The fundamental multiple conjugation quandle of a handlebody-link

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Abstract

A handlebody-link is a disjoint union of handlebodies embedded in the 3-sphere S^3. A multiple conjugation quandle is an algebraic system whose axioms are derived from the Reidemeister moves for handlebody-link diagrams. In this paper, we introduce the notion of a presentation of a multiple conjugation quandle and define the fundamental multiple conjugation quandle of a handlebody-link. We also see that the fundamental multiple conjugation quandle is an invariant of handlebody-links.

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1 Introduction

The fundamental group and the fundamental quandle of a knot are fundamental and powerful invariants, where a quandle [7, 9] is an algebraic system whose axioms are derived from the Reidemeister moves for knot diagrams. A group (resp. quandle) coloring of a knot diagram can be identified with a group (resp. quandle) homomorphism from the fundamental group (resp. quandle) of the knot to a group (resp. quandle). In this sense, the fundamental group and the fundamental quandle are universal objects in the categories of groups and quandles, respectively.

The fundamental group of a knot is often represented by using a presentation of a group. The Alexander polynomial [1] and the twisted Alexander invariant [8, 10] can be defined through a matrix that is obtained from the presentation by using the Fox derivative [2]. This construction also works for the fundamental quandle of a knot. Oshiro and the author [6] defined twisted derivatives for quandles and introduced Alexander type invariants of knots.

A handlebody-knot [3] is a handlebody embedded in S^3. A multiple conjugation quandle (MCQ) [4] is an algebraic system whose axioms are derived from the Reidemeister moves for handlebody-knot diagrams. The purpose of this paper is to establish the fundamental multiple conjugation quandle of a handlebody-link, where a handlebody-link is a disjoint union of handlebodies embedded in the 3-sphere S^3.

We introduce the notion of a free multiple conjugation quandle, which is a free object in the category of multiple conjugation quandles. A presentation of a group (resp. quandle) is defined as the quotient by an equivalence relation on a free group (resp. quandle), where the equivalence relation is defined so that it is compatible with the operations of a group (resp. quandle). In the case of a multiple conjugation quandle, we can define the equivalence relation in a similar manner. However the quotient does not form a multiple conjugation quandle. We then introduce the notion of a quasi multiple conjugation quandle and its MCQ closure. Using the MCQ closure, we succeed in defining a presentation of a multiple conjugation quandle.

Using the presentation, we define the fundamental multiple conjugation quandle of a handlebody-link and show that it is an invariant of the handlebody-link. We see that two Wirtinger presentations of the fundamental multiple conjugation quandle of a handlebody-link are related by a finite sequence of certain transformations, which are the so-called Tietze transformations. We also see that the fundamental multiple conjugation quandle satisfies the desired property with respect to multiple conjugation quandle colorings.

This paper is organized as follows. In Section 2, we recall the definition of a multiple conjugation quandle and introduce the notion of a free multiple conjugation quandle. In Section 3, we introduce the notion of MCQ words and clarify when two MCQ words represent the same element in a free multiple conjugation quandle. In Section 4, we give the definitions of a quasi multiple conjugation quandle and its MCQ closure. In Section 5, we introduce the notion of a presentation of a multiple conjugation quandle. In Section 6, we define the fundamental multiple conjugation quandle of a handlebody-link and see its invariance. We also see that the fundamental multiple conjugation quandle of a handlebody-link is isomorphic to that of its mirror image.

2 Free multiple conjugation quandles

In this section, we recall the definition of a multiple conjugation quandle. We then introduce the notion of the free multiple conjugation quandle and show that it has the universal property.

Definition 2.1 ([4])**.** A *multiple conjugation quandle (MCQ)* X is a disjoint union of groups G_λ $(\lambda \in \Lambda)$ with a binary operation $* : X \times X \to X$ satisfying the following axioms.

- For any $a, b \in G_\lambda$, $a * b = b^{-1}ab$.
- For any $x \in X$ and $a, b \in G_\lambda$, $x * e_\lambda = x$ and $x * (ab) = (x * a) * b$, where e_λ is the identity of G_λ.
- For any $x, y, z \in X$, $(x * y) * z = (x * z) * (y * z)$.
- For any $x \in X$ and $a, b \in G_\lambda$, $(ab) * x = (a * x)(b * x)$, where $a * x, b * x \in G_\mu$ for some $\mu \in \Lambda$.

We remark that μ in the last axiom depends only on x and λ and denote it by $\lambda * x$. For $a \in X$, we define $*a : X \to X$ to be the map that sends x to $x * a$. The last axiom indicates that the map $*x|_{G_\lambda} : G_\lambda \to G_{\lambda * x}$ is a group homomorphism. From the second axiom, the map $*a : X \to X$ is bijective, since its inverse map $(*a)^{-1} : X \to X$ is given by $(*a)^{-1}(x) = x * a^{-1}$. We denote the map $(*a)^n : X \to X$ by $*^n a$ for $n \in \mathbb{Z}$. We then have $x *^n y = x * y^n$. In this paper, we often omit parentheses. When doing so, we apply binary operations from the left on expressions, except for multiplications, which we always apply first. For example, $a *^{e_1} b *^{e_2} cd *^{e_3} (e *^{e_4} f *^{e_5} g)$ stands for $((a *^{e_1} b) *^{e_2} (cd)) *^{e_3} ((e *^{e_4} f) *^{e_5} g)$.

Let $X = \bigsqcup_{\lambda\in\Lambda} G_\lambda$ and $Y = \bigsqcup_{\mu\in M} H_\mu$ be multiple conjugation quandles. An *MCQ homomorphism* $f : X \to Y$ is a map satisfying the following conditions:

- For any $x, y \in X$, $f(x * y) = f(x) * f(y)$.
- For any $a, b \in G_\lambda$, $f(ab) = f(a)f(b)$, where $f(a), f(b) \in H_\mu$ for some $\mu \in M$.

We remark that μ in the last condition depends only on f and λ and denote it by $f(\lambda)$. The last condition indicates that the map $f|_{G_\lambda} : G_\lambda \to H_{f(\lambda)}$ is a group homomorphism. The map $*a : X \to X$ is an MCQ homomorphism. We denote by $\mathrm{Hom}(X, Y)$ the set of MCQ homomorphisms from X to Y, and we set $\mathrm{End}(X) := \mathrm{Hom}(X, X)$. Then the second axiom of the definition of a multiple conjugation quandle indicates that the map from G_λ to $\mathrm{End}(X)$ that sends a to $*a : X \to X$ is a group homomorphism, that is, G_λ acts on X. An *MCQ isomorphism* is a bijective MCQ homomorphism.

We write $\{A_\lambda \,|\, \lambda \in \Lambda\} \subset_{\mathsf{MS}} \{B_\mu \,|\, \mu \in M\}$ if, for any $\lambda \in \Lambda$, there exists $\mu \in M$ such that $A_\lambda \subset B_\mu$. Let $X = \bigsqcup_{\lambda\in\Lambda} G_\lambda$ be a multiple conjugation quandle. A *subMCQ* Y of $X = \bigsqcup_{\lambda\in\Lambda} G_\lambda$ is a disjoint union of subgroups H_μ of G_μ ($\mu \in M \subset \Lambda$) with a binary operation $* : Y \times Y \to Y$ that is a restriction of the binary operation of X. A *generating set* of $X = \bigsqcup_{\lambda\in\Lambda} G_\lambda$ is a set $\{S_\mu \,|\, \mu \in M\} \subset_{\mathsf{MS}} \{G_\lambda \,|\, \lambda \in \Lambda\}$ such that X is the unique subMCQ of X containing $\bigcup_{\mu\in M} S_\mu$.

We denote by $F_{\mathsf{Grp}}(S)$ the free group on a set S. Let $S_\Lambda = \{S_\lambda \,|\, \lambda \in \Lambda\}$ be a set of pairwise disjoint sets. Put $S := \bigcup_{\lambda\in\Lambda} S_\lambda$. For $(a, x), (b, y) \in \bigcup_{\lambda\in\Lambda} F_{\mathsf{Grp}}(S_\lambda) \times F_{\mathsf{Grp}}(S)$, we write $(a, x) \sim (b, y)$ if there exist $\lambda \in \Lambda$ and $c \in F_{\mathsf{Grp}}(S_\lambda)$ such that $a \in F_{\mathsf{Grp}}(S_\lambda)$, $b = cac^{-1}$ and $y = cx$. Then $\sim$ is an equivalence relation on $\bigcup_{\lambda\in\Lambda} F_{\mathsf{Grp}}(S_\lambda) \times F_{\mathsf{Grp}}(S)$. We define $F_{\mathsf{MCQ}}(S_\Lambda) := \bigcup_{\lambda\in\Lambda} F_{\mathsf{Grp}}(S_\lambda) \times F_{\mathsf{Grp}}(S)/\!\sim$. For $x \in F_{\mathsf{Grp}}(S)$, we set

$$F_{\mathsf{Grp}}(S_\lambda) * x := \{[(a, x)] \in F_{\mathsf{MCQ}}(S_\Lambda) \,|\, a \in F_{\mathsf{Grp}}(S_\lambda)\},$$

where $[(a, x)]$ stands for the equivalence class of (a, x). For $a_1, \dots, a_n \in S$ and $\varepsilon_1, \dots, \varepsilon_n \in \{\pm 1\}$, a word $a_1^{\varepsilon_1} \cdots a_n^{\varepsilon_n}$ is said to be *irreducible* if $a_i = a_{i+1}$ implies $\varepsilon_i = \varepsilon_{i+1}$ for any i. We denote by $F_{\mathsf{Grp}}(S; S_\lambda)$ the subset of $F_{\mathsf{Grp}}(S)$ consisting of elements represented by irreducible words $a_1^{\varepsilon_1} \cdots a_n^{\varepsilon_n}$ such that $a_1 \notin S_\lambda$. That is,

$$F_{\mathsf{Grp}}(S; S_\lambda) = \left\{ a_1^{\varepsilon_1} \cdots a_n^{\varepsilon_n} \in F_{\mathsf{Grp}}(S) \;\middle|\; \begin{array}{l} n \geq 0,\ \varepsilon_1, \dots, \varepsilon_n \in \{\pm 1\}, \\ a_1 \in S - S_\lambda,\ a_2, \dots, a_n \in S, \\ a_1^{\varepsilon_1} \cdots a_n^{\varepsilon_n} \text{ is irreducible} \end{array} \right\}.$$

