha\beta}^0\})$ gives a class

$$(59) \quad [(-\{\mathrm{Tr}(\eta(-u_{\alpha\beta}^{\hat{\mu}_k \nabla^0}) \circ \eta(u'_{\beta\gamma}))\})_{\alpha\beta\gamma}, \{\omega_{\alpha\beta}^0\}_{\alpha\beta}]]$$

of $\mathbf{H}^2(\mathcal{C}_M, \Omega_{\mathcal{C}_M/M}^\bullet)$. If the class (59) vanishes, then we obtain the equality (54) by the equality (55).

We claim that the class (59) of $\mathbf{H}^2(\mathcal{C}_M, \Omega_{\mathcal{C}_M/M}^\bullet)$ vanishes. To show this claim, we consider replacement of $(\{\eta(-u_{\alpha\beta}^{\hat{\mu}_k \nabla^0})\}, \{v_\alpha^{\hat{\mu}_k \nabla^0}\})$ in the class (59). We recall the definition of $v_\alpha^{\hat{\mu}_k \nabla^0}$. We set $E_\epsilon := (\mathrm{IM}_M^{\mu_k})^* \tilde{E}$ and $\nabla_{0,\epsilon}^{\sigma_M, \mathrm{IMD}} := (\mathrm{IM}_M^{\mu_k})^* \tilde{\nabla}_0^{\sigma_M}$. If we take trivializations of E_ϵ which satisfies (31), then we can define $v_\alpha^{\hat{\mu}_k \nabla^0}$ by (32). The tuple $\{v_\alpha^{\hat{\mu}_k \nabla^0}\}_\alpha$ satisfies the equality (34). We apply Lemma 4.9 to $\mathrm{IM}_M^{\mu_k} : \mathcal{C}_\epsilon^k \rightarrow \mathcal{C}_M$ and the lift $\hat{\mu}_k \in H^1(\mathcal{C}_M, \Theta_{\mathcal{C}_M/M}(-D(\hat{\mathbf{t}}) - D(\hat{\mathbf{p}})))$ as follows. Let φ'_α be the composition (42). By the definition (32) and the equality (43), we have

$$\begin{aligned} v_\alpha^{\hat{\mu}_k \nabla^0} & = (\varphi_\alpha^{-1} \otimes \mathrm{id}) \circ (\nabla_{0,\epsilon}^{\sigma_M, \mathrm{IMD}}) \circ \varphi_\alpha - \tilde{\nabla}_0^{\sigma_M} \\ & = (\varphi_\alpha^{-1} \otimes \mathrm{id}) \circ ((\varphi'_\alpha)^{-1} \otimes \mathrm{id})^{-1} \circ ((\varphi'_\alpha)^{-1} \otimes \mathrm{id}) \circ (\nabla_{0,\epsilon}^{\sigma_M, \mathrm{IMD}}) \circ \varphi'_\alpha \circ (\varphi'_\alpha)^{-1} \circ \varphi_\alpha - \tilde{\nabla}_0^{\sigma_M} \\ & = (\varphi_\alpha^{-1} \otimes \mathrm{id}) \circ ((\varphi'_\alpha)^{-1} \otimes \mathrm{id})^{-1} \circ \tilde{\nabla}_0^{\sigma_M} \circ (\varphi'_\alpha)^{-1} \circ \varphi_\alpha - \tilde{\nabla}_0^{\sigma_M}. \end{aligned}$$

We define $\chi_\alpha \in \mathcal{A}_{\tilde{E}}(U_\alpha)$ (for any α) as $(\varphi'_\alpha)^{-1} \circ \varphi_\alpha = \text{id} + \epsilon\chi_\alpha$. Remark that $\varphi_\alpha^{-1} \circ \varphi'_\alpha = \text{id} - \epsilon\chi_\alpha$. Then we have the following equality:

$$\begin{aligned}
 (60) \quad v_\alpha^{\hat{\mu}_k \nabla_0} &= (\varphi_\alpha^{-1} \otimes \text{id}) \circ ((\varphi'_\alpha)^{-1} \otimes \text{id})^{-1} \circ \tilde{\nabla}_0^{\sigma_M} \circ (\varphi'_\alpha)^{-1} \circ \varphi_\alpha - \tilde{\nabla}_0^{\sigma_M} \\
 &= (((\text{id} - \epsilon\chi_\alpha) \otimes \text{id}) \circ \tilde{\nabla}_0^{\sigma_M} \circ (\text{id} + \epsilon\chi_\alpha) - \tilde{\nabla}_0^{\sigma_M}) \\
 &= \tilde{\nabla}_0^{\sigma_M} \circ \chi_\alpha - \chi_\alpha \circ \tilde{\nabla}_0^{\sigma_M}
 \end{aligned}$$

