

EXISTENCE OF A CONJUGATE POINT IN THE INCOMPRESSIBLE EULER FLOW ON AN ELLIPSOID

TAITO TAUCHI AND TSUYOSHI YONEDA

ABSTRACT. Existence of a conjugate point in the incompressible Euler flow on a sphere and an ellipsoid is considered. Misiólek (1996) formulated a differential-geometric criterion (we call the M-criterion) for the existence of a conjugate point in a fluid flow. In this paper, it is shown that no zonal flow (stationary Euler flow) satisfies the M-criterion if the background manifold is a sphere, on the other hand, there are zonal flows satisfy the M-criterion if the background manifold is an ellipsoid (even it is sufficiently close to the sphere). The conjugate point is created by the fully nonlinear effect of the inviscid fluid flow with differential geometric mechanism.

1. INTRODUCTION

A remarkable example of a stable multiple zonal jet flow can be observed in Jupiter even against the perturbation by, for example, the famous Great Red Spot. Despite attracting considerable attention over the years, its mechanism is not yet well understood. The incompressible 2D-Navier-Stokes equations on a rotating sphere are one of the simplest models of it, and many researchers have been extensively studying this models. Williams [27] was the first to find that turbulent flow becomes multiple jet flows on such a model. However, he was assuming high symmetry to the flow field. After that Yoden-Yamada [28] and Nozawa-Yoden [18] made further progress. In particular, Obuse-Takehiro-Yamada [19] calculated non-forced 2D-Navier-Stokes flow (without the symmetry assumption) on a rotating sphere, and observed multiple zonal jet flows merging with each other and finally, only two or three broad zonal jets remain. Thus, it seems that we need to find a totally different idea to clarify the existence of stable multiple zonal jet flow in Jupiter. For the recent development in this study field, see Sasaki-Takehiro-Yamada [23, 24], using a spectral method to the linearized fluid equations.

However, as far as the authors are aware, none of the numerous works to date attempted to investigate the effect of the “background manifold” itself. In the above simplest model, the background manifold is a “sphere”, even though in reality Jupiter is not a sphere. It has a perceptible bulge around its equatorial middle and is flattened at the poles (see [9]). In this paper, we investigate the effect of the background manifold, in particular, clarify the crucial difference between sphere and ellipsoid. Let us explain more precisely. Misiólek [13] showed Lagrangian instability of the stationary Euler flow with zero pressure term on a manifold with non-positive curvature. He proved it using differential geometric techniques based on Jacobi fields analysis. From the pioneering work of Arnold (see [1, 2, 3, 25] for example)

Date: June 29, 2021.

2010 Mathematics Subject Classification. Primary 35Q35; Secondary 58B20.

Key words and phrases. inviscid fluid flow, diffeomorphism group, conjugate point.

it is known that solutions to the incompressible Euler equations can be seen as geodesics on the configuration space of diffeomorphisms of the background manifold. Furthermore, negative curvature of the configuration space as well as absence of conjugate points along these geodesics can be regarded as suggesting Lagrangian instability of the corresponding fluid flows. In this study, we will thus view existence of a conjugate point as a suggestion of Lagrangian stability. More precisely, the existence of conjugate points should imply geodesics are certainly less unstable and the strong positivity of the sectional curvature of the configuration space. See Definition 1 for the definition of the conjugate point and see also Nakamura-Hattori-Kambe [17] for the explanation of Lagrangian instability. (There exists an approach using numerical simulations to an Euler-Lagrangian analysis of the Navier-Stokes equations, for example [20], in which the author considered the time evolution of the sectional curvatures and some solutions to the incompressible Euler equations.) Subsequently, Misiolek [14] formulated a geometric criterion (we call the M-criterion, see (1.3) and (2.14)) which is a sufficient condition for the existence of a conjugate point in a fluid flow. Moreover, he also showed that there exists a conjugate point along a geodesic of the diffeomorphism group $\mathcal{D}_\mu^s(\mathbb{T}^2)$ of the 2-dimensional flat torus \mathbb{T}^2 . Note that the conjugate point is created by the fully nonlinear effect of the inviscid fluid flow with differential geometric mechanism. In this paper, we show that no zonal flow (a stationary Euler flow) satisfies the M-criterion if the background manifold is a sphere but that some zonal flows satisfy the M-criterion if the background manifold is an ellipsoid (even it is sufficiently close to the sphere), in particular, having a bulge around its equatorial middle and is flattened at the poles.

For the precise statement of our main theorems, we briefly recall the theory of “diffeomorphism groups” in the context of inviscid fluid flows and the M-criterion. See Section 2 for the details.

Let (M, g) be a compact n -dimensional Riemannian manifold without boundary. Write $\mathcal{D}^s(M)$ for the group of Sobolev H^s diffeomorphisms of M and $\mathcal{D}_\mu^s(M)$ for the subgroup of $\mathcal{D}^s(M)$ consisting volume preserving elements, where μ is the volume form on M defined by g . If $s > \frac{n}{2} + 1$, the group $\mathcal{D}^s(M)$ can be given a structure of an infinite-dimensional weak Riemannian manifold (see [5]) and $\mathcal{D}_\mu^s(M)$ is its weak Riemannian submanifold. This weak Riemannian metric on $\mathcal{D}_\mu^s(M)$ is given by

$$(1.1) \quad (V, W) := \int_M g(V, W)\mu,$$

where $V, W \in T_\eta \mathcal{D}_\mu^s(M)$. Here, we identify the tangent space $T_\eta \mathcal{D}_\mu^s(M)$ of $\mathcal{D}_\mu^s(M)$ at a point $\eta \in \mathcal{D}_\mu^s(M)$ with the space of all H^s divergence-free sections of the pullback bundle η^*TM of the tangent bundle TM . Then, if $\eta(t)$ is a geodesic with respect to this metric in $\mathcal{D}_\mu^s(M)$ joining e and $\eta(t_0)$, a time dependent vector field on M defined by $u(t) := \dot{\eta}(t) \circ \eta^{-1}(t)$ is a solution to the Euler equations on M :

$$(1.2) \quad \begin{aligned} \partial_t u + \nabla_u u &= -\text{grad } p & t \in [0, t_0], \\ \text{div } u &= 0, \\ u|_{t=0} &= \dot{\eta}(0), \end{aligned}$$

with a scalar function (pressure) $p(t)$ determined by $u(t)$. In this context, the existence of conjugate points along a geodesic η on $\mathcal{D}_\mu^s(M)$ corresponds to the

stability of a fluid flow $u = \dot{\eta} \circ \eta^{-1}$. We recall that the definition of a *conjugate point*.

Definition 1. (Conjugate point.) Let D be a Riemannian manifold and $\eta(t) := \exp_p(tV)$ a geodesic for some $V \in T_p D$, where $\exp_p : T_p D \rightarrow D$ is the exponential map at $p \in D$. Then we say that $\eta(1)$ is a *conjugate point* or *conjugate* to p along η if the differential $T_V \exp_p : T_V(T_p D) \rightarrow T_{\eta(1)} D$ of the exponential map at V is not bijective. (In the case of $\dim D = \infty$, there are two reasons for a point to be conjugate to another. See Remark 4 and Appendix 2.)

We define the crucial value for the existence of conjugate points by

$$(1.3) \quad MC_{V,W} := (\nabla_V [V, W] + \nabla_{[V,W]} V, W)$$

for $V, W \in T_e \mathcal{D}_\mu^s(M)$. We call this value the *Misiolek curvature* for V and W .

Remark 1. If $[V, W]$ is an only of class C^0 , we cannot define $MC_{V,W}$ for $V, W \in T_e \mathcal{D}_\mu^s(M)$ since we have one more derivative in $[V, W]$ by ∇_V . Therefore, we require $V, W \in T_e \mathcal{D}_\mu^s(M)$ for $s > 2 + \frac{n}{2}$, which implies that V and W are of class C^2 by Sobolev embedding theorem.

The importance of the Misiolek curvature is the following criterion for the existence of conjugate points, which we call the M-criterion.

Fact 1.1 ([14, Lemmas 2 and 3]). Let M be a compact n -dimensional Riemannian manifold without boundary and $s > 2 + \frac{n}{2}$. Suppose that $V \in T_e \mathcal{D}_\mu^s(M)$ is a time independent solution of the Euler equations (1.2) on M and take a geodesic $\eta(t)$ on $\mathcal{D}_\mu^s(M)$ satisfying $V = \dot{\eta} \circ \eta^{-1}$. Then if $W \in T_e \mathcal{D}_\mu^s(M)$ satisfies $MC_{V,W} > 0$, there exists a point conjugate to $e \in \mathcal{D}_\mu^s(M)$ along $\eta(t)$ on $0 \leq t \leq t^*$ for some $t^* > 0$.

Remark 2. This fact is not explicitly stated but essentially proved in [14, Lemmas 2 and 3]. See Appendix 1 for the proof of the case that $\dim M = 2$. In Appendix 1, we clarify more the meaning of $W \in T_e \mathcal{D}_\mu^s(M)$ satisfying $MC_{V,W} > 0$.

We are ready to state our main theorems: Let M be a 2-dimensional ellipsoid or a sphere, more precisely, $M = M_a := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = a^2(1 - z^2)\}$ for some $a > 1$ (having a bulge around its equatorial middle and is flattened at the poles) and $a = 1$ (sphere). We regard M as a Riemannian manifold by the induced metric g from \mathbb{R}^3 . We say that a vector field V on M is a *zonal flow* if V has the following form:

$$(1.4) \quad V = F(z)(y\partial_x - x\partial_y)$$

for some function $F : [-1, 1] \rightarrow \mathbb{R}$. In other words, V is a product of a function $F(z)$ and the flow of the rotation around the z -axis (This flow is nothing more than a Killing vector field on M_a). Recall that the support of a vector field of V on M is the closure of $\{x \in M \mid V(x) \neq 0\}$.

Theorem 1.2. *Suppose $s > 3$ and $a > 1$. For any zonal flow $V \in T_e \mathcal{D}_\mu^s(M_a)$ whose support is contained in $M_a \setminus \{(0, 0, 1), (0, 0, -1)\}$, then there exists $W \in T_e \mathcal{D}_\mu^s(M_a)$ satisfying $MC_{V,W} > 0$.*

On the other hand, in the sphere case, we have the following (cf. [12]):

Theorem 1.3. *Suppose $s > 3$. For any zonal flow $V \in T_e \mathcal{D}_\mu^s(S^2)$ and any $W \in T_e \mathcal{D}_\mu^s(S^2)$, we have $MC_{V,W} \leq 0$.*

Remark 3. The M-criterion itself cannot be a necessary condition for ensuring the existence of a conjugate point. If both V and W are Killing vector fields on a sphere, then this combination induces the existence of a conjugate point (see Remark 2 in Section 3 in [14]). Thus it would be important to clarify the relation between these Killing vector fields and the M-criterion.

Since this study is interdisciplinary, we first try to explain differential geometry step by step, and then finally we prove the main theorems. Therefore, we briefly recall basic facts and prove some results of the theory of diffeomorphism groups in the context of inviscid fluid flows in Section 2. We discuss about our background manifolds, which we call rotationally symmetric manifolds, in Section 3 and apply the facts of Section 2 to our problem in Sections 4 and 5. Moreover, we sophisticate the meaning of $W \in T_e \mathcal{D}_\mu^s(M)$ satisfying $MC_{V,W} > 0$ and prove the M-criterion in the case $\dim M = 2$ in Appendix 1 and solve an apparent paradox concerning the Fredholmness of the exponential map in the 2D case and the M-criterion in Appendix 2.

2. PRELIMINARY

In this section, we recall the theory of diffeomorphism groups in the context of inviscid fluid flows. Our main references are [5] and [13]. We also refer to [8] for a well-organized review of this field. Moreover, the same theory is applied in [26] for the SQG equation.