We say that $(a, x) \in \bigcup_{\lambda\in\Lambda} F_{\mathsf{Grp}}(S_\lambda) \times F_{\mathsf{Grp}}(S)$ is *irreducible* if $a \in F_{\mathsf{Grp}}(S_\lambda)$ and $x \in F_{\mathsf{Grp}}(S; S_\lambda)$ for some $\lambda \in \Lambda$. If (a, x) and (b, y) are irreducible, then $[(a, x)] = [(b, y)]$ implies $a = b$ and $x = y$. Set $\overline{\Lambda} := \bigcup_{\lambda\in\Lambda}(\{\lambda\} \times F_{\mathsf{Grp}}(S; S_\lambda))$. Then we have $F_{\mathsf{MCQ}}(S_\Lambda) = \bigsqcup_{(\lambda,x)\in\overline{\Lambda}} F_{\mathsf{Grp}}(S_\lambda) * x$.

For $a, b \in F_{\mathsf{Grp}}(S_\lambda)$ and $x \in F_{\mathsf{Grp}}(S)$, we define $[(a, x)][(b, x)] := [(ab, x)]$. For $[(a, x)], [(b, y)] \in F_{\mathsf{MCQ}}(S_\Lambda)$, we define $[(a, x)] * [(b, y)] := [(a, xy^{-1}by)]$. The two operations are well-defined, since we have

$$[(cac^{-1}, cx)][(cbc^{-1}, cx)] = [(cabc^{-1}, cx)],$$
$$[(cac^{-1}, cx)] * [(dbd^{-1}, dy)] = [(cac^{-1}, cxy^{-1}by)].$$

Then $F_{\mathsf{Grp}}(S_\lambda)*x$ is a group with the multiplication for each $(\lambda, x) \in \Lambda \times F_{\mathsf{Grp}}(S)$. The identity and inverse operation of $F_{\mathsf{Grp}}(S_\lambda) * x$ are given by $[(1_{F_{\mathsf{Grp}}(S_\lambda)}, x)]$ and $[(a, x)]^{-1} = [(a^{-1}, x)]$, respectively. We note that $F_{\mathsf{Grp}}(S_\lambda) * x$ is isomorphic to $F_{\mathsf{Grp}}(S_\lambda)$.

Proposition 2.2. *Let $S_\Lambda = \{S_\lambda \mid \lambda \in \Lambda\}$ be a set of pairwise disjoint sets. Then $F_{\mathsf{MCQ}}(S_\Lambda) = \bigsqcup_{(\lambda,x)\in\overline{\Lambda}} F_{\mathsf{Grp}}(S_\lambda) * x$ is a multiple conjugation quandle.*

Proof. For $[(a, x)], [(b, x)] \in F_{\mathsf{Grp}}(S_\lambda) * x$, we have

$$[(a, x)] * [(b, x)] = [(a, bx)] = [(b^{-1}ab, x)] = [(b, x)]^{-1}[(a, x)][(b, x)].$$

For $[(a, x)], [(b, x)] \in F_{\mathsf{Grp}}(S_\lambda) * x$ and $[(c, y)] \in F_{\mathsf{MCQ}}(S_\Lambda)$, we have

$$[(c, y)] * [(1_{F_{\mathsf{Grp}}(S_\lambda)}, x)] = [(c, y)],$$
$$[(c, y)] * ([(a, x)][(b, x)]) = [(c, yx^{-1}abx)] = ([(c, y)] * [(a, x)]) * [(b, x)].$$

For $[(a, x)], [(b, y)], [(c, z)] \in F_{\mathsf{MCQ}}(S_\Lambda)$, we have

$$\begin{aligned}([(a, x)] * [(b, y)]) * [(c, z)] &= [(a, xy^{-1}byz^{-1}cz)] \\ &= ([(a, x)] * [(c, z)]) * ([(b, y)] * [(c, z)]).\end{aligned}$$

For $[(a, x)], [(b, x)] \in F_{\mathsf{Grp}}(S_\lambda) * x$ and $[(c, y)] \in F_{\mathsf{MCQ}}(S_\Lambda)$, we have

$$\begin{aligned}([(a, x)][(b, x)]) * [(c, y)] &= [(ab, xy^{-1}cy)] \\ &= [(a, xy^{-1}cy)][(b, xy^{-1}cy)] \\ &= ([(a, x)] * [(c, y)])([(b, x)] * [(c, y)]).\end{aligned}$$

This completes the proof. □

We call $F_{\mathsf{MCQ}}(S_\Lambda)$ the *free multiple conjugation quandle* on S_Λ. By the injection from S to $F_{\mathsf{MCQ}}(S_\Lambda)$ that sends a to $[(a, 1_{F_{\mathsf{Grp}}(S)})]$, we regard S as a subset of $F_{\mathsf{MCQ}}(S_\Lambda)$ and often denote $[(a, 1_{F_{\mathsf{Grp}}(S)})]$ by a. Then any element $[(a, a_1^{\varepsilon_1} \cdots a_n^{\varepsilon_n})]$ in $F_{\mathsf{MCQ}}(S_\Lambda)$ can be represented as $(\cdots(a * a_1^{\varepsilon_1}) * \cdots) * a_n^{\varepsilon_n}$, since we have

$$[(a, xb)] = [(a, x)] * [(b, 1_{F_{\mathsf{Grp}}(S)})],$$
$$[(ab, 1_{F_{\mathsf{Grp}}(S)})] = [(a, 1_{F_{\mathsf{Grp}}(S)})][(b, 1_{F_{\mathsf{Grp}}(S)})],$$
$$[(a^{-1}, 1_{F_{\mathsf{Grp}}(S)})] = [(a, 1_{F_{\mathsf{Grp}}(S)})]^{-1}.$$

A free multiple conjugation quandle satisfies the universal property as in the following proposition.

Proposition 2.3. *Let $S_\Lambda = \{S_\lambda \mid \lambda \in \Lambda\}$ be a set of pairwise disjoint sets. Put $S := \bigcup_{\lambda\in\Lambda} S_\lambda$. Let $X = \bigsqcup_{\mu\in M} G_\mu$ be a multiple conjugation quandle. Let $f : S \to X$ be a map satisfying $\{f(S_\lambda) \mid \lambda \in \Lambda\} \subset_{\mathsf{MS}} \{G_\mu \mid \mu \in M\}$. Then, there exists a unique MCQ homomorphism $\tilde{f} : F_{\mathsf{MCQ}}(S_\Lambda) \to X$ such that $\tilde{f}|_S = f$.*

Proof. For any $\lambda \in \Lambda$, there exist $\mu \in M$ and a group homomorphism $f_\lambda : F_{\mathsf{Grp}}(S_\lambda) \to G_\mu$ such that $f_\lambda|_{S_\lambda} = f|_{S_\lambda}$, since $F_{\mathsf{Grp}}(S_\lambda)$ is free on S_λ. We define a map $\tilde{f} : F_{\mathsf{MCQ}}(S_\Lambda) \to X$ by $\tilde{f}([(a, a_1^{\varepsilon_1} \cdots a_n^{\varepsilon_n})]) = f_\lambda(a) * f(a_1)^{\varepsilon_1} * \cdots * f(a_n)^{\varepsilon_n}$ for $a \in F_{\mathsf{Grp}}(S_\lambda)$, $a_1, \dots, a_n \in S$ and $\varepsilon_1, \dots, \varepsilon_n \in \{\pm 1\}$. The map $\tilde{f}$ is well-defined, since we have

$$\begin{aligned}
&\tilde{f}([(a, a_1^{\varepsilon_1} \cdots a_{i-1}^{\varepsilon_{i-1}} a_i^{\varepsilon_i} a_i^{-\varepsilon_i} a_{i+1}^{\varepsilon_{i+1}} \cdots a_n^{\varepsilon_n})]) \\
&= f_\lambda(a) * f(a_1)^{\varepsilon_1} * \cdots * f(a_i)^{\varepsilon_i} * f(a_i)^{-\varepsilon_i} * \cdots * f(a_n)^{\varepsilon_n} \\
&= f_\lambda(a) * f(a_1)^{\varepsilon_1} * \cdots * f(a_i)^0 * \cdots * f(a_n)^{\varepsilon_n} \\
&= \tilde{f}([(a, a_1^{\varepsilon_1} \cdots a_{i-1}^{\varepsilon_{i-1}} a_{i+1}^{\varepsilon_{i+1}} \cdots a_n^{\varepsilon_n})])
\end{aligned}$$