for any α . The symbol $\text{symb}_1(\chi_\alpha)$ of $\chi_\alpha \in \mathcal{A}_{\tilde{E}}(U_\alpha)$ has zero along the components of $D(\tilde{\mathbf{p}})$, since the class $[\{\text{symb}_1(\varphi_\alpha^{-1} \circ \varphi_\beta - \text{id})\}_{\alpha\beta}]$ coincides with the class $[\{\text{symb}_1((\varphi'_\alpha)^{-1} \circ \varphi'_\beta - \text{id})\}_{\alpha\beta}]$ in $H^1(\mathcal{C}_M, \Theta_{\mathcal{C}_M/M}(-D(\tilde{\mathbf{t}}) - D(\tilde{\mathbf{p}})))$. Then $\eta_0(\chi_\alpha)$ has no pole along the components of $D(\tilde{\mathbf{p}})$. As in the proof of Lemma 3.4, we may check the equality

$$\begin{aligned}
 (61) \quad \tilde{\nabla}_0^{\sigma_M} \circ \eta_0(\chi_\alpha) - \eta_0(\chi_\alpha) \circ \tilde{\nabla}_0^{\sigma_M} &= \tilde{\nabla}_0^{\sigma_M} \circ \chi_\alpha - \chi_\alpha \circ \tilde{\nabla}_0^{\sigma_M} \\
 &= v_\alpha^{\hat{\mu}_k \nabla_0}
 \end{aligned}$$

by the equality (60). Then we may replace $(\{\eta(-u_{\alpha\beta}^{\hat{\mu}_k \nabla_0})\}, \{v_\alpha^{\hat{\mu}_k \nabla_0}\})$ for

$$(62) \quad (\{\eta(-u_{\alpha\beta}^{\hat{\mu}_k \nabla_0}) - \eta_0(\chi_\beta) + \eta_0(\chi_\alpha)\}_{\alpha\beta}, \{0\}_\alpha).$$

We set

$$x_{\alpha\beta}^{\nabla_0} := \eta(-u_{\alpha\beta}^{\hat{\mu}_k \nabla_0}) - \eta_0(\chi_\beta) + \eta_0(\chi_\alpha).$$

Second, we consider the pair $(\{\eta(u'_{\alpha\beta})\}, \{(v_\alpha^0)'\})$. We take a lift

$$\hat{\mu}(0) \in H^1(\mathcal{C}_M, \Theta_{\mathcal{C}_M/M}(-D(\tilde{\mathbf{t}}) - D(\tilde{\mathbf{p}})))$$

of $0 \in H^1(\mathcal{C}_M, \Theta_{\mathcal{C}_M/M}(-D(\tilde{\mathbf{t}})))$ by a flat family $(\mathcal{C}_M \times_M M[\epsilon], \text{Supp}(F_{v'_\eta}^*(D(\tilde{\mathbf{t}}) + D(\tilde{\mathbf{p}}))))$ of curves with points as in the definition of the lift (27) of μ . Let $\tilde{F}_{v'_\eta}: \mathcal{C}_{v'_\eta}^{\hat{\mu}(0)} \rightarrow \mathcal{C}_M$ be the morphism corresponding a representative of $\hat{\mu}(0)$. We consider the pull-back of $(\tilde{E}, \tilde{\nabla}_0^{\sigma_M})$ by $\tilde{F}_{v'_\eta}: \mathcal{C}_{v'_\eta}^{\hat{\mu}(0)} \rightarrow \mathcal{C}_M$ and we apply Lemma 4.9 to $\tilde{F}_{v'_\eta}: \mathcal{C}_{v'_\eta}^{\hat{\mu}(0)} \rightarrow \mathcal{C}_M$ and the lift $\hat{\mu}(0)$. Then we can take $\chi'_\alpha \in \mathcal{A}_{\tilde{E}}(U_\alpha)$ (for any α) such that $\tilde{\nabla}_0^{\sigma_M} \circ \chi'_\alpha - \chi'_\alpha \circ \tilde{\nabla}_0^{\sigma_M} = (v_\alpha^0)'$ for any α . As in the proof of Lemma 3.4, we can check the following equality:

$$(63) \quad \tilde{\nabla}_0^{\sigma_M} \circ \eta_0(\chi'_\alpha) - \eta_0(\chi'_\alpha) \circ \tilde{\nabla}_0^{\sigma_M} = (v_\alpha^0)'.$$

Remark that $\eta_0(\chi'_\alpha)$ may have poles on the components of $D(\tilde{\mathbf{p}})$, since $\tilde{\nabla}_0^{\sigma_M}$ has poles on the components of $D(\tilde{\mathbf{p}})$. Now we consider the class (59). By (62), the class (59) coincides with

$$(64) \quad \left[\left(- \left\{ \text{Tr} \left(x_{\alpha\beta}^{\nabla_0} \circ \eta(u'_{\beta\gamma}) \right) \right\}_{\alpha\beta\gamma}, \left\{ \text{Tr} \left(x_{\alpha\beta}^{\nabla_0} \circ (v_\beta^0)' \right) \right\}_{\alpha\beta} \right) \right].$$