Let (M, g) be a compact n -dimensional Riemannian manifold without boundary and $\mathcal{D}^s(M)$ the group of Sobolev H^s diffeomorphisms of M and $\mathcal{D}_\mu^s(M)$ the subgroup of $\mathcal{D}^s(M)$ consisting volume preserving elements, where μ is the volume form on M defined by g . If $s > 1 + \frac{n}{2}$, the group $\mathcal{D}^s(M)$ can be given a structure of an infinite-dimensional weak Riemannian manifold (see [5]) and $\mathcal{D}_\mu^s(M)$ become its weak Riemannian submanifold (The term “weak” means that the topology induced from the metric is weaker than the original topology of $\mathcal{D}^s(M)$ or $\mathcal{D}_\mu^s(M)$). This weak Riemannian metric is given as follows: The tangent space $T_\eta \mathcal{D}^s(M)$ of $\mathcal{D}^s(M)$ at a point $\eta \in \mathcal{D}^s(M)$ consists of all H^s vector fields on M which cover η , namely, all H^s sections of the pullback bundle $\eta^* TM$. Thus for $x \in M$ and $V, W \in T_\eta \mathcal{D}^s(M)$, we have $V(x), W(x) \in T_{\eta(x)} M$. Then we define an inner product on $T_\eta \mathcal{D}^s(M)$ by

$$(2.1) \quad (V, W) := \int_M g(V(x), W(x)) \mu(x)$$

and set $|V| := \sqrt{(V, V)}$. Similarly, $T_\eta \mathcal{D}_\mu^s(M)$ consists of all H^s divergence-free vector fields on M which cover $\eta \in \mathcal{D}_\mu^s(M)$. Therefore, the metric (2.1) induces a direct sum:

$$(2.2) \quad T_\eta \mathcal{D}^s(M) = T_\eta \mathcal{D}_\mu^s(M) \oplus \{(\text{grad} f) \circ \eta \mid f \in H^{s+1}(M)\},$$

which follows from the fact that the gradient is the adjoint of the negative divergence. Let

$$\begin{aligned} P_\eta &: T_\eta \mathcal{D}^s(M) \rightarrow T_\eta \mathcal{D}_\mu^s(M) \\ Q_\eta &: T_\eta \mathcal{D}^s(M) \rightarrow \{(\text{grad} f) \circ \eta \mid f \in H^{s+1}(M)\} \end{aligned}$$

be the projection to the first and second components of (2.2), respectively. Moreover, we write $e \in \mathcal{D}^s(M)$ for the identity element of $\mathcal{D}^s(M)$.

Lemma 1. For $X, Y \in T_e \mathcal{D}^s(M)$, we have

$$\begin{aligned} (P_e X, P_e Y) &= (P_e X, Y) = (X, P_e Y), \\ (Q_e X, Q_e Y) &= (Q_e X, Y) = (X, Q_e Y). \end{aligned}$$

Proof. This is clear by the direct sum (2.2). \square

The metric (2.1) also induces the right invariant Levi-Civita connections $\bar{\nabla}$ and $\tilde{\nabla}$ on $\mathcal{D}^s(M)$ and $\mathcal{D}_\mu^s(M)$, respectively. This is defined as follows: Let V, W be vector fields on $\mathcal{D}^s(M)$. We write $V_\eta \in T_\eta \mathcal{D}^s(M)$ for the value of V at $\eta \in \mathcal{D}^s(M)$. Then we have $V_\eta \circ \eta^{-1}, W_\eta \circ \eta^{-1} \in T_e \mathcal{D}^s(M)$, namely, $V_\eta \circ \eta^{-1}$ and $W_\eta \circ \eta^{-1}$ are vector fields on M . Moreover, we have $W_\eta \circ \eta^{-1}$ is a vector field of class C^1 on M by Sobolev embedding theorem and the assumption $s > 1 + \frac{n}{2}$. Thus we can consider $\nabla_{V_\eta \circ \eta^{-1}} W_\eta \circ \eta^{-1}$, where ∇ is the Levi-Civita connection on M . Take a path φ on $\mathcal{D}^s(M)$ satisfying $\varphi(0) = \eta$ and $V_\eta = \partial_t \varphi(0) \in T_\eta \mathcal{D}_\mu^s(M)$, then we define

$$(2.3) \quad (\bar{\nabla}_V W)_\eta := \frac{d}{dt} (W_{\varphi(t)} \circ \varphi^{-1}(t))|_{t=0} \circ \eta + (\nabla_{V_\eta \circ \eta^{-1}} W_\eta \circ \eta^{-1}) \circ \eta.$$

Moreover, if V and W are vector fields on $\mathcal{D}_\mu^s(M)$, we define

$$(2.4) \quad (\tilde{\nabla}_V W)_\eta := P_\eta (\bar{\nabla}_V W)_\eta.$$

These definitions are independent of the particular choice of $\varphi(t)$. We note that $(\bar{\nabla}_V W)_\eta = (\bar{\nabla}_V W)_e \circ \eta$ if V and W are right invariant vector fields on $\mathcal{D}^s(M)$ (i.e., $\bar{\nabla}$ is right invariant). This is because if W is right invariant, or equivalently, if W satisfies $W_\eta = W_e \circ \eta$ for any $\eta \in \mathcal{D}_\mu^s(M)$, the first term of (2.3) vanishes.

Moreover, the right invariant Levi-Civita connection $\bar{\nabla}$ induces the curvature tensor \bar{R} on $\mathcal{D}^s(M)$, which is given by

$$\bar{R}_\eta(X, Y)Z = (\bar{\nabla}_X \bar{\nabla}_Y Z)_\eta - (\bar{\nabla}_Y \bar{\nabla}_X Z)_\eta - (\bar{\nabla}_{[X, Y]} Z)_\eta$$

for vector fields X, Y and Z on $\mathcal{D}^s(M)$. As in the case of finite-dimensional Riemannian manifolds, this depends only on the values of X, Y and Z at η , in other words, we can define $\bar{R}_\eta(X_\eta, Y_\eta)Z_\eta$ for $X_\eta, Y_\eta, Z_\eta \in T_\eta \mathcal{D}^s(M)$. Therefore the right invariance of $\bar{\nabla}$ implies

$$\bar{R}_\eta(X_\eta, Y_\eta)Z_\eta = (R(X_\eta \circ \eta^{-1}, Y_\eta \circ \eta^{-1})(Z_\eta \circ \eta^{-1})) \circ \eta,$$

where R is the curvature of M . Similarly, the right invariant Levi-Civita connection $\tilde{\nabla}$ induces the curvature tensor \tilde{R} on $\mathcal{D}_\mu^s(M)$, which is given by

$$\tilde{R}_\eta(X_\eta, Y_\eta)Z_\eta = (P_e \nabla_{X_e} P_e \nabla_{Y_e} Z_e - P_e \nabla_{Y_e} P_e \nabla_{X_e} Z_e - P_e \nabla_{[X_e, Y_e]} Z_e) \circ \eta,$$

where $X_e = X_\eta \circ \eta^{-1}$. These curvatures \bar{R} and \tilde{R} are related by the Gauss-Codazzi equations:

$$(2.5) \quad (\bar{R}(X, Y)Z, W) = (\tilde{R}(X, Y)Z, W) + (Q \nabla_X Z, Q \nabla_Y W) - (Q \nabla_Y Z, Q \nabla_X W)$$

for any vector fields X, Y, Z and W on $\mathcal{D}_\mu^s(M)$.

A geodesic joining the identity element $e \in \mathcal{D}_\mu^s(M)$ and $p \in \mathcal{D}_\mu^s(M)$ can be obtained from a variational principle as a stationary point of the energy function:

$$(2.6) \quad E(\eta)_0^{t_0} := \frac{1}{2} \int_0^{t_0} |\dot{\eta}(t)|^2 dt,$$

where η is a curve on $\mathcal{D}_\mu^s(M)$ satisfying $\eta(0) = e$ and $\eta(t_0) = p$ and we set $\dot{\eta}(t) := \partial_t \eta(t) \in T_{\eta(t)} \mathcal{D}_\mu^s(M)$. Let $\xi(r, t) : (-\varepsilon, \varepsilon) \times [0, t_0] \rightarrow \mathcal{D}_\mu^s(M)$ be a variation of a

geodesic $\eta(t)$ with fixed end points, namely, it satisfies $\xi(r, 0) = \eta(0)$, $\xi(r, t_0) = \eta(t_0)$ and $\xi(0, t) = \eta(t)$ for $t \in [0, t_0]$. We sometimes write $\xi_r(t)$ for $\xi(r, t)$. Let $X(t) := \partial_r \xi(r, t)|_{r=0} \in T_{\eta(t)} \mathcal{D}_\mu^s(M)$ be the associated vector field on $\mathcal{D}_\mu^s(M)$. Then the first and the second variations of the above integral are given by

$$(2.7) \quad \begin{aligned} 0 &= E'(\eta)_0^{t_0}(X) = (X(t_0), \dot{\eta}(t_0)) - (X(0), \dot{\eta}(0)) \\ &\quad - \int_0^{t_0} (X(t), \tilde{\nabla}_{\dot{\eta}(t)} \dot{\eta}(t)) dt, \\ E''(\eta)_0^{t_0}(X, X) &= \int_0^{t_0} \{(\tilde{\nabla}_{\dot{\eta}} X, \tilde{\nabla}_{\dot{\eta}} X) - (\tilde{R}_\eta(X, \dot{\eta})\dot{\eta}, X)\} dt. \end{aligned}$$

The reason why the geometry of $\mathcal{D}_\mu^s(M)$ is important is that geodesics in $\mathcal{D}_\mu^s(M)$ correspond to inviscid fluid flows on M , which was first remarked by V. I. Arnol'd [1]. This correspondence is accomplished in the following way: If $\eta(t)$ is a geodesic on $\mathcal{D}_\mu^s(M)$ (i.e., $\tilde{\nabla}_{\dot{\eta}} \eta = 0$) joining e and $\eta(t_0)$, a time dependent vector field on M defined by $u(t) := \dot{\eta}(t) \circ \eta^{-1}(t)$ is a solution to the Euler equations on M :

$$(2.8) \quad \begin{aligned} \partial_t u + \nabla_u u &= -\text{grad } p \quad t \in [0, t_0], \\ \text{div } u &= 0, \\ u|_{t=0} &= \dot{\eta}(0), \end{aligned}$$

with a scalar function (pressure) $p(t)$ determined by $u(t)$. Here $\text{grad } p$ (resp. $\text{div } u$) is the gradient (resp. divergent) of p (resp. u) with respect to the Riemannian metric g of M . In this context, the existence of conjugate points along a geodesic η (see Definition 1) corresponds to the stability (in a short time) of a fluid flow $u = \dot{\eta} \circ \eta^{-1}$.

Remark 4. For an infinite-dimensional Riemannian manifold D , there are two reasons to be a conjugate point [7]. Let $\widetilde{\text{exp}}_{\eta(0)} : T_{\eta(0)} D \rightarrow D$ be the exponential map of D and $\eta(t) := \widetilde{\text{exp}}_{\eta(0)} tV$ a geodesic for some $V \in T_{\eta(0)} D$. Then, we say that $\eta(1)$ is *monoconjugate* (resp. *epiconjugate*) if the differential $T_V \widetilde{\text{exp}}_{\eta(0)}$ of the exponential map at V is not injective (resp. not surjective). Of course, monoconjugate points are important from the view point of the stability of a fluid flow. However, the following fact implies that monoconjugate points and epiconjugate points along any geodesic on $\mathcal{D}_\mu^s(M)$ coincide in the 2D case.

Fact 2.1 ([6, Theorem 1]). Let M be a compact 2-dimensional Riemannian manifold without boundary. Then, the exponential map $\widetilde{\text{exp}}_e : T_e \mathcal{D}_\mu^s(M) \rightarrow \mathcal{D}_\mu^s(M)$, which is induced by the Levi-Civita connection $\tilde{\nabla}$, is a nonlinear Fredholm map. More precisely, for any $V \in T_e \mathcal{D}_\mu^s$, the derivative $T_V \widetilde{\text{exp}}_e : T_V(T_e \mathcal{D}_\mu^s) \simeq T_e \mathcal{D}_\mu^s \rightarrow T_{\widetilde{\text{exp}}_e(V)} \mathcal{D}_\mu^s$ is a bounded Fredholm operator of index zero.

Remark 5. For example, see [4, 11, 22] for further studies of singularities of the exponential map.

In order to consider the existence of a conjugate point, we start with the following proposition, which is proved by Misiólek [14, Lemma 2] in the case of $M = \mathbb{T}^2$. Although Misiólek's proof can be applied to the case that M is arbitrary compact n -dimensional manifold without boundary, we prove the proposition in such case for the sake of completeness.