and

$$\begin{aligned}
\tilde{f}([(a, b^\varepsilon a_1^{\varepsilon_1} \cdots a_n^{\varepsilon_n})]) &= f_\lambda(a) * f(b)^\varepsilon * f(a_1)^{\varepsilon_1} * \cdots * f(a_n)^{\varepsilon_n} \\
&= f(b)^{-\varepsilon} f_\lambda(a) f(b)^\varepsilon * f(a_1)^{\varepsilon_1} * \cdots * f(a_n)^{\varepsilon_n} \\
&= f_\lambda(b^{-\varepsilon} a b^\varepsilon) * f(a_1)^{\varepsilon_1} * \cdots * f(a_n)^{\varepsilon_n} \\
&= \tilde{f}([(b^{-\varepsilon} a b^\varepsilon, a_1^{\varepsilon_1} \cdots a_n^{\varepsilon_n})])
\end{aligned}$$

for $a \in F_{\mathsf{Grp}}(S_\lambda)$, $b \in S_\lambda$ and $\varepsilon \in \{\pm 1\}$. Then, the map $\tilde{f}$ is an MCQ homomorphism, since we can verify

$$\begin{aligned}
\tilde{f}([(a, a_1^{\varepsilon_1} \cdots a_n^{\varepsilon_n})] * [(b, b_1^{\delta_1} \cdots b_m^{\delta_m})]) &= \tilde{f}([(a, a_1^{\varepsilon_1} \cdots a_n^{\varepsilon_n})]) * \tilde{f}([(b, b_1^{\delta_1} \cdots b_m^{\delta_m})]), \\
\tilde{f}([(a, a_1^{\varepsilon_1} \cdots a_n^{\varepsilon_n})][(b, a_1^{\varepsilon_1} \cdots a_n^{\varepsilon_n})]) &= \tilde{f}([(a, a_1^{\varepsilon_1} \cdots a_n^{\varepsilon_n})]) \tilde{f}([(b, a_1^{\varepsilon_1} \cdots a_n^{\varepsilon_n})])
\end{aligned}$$

by direct calculation. The equality $\tilde{f}|_S = f$ follows from $\tilde{f}([(a, 1_{F_{\mathsf{Grp}}(S)})]) = f_\lambda(a) = f(a)$ for $a \in S_\lambda$. If $\tilde{f}' : F_{\mathsf{MCQ}}(S_\Lambda) \to X$ is an MCQ homomorphism satisfying $\tilde{f}'|_S = f$, we have $\tilde{f}'([(a, a_1^{\varepsilon_1} \cdots a_n^{\varepsilon_n})]) = f_\lambda(a) * f(a_1)^{\varepsilon_1} * \cdots * f(a_n)^{\varepsilon_n}$ for $a \in F_{\mathsf{Grp}}(S_\lambda)$ and $a_1, \dots, a_n \in S$. Therefore, the map $\tilde{f}$ is unique. □

3 MCQ words

In this section, we introduce the notion of an MCQ word, which represents an element of $F_{\mathsf{MCQ}}(S_\Lambda)$. We also introduce transformations on MCQ words and describe when two MCQ words represent the same element in $F_{\mathsf{MCQ}}(S_\Lambda)$.

Let $S_\Lambda = \{S_\lambda \mid \lambda \in \Lambda\}$ be a set of pairwise disjoint sets. Put $S := \bigcup_{\lambda \in \Lambda} S_\lambda$. We set $\mathcal{W}_{\mathsf{MCQ}}(S_\Lambda; 0) := S$. For $k \geq 0$, we define

$$\begin{aligned}
&\mathcal{W}_{\mathsf{MCQ}}(S_\Lambda; k+1) \\
&= \left\{ a_1^{\varepsilon_1} \cdots a_n^{\varepsilon_n} \;\middle|\; \begin{array}{l} n \geq 1,\ \varepsilon_1, \dots, \varepsilon_n \in \{-1, 0, 1\},\ a_1, \dots, a_n \in \mathcal{W}_{\mathsf{MCQ}}(S_\Lambda; k), \\ a_1, \dots, a_n \text{ represent elements in the same group of } F_{\mathsf{MCQ}}(S_\Lambda) \end{array} \right\} \\
&\quad \cup \{x * y \mid x, y \in \mathcal{W}_{\mathsf{MCQ}}(S_\Lambda; k)\},
\end{aligned}$$

where $a_1^{\varepsilon_1} \cdots a_n^{\varepsilon_n}$ and $x * y$ are symbols, and we put parentheses in appropriate places. For example,

$$a * a, a * b, b * a, b * b, a, a^0, a^{-1}, b, b^0, b^{-1}, aa, aa^{-1}, a^{-1}aa, bb^{-1}, bb^0b^{-1}$$

are elements of $\mathcal{W}_{\mathsf{MCQ}}(\{\{a\},\{b\}\};1)$, and

$$(a*b)*b^{-1}, aa*(a*b), (a*b)*aa, b^{-1}*(a*b), (a*b)^0, (b^{-1})^{-1}, (a*b)(a*b)$$

are elements of $\mathcal{W}_{\mathsf{MCQ}}(\{\{a\},\{b\}\};2)$. Define $\mathcal{W}_{\mathsf{MCQ}}(S_\Lambda) := \bigcup_{k=0}^{\infty} \mathcal{W}_{\mathsf{MCQ}}(S_\Lambda;k)$. We call its elements *MCQ words in* S_Λ. Two words $a, b \in \mathcal{W}_{\mathsf{MCQ}}(S_\Lambda)$ are *multiplicable* if they represent elements in the same group of $F_{\mathsf{MCQ}}(S_\Lambda)$. We say that $(\cdots(a_1^{\varepsilon_1}\cdots a_n^{\varepsilon_n} * x_1^{\delta_1})\cdots) * x_m^{\delta_m} \in \mathcal{W}_{\mathsf{MCQ}}(S_\Lambda)$ ($n \geq 1$, $a_1, \ldots, a_n \in S_\lambda$, $m \geq 0$, $x_1, \ldots, x_m \in S$) is *irreducible* if

$$\begin{aligned}
&a_i = a_{i+1} \text{ implies } \varepsilon_i = \varepsilon_{i+1} \in \{\pm 1\},\\
&x_i = x_{i+1} \text{ implies } \delta_i = \delta_{i+1} \in \{\pm 1\},\\
&a_1, \ldots, a_n \in S_\lambda \text{ implies } x_1 \notin S_\lambda,\\
&\varepsilon_i = 0 \text{ implies } n = 1, \text{ and}\\
&\delta_i \neq 0
\end{aligned}$$

for any i.

Lemma 3.1. *Let $S_\Lambda = \{S_\lambda \mid \lambda \in \Lambda\}$ be a set of pairwise disjoint sets. Put $S := \bigcup_{\lambda\in\Lambda} S_\lambda$. Let*

$$\begin{aligned}
w &= (\cdots(a_1^{\varepsilon_1}\cdots a_n^{\varepsilon_n} * x_1^{\delta_1})\cdots) * x_m^{\delta_m} \quad \textit{and}\\
w' &= (\cdots(b_1^{\varepsilon'_1}\cdots b_{n'}^{\varepsilon'_{n'}} * y_1^{\delta'_1})\cdots) * y_{m'}^{\delta'_{m'}}
\end{aligned}$$

be irreducible MCQ words in S_Λ, where $n, n' \geq 1$, $a_1, \ldots, a_n \in S_\lambda$, $b_1, \ldots, b_{n'} \in S_{\lambda'}$, $m, m' \geq 0$, and $x_1, \ldots, x_m, y_1, \ldots, y_{m'} \in S$. If w and w' are multiplicable, then $\lambda = \lambda'$, $m = m'$, $x_1 = y_1, \ldots, x_m = y_{m'}$, and $\delta_1 = \delta'_1, \ldots, \delta_m = \delta'_{m'}$.