We may check this coincidence as follows. We take the image of cochain $(\{\text{Tr}(\eta_0(\chi_\alpha) \circ \eta(u'_{\alpha\beta}))\}_{\alpha\beta}, \{-\text{Tr}(\eta_0(\chi_\alpha) \circ (v_\alpha^0)')\}_\alpha)$:

$$\begin{array}{ccc}
 \{\text{Tr}((\eta_0(\chi_\beta) - \eta_0(\chi_\alpha)) \circ \eta(u'_{\beta\gamma}))\}_{\alpha\beta\gamma} & & \\
 \uparrow \delta_1 & & \\
 \{\text{Tr}(\eta_0(\chi_\alpha) \circ \eta(u'_{\alpha\beta}))\}_{\alpha\beta} & \xrightarrow{d} & \{d \text{Tr}(\eta_0(\chi_\alpha) \circ \eta(u'_{\alpha\beta}))\}_{\alpha\beta} - \{\text{Tr}(\eta_0(\chi_\beta)(v_\beta^0)') - \text{Tr}(\eta_0(\chi_\alpha)(v_\alpha^0)')\}_{\alpha\beta} \\
 & & \uparrow \delta_0 \\
 & & \{-\text{Tr}(\eta_0(\chi_\alpha) \circ (v_\alpha^0)')\}_\alpha \xrightarrow{d} 0
 \end{array}$$

Here

$$\begin{aligned}
 \delta_0: \prod_{\alpha} \Omega_{\mathcal{C}_M/M}^1(U_\alpha) &\longrightarrow \prod_{\alpha \neq \beta} \Omega_{\mathcal{C}_M/M}^1(U_\alpha \cap U_\beta) \text{ and} \\
 \delta_1: \prod_{\alpha \neq \beta} \mathcal{O}_{\mathcal{C}_M}(U_\alpha \cap U_\beta) &\longrightarrow \prod_{\alpha \neq \beta \neq \gamma} \mathcal{O}_{\mathcal{C}_M}(U_\alpha \cap U_\beta \cap U_\gamma)
 \end{aligned}$$

are the coboundary operators, and d is the (relative) exterior derivative on $\mathcal{C}_M \rightarrow M$. Then we have a coboundary. We add this coboundary to the cocycle of (59):

$$-\mathrm{Tr}(\eta(-u_{\alpha\beta}^{\hat{\mu}_k \nabla^0}) \circ \eta(u'_{\beta\gamma})) + \mathrm{Tr}((\eta_0(\chi_\beta) - \eta_0(\chi_\alpha)) \circ \eta(u'_{\beta\gamma})) = -\mathrm{Tr}\left(x_{\alpha\beta}^{\nabla^0} \circ \eta(u'_{\beta\gamma})\right)$$

and

$$\begin{aligned} & \omega_{\alpha\beta}^0 + d\mathrm{Tr}(\eta_0(\chi_\alpha) \circ \eta(u'_{\alpha\beta})) - \mathrm{Tr}(\eta_0(\chi_\beta) \circ (v_\beta^0)') + \mathrm{Tr}(\eta_0(\chi_\alpha) \circ (v_\alpha^0)') \\ &= \mathrm{Tr}(-\eta(u_{\alpha\beta}^{\hat{\mu}_k \nabla^0}) \circ (v_\beta^0)') - \mathrm{Tr}\left(\left(\tilde{\nabla}_0^{\sigma M} \circ \eta_0(\chi_\alpha) - \eta_0(\chi_\alpha) \circ \tilde{\nabla}_0^{\sigma M}\right) \circ \eta(u'_{\alpha\beta})\right) \\ (65) \quad & + d\mathrm{Tr}(\eta_0(\chi_\alpha) \circ \eta(u'_{\alpha\beta})) - \mathrm{Tr}(\eta_0(\chi_\beta) \circ (v_\beta^0)') \\ & + \mathrm{Tr}(\eta_0(\chi_\alpha) \circ \left((v_\beta^0)' - \tilde{\nabla}_0^{\sigma M} \circ (\eta(u'_{\alpha\beta})) + (\eta(u'_{\alpha\beta})) \circ \tilde{\nabla}_0^{\sigma M}\right)) \\ &= \mathrm{Tr}\left(x_{\alpha\beta}^{\nabla^0} \circ (v_\beta^0)'\right). \end{aligned}$$

Here the first equality of (65) follows from (53) and (61). The second equality of (65) is checked as in (58). Then we have this coincidence between (59) and (64). By the equality (63) and the cocycle condition $\tilde{\nabla}_0^{\sigma M} \circ x_{\alpha\beta}^{\nabla^0} - x_{\alpha\beta}^{\nabla^0} \circ \tilde{\nabla}_0^{\sigma M} = 0$ of (62), we have

$$(66) \quad d\left(\mathrm{Tr}\left(x_{\alpha\beta}^{\nabla^0} \circ \eta_0(\chi'_\beta)\right)\right) = \mathrm{Tr}\left(x_{\alpha\beta}^{\nabla^0} \circ (v_\beta^0)'\right).$$