Proposition 2.2. *Let M be a compact n -dimensional Riemannian manifold without boundary and $V, W \in T_e \mathcal{D}_\mu^s(M)$. Suppose that $s > 2 + \frac{n}{2}$ and that V is a*

time independent solution of the Euler equations (2.8) on M . Take a geodesic $\eta(t)$ on $\mathcal{D}_\mu^s(M)$ satisfying $V = \dot{\eta} \circ \eta^{-1}$ as a vector field on M and a smooth function $f : [0, t_0] \rightarrow \mathbb{R}$ satisfying $f(0) = f(t_0) = 0$ for some $t_0 > 0$. Then, we have

$$E''(\eta)_0^{t_0}(\widetilde{W}, \widetilde{W}) = \int_0^{t_0} \left(\dot{f}^2 |W|^2 - f^2 (\nabla_V [V, W] + \nabla_{[V, W]} V, W) \right) dt,$$

where $|W|^2 := (W, W)_{T_e \mathcal{D}_\mu^s(M)}$ and \widetilde{W} is a vector field on $\mathcal{D}_\mu^s(M)$ along η defined by $\widetilde{W}_{\eta(t)} := f(t)(W \circ \eta(t)) \in T_{\eta(t)} \mathcal{D}_\mu^s(M)$.

For the proof of this proposition, we need the following three lemmas.

Lemma 2. Let $X, Y \in T_e \mathcal{D}^s(M)$ and $W \in T_e \mathcal{D}_\mu^s(M)$. Then, we have

$$(\nabla_W X, Y) = -(X, \nabla_W Y).$$

We omit the proof of this lemma, because this is easy.

Lemma 3. Let $V, W \in T_e \mathcal{D}_\mu^s(M)$ and $X \in T_e \mathcal{D}^s(M)$. Then, we have

$$(\nabla_V W, Q_e X)_{T_e \mathcal{D}^s(M)} = (\nabla_W V, Q_e X)_{T_e \mathcal{D}^s(M)}.$$

Proof. This is an easy consequence of $\nabla_V W - \nabla_W V = [V, W] \in T_e \mathcal{D}_\mu^s(M)$. \square

Lemma 4. For any $V, W \in T_e \mathcal{D}_\mu^s(M)$ and $\eta \in \mathcal{D}_\mu^s(M)$, we have

$$(V, W) = (V \circ \eta, W \circ \eta)$$

Proof. This follows from the definition of the metric on $\mathcal{D}_\mu^s(M)$ and $\eta \in \mathcal{D}_\mu^s(M)$. \square

Proof of Proposition 2.2. We follow the same strategy in [14, Lemma 2].

The second variation E'' along \widetilde{W} can be expressed as

$$(2.9) \quad E''(\eta)_0^{t_0}(\widetilde{W}, \widetilde{W}) = \int_0^{t_0} \{ (\widetilde{\nabla}_{\dot{\eta}} \widetilde{W}, \widetilde{\nabla}_{\dot{\eta}} \widetilde{W}) - (\widetilde{R}_\eta(\widetilde{W}, \dot{\eta}) \dot{\eta}, \widetilde{W}) \} dt.$$

For the first term, we have

$$\begin{aligned} \widetilde{\nabla}_{\dot{\eta}} \widetilde{W} &= P_\eta \bar{\nabla}_{\dot{\eta}} \widetilde{W} = P_\eta \left(\frac{d}{dt} (\widetilde{W}_\eta \circ \eta^{-1}) \circ \eta + (\nabla_{\dot{\eta} \circ \eta^{-1}} \widetilde{W} \circ \eta^{-1}) \circ \eta \right) \\ &= P_\eta \left(\frac{d}{dt} (fW) \circ \eta + (\nabla_V (fW)) \circ \eta \right) \\ &= P_\eta \left(\dot{f} \cdot (W \circ \eta) + (f \nabla_V W) \circ \eta \right) \end{aligned}$$

by (2.3), (2.4). We note that $\nabla_V (fW) = f \nabla_V W$ follows from the fact that f depends only on the time variable $t \in [0, t_0]$. Moreover, we have

$$\widetilde{\nabla}_{\dot{\eta}} \widetilde{W} = (\dot{f} \cdot W + f \cdot P_e \nabla_V W) \circ \eta$$

by $P_\eta(W \circ \eta) = (P_e W) \circ \eta = W \circ \eta$. Thus, Lemma 4 implies

$$(\widetilde{\nabla}_{\dot{\eta}} \widetilde{W}, \widetilde{\nabla}_{\dot{\eta}} \widetilde{W}) = \dot{f}^2 |W|^2 + 2f \dot{f} (W, P_e \nabla_V W) + f^2 |P_e \nabla_V W|^2,$$

where $|W|^2 := (W, W)$. The direct sum (2.2) and $\operatorname{div} W = 0$ imply

$$(W, P_e \nabla_V W) = (W, (P_e + Q_e) \nabla_V W) = (W, \nabla_V W),$$

which vanishes because $(W, \nabla_V W) = -(W, \nabla_V W)$ by Lemma 2. Thus, we have

$$(2.10) \quad \begin{aligned} (\widetilde{\nabla}_{\dot{\eta}} \widetilde{W}, \widetilde{\nabla}_{\dot{\eta}} \widetilde{W}) &= \dot{f}^2 |W|^2 + f^2 |P_e \nabla_V W|^2 \\ &= \dot{f}^2 |W|^2 + f^2 (\nabla_V W, P_e \nabla_V W) \end{aligned}$$

by Lemma 1. For the second term of (2.9), we have

$$(2.11) \quad \begin{aligned} (\tilde{R}_\eta(\tilde{W}, \dot{\eta})\dot{\eta}, \tilde{W}) &= (\tilde{R}_\eta(f \cdot (W \circ \eta), (V \circ \eta))(V \circ \eta), f \cdot (W \circ \eta)) \\ &= f^2(\tilde{R}_e(W, V)V, W). \end{aligned}$$

by the right invariance of \tilde{R} . The Gauss-Codazzi equations (2.5) imply

$$(\bar{R}_e(W, V)V, W) = (\tilde{R}_e(W, V)V, W) + (Q_e \nabla_W V, Q_e \nabla_V W) - (Q_e \nabla_V V, Q_e \nabla_W W).$$

Therefore, by Lemmas 1, 2 and 3, we have

$$\begin{aligned} (\tilde{R}_e(W, V)V, W) &= (\bar{R}_e(W, V)V, W) - (\nabla_W V, Q_e \nabla_V W) + (Q_e \nabla_V V, \nabla_W W) \\ &= (\nabla_W \nabla_V V - \nabla_V \nabla_W V - \nabla_{[W, V]} V, W) \\ &\quad - (\nabla_V W, Q_e \nabla_V W) + (Q_e \nabla_V V, \nabla_W W). \end{aligned}$$

We note that V is a time independent solution of (2.8), namely, $Q_e \nabla_V V = \nabla_V V$. Thus, Lemma 2 and $\nabla_V W - \nabla_W V = [V, W]$ imply

$$(2.12) \quad \begin{aligned} &-(\tilde{R}_e(W, V)V, W) + (\nabla_V W, P_e \nabla_V W) \\ &= -(\nabla_W \nabla_V V - \nabla_V \nabla_W V - \nabla_{[W, V]} V, W) + (\nabla_V W, \nabla_V W) - (\nabla_V V, \nabla_W W) \\ &= -(\nabla_V [V, W] + \nabla_{[V, W]} V, W). \end{aligned}$$

Therefore, by (2.10), (2.11) and (2.12), we have

$$(2.13) \quad \begin{aligned} E''(\eta)_0^{t_0}(\tilde{W}, \tilde{W}) &= \int_0^{t_0} \left(f^2 |W|^2 + f^2 \left((\nabla_V W, P_e \nabla_V W) - (\tilde{R}_e(W, V)V, W) \right) \right) \\ &= \int_0^{t_0} \left(f^2 |W|^2 - f^2 (\nabla_V [V, W] + \nabla_{[V, W]} V, W) \right) dt. \end{aligned}$$

This completes the proof. \square

From the above lemma, we can naturally extract the key value $MC_{V, W}$:

$$(2.14) \quad \begin{aligned} MC_{V, W} &:= (\nabla_V [V, W] + \nabla_{[V, W]} V, W) \\ &= (\tilde{R}_e(W, V)V, W) - |\tilde{\nabla}_V W|_{T_e \mathcal{D}_\mu^s}^2 \end{aligned}$$

for $W \in T_e \mathcal{D}_\mu^s(M)$ and a time independent solution $V \in T_e \mathcal{D}_\mu^s(M)$ of the Euler equations (2.8) on M . The second equality follows from (2.4) and the calculation in (2.13). We call $MC_{V, W}$ the ‘‘Misiol ek curvature’’. This value is the crucial in this paper, since $MC_{V, W} > 0$ ensures the existence of a conjugate point (see Fact 1.1 and Corollary 6.4).

Remark 6. By (2.14), it is obvious that $MC_{V, W} > 0$ implies the sectional curvature $(\tilde{R}_e(W, V)V, W)$ is positive. Moreover, we have $MC_{V, W+cV} = MC_{V, W}$ for any $c \in \mathbb{R}$. Thus $MC_{V, *}: T_e \mathcal{D}_\mu^s(M) \rightarrow \mathbb{R}$ should be defined on $V^\perp := \{W \in T_e \mathcal{D}_\mu^s(M) \mid (V, W) = 0\}$.

Corollary 2.3. Let M be a compact n -dimensional Riemannian manifold without boundary and $s > 2 + \frac{n}{2}$. Suppose that $V \in T_e \mathcal{D}_\mu^s(M)$ is a time independent solution of the Euler equations (2.8) on M and that $W \in T_e \mathcal{D}_\mu^s(M)$ satisfies $MC_{V, W} > 0$. Take a geodesic $\eta(t)$ on $\mathcal{D}_\mu^s(M)$ satisfying $V = \dot{\eta} \circ \eta^{-1}$ as a vector field on M and

$k \in \mathbb{R}_{>0}$. Define a positive number $t_{V,W,k} > 0$ and a smooth function $f_{V,W,k} : [0, t_{V,W,k}] \rightarrow \mathbb{R}$ satisfying $f_{V,W,k}(0) = f_{V,W,k}(t_{V,W,k}) = 0$ by

$$t_{V,W,k} := \pi|W|\sqrt{\frac{k}{MC_{V,W}}}, \quad f_{V,W,k}(t) := \sin\left(\frac{t}{|W|}\sqrt{\frac{MC_{V,W}}{k}}\right).$$

Then we have

$$E''(\eta)_0^{t_{V,W,k}}(\widetilde{W}^k, \widetilde{W}^k) = \frac{\pi}{2}(1-k)\sqrt{\frac{MC_{V,W}}{k}},$$

where \widetilde{W}^k is a vector field on $\mathcal{D}_\mu^s(M)$ along η defined by

$$\widetilde{W}_{\eta(t)}^k := f_{V,W,k}(t)(W \circ \eta(t)) \in T_{\eta(t)}\mathcal{D}_\mu^s(M).$$

In particular, if $k > 1$ we have $E''(\eta)_0^{t_{V,W,k}}(\widetilde{W}^k, \widetilde{W}^k) < 0$ and if $k = 1$ we have $E''(\eta)_0^{t_{V,W,k}}(\widetilde{W}^k, \widetilde{W}^k) = 0$.

Proof. Proposition 2.2 implies

$$\begin{aligned} & E''(\eta)_0^{t_{V,W,k}}(\widetilde{W}^k, \widetilde{W}^k) \\ &= \int_0^{t_{V,W,k}} \left(\dot{f}_{V,W,k}^2 |W|^2 - MC_{V,W} f_{V,W,k}^2 \right) dt \\ &= MC_{V,W} \int_0^{\pi|W|\sqrt{\frac{k}{MC_{V,W}}}} \left(\frac{1}{k} \cos^2\left(\frac{t}{|W|}\sqrt{\frac{MC_{V,W}}{k}}\right) - \sin^2\left(\frac{t}{|W|}\sqrt{\frac{MC_{V,W}}{k}}\right) \right) dt \\ &= MC_{V,W} \int_0^\pi \left(\frac{1}{k} \cos^2 x - \sin^2 x \right) |W|\sqrt{\frac{k}{MC_{V,W}}} dx \\ &= \frac{\pi}{2}|W|(1-k)\sqrt{\frac{MC_{V,W}}{k}}. \end{aligned}$$

This completes the proof. \square

3. ROTATIONALLY SYMMETRIC MANIFOLDS

In this section, we define the notion of *rotationally symmetric manifolds*, which we take as our “back ground manifolds” in the later sections. Our background manifold is a sphere or an ellipsoid in the main application. We refer to [21, Section 1.3] for the contents of this section.