Proof. The MCQ words w and w' represent

$$\begin{aligned}
&[(a_1^{\varepsilon_1}\cdots a_n^{\varepsilon_n}, x_1^{\delta_1}\cdots x_m^{\delta_m})] \in F_{\mathsf{Grp}}(S_\lambda) * x_1^{\delta_1}\cdots x_m^{\delta_m} \quad \text{and}\\
&[(b_1^{\varepsilon'_1}\cdots b_{n'}^{\varepsilon'_{n'}}, y_1^{\delta'_1}\cdots y_{m'}^{\delta'_{m'}})] \in F_{\mathsf{Grp}}(S_\lambda) * y_1^{\delta'_1}\cdots y_{m'}^{\delta'_{m'}},
\end{aligned}$$

respectively. Since w and w' are irreducible, we have

$$x_1^{\delta_1}\cdots x_m^{\delta_m} \in F_{\mathsf{Grp}}(S;S_\lambda) \qquad \text{and} \qquad y_1^{\delta'_1}\cdots y_{m'}^{\delta'_{m'}} \in F_{\mathsf{Grp}}(S;S_{\lambda'}).$$

Since w and w' are multiplicable, we have $\lambda = \lambda'$ and $x_1^{\delta_1}\cdots x_m^{\delta_m} = y_1^{\delta'_1}\cdots y_{m'}^{\delta'_{m'}}$. Since w and w' are irreducible, so are $x_1^{\delta_1}\cdots x_m^{\delta_m}$ and $y_1^{\delta'_1}\cdots y_{m'}^{\delta'_{m'}}$, which implies $m = m'$, $x_1 = y_1, \ldots, x_m = y_{m'}$, and $\delta_1 = \delta'_1, \ldots, \delta_m = \delta'_{m'}$. □

Proposition 3.2. *Let $S_\Lambda = \{S_\lambda \mid \lambda \in \Lambda\}$ be a set of pairwise disjoint sets. Put $S := \bigcup_{\lambda\in\Lambda} S_\lambda$. For $w_1, w_2 \in \mathcal{W}_{\mathsf{MCQ}}(S_\Lambda)$, w_1 and w_2 represent the same element in $F_{\mathsf{MCQ}}(S_\Lambda)$ if and only if w_1 and w_2 are related by a finite sequence of the following local replacements on MCQ words:*

$$ab^{\varepsilon}b^{-\varepsilon}c \leftrightarrow ab^0c \leftrightarrow ac, \tag{1}$$

$$a*b \leftrightarrow b^{-1}ab, \tag{2}$$

$$x*a^0 \leftrightarrow x,\ x*ab \leftrightarrow (x*a)*b, \tag{3}$$

$$(x*y)*z \leftrightarrow (x*z)*(y*z), \tag{4}$$

$$ab*x \leftrightarrow (a*x)(b*x), \tag{5}$$

where $\varepsilon \in \{-1, 0, 1\}$ and a, b, c, x, y, z are elements of $\mathcal{W}_{\mathsf{MCQ}}(S_\Lambda)$ such that a, b, c are multiplicable.

Proof. It is easy to see the "if" part. We show the "only if" part. By the transformation (1), we have

$$(a^\varepsilon)^0 \leftrightarrow (a^\varepsilon)^0 a^0 \leftrightarrow a^0, \tag{6}$$
$$(a^\varepsilon)^{-1} \leftrightarrow (a^\varepsilon)^{-1} a^\varepsilon a^{-\varepsilon} \leftrightarrow a^{-\varepsilon}, \tag{7}$$
$$(ab)^0 \leftrightarrow (ab)^0 a^0 \leftrightarrow (ab)^0 a^0 b^0 \leftrightarrow a^0 b^0, \tag{8}$$
$$(ab)^{-1} \leftrightarrow (ab)^{-1} aa^{-1} \leftrightarrow (ab)^{-1} abb^{-1} a^{-1} \leftrightarrow b^{-1} a^{-1} \tag{9}$$

for $\varepsilon \in \{-1, 0, 1\}$. By the transformations (1) and (5), we have

$$\begin{aligned}(x * y)^0 &\leftrightarrow (x * y)(x * y)^{-1} \leftrightarrow (x^0 x * y)(x * y)^{-1} \\ &\leftrightarrow (x^0 * y)(x * y)(x * y)^{-1} \leftrightarrow x^0 * y. \end{aligned}\tag{10}$$

By the transformations (1), (5) and (10), we have

$$\begin{aligned}(x * y)^{-1} &\leftrightarrow (x * y)^0 (x * y)^{-1} \leftrightarrow (x^0 * y)(x * y)^{-1} \leftrightarrow (x^{-1} x * y)(x * y)^{-1} \\ &\leftrightarrow (x^{-1} * y)(x * y)(x * y)^{-1} \leftrightarrow x^{-1} * y. \end{aligned}\tag{11}$$

By using the transformations (6)–(11), any word in $\mathcal{W}_{\mathsf{MCQ}}(S_\Lambda)$ can be transformed into an MCQ word in which a^ε implies $a \in S$ for $\varepsilon \in \{-1, 0\}$. Furthermore, by using the transformations (3) and (5), the MCQ word can be transformed into the form $w_1 \cdots w_n$, where w_i is an MCQ word in which there are no multiplications.

By the transformations (1), (3) and (4), we have

$$\begin{aligned}x * (y * z) &\leftrightarrow (x * z^0) * (y * z) \leftrightarrow (x * z^{-1} z) * (y * z) \\ &\leftrightarrow ((x * z^{-1}) * z) * (y * z) \leftrightarrow ((x * z^{-1}) * y) * z. \end{aligned}\tag{12}$$

By using the transformations (1) and (3)–(12), any element in $\mathcal{W}_{\mathsf{MCQ}}(S_\Lambda)$ can be transformed into the following form

$$((\cdots(a_1^{\varepsilon_1} * x_{1,1}^{\delta_{1,1}})\cdots) * x_{1,m_1}^{\delta_{1,m_1}}) \cdots ((\cdots(a_n^{\varepsilon_n} * x_{n,1}^{\delta_{n,1}})\cdots) * x_{n,m_n}^{\delta_{n,m_n}}), \tag{13}$$

where $a_1, \ldots, a_n, x_{1,1}, \ldots, x_{n,m_n} \in S$. We may assume that

$$(\cdots(a_1^{\varepsilon_1} * x_{1,1}^{\delta_{1,1}})\cdots) * x_{1,m_1}^{\delta_{1,m_1}}, \ldots, (\cdots(a_n^{\varepsilon_n} * x_{n,1}^{\delta_{n,1}})\cdots) * x_{n,m_n}^{\delta_{n,m_n}} \tag{14}$$

are irreducible, since we have (1), (2), (5) and

$$(x * a^\varepsilon) * a^{-\varepsilon} \leftrightarrow x * a^\varepsilon a^{-\varepsilon} \leftrightarrow x * a^0 \leftrightarrow x,$$

which follows from (1) and (3). Since MCQ words of (14) are multiplicable, by Lemma 3.1, we have $m_1 = \cdots = m_n$, $x_{1,j}^{\delta_{1,j}} = \cdots = x_{n,j}^{\delta_{n,j}}$ and $a_1, \ldots, a_n \in S_\lambda$ for some $\lambda \in \Lambda$. Then, by using (5), an MCQ word of the form (13) can be transformed into an irreducible word

$$(\cdots(a_1^{\varepsilon_1} \cdots a_n^{\varepsilon_n} * x_1^{\delta_1})\cdots) * x_m^{\delta_m}.$$

Therefore, by using the transformations (1)–(5), w_1 and w_2 are respectively transformed into irreducible words

$$(\cdots(a_1^{\varepsilon_1}\cdots a_n^{\varepsilon_n}*x_1^{\delta_1})\cdots)*x_m^{\delta_m} \quad \text{and} \quad (\cdots(b_1^{\varepsilon'_1}\cdots b_{n'}^{\varepsilon'_{n'}}*y_1^{\delta'_1})\cdots)*y_{m'}^{\delta'_{m'}}$$

for some $a_1,\dots,a_n,b_1,\dots,b_{n'},x_1,\dots,x_m,y_1,\dots,y_{m'}\in S$. Since w_1 and w_2 represent the same element in $F_{\mathsf{MCQ}}(S_\Lambda)$, we have

$$[(a_1^{\varepsilon_1}\cdots a_n^{\varepsilon_n},x_1^{\delta_1}\cdots x_m^{\delta_m})]=[(b_1^{\varepsilon'_1}\cdots b_{n'}^{\varepsilon'_{n'}},y_1^{\delta'_1}\cdots y_{m'}^{\delta'_{m'}})].$$

Since $(a_1^{\varepsilon_1}\cdots a_n^{\varepsilon_n},x_1^{\delta_1}\cdots x_m^{\delta_m})$ and $(b_1^{\varepsilon'_1}\cdots b_{n'}^{\varepsilon'_{n'}},y_1^{\delta'_1}\cdots y_{m'}^{\delta'_{m'}})$ are irreducible, we have

$$a_1^{\varepsilon_1}\cdots a_n^{\varepsilon_n}=b_1^{\varepsilon'_1}\cdots b_{n'}^{\varepsilon'_{n'}} \qquad \text{and} \qquad x_1^{\delta_1}\cdots x_m^{\delta_m}=y_1^{\delta'_1}\cdots y_{m'}^{\delta'_{m'}}.$$

The last equality implies $m=m'$, $x_i=y_i$, $\delta_i=\delta'_i$ for any i. The first equality implies $n=n'=1$, $\varepsilon_1=\varepsilon'_1=0$ or $n=n'$, $a_i=b_i$, $\varepsilon_i=\varepsilon'_i\neq 0$ for any i. If $n=n'=1$ and $\varepsilon_1=\varepsilon'_1=0$, then we have

$$\begin{aligned}(\cdots(a_1^0*x_1^{\delta_1})\cdots)*x_m^{\delta_m} &\leftrightarrow (\cdots(a_1^0b_1^0*x_1^{\delta_1})\cdots)*x_m^{\delta_m}\\ &\leftrightarrow (\cdots(b_1^0*x_1^{\delta_1})\cdots)*x_m^{\delta_m}=(\cdots(b_1^0*y_1^{\delta'_1})\cdots)*y_{m'}^{\delta'_{m'}}\end{aligned}$$

by using (1). This completes the proof. □

4 A quasi MCQ and its MCQ closure

In this section, we introduce the notion of a quasi multiple conjugation quandle, whose axioms are obtained by weakening those of a multiple conjugation quandle. We then introduce the MCQ closure of a quasi multiple conjugation quandle and show that it is a multiple conjugation quandle.