Indeed,

$$\begin{aligned} \mathrm{Tr}\left(x_{\alpha\beta}^{\nabla^0} \circ (v_\beta^0)'\right) &= \mathrm{Tr}\left(x_{\alpha\beta}^{\nabla^0} \circ \left(\tilde{\nabla}_0^{\sigma M} \circ \eta_0(\chi'_\beta) - \eta_0(\chi'_\beta) \circ \tilde{\nabla}_0^{\sigma M}\right)\right) \\ &= \mathrm{Tr}\left(x_{\alpha\beta}^{\nabla^0} \circ d(\eta_0(\chi'_\beta))\right) + \mathrm{Tr}\left(d\left(x_{\alpha\beta}^{\nabla^0}\right) \circ \eta_0(\chi'_\beta)\right) \\ &\quad - \mathrm{Tr}\left(\left(\tilde{\nabla}_0^{\sigma M} \circ x_{\alpha\beta}^{\nabla^0} - x_{\alpha\beta}^{\nabla^0} \circ \tilde{\nabla}_0^{\sigma M}\right) \circ \eta_0(\chi'_\beta)\right) \\ &= d\left(\mathrm{Tr}\left(x_{\alpha\beta}^{\nabla^0} \circ \eta_0(\chi'_\beta)\right)\right). \end{aligned}$$

Note that the trace of $x_{\alpha\beta}^{\nabla^0} \circ \eta_0(\chi'_\beta)$ has no pole on $U_{\alpha\beta}$. That is, the trace of $x_{\alpha\beta}^{\nabla^0} \circ \eta_0(\chi'_\beta)$ is an element of $\mathcal{O}_{\mathcal{C}_M}(U_{\alpha\beta})$. We may check that

$$x_{\beta\gamma}^{\nabla^0} \circ \eta_0(\chi'_\gamma) - x_{\alpha\gamma}^{\nabla^0} \circ \eta_0(\chi'_\gamma) + x_{\alpha\beta}^{\nabla^0} \circ \eta_0(\chi'_\beta) = x_{\alpha\beta}^{\nabla^0} \circ (\eta_0(\chi'_\beta) - \eta_0(\chi'_\gamma))$$

by the cocycle condition of $\{x_{\alpha\beta}^{\nabla^0}\}$. By this equality and the equality (66), we may check the class (64) coincides with

$$(67) \quad \left[\left(- \left\{ \mathrm{Tr}\left(x_{\alpha\beta}^{\nabla^0} \circ (\eta(u'_{\beta\gamma}) - \eta_0(\chi'_\beta) + \eta_0(\chi'_\gamma))\right) \right\}_{\alpha\beta\gamma}, \{0\}_{\alpha\beta} \right) \right] \in \mathbf{H}^2(\mathcal{C}_M, \Omega_{\mathcal{C}_M/M}^\bullet).$$

Remark that $d(\mathrm{Tr}\left(x_{\alpha\beta}^{\nabla^0} \circ (\eta(u'_{\beta\gamma}) - \eta_0(\chi'_\beta) + \eta_0(\chi'_\gamma))\right)) = 0$, by the cocycle condition of $\mathbf{H}^2(\mathcal{C}_M, \Omega_{\mathcal{C}_M/M}^\bullet)$.

That is $\mathrm{Tr}\left(x_{\alpha\beta}^{\nabla^0} \circ (\eta(u'_{\beta\gamma}) - \eta_0(\chi'_\beta) + \eta_0(\chi'_\gamma))\right)$ is constant. Then we may consider this class in $\mathbf{H}^2(\mathcal{C}_M, \Omega_{\mathcal{C}_M/M}^\bullet)$ as a class in a second cohomology of a locally constant sheaf. Since this locally constant sheaf is with the Zariski topology, we obtain that the class (67) vanishes. Finally, we have that the class (59) vanishes. We obtain the equality (54) by the equality (55). \square

Theorem 4.15. *Let ω_M be the restriction of (45) on the affine open subset $M \subset M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)_\nu$. We define $dH_k \in \mathrm{Hom}_{\mathcal{O}_M}(\mathbf{H}^1(\mathcal{G}_M^\bullet), \mathcal{O}_M)$ as*

$$dH_k: \mathbf{H}^1(\mathcal{G}_M^\bullet) \rightarrow \mathcal{O}_M; [(\{u_{\alpha\beta}\}, \{v_\alpha\})] \mapsto d_{M/T}H_k[(\{\eta(u_{\alpha\beta})\}, \{v_\alpha\})].$$

Then the 1-form $\omega_M(v_{\hat{\mu}_k}, \cdot) \in \mathrm{Hom}_{\mathcal{O}_M}(\mathbf{H}^1(\mathcal{G}_M^\bullet), \mathcal{O}_M)$ coincides with the 1-form dH_k .

Proof. We claim that the 1-form $dH_i \in \mathbf{H}^1(\mathcal{G}_M^\bullet)^\vee$ is described as

$$(68) \quad dH_k: \mathbf{H}^1(\mathcal{G}_M^\bullet) \rightarrow H^1(\Omega_{\mathcal{C}_M/M}^1) \cong H^0(\mathcal{O}_M) \\ [(\{u'_{\alpha\beta}\}, \{v'_\alpha\})] \mapsto \left[\left\{ \mathrm{Tr} \left(\left(\bar{\phi}_\beta^{-1} \circ \left\langle (\hat{d}_k)_{\alpha\beta}, \left(\tilde{A}_\beta \frac{df_\beta}{f_\beta} - \tilde{A}_\beta^0 \frac{dz_\beta}{z_\beta} \right) \right\rangle \circ \bar{\phi}_\beta \right) (v'_\beta - (v_\beta^0)') \right\} \right].$$