Let $I_d := (-d, d) \subset \mathbb{R}$ be an open interval for some $d \in \mathbb{R}$ and

$$c(r) := (c_1(r), 0, c_2(r)) : [-d, d] \rightarrow \mathbb{R}^3$$

a smooth curve. Suppose that $c(r)$ satisfies

- (1) $c_1(r) > 0$ for all $r \in I_d$,
- (2) $c_1(d) = c_1(-d) = 0$,
- (3) $\dot{c}_2(r) := \frac{dc_2}{dr}(r) > 0$ for all $r \in I_d$,
- (4) r is a length parameter (i.e., $\dot{c}_1^2 + \dot{c}_2^2 = 1$).

The condition (3) of (3.1) means that $c_2(r)$ is a monotonically increasing function. Rotating this curve $c(r)$ with respect to the z -axis, we obtain a surface of revolution:

$$(3.2) \quad R'(c) := \{(c_1(r) \cos \theta, c_1(r) \sin \theta, c_2(r)) \mid r \in I_d, \theta \in \mathbb{R}\} \subset \mathbb{R}^3.$$

We want to obtain a sufficient (and in fact necessary) condition so that the closure $R(c) := cl(R'(c))$ has a smooth Riemannian manifold structure induced from the usual Riemannian structure of \mathbb{R}^3 .

Lemma 5. *Suppose that*

$$(3.3) \quad \frac{dc_1}{dr}(d) = \frac{dc_1}{dr}(-d) = 1, \text{ and } \frac{d^{2n}c_1}{dr^{2n}}(d) = \frac{d^{2n}c_1}{dr^{2n}}(-d) = 0 \text{ for any } n \in \mathbb{Z}_{>0}$$

Then $R(c) := cl(R'(c))$ has a smooth Riemannian manifold structure with the induced metric from \mathbb{R}^3 .

Proof. By the definition of $R(c)$, it is clear that

$$R(c) \setminus R'(c) = \{c(d), c(-d)\}.$$

We only prove that $c(-d)$ is not singular point of $R(c)$. The case for $c(d)$ can be proved in the similar way.

We first calculate the Riemannian metric $g_{R'}$ on $R'(c)$ induced from the usual Riemannian metric of \mathbb{R}^3 in the coordinate system (r, θ) . Define

$$\begin{aligned} \phi : I_d \times \mathbb{R} &\rightarrow R'(c) && \subset R(c) = M \\ (r, \theta) &\mapsto (c_1(r) \cos \theta, c_1(r) \sin \theta, c_2(r)) \end{aligned}$$

Then, we have

$$\begin{aligned} \phi_*(\partial_r) &= \dot{c}_1(r) \cos \theta \partial_x + \dot{c}_1(r) \sin \theta \partial_y + \dot{c}_2(r) \partial_z, \\ \phi_*(\partial_\theta) &= -c_1(r) \sin \theta \partial_x + c_1(r) \cos \theta \partial_y, \end{aligned}$$

where ϕ_* denotes the push out. Then, it follows from an easy calculation that

$$(3.4) \quad g_{R'}(\partial_r, \partial_r) = \dot{c}_1^2 + \dot{c}_2^2 = 1, \quad g_{R'}(\partial_r, \partial_\theta) = 0, \quad g_{R'}(\partial_\theta, \partial_\theta) = c_1^2.$$

Next, we introduce a coordinate system

$$(a, b) := (t \cos \theta, t \sin \theta) \quad \text{where } t := r + d.$$

Note that $t^2 = a^2 + b^2$ and $r \rightarrow -d$ corresponds to $t \rightarrow 0$. Then, we have

$$(3.5) \quad \partial_r = \cos \theta \partial_a + \sin \theta \partial_b, \quad \partial_\theta = -t \sin \theta \partial_a + t \cos \theta \partial_b,$$

or equivalently,

$$(3.6) \quad \partial_a = \cos \theta \partial_r - \frac{\sin \theta}{t} \partial_\theta, \quad \partial_b = \sin \theta \partial_r + \frac{\cos \theta}{t} \partial_\theta.$$

Combining (3.4) and (3.6), we have

$$(3.7) \quad \begin{aligned} g_{R'}(\partial_a, \partial_a) &= 1 + \left(\frac{c_1^2}{t^2} - 1 \right) \sin^2 \theta = 1 + \frac{c_1^2 - t^2}{t^4} b^2, \\ g_{R'}(\partial_a, \partial_b) &= \left(1 - \frac{c_1^2}{t^2} \right) \sin \theta \cos \theta = \frac{c_1^2 - t^2}{t^4} ab, \\ g_{R'}(\partial_b, \partial_b) &= 1 + \left(\frac{c_1^2}{t^2} - 1 \right) \cos^2 \theta = 1 + \frac{c_1^2 - t^2}{t^4} a^2. \end{aligned}$$

Then, considering a Taylor expansion of c_1 , we obtain that all functions of (3.7) is smooth at $(a, b) = 0$ if c_1 satisfies (2) of (3.1) and (3.3). This completes the proof. \square

Definition 2. Let M be a 2-dimensional Riemannian submanifold of \mathbb{R}^3 . We say M is a *rotationally symmetric manifold* if M is isometric to $R(c)$ (see (3.2) for the definition) for some smooth curve $c(r) : [-d, d] \rightarrow \mathbb{R}^3$ satisfying (3.1) and (3.3).

4. COMPUTATIONS ON ROTATIONALLY SYMMETRIC MANIFOLDS

In this section, we apply the results in Section 2 to the case that M is a compact 2-dimensional rotationally symmetric manifold, which is defined in Section 3. Our main background manifold is a sphere or an ellipsoid.

Let M be a rotationally symmetric manifold with a Riemannian metric g_M . See Definition 2. We use the same notations in Section 3. In particular, $I_d := (-d, d) \subset \mathbb{R}$ is an open interval, where $d \in \mathbb{R}_{>0}$ and

$$\begin{aligned} \phi : I_d \times I_\pi &\xrightarrow{\simeq} R'(c) \subset R(c) = M \\ (r, \theta) &\mapsto (c_1(r) \cos \theta, c_1(r) \sin \theta, c_2(r)) \end{aligned}$$

is a local coordinate of M . Note that $c(r)$ satisfies $\dot{c}_1(r)^2 + \dot{c}_2(r)^2 = 1$ for any $r \in I_d$, namely, $c(r)$ is parameterized by arc length. Then, we obtain (see (3.4))

$$g_M(\partial_r, \partial_r) = 1, \quad g_M(\partial_r, \partial_\theta) = 0, \quad g_M(\partial_\theta, \partial_\theta) = c_1^2$$

and

$$\mu = c_1(r) dr \wedge d\theta.$$

This implies

$$(4.1) \quad (V, W) = \int_{-d}^d \int_{-\pi}^{\pi} (V_1 W_1 + V_2 W_2 c_1^2) c_1 d\theta dr$$

for $V = V_1 \partial_r + V_2 \partial_\theta$ and $W = W_1 \partial_r + W_2 \partial_\theta$, which are elements of $T_e \mathcal{D}_\mu^s(M)$.

For a time dependent vector field u and a time dependent scalar valued function p , the Euler equations of an incompressible and inviscid fluid on M are as follows:

$$(4.2) \quad \begin{aligned} \partial_t u + \nabla_u u &= -\text{grad } p \quad t \geq 0, \\ \text{div } u &= 0, \\ u|_{t=0} &= u_0, \end{aligned}$$

where $\text{grad } p$ (resp. $\text{div } u$) is the gradient (resp. divergent) of p (resp. u) with respect to g_M and ∇ is the Levi-Civita connection of g_M . In the local coordinate system (r, θ) , these are given by

$$\begin{aligned} \text{grad } p &= \partial_r p \partial_r + c_1^{-2} \partial_\theta p \partial_\theta, \\ \text{div } u &= (\partial_r + c_1^{-1} \partial_r c_1) u_1 + \partial_\theta u_2 \end{aligned}$$

for $u = u_1 \partial_r + u_2 \partial_\theta$.

Recall that we call a vector field V on M a *zonal flow* if V has the following form:

$$(4.3) \quad V = F(r) \partial_\theta$$

for some function $F : I_d \rightarrow \mathbb{R}$. See also (1.4). Take a geodesic $\eta(t)$ of $\mathcal{D}_\mu^s(M)$ such that

$$\dot{\eta}(t) \circ \eta^{-1}(t) = V$$

as a vector field on M . Because V is a time independent solution of (4.2), we have $\eta(t) = \widetilde{\text{exp}}_e(tV)$. We now compute the Misiólek curvature, namely, $MC_{V,W} := (\nabla_V[V, W] + \nabla_{[V,W]}V, W)$.

Proposition 4.1. *Let $s > 3$ and $V \in T_e \mathcal{D}_\mu^s(M)$ a zonal flow. For $W \in T_e \mathcal{D}_\mu^s(M)$, we have*

$$MC_{V,W} = \int_{-d}^d \int_{-\pi}^{\pi} F^2 c_1 \left(-(\partial_\theta W_1)^2 - c_1^2 (\partial_r W_1)^2 + \left((\partial_r c_1)^2 - c_1 \partial_r^2 c_1 \right) W_1^2 \right) d\theta dr,$$

where $V = F(r) \partial_\theta$ and $W = W_1 \partial_r + W_2 \partial_\theta$.

Proof of Proposition 4.1. Recall that the suffix 1 is corresponding to r and 2 is corresponding to θ . Let Γ_{ij}^k ($1 \leq i, j, k \leq 2$) be the Christoffel symbols, which is given by

$$(4.4) \quad \Gamma_{ij}^k = \frac{1}{2} (g^{k1} (\partial_i g_{j1} + \partial_j g_{i1} - \partial_1 g_{ij}) + g^{k2} (\partial_i g_{j2} + \partial_j g_{i2} - \partial_2 g_{ij})).$$

Here we write $g^{-1} = (g^{ij})$ for the inverse of g . In our setting, we have

$$(4.5) \quad \Gamma_{22}^1 = -c_1 \partial_r c_1, \quad \Gamma_{12}^2 = \frac{\partial_r c_1}{c_1}, \quad \Gamma_{21}^2 = \frac{\partial_r c_1}{c_1}.$$

The other symbols are zero. Then, by the definition, we have

$$\nabla_v w = \sum_k \left\{ v w_k + \sum_{ij} \Gamma_{ij}^k v_i w_j \right\} \partial_k$$

for $v = \sum_i v_i \partial_i$ and $w = \sum_j w_j \partial_j$.