Definition 4.1. A *quasi multiple conjugation quandle (quasi MCQ)* X is a union of groups G_λ ($\lambda\in\Lambda$) with a binary operation $*:X\times X\to X$ satisfying the following axioms.

- For any $\lambda,\mu\in\Lambda$, the multiplications and inverse operations of G_λ and G_μ respectively coincide on $G_\lambda\cap G_\mu$.
- For any $a,b\in G_\lambda$, $a*b=b^{-1}ab$.
- For any $x\in X$ and $a,b\in G_\lambda$, $x*e_\lambda=x$ and $x*(ab)=(x*a)*b$, where e_λ is the identity of G_λ.
- For any $x,y,z\in X$, $(x*y)*z=(x*z)*(y*z)$.
- For any $x\in X$ and $\lambda\in\Lambda$, there exists $\mu\in\Lambda$ such that $G_\lambda*x\subset G_\mu$ and the map $*x:G_\lambda\to G_\mu$ defined by $(*x)(a)=a*x$ is a group homomorphism, where $G_\lambda*x:=\{a*x\,|\,a\in G_\lambda\}$.

A multiple conjugation quandle is a quasi multiple conjugation quandle. From the first axiom, we may use the same notations ab and a^{-1} to represent the multiplications and inverse operations, respectively. In other words, the multiplications define a map from $\bigcup_{\lambda\in\Lambda} G_\lambda^2$ to X that sends (a,b) to ab, and the inverse operations define a map from X to X that sends a to a^{-1}. We remark that, if $G_\lambda \cap G_\mu \neq \emptyset$, then $G_\lambda \cap G_\mu$ is a subgroup of G_λ and G_μ. We also remark that $G_\lambda \cap G_\mu \neq \emptyset$ if and only if $e_\lambda = e_\mu$.

Let $X = \bigcup_{\lambda\in\Lambda} G_\lambda$ and $Y = \bigcup_{\mu\in M} H_\mu$ be quasi multiple conjugation quandles. A *quasi MCQ homomorphism* $f : X \to Y$ is a map satisfying the following conditions:

- For any $x, y \in X$, $f(x * y) = f(x) * f(y)$.
- For any $\lambda \in \Lambda$, there exists $\mu \in M$ such that $f(G_\lambda) \subset H_\mu$ and that $f|_{G_\lambda} : G_\lambda \to H_\mu$ is a group homomorphism.

We note that an MCQ homomorphism is a quasi MCQ homomorphism and that a quasi MCQ homomorphism between two multiple conjugation quandles is an MCQ homomorphism. A quasi MCQ homomorphism $f : X \to Y$ is a *quasi MCQ isomorphism* if there exists a quasi MCQ homomorphism $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. The map $*a : X \to X$ that sends x to $x * a$ is a quasi MCQ isomorphism.

Lemma 4.2. *Let $X = \bigcup_{\lambda\in\Lambda} G_\lambda$ be a quasi multiple conjugation quandle. Let $a \in G_{\lambda_1}$ and $b \in G_{\lambda_2}$. We choose $\lambda_3 \in \Lambda$ so that $a * b \in G_{\lambda_3}$. If $G_{\lambda_1} \cap G_{\lambda_2} \neq \emptyset$, then $G_{\lambda_1} \cap G_{\lambda_2} \cap G_{\lambda_3} \neq \emptyset$.*

Proof. It is sufficient to show that $e_{\lambda_1} = e_{\lambda_2}$ implies $e_{\lambda_1} = e_{\lambda_2} = e_{\lambda_3}$. There exists $\mu \in \Lambda$ such that $G_{\lambda_1} * b \subset G_\mu$ and the map $*b : G_{\lambda_1} \to G_\mu$ is a group homomorphism. We then have $e_{\lambda_1} * b = e_\mu$. Since $a * b \in G_\mu \cap G_{\lambda_3} \neq \emptyset$, we have $e_\mu = e_{\lambda_3}$. Hence we have $e_{\lambda_3} = e_\mu = e_{\lambda_1} * b = e_{\lambda_2} * b = b^{-1} e_{\lambda_2} b = e_{\lambda_2}$, which completes the proof. □

Let $X = \bigcup_{\lambda\in\Lambda} G_\lambda$ be a quasi multiple conjugation quandle. Put $E := \{e_\lambda \mid \lambda \in \Lambda\}$. For $e \in E$, we set $\Lambda_e := \{\lambda \in \Lambda \mid e_\lambda = e\}$. A *word* in $\bigcup_{\lambda\in\Lambda_e} G_\lambda$ is an expression of the form $a_1 \cdots a_n$, where $n \geq 1$ and $a_1, \ldots, a_n \in \bigcup_{\lambda\in\Lambda_e} G_\lambda$. Two words in $\bigcup_{\lambda\in\Lambda_e} G_\lambda$ are *equivalent* if they are related by a finite sequence of the following transformations:

- $a_1 \cdots a_{i-1} a_i b_i a_{i+1} \cdots a_n \leftrightarrow a_1 \cdots a_{i-1} (a_i b_i) a_{i+1} \cdots a_n$,
  where $a_1, \ldots, a_n, b_1, \ldots, b_n \in \bigcup_{\lambda\in\Lambda_e} G_\lambda$, $a_i, b_i \in G_\lambda$ and $(a_i b_i)$ stands for the product $a_i b_i$ in G_λ.
- $a_1 \cdots a_{i-1} (a_i * b_i) a_{i+1} \cdots a_n \leftrightarrow a_1 \cdots a_{i-1} b_i^{-1} a_i b_i a_{i+1} \cdots a_n$,
  where $a_1, \ldots, a_n, b_1, \ldots, b_n \in \bigcup_{\lambda\in\Lambda_e} G_\lambda$.

Here, we remark that $a_i * b_i \in \bigcup_{\lambda\in\Lambda_e} G_\lambda$ by Lemma 4.2. We denote by $\mathrm{Amg}(\bigcup_{\lambda\in\Lambda_e} G_\lambda)$ the set of equivalence classes of words in $\bigcup_{\lambda\in\Lambda_e} G_\lambda$, where the symbol "Amg" indicates "amalgamate." We denote the equivalence class of a word $a_1 \cdots a_n$ by $\overline{a_1 \cdots a_n}$ or simply by $a_1 \cdots a_n$. Then $\mathrm{Amg}(\bigcup_{\lambda\in\Lambda_e} G_\lambda)$ is a group with the concatenation operation. We define

$$\overline{X} := \bigsqcup_{e\in E} \mathrm{Amg}\left(\bigcup_{\lambda\in\Lambda_e} G_\lambda\right)$$

and a binary operation $* : \overline{X} \times \overline{X} \to \overline{X}$ by

$$\overline{a_1 \cdots a_n} * \overline{b_1 \cdots b_m} = \overline{(a_1 * b_1 * \cdots * b_m) \cdots (a_n * b_1 * \cdots * b_m)}$$

for $a_1, \ldots, a_n \in \bigcup_{\lambda \in \Lambda_e} G_\lambda$ and $b_1, \ldots, b_m \in \bigcup_{\lambda \in \Lambda_{e'}} G_\lambda$. For a quasi MCQ homomorphism $f : X \to Y$, we define $\bar{f} : \overline{X} \to \overline{Y}$ by $\bar{f}(\overline{a_1 \cdots a_n}) = \overline{f(a_1) \cdots f(a_n)}$ for $a_1, \ldots, a_n \in \bigcup_{\lambda \in \Lambda_e} G_\lambda$. We note that, if X is a multiple conjugation quandle, then $\overline{X} = X$.