We show this claim. We consider the pull-back of $\tilde{\nabla} - \tilde{\nabla}_0^{\sigma_m}$ under the morphism $F_{v'_\eta} : \mathcal{C}_M \times_M M[\epsilon] \rightarrow \mathcal{C}_M$. We have the following equalities

$$\begin{aligned} \mathrm{Tr} \left(F_{v'_\eta}^* \tilde{\nabla} - F_{v'_\eta}^* \tilde{\nabla}_0^{\sigma_m} \right)^2 &= \mathrm{Tr} \left(\varphi_\beta(v'_\eta) \circ \left(F_{v'_\eta}^* \tilde{\nabla} - F_{v'_\eta}^* \tilde{\nabla}_0^{\sigma_m} \right) \circ \varphi_\beta(v'_\eta)^{-1} \right) \\ &= \mathrm{Tr} \left(\left(\tilde{\nabla} + \epsilon v'_\beta - \tilde{\nabla}_0^{\sigma_m} - \epsilon(v'_\beta)' \right)^2 \right) \\ &= \mathrm{Tr} \left(\tilde{\nabla} - \tilde{\nabla}_0^{\sigma_m} \right)^2 + 2\epsilon \mathrm{Tr} \left(\left(\tilde{\nabla} - \tilde{\nabla}_0^{\sigma_m} \right) (v'_\beta - (v'_\beta)') \right) \\ &= \mathrm{Tr} \left(\tilde{\nabla} - \tilde{\nabla}_0^{\sigma_m} \right)^2 + 2\epsilon \mathrm{Tr} \left(\left(\bar{\phi}_\beta^{-1} \circ \left(\tilde{A}_\beta \frac{df_\beta}{f_\beta} - \tilde{A}_\beta^0 \frac{dz_\beta}{z_\beta} \right) \circ \bar{\phi}_\beta \right) (v'_\beta - (v'_\beta)') \right). \end{aligned}$$

Let $D_{v'_\eta} : \mathcal{O}_M \rightarrow \mathcal{O}_M$ be the derivative corresponding to the vector field v'_η . We compute $D_{v'_\eta} \left(\frac{1}{2} \{ \mathrm{Tr}((H_k)_{\alpha\beta}) \} \right)$. By the equalities (49) and (50), we have

$$\begin{aligned} (69) \quad & D_{v'_\eta} \left(\frac{1}{2} \{ \mathrm{Tr}((H_k)_{\alpha\beta}) \} \right) \\ &= \left[\left\{ \mathrm{Tr} \left(\left(\bar{\phi}_\beta^{-1} \circ \left\langle (d_k)_{\alpha\beta}, \left(\tilde{A}_\beta \frac{df_\beta}{f_\beta} - \tilde{A}_\beta^0 \frac{dz_\beta}{z_\beta} \right) \right\rangle \circ \bar{\phi}_\beta \right) (v'_\beta - (v'_\beta)') \right\} \right] \\ &+ \left[\left\{ \mathrm{Tr} \left(\left(\bar{\phi}_\beta^{-1} \circ \left\langle \left(\tilde{\delta}_{k,\beta}^{(0)} \frac{\partial}{\partial z_\beta} - \tilde{\delta}_{k,\alpha}^{(0)} \frac{\partial}{\partial z_\alpha} \right), \left(\tilde{A}_\beta \frac{df_\beta}{f_\beta} - \tilde{A}_\beta^0 \frac{dz_\beta}{z_\beta} \right) \right\rangle \circ \bar{\phi}_\beta \right) (v'_\beta - (v'_\beta)') \right\} \right] \\ &\in H^1(\Omega_{\mathcal{C}_M/M}^1). \end{aligned}$$

Since $D_{v'_\eta} \left(\frac{1}{2} \{ \mathrm{Tr}((H_k)_{\alpha\beta}) \} \right) = d_{M/T} H_k([\{\eta(u'_{\alpha\beta})\}, \{v'_\alpha\}])$ and

$$d_{M/T} H_k([\{\eta(u'_{\alpha\beta})\}, \{v'_\alpha\}]) = dH_k([\{u'_{\alpha\beta}\}, \{v'_\alpha\}]),$$

we have the description (68).

By the definition (29) of $u_{\alpha\beta}^{\hat{\mu}_k \nabla^0}$ for $\hat{\mu} = \hat{\mu}_k$ and the definition (44) of η , we have

$$\eta(u_{\alpha\beta}^{\hat{\mu}_k \nabla^0}) = \bar{\phi}_\beta^{-1} \circ \left\langle (\hat{d}_k)_{\alpha\beta}, \left(\tilde{A}_\beta^0 \frac{dz_\beta}{z_\beta} - \tilde{A}_\beta \frac{df_\beta}{f_\beta} \right) \right\rangle \circ \bar{\phi}_\beta.$$