By direct calculation, we have

$$[V, W] = [F \partial_\theta W_1] \partial_r + [F \partial_\theta W_2 - W_1 \partial_r F] \partial_\theta.$$

Also, we have

$$\begin{aligned} \nabla_{[V,W]} V &= [\Gamma_{22}^1 (F \partial_\theta W_2 - W_1 \partial_r F) F] \partial_r \\ &\quad + [(F \partial_\theta W_1) \partial_r F + \Gamma_{12}^2 (F \partial_\theta W_1) F] \partial_\theta, \\ \nabla_V [V, W] &= [F \partial_\theta (F \partial_\theta W_1) + \Gamma_{22}^1 F (F \partial_\theta W_2 - W_1 \partial_r F)] \partial_r \\ &\quad + [F \partial_\theta (F \partial_\theta W_2 - W_1 \partial_r F) + \Gamma_{21}^2 F (F \partial_\theta W_1)] \partial_\theta. \end{aligned}$$

By $\partial_\theta F = 0$, we have

$$(4.6) \quad \begin{aligned} &\nabla_{[V,W]} V + \nabla_V [V, W] \\ &= [F^2 (\partial_\theta^2 W_1 + 2\Gamma_{22}^1 \partial_\theta W_2) - 2\Gamma_{22}^1 W_1 F \partial_r F] \partial_r \\ &\quad + [F^2 (\partial_\theta^2 W_2 + 2\Gamma_{21}^2 \partial_\theta W_1)] \partial_\theta. \end{aligned}$$

Then, (4.1) and (4.5) imply

$$\begin{aligned} MC_{V,W} &= \int_M \left(F^2 c_1 W_1 \partial_\theta^2 W_1 - 2F^2 c_1^2 \partial_r c_1 W_1 \partial_\theta W_2 + c_1^2 \partial_r c_1 W_1^2 \partial_r (F^2) \right. \\ &\quad \left. + F^2 c_1^3 W_2 \partial_\theta^2 W_2 + 2F^2 c_1^2 \partial_r c_1 W_2 \partial_\theta W_1 \right) dr d\theta. \end{aligned}$$

We note that $F = F(r)$ and $c_1 = c_1(r)$ are independent of the variable θ . Thus, applying Stokes theorem to the first, fourth, and fifth terms, we have

$$= \int_M \left(-F^2 c_1 (\partial_\theta W_1)^2 - 4F^2 c_1^2 \partial_r c_1 W_1 \partial_\theta W_2 + c_1^2 \partial_r c_1 \partial_r (F^2) W_1^2 - F^2 c_1^3 (\partial_\theta W_2)^2 \right) dr d\theta.$$

Recall that

$$\operatorname{div} W = \partial_r W_1 + c_1^{-1} \partial_r c_1 W_1 + \partial_\theta W_2,$$

which implies $\partial_\theta W_2 = -\partial_r W_1 - c_1^{-1} \partial_r c_1 W_1$ by the assumption $\operatorname{div} W = 0$. Therefore, we have

$$= \int_M \left(-F^2 c_1 (\partial_\theta W_1)^2 + 4F^2 c_1^2 \partial_r c_1 W_1 \partial_r W_1 + 4F^2 c_1 (\partial_r c_1)^2 W_1^2 + c_1^2 \partial_r c_1 \partial_r (F^2) W_1^2 - F^2 c_1^3 (\partial_r W_1)^2 - 2F^2 c_1^2 \partial_r c_1 W_1 \partial_r W_1 - F^2 c_1 (\partial_r c_1)^2 W_1^2 \right) dr d\theta.$$

This is equal to

$$= \int_M \left(-F^2 c_1 (\partial_\theta W_1)^2 - F^2 c_1^3 (\partial_r W_1)^2 + (2F^2 c_1^2 \partial_r c_1) W_1 \partial_r W_1 + \left(3F^2 c_1 (\partial_r c_1)^2 + c_1^2 \partial_r c_1 \partial_r (F^2) \right) W_1^2 \right) dr d\theta.$$

We note that the values of $F^2 c_1^2 \partial_r c_1$ at $r = d$ and $r = -d$ are zero by $\lim_{r \rightarrow d} c_1(r) = \lim_{r \rightarrow -d} c_1(r) = 0$. (The assumption $\dot{c}_1(r)^2 + \dot{c}_2(r)^2 = 1$ implies that $\partial_r c_1$ is bounded.) Thus, applying the Stokes theorem to the term $c_1^2 \partial_r c_1 \partial_r (F^2) (W_1)^2$, we have

$$= \int_M \left(-F^2 c_1 (\partial_\theta W_1)^2 - F^2 c_1^3 (\partial_r W_1)^2 + (2F^2 c_1^2 \partial_r c_1 - 2F^2 c_1^2 \partial_r c_1) W_1 \partial_r W_1 + \left(3F^2 c_1 (\partial_r c_1)^2 - 2F^2 c_1 (\partial_r c_1)^2 - F^2 c_1^2 \partial_r^2 c_1 \right) W_1^2 \right) dr d\theta \\ = \int_M F^2 c_1 \left(-(\partial_\theta W_1)^2 - c_1^2 (\partial_r W_1)^2 + \left((\partial_r c_1)^2 - c_1 \partial_r^2 c_1 \right) W_1^2 \right) dr d\theta.$$

This completes the proof. \square

Recall that $MC_{V,W} := (\nabla_{[V,W]} V + \nabla_V [V, W], W)$. For the existence of $W \in T_e \mathcal{D}_\mu^s(M)$ satisfying $MC_{V,W} > 0$, we have the following:

Proposition 4.2. *Suppose $s > 3$ and $(\partial_r c_1)^2 - c_1 \partial_r^2 c_1 > 1$. Then for any zonal flow $V \in T_e \mathcal{D}_\mu^s(M)$ whose support is contained in $R'(c)$ (see (3.2) for the definition), there exists $W_0 \in T_e \mathcal{D}_\mu^s(M)$ satisfying $MC_{V,W_0} > 0$.*

Remark 7. We can easily relax the condition on V . However we omit its detail here, since we would like to keep the simple statement.

Proof. Set $\epsilon(r) := \sqrt{(\partial_r c_1)^2 - c_1 \partial_r^2 c_1 - 1}$ and write $V = F(r) \partial_\theta$. The assumption of the support of V implies that the support of F is properly contained in I_d . Define a divergence-free vector field $W_0 = W_{01} \partial_r + W_{02} \partial_\theta$ on $I_d \times S^1$ by

$$(4.7) \quad W_0 := h(r) \sin \theta \partial_r + \left(\partial_r h(r) + \frac{h(r) \partial_r c_1(r)}{c_1(r)} \right) \cos \theta \partial_\theta$$

for some smooth real valued function $h = h(r)$ on $r \in I_d$. Moreover, by Proposition 4.1, we have

$$(4.8) \quad \begin{aligned} & MC_{V, W_0} \\ &= \int_{-d}^d \int_{-\pi}^{\pi} -F^2 c_1 \left((\partial_\theta W_{01})^2 + c_1^2 (\partial_r W_{01})^2 - (1 + \epsilon^2(r)) W_{01}^2 \right) d\theta dr \\ &= \int_{-d}^d -F^2 c_1 \int_{-\pi}^{\pi} \left(h^2 (\cos^2 \theta - \sin^2 \theta - \epsilon^2 \sin^2 \theta) + c_1^2 (\partial_r h)^2 \sin^2 \theta \right) d\theta dr \\ &= \pi \int_{-d}^d F^2 c_1 \left(h^2 \epsilon^2 - c_1^2 (\partial_r h)^2 \right) dr. \end{aligned}$$

Because the support of F is contained in I_d , there exists a smooth real valued function $h = h(r)$ on $r \in I_d$ satisfying the following properties:

- (i) $\partial_r h = 0$ on the support of F ,
- (ii) $h \neq 0$ on the support of F ,
- (iii) h is identically zero near the points $r = -d$ and $r = d$.

For such h , the last term of (4.8) is positive and W_0 defines an element of $T_e \mathcal{D}_\mu^s(M)$. This completes the proof. \square

Remark 8. The proof of Proposition 4.2 implies that $\#\{W \in V^\perp \mid MC_{V, W} = MC_{V, W_0}, |W| = 1\} = \infty$.

Corollary 4.3. Suppose that $s > 3$ and $(\partial_r c_1)^2 - c_1 \partial_r^2 c_1 > 1$. Then for any zonal flow $V \in T_e \mathcal{D}_\mu^s(M)$ whose support is contained in $R'(c)$, there exists a point conjugate to $e \in \mathcal{D}_\mu^s(M)$ along $\eta(t) = \exp_e(tV)$ on $0 \leq t \leq t^*$ for some $t^* > 0$.

Proof. It is obvious by Fact 1.1 and Proposition 4.2. \square

5. THE MAIN THEOREMS: ELLIPSOID AND SPHERE CASES

In this section, we investigate the case that M is a 2-dimensional ellipsoid and the case M is a sphere, more precisely, for $M = M_a := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = a^2(1 - z^2)\}$, the case of $a > 1$ (having a bulge around its equatorial middle and is flattened at the poles), and the case of $a = 1$ (sphere).

Let $E_a := \{(x, z) \in \mathbb{R}^2 \mid x^2 = a^2(1 - z^2)\}$ be an ellipse in \mathbb{R}^2 and ℓ the arc length of E_a . Set $d := \ell/4$ and take a curve

$$c(r) := (c_1(r), c_2(r)) : I_d = (-d, d) \rightarrow E_a$$

satisfying $\lim_{r \rightarrow -d} c(r) = (0, -1)$, $\lim_{r \rightarrow d} c(r) = (0, 1)$, $c_1(r) > 0$ and $\dot{c}_1(r)^2 + \dot{c}_1(r)^2 = 1$ on $r \in I_d$. Then, we have $M_a = R(c)$ (see Lemma 5). We note that $c_1(r)$ is a positive even function by the definition.

Therefore, we can apply the results of Section 4 to the ellipsoid case. For this purpose, we firstly show the following:

Proposition 5.1. *If $a > 1$, then $(\dot{c}_1)^2 - c_1 \ddot{c}_1 - 1 > 0$.*

Remark 9. In contrast to this, we have $(\dot{c})^2 - c_1\ddot{c}_1 \equiv 1$ in the case of $a = 1$ (i.e., sphere case).

Proof. Recall that $E_a := \{(x, z) \in \mathbb{R}^2 \mid x^2 = a^2(1 - z^2)\}$. We note that the gradient of the function $x^2 - a^2(1 - z^2)$ on \mathbb{R}^2 is equal to $2x\partial_x + 2a^2z\partial_z$. Therefore $x\partial_x + a^2z\partial_z$ is a normal vector field of E_a . Thus $-a^2z\partial_x + x\partial_z$ is tangent to E_a . This implies

$$(\dot{c}_1, \dot{c}_2) = \frac{1}{\sqrt{c_1^2 + a^4c_2^2}}(-a^2c_2, c_1).$$

Thus we have

$$\begin{aligned} \ddot{c}_1 &= \frac{-a^2}{\sqrt{c_1^2 + a^4c_2^2}}\dot{c}_2 + (-a^2c_2) \left(-\frac{1}{2}\right) \frac{2c_1\dot{c}_1 + 2a^4c_2\dot{c}_2}{(c_1^2 + a^4c_2^2)^{\frac{3}{2}}} \\ &= \frac{-a^2c_1^3 - a^4c_1c_2^2}{(c_1^2 + a^4c_2^2)^2}. \end{aligned}$$

Therefore

$$(\dot{c}_1)^2 - c_1\ddot{c}_1 - 1 = \frac{(a^2 - 1)c_1^4}{(c_1^2 + a^4c_2^2)^2}.$$

This and the assumption $a > 1$ imply the proposition. \square

We now recall the first main theorem:

Theorem 1.2. *Let $s > 3$ and $M_a := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = a^2(1 - z^2)\}$ be an ellipsoid with $a > 1$. For any zonal flow $V \in T_e\mathcal{D}_\mu^s(M)$ whose support is contained in $M_a \setminus \{(0, 0, 1), (0, 0, -1)\}$, there exists $W \in T_e\mathcal{D}_\mu^s(M)$ satisfying $MC_{V,W} > 0$.*

Proof. This is a consequence of Corollary 4.3 and Proposition 5.1. \square

Now we investigate the case that M is a 2-dimensional sphere, namely, the case of $a = 1$. Therefore we have $M := M_1 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = (1 - z^2)\} = S^2$, $d = \frac{\pi}{2}$ and $c_1(r) := \cos r$. By Proposition 4.1, we have

$$MC_{V,W} = \int_{-\pi/2}^{\pi/2} \int_{-\pi}^{\pi} -F^2c_1 \left((\partial_\theta W_1)^2 + c_1^2 (\partial_r W_1)^2 - W_1^2 \right) d\theta dr$$

for $V = F\partial_\theta$ and $W = W_1\partial_r + W_2\partial_\theta$. Also we now recall the second main theorem:

Theorem 1.3. *Suppose $s > 3$. For any zonal flow $V \in T_e\mathcal{D}_\mu^s(S^2)$ and any $W \in T_e\mathcal{D}_\mu^s(S^2)$, we have $MC_{V,W} \leq 0$.*