Proposition 4.3. *Let $X = \bigcup_{\lambda \in \Lambda} G_\lambda$ be a quasi multiple conjugation quandle. Then $\overline{X}$ is a multiple conjugation quandle and $\bar{f}$ is an MCQ homomorphism.*

Proof. Let $a_1, \ldots, a_n, b_1, \ldots, b_n \in \bigcup_{\lambda \in \Lambda_e} G_\lambda$ and let $c_1, \ldots, c_m, d_1, \ldots, d_m \in \bigcup_{\lambda \in \Lambda_{e'}} G_\lambda$. If $a_i, b_i \in G_\lambda$, then we have

$$\begin{aligned}
&(a_1 \cdots a_i b_i \cdots a_n) * (c_1 \cdots c_m) \\
&= (a_1 * c_1 * \cdots * c_m) \cdots (a_i * c_1 * \cdots * c_m)(b_i * c_1 * \cdots * c_m) \cdots (a_n * c_1 * \cdots * c_m) \\
&= (a_1 * c_1 * \cdots * c_m) \cdots ((a_i * c_1 * \cdots * c_m)(b_i * c_1 * \cdots * c_m)) \cdots (a_n * c_1 * \cdots * c_m) \\
&= (a_1 * c_1 * \cdots * c_m) \cdots ((a_i b_i) * c_1 * \cdots * c_m) \cdots (a_n * c_1 * \cdots * c_m) \\
&= (a_1 \cdots (a_i b_i) \cdots a_n) * (c_1 \cdots c_m),
\end{aligned}$$

where we remark that $a_i * c_1 * \cdots * c_m, b_i * c_1 * \cdots * c_m \in G_\mu$ for some $\mu \in \Lambda$. If $c_i, d_i \in G_\lambda$, then we have

$$\begin{aligned}
&(a_1 \cdots a_n) * (c_1 \cdots c_i d_i \cdots c_m) \\
&= (a_1 * c_1 * \cdots * c_i * d_i * \cdots * c_m) \cdots (a_n * c_1 * \cdots * c_i * d_i * \cdots * c_m) \\
&= (a_1 * c_1 * \cdots * (c_i d_i) * \cdots * c_m) \cdots (a_n * c_1 * \cdots * (c_i d_i) * \cdots * c_m) \\
&= (a_1 \cdots a_n) * (c_1 \cdots (c_i d_i) \cdots c_m).
\end{aligned}$$

In a similar manner, we have

$$\begin{aligned}
(a_1 \cdots (a_i * b_i) \cdots a_n) * (c_1 \cdots c_m) &= (a_1 \cdots b_i^{-1} a_i b_i \cdots a_n) * (c_1 \cdots c_m), \\
(a_1 \cdots a_n) * (c_1 \cdots (c_i * d_i) \cdots c_m) &= (a_1 \cdots a_n) * (c_1 \cdots d_i^{-1} c_i d_i \cdots c_m).
\end{aligned}$$

Hence, the binary operation $* : \overline{X} \times \overline{X} \to \overline{X}$ is well-defined.

For $a_1, \ldots, a_n, b_1, \ldots, b_m \in \bigcup_{\lambda \in \Lambda_e} G_\lambda$, we have

$$\begin{aligned}
(a_1 \cdots a_n) * (b_1 \cdots b_m) &= (a_1 * b_1 * \cdots * b_m) \cdots (a_n * b_1 * \cdots * b_m) \\
&= (b_m^{-1} \cdots b_1^{-1} a_1 b_1 \cdots b_m) \cdots (b_m^{-1} \cdots b_1^{-1} a_n b_1 \cdots b_m) \\
&= (b_1 \cdots b_m)^{-1} (a_1 \cdots a_n)(b_1 \cdots b_m)
\end{aligned}$$

and

$$(a_1 \cdots a_n) * e_\mu = (a_1 * e_\mu) \cdots (a_n * e_\mu) = a_1 \cdots a_n.$$

In a similar manner, we have

$$\begin{aligned}
&(a_1 \cdots a_n) * ((b_1 \cdots b_i)(b_{i+1} \cdots b_m)) \\
&= ((a_1 \cdots a_n) * (b_1 \cdots b_i)) * (b_{i+1} \cdots b_m), \\
&((a_1 \cdots a_i)(a_{i+1} \cdots a_n)) * (b_1 \cdots b_m) \\
&= ((a_1 \cdots a_i) * (b_1 \cdots b_m))((a_{i+1} \cdots a_n) * (b_1 \cdots b_m)), \\
&((a_1 \cdots a_n) * (b_1 \cdots b_m)) * (c_1 \cdots c_l) \\
&= ((a_1 \cdots a_n) * (c_1 \cdots c_l)) * ((b_1 \cdots b_m) * (c_1 \cdots c_l))
\end{aligned}$$

for $a_1, \ldots, a_n \in \bigcup_{\lambda\in\Lambda_e} G_\lambda$, $b_1, \ldots, b_m \in \bigcup_{\lambda\in\Lambda_{e'}} G_\lambda$, $c_1, \ldots, c_l \in \bigcup_{\lambda\in\Lambda_{e''}} G_\lambda$. Therefore, $\overline{X}$ is a multiple conjugation quandle.

For $a_1, \ldots, a_n \in \bigcup_{\lambda\in\Lambda_e} G_\lambda$ and $b_1, \ldots, b_m \in \bigcup_{\lambda\in\Lambda_{e'}} G_\lambda$, we have

$$\begin{aligned}
&\bar{f}((a_1 \cdots a_n) * (b_1 \cdots b_m)) \\
&= \bar{f}((a_1 * b_1 * \cdots * b_m) \cdots (a_n * b_1 * \cdots * b_m)) \\
&= f(a_1 * b_1 * \cdots * b_m) \cdots f(a_n * b_1 * \cdots * b_m) \\
&= (f(a_1) * f(b_1) * \cdots * f(b_m)) \cdots (f(a_n) * f(b_1) * \cdots * f(b_m)) \\
&= (f(a_1) \cdots f(a_n)) * (f(b_1) \cdots f(b_m)) \\
&= \bar{f}(a_1 \cdots a_n) * \bar{f}(b_1 \cdots b_m)
\end{aligned}$$

and

$$\bar{f}((a_1 \cdots a_i)(a_{i+1} \cdots a_n)) = f(a_1) \cdots f(a_n) = \bar{f}(a_1 \cdots a_i)\bar{f}(a_{i+1} \cdots a_n).$$

Therefore, $\bar{f}$ is an MCQ homomorphism. □

We note that $\overline{\cdot}$ gives a functor from the category of quasi multiple conjugation quandles to that of multiple conjugation quandles, since we have $\overline{g \circ f} = \overline{g} \circ \overline{f}$ and $\overline{\mathrm{id}_X} = \mathrm{id}_{\overline{X}}$.

5 A presentation of a multiple conjugation quandle

In this section, we introduce an equivalence relation on a multiple conjugation quandle to define its quotient, which is a quasi multiple conjugation quandle. A presentation of the multiple conjugation quandle is defined as the MCQ closure of the quasi multiple conjugation quandle. We see that every multiple conjugation quandle has a presentation.

Let $X = \bigcup_{\lambda\in\Lambda} G_\lambda$ be a quasi multiple conjugation quandle, and P a subset of $X \times X$. For $a, b \in X$, we write $a \sim_P b$ if $(a, b) \in P$. An *MCQ congruence relation* on X is an equivalence relation $\sim_P$ on X satisfying the following conditions:

- If $a_1 \sim_P a_2$ and $b_1 \sim_P b_2$, then $a_1 * b_1 \sim_P a_2 * b_2$.
- If $a_1 \sim_P a_2$ and $b_1 \sim_P b_2$ ($a_i, b_i \in G_{\lambda_i}$), then $a_1 b_1^{-1} \sim_P a_2 b_2^{-1}$.

For $a, b \in X$, we define $[a] * [b] := [a * b]$, where $[a]$ stands for the equivalence class of a. For $a, b \in G_\lambda$, we define $[a][b] := [ab]$. We set $[G_\lambda] := \{[a] \mid a \in G_\lambda\}$, which is a group with the multiplication. We note that the identity of $[G_\lambda]$ is $[e_\lambda]$ and $[a]^{-1} = [a^{-1}]$. We then have $X/{\sim_P} = \bigcup_{\lambda\in\Lambda} [G_\lambda]$.

Proposition 5.1. *Let $X = \bigcup_{\lambda\in\Lambda} G_\lambda$ be a quasi multiple conjugation quandle, and let $\sim_P$ be an MCQ congruence relation on X. Then $X/{\sim_P} = \bigcup_{\lambda\in\Lambda} [G_\lambda]$ is a quasi multiple conjugation quandle with the binary operation $* : X/{\sim_P} \times X/{\sim_P} \to X/{\sim_P}$ defined by $[a] * [b] = [a * b]$.*

Proof. For $a_1, b_1 \in G_{\lambda_1}$ and $a_2, b_2 \in G_{\lambda_2}$ such that $[a_1] = [a_2]$ and $[b_1] = [b_2]$, we have $[a_1][b_1]^{-1} = [a_1 b_1^{-1}] = [a_2 b_2^{-1}] = [a_2][b_2]^{-1}$. Hence the multiplications and the inverse operations of $[G_{\lambda_1}]$ and $[G_{\lambda_2}]$ respectively coincide on $[G_{\lambda_1}] \cap [G_{\lambda_2}]$. By direct calculation, we can check the remaining axioms of a quasi multiple conjugation quandle. □

For a quasi MCQ homomorphism $f : X \to Y$, we define $R_f := \{(a,b) \in X \times X \mid f(a) = f(b)\}$ and set $\sim_f := \sim_{R_f}$. It is easy to see that $\sim_f$ is an MCQ congruence relation on X.

Proposition 5.2. *Let $X = \bigcup_{\lambda\in\Lambda} G_\lambda$ and $Y = \bigcup_{\mu\in M} H_\mu$ be quasi multiple conjugation quandles. Let $f : X \to Y$ be a quasi MCQ homomorphism. Then $\operatorname{Im} f = \bigcup_{\lambda\in\Lambda} f(G_\lambda)$ is a quasi multiple conjugation quandle whose operations are restrictions of those of Y, and $\tilde{f} : X/{\sim_f} \to \operatorname{Im} f$ defined by $\tilde{f}([x]) = f(x)$ is a quasi MCQ isomorphism.*

Proof. Since f is a quasi MCQ homomorphism, $f(G_\lambda)$ is a subgroup of H_μ for some $\mu \in M$. For $x, y \in X$, we have $f(x) * f(y) = f(x * y) \in \operatorname{Im} f$. Then $\operatorname{Im} f = \bigcup_{\lambda\in\Lambda} f(G_\lambda)$ is a quasi multiple conjugation quandle whose operations are restrictions of those of Y.