Here, $\tilde{A}_\alpha^0 z_\alpha^{-1} dz_\alpha$ and $\tilde{A}_\alpha f_\alpha^{-1} df_\alpha$ be connection matrices of $\tilde{\nabla}_0^{\sigma_M}$ and $\tilde{\nabla}$ on U_α via the trivialization $\bar{\phi}_\alpha$, respectively. By the description (68) of dH_k , we have the following equalities:

$$\begin{aligned} dH_k([\{u'_{\alpha\beta}\}, \{v'_\alpha\}]) &= \left[\left\{ \mathrm{Tr} \left(\left(\bar{\phi}_\beta^{-1} \circ \left\langle (\hat{d}_k)_{\alpha\beta}, \left(\tilde{A}_\beta \frac{df_\beta}{f_\beta} - \tilde{A}_\beta^0 \frac{dz_\beta}{z_\beta} \right) \right\rangle \circ \bar{\phi}_\beta \right) (v'_\beta - (v'_\beta)') \right\} \right] \\ &= \left[\left\{ \mathrm{Tr} \left(-\eta(u_{\alpha\beta}^{\hat{\mu}_k \nabla^0}) \circ (v'_\beta - (v'_\beta)') \right) \right\}_{\alpha\beta} \right] \in H^1(\Omega_{\mathcal{C}_M/M}^1). \end{aligned}$$

We take the image of this element of $H^1(\Omega_{\mathcal{C}_M/M}^1)$ under the natural morphism $H^1(\Omega_{\mathcal{C}_M/M}^1) \rightarrow \mathbf{H}^2(\Omega_{\mathcal{C}_M/M}^\bullet)$.

This image is

$$\left[\left(\{0\}_{\alpha\beta\gamma}, \left\{ \mathrm{Tr} \left(-\eta(u_{\alpha\beta}^{\hat{\mu}_k \nabla^0}) \circ (v'_\beta - (v'_\beta)') \right) \right\}_{\alpha\beta} \right) \right] \in \mathbf{H}^2(\Omega_{\mathcal{C}_M/M}^\bullet).$$

By Lemme 4.14, this image coincides with $\omega(v_{\hat{\mu}_k}, [\{\eta(u'_{\alpha\beta})\}, \{v'_\alpha\}])$. For the natural morphism $H^1(\Omega_{\mathcal{C}_M/M}^1) \rightarrow \mathbf{H}^2(\Omega_{\mathcal{C}_M/M}^\bullet)$, the diagram

$$\begin{array}{ccc} H^1(\Omega_{\mathcal{C}_M/M}^1) & \longrightarrow & \mathbf{H}^2(\Omega_{\mathcal{C}_M/M}^\bullet) \\ & \searrow \cong & \downarrow \cong \\ & & H^0(\mathcal{O}_M) \end{array}$$

is commutative. Then the image of $dH_k([\{u'_{\alpha\beta}\}, \{v'_\alpha\}])$ under the isomorphism $H^1(\Omega_{\mathcal{C}_M/M}^1) \xrightarrow{\cong} H^0(\mathcal{O}_M)$ coincides with the image of $\omega(v_{\hat{\mu}_k}, [\{\eta(u'_{\alpha\beta})\}, \{v'_\alpha\}])$ under the isomorphism $\mathbf{H}^2(\Omega_{\mathcal{C}_M/M}^\bullet) \xrightarrow{\cong} H^0(\mathcal{O}_M)$. This coincidence means the coincidence in the statement of this theorem. \square

In the proof of Lemma 4.14, we have the equality (57). Now we consider meaning of this equality by using the morphism h_{∇_0} . We take μ_k and the lift $\hat{\mu}_k$ of μ_k . We take the vector field

$$[(\{\hat{u}_{\alpha\beta}^{\text{IMD}}\}, \{\hat{v}_\alpha^{\text{IMD}}\})]$$

of isomonodromic deformations in M associated to μ_k . We take the image of $[(\{\hat{u}_{\alpha\beta}^{\text{IMD}}\}, \{\hat{v}_\alpha^{\text{IMD}}\})]$ under the morphism $\Theta_M \rightarrow h_{\nabla_0}^* \Theta_{M'}$. Then the infinitesimal deformation of $(\tilde{E}, \tilde{\nabla}_0^{\sigma M}, \mathbf{l}^{\sigma m})$ associated to this image corresponds to the pair

$$(70) \quad (\{\hat{u}_{\alpha\beta}^{\text{IMD}}\}_{\alpha\beta}, \{v_\alpha^{\hat{\mu}_k \nabla_0}\}_\alpha).$$

We consider the vector field of the isomonodromic deformation for $\varpi': M' \rightarrow \mathcal{M}_{g,n+N}$. That is, we define a splitting $(\varpi')^* \Theta_{\mathcal{M}_{g,n+N}} \rightarrow \Theta_{M'}$. By h_{∇_0} , we have a morphism $h_{\nabla_0}^* (\varpi')^* \Theta_{\mathcal{M}_{g,n+N}} \rightarrow h_{\nabla_0}^* \Theta_{M'}$. The lift $\hat{\mu}_k$ of μ_k gives a section of $h_{\nabla_0}^* (\varpi')^* \Theta_{\mathcal{M}_{g,n+N}}$. We take the image of $\hat{\mu}_k$ by this morphism $h_{\nabla_0}^* (\varpi')^* \Theta_{\mathcal{M}_{g,n+N}} \rightarrow h_{\nabla_0}^* \Theta_{M'}$. This image gives an infinitesimal deformation of $\tilde{\nabla}_0^{\sigma M}$ parametrized by $M[\epsilon]$. By this infinitesimal deformation of $(\tilde{E}, \tilde{\nabla}_0^{\sigma M}, \mathbf{l}^{\sigma m})$, we have a pair

$$(71) \quad (\{u_{\alpha\beta}^{\hat{\mu}_k \nabla_0}\}_{\alpha\beta}, \{0\}_\alpha).$$