Proof. By Sobolev embedding theorem, W_1 and W_2 are of class C^2 (see Remark 1). Thus, we can consider the Fourier series of $W_j(r, \theta) := \sum_{k \in \mathbb{Z}} w_j^{(k)}(r)e^{ik\theta}$ for $j \in \{1, 2\}$, where $w_j^{(k)}(r) = \int_{-\pi}^{\pi} W_j(r, \theta)e^{-ik\theta} d\theta$. By Green-Stokes theorem and $\operatorname{div} W = 0$, we have

$$\begin{aligned} c_1(r)w_1^{(0)}(r) &= \int_{-\pi}^{\pi} c_1(r)W_1(r, \theta)d\theta \\ &= \int_r^{\pi/2} \int_{-\pi}^{\pi} c_1(r) \operatorname{div} W(r', \theta)d\theta dr' \\ &= 0, \end{aligned}$$

which implies $w_1^{(0)} \equiv 0$. Note that the complex conjugate of $w_1^{(k)}$ is equal to $w_1^{(-k)}$ because W_1 is a real valued function. Then,

$$\begin{aligned} & W_1^2 - (\partial_\theta W_1)^2 \\ &= \sum_{k \in \mathbb{Z}} \sum_{n+m=k} (1+nm) w_1^{(n)} w_1^{(m)} e^{ik\theta} \\ &= \sum_{k \neq 0} \sum_{n+m=k} (1+nm) w_1^{(n)} w_1^{(m)} e^{ik\theta} + \sum_{l \neq 0} (1-l^2) |w_1^{(l)}|^2. \end{aligned}$$

Therefore, we have

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \left(W_1^2 - (\partial_\theta W_1)^2 \right) d\theta = \sum_{l \neq 0} (1-l^2) |w_1^{(l)}|^2 \leq 0.$$

Then

$$\begin{aligned} & \int_{-\pi/2}^{\pi/2} \int_{-\pi}^{\pi} -F^2 c_1 \left((\partial_\theta W_1)^2 + c_1^2 (\partial_r W_1)^2 - W_1^2 \right) d\theta dr \\ & \leq \int_{-\pi/2}^{\pi/2} \int_{-\pi}^{\pi} -F^2 c_1 \left((\partial_\theta W_1)^2 - W_1^2 \right) d\theta dr \leq 0. \end{aligned}$$

This completes the proof. \square

Remark 10. A geometric meaning of Theorems 1.2 and 1.3 is the following: Let $V = F(r)\partial_\theta$ be a zonal flow and $\eta(t)$ a corresponding geodesic on $\mathcal{D}_\mu^s(M_a)$. Then, its length function is

$$E(\eta)_0^{t_0} = 2\pi t_0 \int_{-\pi}^{\pi} F^2(r) c_1^3(r) dr$$

by (2.6) and (4.1). We note that the value $2\pi c_1^2(r)$ is equal to the length of the horizontal circle $M_a^r := \{(x, y, z) \in M_a \mid z = c_2(r)\}$.

On the other hand, if the horizontal circle M_a^r is slightly tilted in such a way that the area of the enclosed region of M_a^r is invariant, the length of M_a^r can be smaller (resp. greater) in the $a > 1$ (resp. $a < 1$) case by a comparison theorem.

For $W \in T_e \mathcal{D}_\mu^s(M_a)$, we have $\exp(tW) \in \mathcal{D}_\mu^s(M_a)$, namely, $\exp(tW)$ preserves the volume element μ of M_a . Thus, a deformation of W preserves the area of the enclosed region of M_a^r , which implies that the second variation of $E(\eta)$ by W can be negative in the $a > 1$ case.

6. APPENDIX 1: EXISTENCE OF A CONJUGATE POINT AND THE M-CRITERION

In Section 4, it is observed that there are many $W \in T_e \mathcal{D}_\mu^s(M)$ satisfying $MC_{V,W} > 0$ for some fixed zonal flow V (see Proposition 4.2 and Remark 8), where M is a compact 2-dimensional rotationally symmetric manifold. Therefore, it seems to be worthwhile clarify more the meaning of $W \in T_e \mathcal{D}_\mu^s(M)$ satisfying $MC_{V,W} > 0$ in the case that $\dim M = 2$. This is the main purpose of this section. Moreover, for the completeness, we also give a proof of the M-criterion (Fact 1.1) in the 2D case, which is essentially already proved by Misiólek. We suppose that M is a compact 2-dimensional Riemannian manifold without boundary in this section.

For a positive number $t_0 > 0$, we define a subspace $K_\eta^{t_0}$ of $T_e \mathcal{D}_\mu^s(M)$ by

$$K_\eta^{t_0} := \sum_{t \in [0, t_0]} \text{Ker} \left(T_t V \widehat{\text{exp}}_e : T_t V (T_e \mathcal{D}_\mu^s(M)) \simeq T_e \mathcal{D}_\mu^s(M) \rightarrow T_{\eta(t)} \mathcal{D}_\mu^s(M) \right).$$

We write $K_\eta^{t_0, \perp}$ is the orthogonal complement of $K_\eta^{t_0}$ with respect to the Sobolev inner product, namely, the inner product defines the original topology of $T_e \mathcal{D}_\mu^s(M)$. In particular, $K_\eta^{t_0, \perp}$ is closed in $T_e \mathcal{D}_\mu^s(M)$ with respect to the original topology. We define a subset of $\mathcal{D}_\mu^s(M)$ by

$$(6.1) \quad E_\eta^{t_0, \perp} := \widetilde{\text{exp}}_e(K_\eta^{t_0, \perp}) \subset \mathcal{D}_\mu^s(M).$$

The finite-dimensionality of $K_\eta^{t_0}$ and finite-codimensionality of $K_\eta^{t_0, \perp}$ in the 2D case follow from Facts 2.1 and 6.1.

Fact 6.1 ([15, Lemma 3]). Let M be a compact 2-dimensional Riemannian manifold without boundary. Then any finite geodesic segment in $\mathcal{D}_\mu^s(M)$ contains at most finitely many conjugate points.

Remark 11. Fact 6.1 implies that for any $t_0 > 0$, there exist $N \in \mathbb{N}$ and $t_1, \dots, t_N \in [0, t_0]$ such that $\eta(t_1), \dots, \eta(t_N)$ exhaust all points conjugate to $e \in \mathcal{D}_\mu^s(M)$ along $\eta(t)$ for $0 \leq t \leq t_0$. Then we have

$$K_\eta^{t_0} = \bigoplus_{j=1}^N \text{Ker} (T_{t_j V} \widetilde{\text{exp}}_e : T_{t_j V} (T_e \mathcal{D}_\mu^s(M)) \simeq T_e \mathcal{D}_\mu^s(M) \rightarrow T_{\eta(t_j)} \mathcal{D}_\mu^s(M)).$$

Lemma 6. *Let M be a compact 2-dimensional Riemannian manifold without boundary. Then for any $t \in [0, t_0]$, we have an isomorphism*

$$T_{tV}(K_\eta^{t_0, \perp}) \xrightarrow{\simeq} T_{tV} \widetilde{\text{exp}}_e(T_{tV}(K_\eta^{t_0, \perp})),$$

which is induced by $T_{tV} \widetilde{\text{exp}}_e : T_{tV}(T_e \mathcal{D}_\mu^s(M)) \rightarrow T_{\eta(t)} \mathcal{D}_\mu^s(M)$.

Proof. Recall that $T_{tV} \widetilde{\text{exp}}_e : T_{tV}(T_e \mathcal{D}_\mu^s(M)) \rightarrow T_{\eta(t)} \mathcal{D}_\mu^s(M)$ is a nonlinear Fredholm map by Fact 2.1. In particular, it has a closed range, namely, $\text{Image}(T_{tV} \widetilde{\text{exp}}_e)$ is a closed subspace of $T_{\eta(t)} \mathcal{D}_\mu^s(M)$. Then, we have an isomorphism

$$(6.2) \quad (\text{Ker}(T_{tV} \widetilde{\text{exp}}_e))^\perp \xrightarrow{\simeq} T_{tV} \widetilde{\text{exp}}_e((\text{Ker}(T_{tV} \widetilde{\text{exp}}_e))^\perp) \stackrel{\text{closed}}{\subset} T_{\eta(t)} \mathcal{D}_\mu^s(M)$$

by the open mapping theorem and the following diagram:

$$\begin{array}{ccc} T_{tV} \widetilde{\text{exp}}_e : T_{tV}(T_e \mathcal{D}_\mu^s(M)) & \rightarrow & T_{\eta(t)} \mathcal{D}_\mu^s(M) \\ \parallel & & \cup \\ (\text{Ker}(T_{tV} \widetilde{\text{exp}}_e))^\perp & \xrightarrow{\simeq} & T_{tV} \widetilde{\text{exp}}_e((\text{Ker}(T_{tV} \widetilde{\text{exp}}_e))^\perp) = \text{Image}(T_{tV} \widetilde{\text{exp}}_e) \\ \oplus & & \\ \text{Ker}(T_{tV} \widetilde{\text{exp}}_e) & \rightarrow & 0 \end{array}$$

Next, we show that $K_\eta^{t_0, \perp} \subset T_e \mathcal{D}_\mu^s(M)$ satisfies the following properties for any $t \in [0, t_0]$:

- (i) $T_{tV}(K_\eta^{t_0, \perp}) \simeq K_\eta^{t_0, \perp}$ is a closed subspace of $T_{tV}(T_e \mathcal{D}_\mu^s(M)) \simeq T_e \mathcal{D}_\mu^s(M)$,
- (ii) $T_{tV}(K_\eta^{t_0, \perp})$ is contained in $(\text{Ker}(T_{tV} \widetilde{\text{exp}}_e))^\perp$.

Indeed, (i) follows from the fact that $K_\eta^{t_0, \perp}$ is the orthogonal complement of $K_\eta^{t_0}$ and (ii) is a consequence of the definition of $K_\eta^{t_0, \perp}$. The following diagram describes the relationship among regarding spaces:

$$\begin{array}{ccc} T_{tV} \widetilde{\text{exp}}_e : T_{tV}(T_e \mathcal{D}_\mu^s(M)) & \rightarrow & T_{\eta(t)} \mathcal{D}_\mu^s(M) \\ \parallel & & \cup \\ (\text{Ker}(T_{tV} \widetilde{\text{exp}}_e))^\perp & \stackrel{\text{closed}}{\supset} & T_{tV}(K_\eta^{t_0, \perp}) \xrightarrow{\simeq} T_{tV} \widetilde{\text{exp}}_e(T_{tV}(K_\eta^{t_0, \perp})) \\ \oplus & & \\ \text{Ker}(T_{tV} \widetilde{\text{exp}}_e) & \subset & T_{tV}(K_\eta^{t_0}) \rightarrow 0 \end{array}$$

Therefore, the isomorphism (6.2) induces the desired isomorphism. \square

Remark 12. This lemma is not true in the case that $\dim M = 3$, see [6, Section 4].

Recall that we say $\xi(r, t) : (-\varepsilon, \varepsilon) \times [0, t_0] \rightarrow \mathcal{D}_\mu^s(M)$ is a variation of a geodesic $\eta(t)$ on $\mathcal{D}_\mu^s(M)$ with fixed endpoints, if it satisfies $\xi(r, 0) \equiv \eta(0)$, $\xi(r, t_0) \equiv \eta(t_0)$ and $\xi(0, t) = \eta(t)$. We sometimes write $\xi_r(t)$ for $\xi(r, t)$.

Proposition 6.2. *Let M be a compact 2-dimensional Riemannian manifold without boundary, $V \in T_e \mathcal{D}_\mu^s(M)$ a time independent solution of Euler equations (2.8) on M and $\eta(t)$ the geodesic on $\mathcal{D}_\mu^s(M)$ corresponding to V . Let $\xi(r, t) : (-\varepsilon, \varepsilon) \times [0, t_0] \rightarrow \mathcal{D}_\mu^s(M)$ be a variation of $\eta(t)$ with fixed endpoints satisfying $\text{Image}(\xi) \subset E_\eta^{t_0, \perp}$. Then we have $E''(\eta)_0^{t_0}(X, X) \geq 0$, where $X = \partial_r \xi(r, t)|_{r=0}$.*

Proof. We almost follow the same strategy in [14, Lemma 3].