We show that $\tilde{f}$ is a quasi MCQ isomorphism. Let $\tilde{g} : \operatorname{Im} f \to X/{\sim_f}$ be the map defined by $\tilde{g}(f(x)) = [x]$. Since $\tilde{g} \circ \tilde{f} = \mathrm{id}_{X/\sim_f}$ and $\tilde{g} \circ \tilde{f} = \mathrm{id}_{\operatorname{Im} f}$, it is sufficient to show that $\tilde{f}$ and $\tilde{g}$ are quasi MCQ homomorphisms. For $[x], [y] \in X/{\sim_f}$, we have

$$\tilde{f}([x] * [y]) = \tilde{f}([x * y]) = f(x * y) = f(x) * f(y) = \tilde{f}([x]) * \tilde{f}([y]).$$

For any $\lambda \in \Lambda$, we have $\tilde{f}([G_\lambda]) = \{\tilde{f}([x]) \mid x \in G_\lambda\} = f(G_\lambda)$. For $[a], [b] \in [G_\lambda]$ $(a, b \in G_\lambda)$, we have

$$\tilde{f}([a][b]) = \tilde{f}([ab]) = f(ab) = f(a)f(b) = \tilde{f}([a])\tilde{f}([b]).$$

Then, $\tilde{f}$ is a quasi MCQ homomorphism. In a similar manner, we see that $\tilde{g}$ is a quasi MCQ homomorphism. □

Let $X = \bigsqcup_{\lambda\in\Lambda} G_\lambda$ be a multiple conjugation quandle. For $R \subset X \times X$, we define $N_{\mathsf{MCQ}}(R)$ to be the smallest MCQ congruence relation including R. For $A \subset X \times X$, we define

$$\begin{aligned} N_{\mathsf{mcq}}(A) := A &\cup \{(x,x) \mid x \in X\} \\ &\cup \{(b,a) \mid (a,b) \in A\} \\ &\cup \{(a,c) \mid (a,b),(b,c) \in A\} \\ &\cup \{(a_1 * b_1, a_2 * b_2) \mid (a_1,a_2),(b_1,b_2) \in A\} \\ &\cup \{(a_1 b_1^{-1}, a_2 b_2^{-1}) \mid (a_1,a_2),(b_1,b_2) \in A, a_i, b_i \in G_{\lambda_i}\}. \end{aligned}$$

We then have $N_{\mathsf{MCQ}}(R) = \bigcup_{n=1}^{\infty} N_{\mathsf{mcq}}^n(R)$.

Let $S_\Lambda = \{S_\lambda \mid \lambda \in \Lambda\}$ be a set of pairwise disjoint sets. Put $S := \bigcup_{\lambda\in\Lambda} S_\lambda$. Let $R \subset F_{\mathsf{MCQ}}(S_\Lambda) \times F_{\mathsf{MCQ}}(S_\Lambda)$. We define

$$\langle S_\Lambda \mid R\rangle := \overline{F_{\mathsf{MCQ}}(S_\Lambda)/{\sim_{N_{\mathsf{MCQ}}(R)}}}.$$

We also denote it by $\langle S_\lambda\,(\lambda \in \Lambda) \mid R\rangle$. For example, if $\Lambda = \{1, \ldots, n\}$, then $\langle S_\Lambda \mid R\rangle = \langle S_1, \ldots, S_n \mid R\rangle$. We call S_Λ the *generating set* of $\langle S_\Lambda \mid R\rangle$ and call the elements of R the *relators* of $\langle S_\Lambda \mid R\rangle$. A relator (a,b) is also written as $a = b$. We call $\langle S_\Lambda \mid R\rangle$ a *presentation* of a multiple conjugation quandle X if $X \cong \langle S_\Lambda \mid R\rangle$. A presentation $\langle S_\Lambda \mid R\rangle$ is called a *finite presentation* if S and R are finite. For a finitely presented multiple conjugation quandle, we often write

$$\begin{aligned} &\langle x_{1,1}, \ldots, x_{1,n_1}; \ldots; x_{l,1}, \ldots, x_{l,n_l} \mid r_1, \ldots, r_m\rangle \\ &:= \langle\{\{x_{1,1}, \ldots, x_{1,n_1}\}, \ldots, \{x_{l,1}, \ldots, x_{l,n_l}\}\} \mid \{r_1, \ldots, r_m\}\rangle. \end{aligned} \tag{15}$$

Proposition 5.3. *Let $X = \bigsqcup_{\mu \in M} G_\mu$ be a multiple conjugation quandle. Let $S_\Lambda = \{S_\lambda \,|\, \lambda \in \Lambda\}$ be a generating set of X. Then there exists $R \subset F_{\mathsf{MCQ}}(S_\Lambda) \times F_{\mathsf{MCQ}}(S_\Lambda)$ such that $X \cong \langle S_\Lambda \,|\, R\rangle$.*

Proof. Put $S := \bigcup_{\lambda \in \Lambda} S_\lambda$. Since $F_{\mathsf{MCQ}}(S_\Lambda)$ is free on S_Λ and $\{S_\lambda \,|\, \lambda \in \Lambda\} \subset_{\mathsf{MS}} \{G_\mu \,|\, \mu \in M\}$, there exists a unique MCQ homomorphism $f : F_{\mathsf{MCQ}}(S_\Lambda) \to X$ such that $f(x) = x$ for any $x \in S$. Since S_Λ is a generating set of X, we have $\operatorname{Im} f = X$. By Proposition 5.2, we have a quasi MCQ isomorphism $\tilde{f} : F_{\mathsf{MCQ}}(S_\Lambda)/{\sim_f} \to X$. Since X is a multiple conjugation quandle, $F_{\mathsf{MCQ}}(S_\Lambda)/{\sim_f}$ is a multiple conjugation quandle and $\tilde{f}$ is an MCQ isomorphism. Setting $R := R_f$, we have $X \cong F_{\mathsf{MCQ}}(S_\Lambda)/{\sim_f} = \overline{F_{\mathsf{MCQ}}(S_\Lambda)/{\sim_f}} = \langle S_\Lambda \,|\, R\rangle$. □

Proposition 5.4. *Let $S_\Lambda = \{S_\lambda \,|\, \lambda \in \Lambda\}$ be a set of pairwise disjoint sets. Put $S := \bigcup_{\lambda \in \Lambda} S_\lambda$. Let $R \subset F_{\mathsf{MCQ}}(S_\Lambda) \times F_{\mathsf{MCQ}}(S_\Lambda)$. Let $X = \bigsqcup_{\mu \in M} G_\mu$ be a multiple conjugation quandle. Let $f : S \to X$ be a map satisfying $\{f(S_\lambda) \,|\, \lambda \in \Lambda\} \subset_{\mathsf{MS}} \{G_\mu \,|\, \mu \in M\}$. We denote by the same symbol f the unique MCQ homomorphism from $F_{\mathsf{MCQ}}(S_\Lambda)$ to X that sends x to $f(x)$ for $x \in S$. If $R \subset R_f$, there exists a unique MCQ homomorphism $\tilde{f} : \langle S_\Lambda \,|\, R\rangle \to X$ such that $\tilde{f}([x]) = f(x)$ for any $x \in S$.*

Proof. There exists a unique map $\hat{f} : F_{\mathsf{MCQ}}(S_\Lambda)/{\sim_{N_{\mathsf{MCQ}}(R)}} \to X$ that sends $[x]$ to $f(x)$, since $R \subset R_f$ implies $N_{\mathsf{MCQ}}(R) \subset R_f$. The map $\hat{f} : F_{\mathsf{MCQ}}(S_\Lambda)/{\sim_{N_{\mathsf{MCQ}}(R)}} \to X$ is a quasi MCQ homomorphism, since

$$\begin{aligned}
&\hat{f}([x] * [y]) = \hat{f}([x * y]) = f(x * y) = f(x) * f(y) = \hat{f}([x]) * \hat{f}([y]),\\
&\hat{f}([a][b]) = \hat{f}([ab]) = f(ab) = f(a)f(b) = \hat{f}([a])\hat{f}([b]),\\
&\hat{f}([F_{\mathsf{Grp}}(S_\lambda) * x]) = f(F_{\mathsf{Grp}}(S_\lambda) * x) \subset G_{f((\lambda,x))}.
\end{aligned}$$

Set $\tilde{f} := \bar{\hat{f}}$. Then, $\tilde{f} : \langle S_\Lambda \,|\, R\rangle \to X$ is the desired MCQ homomorphism. Since S_Λ is a generating set of $\langle S_\Lambda \,|\, R\rangle$, $\tilde{f}$ is unique. □

We define the following transformations on presentations.

(T1) $\langle S_\Lambda \,|\, R\rangle \leftrightarrow \langle S_\Lambda \,|\, R, r\rangle$, where $r \in N_{\mathsf{MCQ}}(R)$.