We consider the difference of (70) and (71), which is described by

$$(72) \quad (\{\eta_0(\hat{u}_{\alpha\beta}^{\text{IMD}})\}_{\alpha\beta}, \{v_\alpha^{\hat{\mu}_k \nabla_0}\}_\alpha).$$

Here η_0 is defined in (56). The pair (72) means an infinitesimal deformation of $(\tilde{E}, \tilde{\nabla}_0^{\sigma M}, \mathbf{l}^{\sigma m})$. The equality (57) means the cocycle condition of $(\{\eta_0(\hat{u}_{\alpha\beta}^{\text{IMD}})\}, \{v_\alpha^{\hat{\mu}_k \nabla_0}\})$. Here note that

$$-\eta(u_{\alpha\beta}^{\hat{\mu}_k \nabla_0}) = \eta_0(\hat{u}_{\alpha\beta}^{\text{IMD}}).$$

That is, $(\{\eta_0(\hat{u}_{\alpha\beta}^{\text{IMD}})\}, \{v_\alpha^{\hat{\mu}_k \nabla_0}\})$ gives an element of $h_{\nabla}^* \Theta_{M'}$, which is described by using some hypercohomology. The isomonodromic deformations gives the isomonodromic splitting $\Theta_{M'} \cong \Theta_{M'}/\mathcal{M}_{g,n+N} \oplus (\varpi')^* \Theta_{\mathcal{M}_{g,n+N}}$. Roughly speaking, the class of (72) means the first component of the decomposition of the class of (70) induced by the isomonodromic splitting.

Next we consider meaning of (55) and (59). We may define a relative 2-form

$$\omega_{M'/\mathcal{M}_{g,n+N}}^0 : \Theta_{M'/\mathcal{M}_{g,n+N}} \otimes \Theta_{M'/\mathcal{M}_{g,n+N}} \longrightarrow \mathcal{O}_{M'}$$

on $\Theta_{M'/\mathcal{M}_{g,n+N}}$ as in Section 2.3. By the isomonodromic splitting $\Theta_{M'} \cong \Theta_{M'/\mathcal{M}_{g,n+N}} \oplus (\varpi')^* \Theta_{\mathcal{M}_{g,n+N}}$ and taking the first component of vector fields in $\Theta_{M'}$, we may define a 2-form

$$\omega_{M'}^0 : \Theta_{M'} \otimes \Theta_{M'} \longrightarrow \mathcal{O}_{M'}.$$

We consider the pull-back $h_{\nabla_0}^* \omega_{M'}^0$ of the 2-form $\omega_{M'}^0$ under the morphism $h_{\nabla_0} : M \rightarrow M'$. We put

$$\Phi_0 := h_{\nabla_0}^* \omega_{M'}^0([\{\hat{u}_{\alpha\beta}^{\text{IMD}}\}, \{\hat{v}_\alpha^{\text{IMD}}\}], \cdot),$$

which is a 1-form on M . We denote by Φ_1 the 1-form $\omega_M(v_{\hat{\mu}_k}, \cdot)$ on M . Moreover, we denote by $\tilde{\Phi}_0$ the relative 1-form induced by Φ_0 on $M \rightarrow T \times \{\nu\}$ and we denote by $\tilde{\Phi}_1$ the relative 1-form induced by Φ_1 on $M \rightarrow T \times \{\nu\}$. We consider the relative 1-form $d_{M/T} H_k$ on $\varpi_M : M \rightarrow T$. By the isomonodromic splitting we take the 2-forms $\tilde{\Phi}_1^{\text{IMD}}$, dH_k , and $\tilde{\Phi}_0^{\text{IMD}}$ on M induced by $\tilde{\Phi}_1$, $d_{M/T} H_k$, and $\tilde{\Phi}_0$, respectively. We may check that the class (59) coincides with the value $\tilde{\Phi}_0^{\text{IMD}}([\{u'_{\alpha\beta}\}, \{v'_\alpha\}])$ by the argument as in the proof of Lemma 4.14. The equality (55) means the equality

$$\tilde{\Phi}_1^{\text{IMD}} = dH_k + \tilde{\Phi}_0^{\text{IMD}}.$$

4.6. Hamiltonian description of isomonodromic deformations. Set $N = r^2(g-1) + nr(r-1)/2 + 1$. Note that $\dim M = 2N$. Let $\partial/\partial q_i \in \mathbf{H}^1(\mathcal{F}_M^\bullet)$ and $\partial/\partial p_i \in \mathbf{H}^1(\mathcal{F}_M^\bullet)$ ($i = 1, \dots, N$) be vector fields on M such that the morphism