Lemma 6 implies that there exists a sufficiently small open neighborhood $U_t \subset K_\eta^{t_0, \perp}$ of tV such that U_t is diffeomorphic to $\widetilde{\text{exp}}_e(U_t) \subset E_\eta^{t_0, \perp} = \widetilde{\text{exp}}_e(K_\eta^{t_0, \perp})$ by the inverse function theorem (See [10, Proposition 2.3], for instance) for any $t \in [0, t_0]$. In particular, $\widetilde{\text{exp}}_e(U_t)$ is open in $E_\eta^{t_0, \perp}$ and we can define $\widetilde{\text{log}}_e := \widetilde{\text{exp}}_e^{-1} : \widetilde{\text{exp}}_e(U_t) \rightarrow U_t$. Set $U := \bigcup_{t \in [0, t_0]} U_t$, then we have $tV \in U$ for any $t \in [0, t_0]$ because $tV \in U_t \subset U$. Thus, we have $\eta(t) = \widetilde{\text{exp}}_e(tV) \in \widetilde{\text{exp}}_e(U)$, namely, $\xi(0, t) \in \widetilde{\text{exp}}_e(U)$. Then, we can assume $\text{Image}(\xi) \subset \widetilde{\text{exp}}_e(U)$ by taking smaller $\varepsilon > 0$ because $\widetilde{\text{exp}}_e(U)$ is open in $E_\eta^{t_0, \perp}$ and $\text{Image}(\xi)$ is contained in $E_\eta^{t_0, \perp}$ by the assumption.

$$\begin{array}{lcl} T_{tV} \widetilde{\text{exp}}_e & : & T_{tV}(K_\eta^{t_0, \perp}) \xrightarrow{\simeq} T_{tV} \widetilde{\text{exp}}_e(T_{tV}(K_\eta^{t_0, \perp})) \\ \widetilde{\text{exp}}_e & : & \begin{array}{ccc} K_\eta^{t_0, \perp} & \rightarrow & \widetilde{\text{exp}}_e(K_\eta^{t_0, \perp}) = E_\eta^{t_0, \perp} \\ \cup & & \cup \\ U_t & \xrightarrow{\simeq} & \widetilde{\text{exp}}_e(U_t) \\ \cup & & \cup \\ tV & \mapsto & \widetilde{\text{exp}}_e(tV) = \eta(t) \end{array} \end{array}$$

Therefore we can define a curve $c_r(t) := \widetilde{\text{log}}_e \xi_r(t)$ in $T_e \mathcal{D}_\mu^s(M)$ and $\ell_r(t) := |c_r(t)| = \sqrt{\langle c_r(t), c_r(t) \rangle}$. Then we have $\ell_r(0) = 0$, $\ell_r(t_0) = t_0|V|$ and $c_r(t) = \ell(t) \frac{c_r(t)}{|c_r(t)|}$. Thus, we obtain

$$\dot{c}_r(t) = \dot{\ell}_r(t) \frac{c_r(t)}{|c_r(t)|} + \ell_r(t) \frac{d}{dt} \left(\frac{c_r(t)}{|c_r(t)|} \right).$$

Then, for any $r \in (-\varepsilon, \varepsilon)$, we have

$$|\dot{\xi}_r(t)| = \left| \frac{d}{dt} (\widetilde{\text{exp}}_e c_r(t)) \right| = |T_{c_r(t)} \widetilde{\text{exp}}_e(\dot{c}_r(t))| = |\dot{c}_r(t)| \geq \dot{\ell}_r(t)^2.$$

In the third equality, we used Gauss's lemma or [15, Lemma 2]. Then, by (2.6) and the Cauchy-Schwartz inequality, we have

$$\begin{aligned} E(\xi_r) &\geq \frac{1}{2} \int_0^{t_0} \dot{\ell}_r(t)^2 dt = \frac{1}{2t_0} \left(\int_0^{t_0} \dot{\ell}_r(t)^2 dt \right) \left(\int_0^{t_0} 1^2 dt \right) \\ &\geq \frac{1}{2t_0} \left(\int_0^{t_0} \dot{\ell}_r(t) dt \right)^2 = \frac{t_0}{2} |V|^2 \\ &= E(\eta) \end{aligned}$$

for any $r \in (-\varepsilon, \varepsilon)$. This implies $E''(\eta)_0^{t_0}(X, X) \geq 0$. \square

Recall that

$$t_{V,W,k} := \pi|W|\sqrt{\frac{k}{MC_{V,W}}}, \quad f_{V,W,k}(t) := \sin\left(\frac{t}{|W|}\sqrt{\frac{MC_{V,W}}{k}}\right),$$

$$\widetilde{W}_{\eta(t)}^k := f_{V,W,k}(t)(W \circ \eta(t)) \in T_{\eta(t)}\mathcal{D}_\mu^s(M)$$

for $W \in T_e\mathcal{D}_\mu^s(M)$ satisfying $MC_{V,W} > 0$ and $k \in \mathbb{R}_{>0}$.

Corollary 6.3. Let M be a compact n -dimensional Riemannian manifold without boundary and $s > 2 + \frac{n}{2}$. Suppose that $V \in T_e\mathcal{D}_\mu^s(M)$ is a time independent solution of (2.8) and that $W \in T_e\mathcal{D}_\mu^s(M)$ satisfies $MC_{V,W} > 0$. Take the geodesic $\eta(t)$ on $\mathcal{D}_\mu^s(M)$ corresponding to V and define a variation $\xi^k(r, t) : (-\varepsilon, \varepsilon) \times [0, t_{V,W,k}] \rightarrow \mathcal{D}_\mu^s(M)$ of $\eta(t)$ with fixed endpoints by $\xi_r^k(t) := \widetilde{\text{exp}}_{\eta(t)}(r\widetilde{W}^k)$. Then we have $\{\xi_r^k(t) \mid t \in [0, t_{V,W,k}] \mid r| \ll 1\} \not\subset E_\eta^{t_{V,W,k},k,\perp}$ for any $k > 1$.

Proof. Suppose that the contrary, namely, $\{\xi_r^k(t) \mid t \in [0, t_{V,W,k}] \mid r| \ll 1\} \subset E_\eta^{t_{V,W,k},k,\perp}$. Then we have $E''(\eta)_0^{t_{V,W,k}}(\widetilde{W}^k, \widetilde{W}^k) \geq 0$ by Proposition 6.2. However, this contradicts Corollary 2.3. \square

Corollary 6.4. (Existence of a conjugate point, M-criterion) Let M be a compact 2-dimensional Riemannian manifold without boundary and $s > 2 + \frac{n}{2}$. Suppose that $V \in T_e\mathcal{D}_\mu^s(M)$ is a time independent solution of Euler equations (2.8) on M . Take the geodesic $\eta(t)$ on $\mathcal{D}_\mu^s(M)$ corresponding to V . If there exists a $W_0 \in T_e\mathcal{D}_\mu^s(M)$ satisfying $MC_{V,W_0} > 0$, there exists a point conjugate to $e \in \mathcal{D}_\mu^s(M)$ along $\eta(t)$ for $0 \leq t \leq t_{V,W_0,1}$.

Proof. Suppose that there are no points conjugate to $e \in \mathcal{D}_\mu^s(M)$ along $\eta(t)$ for $0 \leq t \leq t_{V,W_0,k}$ for $k > 1$. Then $K_\eta^{t_{V,W_0,k}} = 0$ and $K_\eta^{t_{V,W_0,k},\perp} = T_e\mathcal{D}_\mu^s(M)$. In particular, $\text{Image}(\xi^k) \subset E_\eta^{t_{V,W_0,k},k,\perp} = \widetilde{\text{exp}}_e(T_e\mathcal{D}_\mu^s(M))$, where $\xi^k(r, t) := \widetilde{\text{exp}}_{\eta(t)}(r\widetilde{W}^k)$. Therefore Proposition 6.2 implies $E''(\eta)_0^{t_{V,W_0,k}}(\widetilde{W}_0^k, \widetilde{W}_0^k) \geq 0$. On the other hand, we have $E''(\eta)_0^{t_{V,W_0,k}}(\widetilde{W}_0^k, \widetilde{W}_0^k) < 0$ by Corollary 2.3. This contradiction implies that there exists a point conjugate to $e \in \mathcal{D}_\mu^s(M)$ along $\eta(t)$ on $0 \leq t \leq t_{V,W_0,k}$ for any $k > 1$. Taking a limit, we have the corollary. \square

Corollary 6.5. Let M be a compact 2-dimensional Riemannian manifold without boundary and $s > 2 + \frac{n}{2}$. Suppose that $V \in T_e\mathcal{D}_\mu^s(M)$ is a time independent solution of Euler equations (2.8) on M and take $\eta(t)$ the geodesic on $\mathcal{D}_\mu^s(M)$ corresponding to V . Then, $\sup\{MC_{V,W} \mid W \in T_e\mathcal{D}_\mu^s(M)\} < \infty$.

Proof. By Fact 6.1, there exists $t^* > 0$ such that there are no points conjugate to $e \in \mathcal{D}_\mu^s(M)$ along $\eta(t)$ for $0 \leq t \leq t^*$. On the other hand, by Corollary 6.4, if $W \in T_e\mathcal{D}_\mu^s(M)$ satisfies $MC_{V,W} > 0$, there exists a point conjugate to $e \in \mathcal{D}_\mu^s(M)$ along $\eta(t)$ for $0 \leq t \leq t_{V,W,1}$. Thus we have $t_{V,W,1} = \pi|W|\sqrt{\frac{1}{MC_{V,W}}} > t^* > 0$. This implies the corollary. \square

7. APPENDIX 2: FREDHOLMNESS OF THE EXPONENTIAL MAP IN 2D CASE

Let M be a compact 2-dimensional Riemannian manifold without boundary and $s > 3$. Suppose that $V \in T_e \mathcal{D}_\mu^s(M)$ is a time independent solution of the Euler equations (2.8) on M and that $W \in T_e \mathcal{D}_\mu^s(M)$ satisfies $MC_{V,W} > 0$. Take a geodesic $\eta(t)$ on $\mathcal{D}_\mu^s(M)$ satisfying $V = \dot{\eta} \circ \eta^{-1}$ as a vector field on M . Then, Corollary 2.3 implies that there exists a vector field \widetilde{W}^k on $\mathcal{D}_\mu^s(M)$ along $\eta(t)$ ($0 \leq t \leq t_{V,W,k}$) such that $E''(\eta)_0^{t_{V,W,k}}(\widetilde{W}^k, \widetilde{W}^k) < 0$ for any $k > 1$. It seems that this contradicts to Fact 6.1 because the dimension of the subspace of the space of all vector fields along $\eta(t)$ on which $E''(\eta)_0^{t_{V,W,k}}(\cdot, \cdot)$ is negative definite is equal to the number of conjugate points to $e \in \mathcal{D}_\mu^s(M)$ along $\eta(t)$ (see Fact 7.1 given below). However, this apparent paradox is not really an issue because $\{\widetilde{W}^k\}$ does not form any infinite-dimensional vector space. The main purpose of this Appendix is to explain this phenomena. For the precise statement, we fix some notations from now on. We refer to [16, Section 8] for the detail of the content of this Appendix.

Let M be a compact 2-dimensional Riemannian manifold without boundary. Take a geodesic $\eta(t)$ ($0 \leq t \leq 1$) joining $e \in \mathcal{D}_\mu^s(M)$ and $\eta(1)$ on $\mathcal{D}_\mu^s(M)$. For $0 \leq t \leq 1$, we consider the space $C_t^\infty := C_t^\infty(\eta)$ of all smooth vector fields on $[0, t]$ along η which are zero at the end points e and $\eta(t)$. Then, we write $H_t := H_t(\eta)$ for the completion of C_t^∞ by the norm induced from the inner product

$$\langle X, Y \rangle_{H_t} := \int_0^t (\widetilde{\nabla}_{\dot{\eta}} X(t'), \widetilde{\nabla}_{\dot{\eta}} Y(t')) dt',$$

where $X, Y \in C_t^\infty$. Extending elements of H_t by zero on $[t, 1]$ along η , we regard $H_t \subset H_{t'}$ for $0 \leq t \leq t' \leq 1$. Then, the second variation $E''(\eta)_0^t(\cdot, \cdot)$ of the length function $E(\eta)_0^t$ (see (2.6)) defines a bounded symmetric bilinear form on H_t , which is given by

$$E''(\eta)_0^t(X, Y) = \int_0^t \{(\widetilde{\nabla}_{\dot{\eta}} X(t'), \widetilde{\nabla}_{\dot{\eta}} Y(t')) - (\widetilde{R}_\eta(X(t'), \dot{\eta})\dot{\eta}, Y(t'))\} dt'.$$

Recall that the index of the form $E''(\eta)_0^t(\cdot, \cdot)$ is the dimension of the largest subspace of H_t on which $E''(\eta)_0^t(\cdot, \cdot)$ is negative definite. We note that the subset of H_t , on which the form $E''(\eta)_0^t(\cdot, \cdot)$ is negative, is not closed under addition. Therefore, even if the index is finite, the subset can contain infinitely many linear independent vectors.