$$\begin{aligned} \mathcal{O}_M^{\oplus 2N} &\longrightarrow \mathbf{H}^1(\mathcal{F}_M^\bullet) \\ (f_1, \dots, f_{2N}) &\longmapsto f_1 \partial/\partial q_1 + \dots + f_N \partial/\partial q_N + f_{N+1} \partial/\partial p_1 + \dots + f_{2N} \partial/\partial p_N \end{aligned}$$

gives a trivialization of $\mathbf{H}^1(\mathcal{F}_M^\bullet)$ and the vector fields satisfy the conditions $\omega_M(\partial/\partial q_i, \partial/\partial q_j) = \omega_M(\partial/\partial p_i, \partial/\partial p_j) = 0$ and $\omega_M(\partial/\partial q_i, \partial/\partial p_j) = \delta_{i,j}$, where $\delta_{i,j}$ is the Kronecker's symbol. Here we also denote by $\partial/\partial q_i$ and $\partial/\partial p_i$ the images of $\partial/\partial q_i \in \mathbf{R}^1\pi_*(\mathcal{F}^\bullet)(M)$ and $\partial/\partial p_i \in \mathbf{R}^1\pi_*(\mathcal{F}_M^\bullet)$ by the tangent morphism

$$\mathbf{H}^1(\mathcal{F}_M^\bullet) \longrightarrow \mathbf{H}^1(\mathcal{G}_M^\bullet).$$

Remark 4.16. We do not have a proof of existence of such algebraic vector fields $\partial/\partial q_i \in \mathbf{H}^1(\mathcal{G}_M^\bullet)$ and $\partial/\partial p_i \in \mathbf{H}^1(\mathcal{G}_M^\bullet)$. For some special cases, there exists studies of giving algebraic Darboux coordinates for some symplectic structures on moduli spaces of connections. For example, one of techniques to give algebraic Darboux coordinates is the *theory of apparent singularities* ([21], [13], [6], [17], [7]). In this paper, we do not discuss that there exist algebraic Darboux coordinates for the symplectic structure. We assume that existence, although this argument is optimistic.

Corollary 4.17. *If we may take vector fields $\partial/\partial q_i \in \mathbf{H}^1(\mathcal{G}_M^\bullet)$ and $\partial/\partial p_i \in \mathbf{H}^1(\mathcal{G}_M^\bullet)$ as above, then the vector field determined by the isomonodromic deformation on M is described as*

$$(73) \quad v_{\hat{\mu}_k} - \sum_{i=1}^N \left(dH_k \left(\frac{\partial}{\partial p_i} \right) \frac{\partial}{\partial q_i} - dH_k \left(\frac{\partial}{\partial q_i} \right) \frac{\partial}{\partial p_i} \right)$$

for $k = 1, \dots, 3g - 3 + n$.

Proof. Let X be the vector field (73). We show that $X \in \text{Ker}(\omega_M)$. Note that

$$\begin{aligned} \omega_M(\partial/\partial q_i, \cdot) : [(\{u_{\alpha\beta}\}, \{v_\alpha\})] &\longmapsto \omega(\partial/\partial q_i, [(\{\eta(u_{\alpha\beta})\}, \{v_\alpha\})]), \\ \omega_M(\partial/\partial p_i, \cdot) : [(\{u_{\alpha\beta}\}, \{v_\alpha\})] &\longmapsto \omega(\partial/\partial p_i, [(\{\eta(u_{\alpha\beta})\}, \{v_\alpha\})]), \\ dH_k : [(\{u_{\alpha\beta}\}, \{v_\alpha\})] &\longmapsto d_{M/(T \times N_r^{(n)}(d))} H_k [(\{\eta(u_{\alpha\beta})\}, \{v_\alpha\})]. \end{aligned}$$

We have

$$dH_k - \sum_i \left(dH_k \left(\frac{\partial}{\partial p_i} \right) \omega_M(\partial/\partial q_i, \cdot) - dH_k \left(\frac{\partial}{\partial q_i} \right) \omega_M(\partial/\partial p_i, \cdot) \right) = 0.$$

By Theorem 4.15, we have

$$\omega_M(v_{\hat{\mu}_k}, \cdot) - \sum_i \left(dH_k \left(\frac{\partial}{\partial p_i} \right) \omega_M(\partial/\partial q_i, \cdot) - dH_k \left(\frac{\partial}{\partial q_i} \right) \omega_M(\partial/\partial p_i, \cdot) \right) = 0.$$

Then we obtain that $X \in \text{Ker}(\omega_M)$. Proposition 4.10 implies that X is the vector field determined by the isomonodromic deformation. By uniqueness of the isomonodromic deformation for a Kodaira–Spencer class, we obtain this corollary. \square

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CENTER FOR MATHEMATICAL AND DATA SCIENCES, KOBE UNIV., 1-1 ROKKODAI-CHO, NADA-KU, KOBE, 657-8501, JAPAN

Email address: akomyo@math.kobe-u.ac.jp