Fact 7.1 ([16, Theorem 8.2]). Let $\eta(t)$ ($0 \leq t \leq 1$) be a geodesic from the identify e to $\eta(1)$ in the group of volume preserving diffeomorphisms $\mathcal{D}_\mu^s(M)$ of a surface without boundary. Then, the index of $E''(\eta)_0^1(\cdot, \cdot)$ is finite and equal to the number of conjugate points to e along η each counted with multiplicity.

It could seem that Corollary 2.3 contradicts to this fact as we explained in the beginning of this section. The proposition given below in this Appendix states that this is not a contradiction:

Proposition 7.2. *For any $1 < k < l$, we have*

$$\int_0^{t_{V,W,l}} (\widetilde{\nabla}_{\dot{\eta}} \widetilde{W}^k, \widetilde{\nabla}_{\dot{\eta}} \widetilde{W}^l) dt > 0.$$

In other words, \widetilde{W}^k and \widetilde{W}^l are not orthogonal.

If $\{\widetilde{W}^k\}$ form an infinite-dimensional vector subspace of H_t , there exist $k_1 > k_2 > \dots$ such that $\int_0^{t_{V,W,k_i}} (\widetilde{\nabla}_{\dot{\eta}} \widetilde{W}^{k_i}, \widetilde{\nabla}_{\dot{\eta}} \widetilde{W}^{k_j}) dt = 0$ ($i > j$). Therefore this proposition means that $\{\widetilde{W}^k\}$ does not form any infinite-dimensional vector subspace of H_t , which solves the apparent paradox. For the proof, let us recall that

$$t_{V,W,k} := \pi |W| \sqrt{\frac{k}{MC_{V,W}}}, \quad f_{V,W,k}(t) := \sin \left(\frac{t}{|W|} \sqrt{\frac{MC_{V,W}}{k}} \right),$$

$$\widetilde{W}_{\eta(t)}^k := f_{V,W,k}(t) (W \circ \eta(t)) \in T_{\eta(t)} \mathcal{D}_\mu^s(M).$$

Proof. By the same calculation of the proof of Proposition 2.2, we have

$$(\widetilde{\nabla}_{\dot{\eta}} \widetilde{W}^k, \widetilde{\nabla}_{\dot{\eta}} \widetilde{W}^l) = \dot{f}_{V,W,k} \dot{f}_{V,W,l} |W|^2 + f_{V,W,k} f_{V,W,l} |P_e \nabla_V W|^2.$$

We note that

$$\int_0^{t_{V,W,l}} \dot{f}_{V,W,k} \dot{f}_{V,W,l} dt$$

$$= \frac{MC_{V,W}}{|W|^2 \sqrt{kl}} \int_0^{t_{V,W,k}} \cos \left(\frac{t}{|W|} \sqrt{\frac{MC_{V,W}}{k}} \right) \cos \left(\frac{t}{|W|} \sqrt{\frac{MC_{V,W}}{l}} \right) dt.$$

Applying Product-Sum identities and calculating the integral, we have

$$= \frac{\sqrt{MC_{V,W}}}{2|W|(\sqrt{l} + \sqrt{k})} \sin \left(\left(1 + \sqrt{\frac{k}{l}}\right) \pi \right) + \frac{\sqrt{MC_{V,W}}}{2|W|(\sqrt{l} - \sqrt{k})} \sin \left(\left(1 - \sqrt{\frac{k}{l}}\right) \pi \right)$$

$$= \frac{-\sqrt{MC_{V,W}}}{2|W|(\sqrt{l} + \sqrt{k})} \sin \left(\pi \sqrt{\frac{k}{l}} \right) + \frac{\sqrt{MC_{V,W}}}{2|W|(\sqrt{l} - \sqrt{k})} \sin \left(\pi \sqrt{\frac{k}{l}} \right).$$

Moreover,

$$\int_0^{t_{V,W,l}} f_{V,W,k} f_{V,W,l} dt = \int_0^{t_{V,W,k}} \sin \left(\frac{t}{|W|} \sqrt{\frac{MC_{V,W}}{k}} \right) \sin \left(\frac{t}{|W|} \sqrt{\frac{MC_{V,W}}{l}} \right) dt$$

By Product-Sum identities and the equalities $\sin(x + \pi) = -\sin(x)$, $\sin(x - \pi) = \sin(x)$, we have

$$= \frac{|W| \sqrt{kl}}{2\sqrt{MC_{V,W}}} \left(\frac{1}{(\sqrt{l} + \sqrt{k})} \sin \left(\pi \sqrt{\frac{k}{l}} \right) + \frac{1}{(\sqrt{l} - \sqrt{k})} \sin \left(\pi \sqrt{\frac{k}{l}} \right) \right)$$

Thus, we have

$$\int_0^{t_{V,W,l}} (\widetilde{\nabla}_{\dot{\eta}} \widetilde{W}^k, \widetilde{\nabla}_{\dot{\eta}} \widetilde{W}^l) dt$$

$$= |W| \left(\frac{\sqrt{MC_{V,W}} \sqrt{k}}{l^2 - k^2} + \frac{|P_e \nabla_V W|^2 l \sqrt{k}}{\sqrt{MC_{V,W}} (l^2 - k^2)} \right) \sin \left(\pi \sqrt{\frac{k}{l}} \right)$$

$$= \frac{|W| \sqrt{k} (MC_{V,W} + |P_e \nabla_V W|^2 l)}{\sqrt{MC_{V,W}} (l^2 - k^2)} \sin \left(\pi \sqrt{\frac{k}{l}} \right) > 0.$$

This completes the proof. \square

Acknowledgments. The authors would like to thank G. Misiolek for his helpful and careful comments. The authors also thank the anonymous referee for his/her careful reading of our manuscript and many insightful comments and suggestions. Research of TT was partially supported by Foundation of Research Fellows, The Mathematical Society of Japan. Research of TY was partially supported by Grant-in-Aid for Young Scientists A (17H04825), Grant-in-Aid for Scientific Research B (15H03621, 17H02860, 18H01136 and 18H01135).

REFERENCES

1. V. I. Arnol'd, *Sur la geometrie differentielle des groupes de Lie de dimension infinie et ses applications a l'hydrodynamique des fluids parfaits*, Ann. Inst. Grenoble **16** (1966).
2. V. I. Arnol'd, *Arnold's problems*, Springer-Verlag, Berlin; PHASIS, Moscow, (2004), xvi+639 pp.
3. V. I. Arnol'd, B. Khesin, *Topological methods in hydrodynamics*, Applied Mathematical Sciences, **125**, Springer-Verlag, New York, (1998), xvi+374 pp.
4. J. Benn, G. Misiolek, S. Preston, *The exponential map of the group of area-preserving diffeomorphisms of a surface with boundary*, Arch. Ration. Mech. Anal, **229** (2018), no. 3, 1015–1035.
5. D. Ebin, J. Marsden, *Groups of diffeomorphisms and the motion of an incompressible fluid*, Ann. of Math. (2) **92** (1970), 102–163.
6. D. Ebin, G. Misiolek, S. C. Preston, *Singularities of the exponential map on the volume-preserving diffeomorphism group*, Geom. Funct. Anal. **16** (2006), 850–868.
7. N. Grossman, *Hilbert manifolds without epicongugate points*, Proc. Amer. Math. Soc. **16** (1965), 1365–1371.
8. T. Kambe, *Geometrical theory of fluid flows and dynamical systems*, Fluid Dynam. Res. **30** (2002), no. 6, 331–378.
9. K. R. Lang, http://ase.tufts.edu/cosmos/view_chapter.asp?id=9&page=3
10. S. Lang, *Fundamentals of differential geometry*, Graduate Texts in Mathematics, **191**, (1999), Springer-Verlag, New York, xviii+535 pp.
11. L. Lichtenfelz, *Normal forms for the L^2 Riemannian exponential map on diffeomorphism groups*, Int. Math. Res. Not. IMRN 2018, no. 6, 1730–1753.
12. A. M. Lukatsky, *On the curvature of the diffeomorphisms group*, Ann. Global Anal. Geom., **11**, (1993), 135–140.
13. G. Misiolek, *Stability of flows of ideal fluids and the geometry of the group of diffeomorphisms*, Indiana Univ. Math. J., **42** (1993), 215–235.
14. G. Misiolek, *Conjugate points in $\mathcal{D}_\mu(T^2)$* , Proc. Amer. Math. Soc., **124** (1996), 977–982.
15. G. Misiolek, *The exponential map near conjugate points in 2D hydrodynamics*, Arnold Math. J. **1** (2015), 243–251.
16. G. Misiolek, S. Preston, *Fredholm properties of Riemannian exponential maps on diffeomorphism groups*, Invent. Math. **179** (2010), no. 1, 191–227.
17. F. Nakamura, Y. Hattori and T. Kambe, *Geodesics and curvature of a group of diffeomorphisms and motion of an ideal fluid*, J. Phys. A: Math. Gen., **25** (1992), L45–L50.
18. T. Nozawa and S. Yoden, *Formation of zonal band structure in forced two-dimensional turbulence on a rotating sphere*, Phys. Fluids, **9** (1997), 2081.
19. K. Obuse, S. Takehiro and M. Yamada, *Long-time asymptotic states of forced two-dimensional barotropic incompressible flows on a rotating sphere*, Phys. Fluids, **22** (2010), 056601.
20. K. Ohkitani, P. Constantin, *Numerical study of the Eulerian-Lagrangian formulation of the Navier-Stokes equations*, Phys. Fluids **15** (2003), no. 10, 3251–3254.
21. P. Petersen, *Riemannian Geometry*, Third edition. Graduate Texts in Mathematics, Springer, Cham, **171** (2016), xviii+499 pp.
22. S. C. Preston, *On the volumorphism group, the first conjugate point is always the hardest*, Comm. Math. Phys. **267** (2006), no. 2, 493–513.
23. E. Sasaki, S. Takehiro and M. Yamada, *Linear stability of viscous zonal jet flow on a rotating sphere*, J. Phys. Soc. Japan, **82** (2013), 094402.
24. E. Sasaki, S. Takehiro and M. Yamada, *Bifurcation structure of two-dimensional viscous zonal flows on a rotating sphere*, J. Fluid Mech., **774** (2015), 224–244.

25. N. K. Smolentsev, *Diffeomorphism groups of compact manifolds* (Russian. Russian summary), *Sovrem. Mat. Prilozh. No. 37, Geometriya* (2006), 3–100; translation in *J. Math. Sci. (N.Y.)* **146** (2007), 6213–6312.
26. P. Washabaugh, *The SQG equation as a geodesic equation*, *Arch. Ration. Mech. Anal.* **222** (2016), no. 3, 1269–1284.
27. G. P. Williams, *Planetary circulations: 1. Barotropic representation of Jovian and terrestrial turbulence*, *J. Atmos. Sci.*, **35** (1978), 1399.
28. S. Yoden, M. Yamada, *A numerical experiment on two-dimensional decaying turbulence on a rotating sphere*, *J. Atmos. Sci.*, **50** (1993), 631.

INSTITUTE OF MATHEMATICS FOR INDUSTRY, KYUSHU UNIVERSITY, 744 MOTOOKA, NISHI-KU, FUKUOKA 819-0395, JAPAN

Email address: `tauchi.taito.342@m.kyushu-u.ac.jp`

GRADUATE SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF TOKYO, KOMABA 3-8-1 ME-GURO, TOKYO 153-8914, JAPAN

Email address: `yoneda@ms.u-tokyo.ac.jp`