(T2) $\langle S_\Lambda \,|\, R\rangle \leftrightarrow \langle S_\Lambda, \{y\} \,|\, R, (y, w)\rangle$, where $y \notin F_{\mathsf{MCQ}}(S_\Lambda)$ and $w \in F_{\mathsf{MCQ}}(S_\Lambda)$.

(T3) $\langle S_\Lambda, S_\mu, S_\nu \,|\, R, (a, b)\rangle \leftrightarrow \langle S_\Lambda, S_\mu \cup S_\nu \,|\, R, (a, b)\rangle$, where $a \in S_\mu$ and $b \in S_\nu$.

In the transformations, $\langle S_\Lambda, S_1, \dots, S_n \,|\, R, r_1, \dots, r_m\rangle$ stands for

$$\langle S_\Lambda \cup \{S_1, \dots, S_n\} \,|\, R \cup \{r_1, \dots, r_m\}\rangle.$$

We define the MCQ homomorphism

$$f_{\mathrm{T1}} : \langle S_\Lambda \,|\, R\rangle \to \langle S_\Lambda \,|\, R, r\rangle$$

by $f_{\mathrm{T1}}([x]) = [x]$ for $x \in \bigcup_{\lambda \in \Lambda} S_\lambda$. We define the MCQ homomorphism

$$f_{\mathrm{T2}} : \langle S_\Lambda \,|\, R\rangle \to \langle S_\Lambda, \{y\} \,|\, R, (y, w)\rangle$$

by $f_{\mathrm{T2}}([x]) = [x]$ for $x \in \bigcup_{\lambda \in \Lambda} S_\lambda$. We define the MCQ homomorphism

$$f_{\mathrm{T3}} : \langle S_\Lambda, S_\mu, S_\nu \,|\, R, (a, b)\rangle \to \langle S_\Lambda, S_\mu \cup S_\nu \,|\, R, (a, b)\rangle$$

by $f_{\mathrm{T3}}([x]) = [x]$ for $x \in \bigcup_{\lambda\in\Lambda} S_\lambda \cup S_\mu \cup S_\nu$. Then these MCQ homomorphisms are MCQ isomorphisms, that is, two multiple conjugation quandles related by the transformations (T1)–(T3) are isomorphic.

We also define the following transformations on presentations.

(T1-1) $\langle S_\Lambda \mid R\rangle \leftrightarrow \langle S_\Lambda \mid R, (x,x)\rangle$, where $x \in F_{\mathsf{MCQ}}(S_\Lambda)$,

(T1-2) $\langle S_\Lambda \mid R, (a,b)\rangle \leftrightarrow \langle S_\Lambda \mid R, (a,b), (b,a)\rangle$,

(T1-3) $\langle S_\Lambda \mid R, (a,b), (b,c)\rangle \leftrightarrow (\langle S_\Lambda \mid R, (a,b), (b,c), (a,c)\rangle$,

(T1-4) $\langle S_\Lambda \mid R, (a_1,a_2), (b_1,b_2)\rangle) \leftrightarrow \langle S_\Lambda \mid R, (a_1,a_2), (b_1,b_2), (a_1 * b_1, a_2 * b_2)\rangle$,

(T1-5) $\langle S_\Lambda \mid R, (a_1,a_2), (b_1,b_2)\rangle \leftrightarrow \langle S_\Lambda \mid R, (a_1,a_2), (b_1,b_2), (a_1 b_1^{-1}, a_2 b_2^{-1})\rangle$, where a_i and b_i are multiplicable,

(T3-1) $\langle S_\Lambda, S_\mu, S_\nu \mid R, (a^0, b^0)\rangle \leftrightarrow \langle S_\Lambda, S_\mu \cup S_\nu \mid R, (a^0, b^0)\rangle$, where $a \in S_\mu$ and $b \in S_\nu$.

In a manner similar to the definitions of $f_{\mathrm{T1}}, f_{\mathrm{T2}}, f_{\mathrm{T3}}$, we define MCQ isomorphisms $f_{\text{T1-1}}, \ldots, f_{\text{T1-5}}$ and $f_{\text{T3-1}}$.

Proposition 5.5. *(1) The transformation (T1) is realized as a finite sequence of the transformations (T1-1)–(T1-5). Furthermore, the MCQ isomorphism f_{T1} is realized as a composition of $f_{\text{T1-1}}, \ldots, f_{\text{T1-5}}$ and their inverses.*

(2) The transformation (T3) is realized as a finite sequence of the transformations (T1-5) and (T3-1). Furthermore, the MCQ isomorphism f_{T3} is realized as a composition of $f_{\text{T1-5}}, f_{\text{T3-1}}$ and their inverses.

Proof. (1) There is an integer $n \geq 1$ such that $r \in N^n_{\mathsf{mcq}}(R) \subset N_{\mathsf{MCQ}}(R)$. Let $R_n := \{r\}$. For $i \in \{1, \ldots, n-1\}$, we choose a finite set $R_i \subset N^i_{\mathsf{mcq}}(R)$ so that $R_{i+1} \subset N_{\mathsf{mcq}}(R_i)$. Then we have

$$\begin{aligned}
\langle S_\Lambda \mid R\rangle &\leftrightarrow \cdots \leftrightarrow \langle S_\Lambda \mid R \cup R_1\rangle \\
&\leftrightarrow \cdots \leftrightarrow \langle S_\Lambda \mid R \cup R_1 \cup R_2\rangle \\
&\leftrightarrow \cdots \leftrightarrow \langle S_\Lambda \mid R \cup R_1 \cup R_2 \cup \cdots \cup R_n\rangle, \\
\langle S_\Lambda \mid R \cup \{r\}\rangle &\leftrightarrow \cdots \leftrightarrow \langle S_\Lambda \mid R \cup \{r\} \cup R_1\rangle \\
&\leftrightarrow \cdots \leftrightarrow \langle S_\Lambda \mid R \cup \{r\} \cup R_1 \cup R_2\rangle \\
&\leftrightarrow \cdots \leftrightarrow \langle S_\Lambda \mid R \cup \{r\} \cup R_1 \cup R_2 \cup \cdots \cup R_{n-1}\rangle,
\end{aligned}$$

where each "$\leftrightarrow$" stands for one of the transformations (T1-1)–(T1-5). Since $R_n = \{r\}$, $\langle S_\Lambda \mid R\rangle$ and $\langle S_\Lambda \mid R, r\rangle$ are transformed into each other by a finite sequence of the transformations (T1-1)–(T1-5). The sequence gives a composition of $f_{\text{T1-1}}, \ldots, f_{\text{T1-5}}$ and their inverses. The composition coincides with f_{T1}, since both MCQ homomorphisms send $[x]$ to $[x]$ for $x \in \bigcup_{\lambda\in\Lambda} S_\lambda$.

(2) We have

$$\begin{aligned}
\langle S_\Lambda, S_\mu, S_\nu \mid R, (a,b)\rangle &\overset{\text{T1-5}}{\leftrightarrow} \langle S_\Lambda, S_\mu, S_\nu \mid R, (a,b), (aa^{-1}, bb^{-1})\rangle \\
&\overset{\text{T3-1}}{\leftrightarrow} \langle S_\Lambda, S_\mu \cup S_\nu \mid R, (a,b), (aa^{-1}, bb^{-1})\rangle \\
&\overset{\text{T1-5}}{\leftrightarrow} \langle S_\Lambda, S_\mu \cup S_\nu \mid R, (a,b)\rangle.
\end{aligned}$$

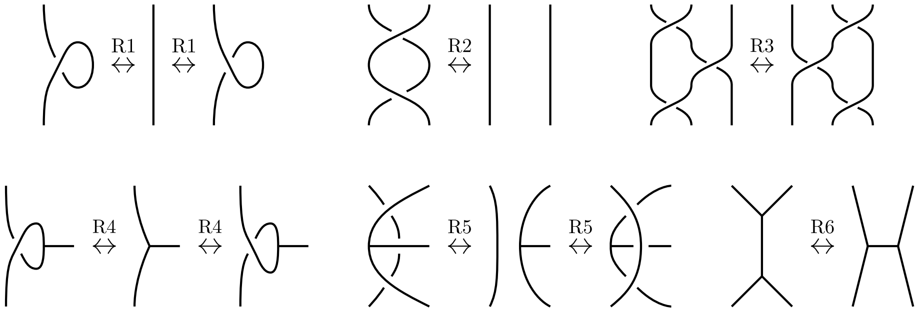


Figure 1: Reidemeister moves for handlebody-links

This sequence gives a composition of $f_{\text{T1-5}}, f_{\text{T3-1}}$ and their inverses. The composition coincides with f_{T3}, since both MCQ homomorphisms send $[x]$ to $[x]$ for $x \in \bigcup_{\lambda\in\Lambda} S_\lambda \cup S_\mu \cup S_\nu$.

□

6 The fundamental multiple conjugation quandle of a handlebody-link

In this section, we recall handlebody-links and the Reidemeister moves for them and define the fundamental multiple conjugation quandle of a handlebody-link. We show that the fundamental multiple conjugation quandle is an invariant of a handlebody-link. We work in the piecewise linear category.

A *handlebody-link* is a disjoint union of handlebodies embedded in the 3-sphere S^3. A *handlebody-knot* is a handlebody embedded in S^3, which is a one component handlebody-link. Two handlebody-links are *equivalent* if there is an orientation-preserving self-homeomorphism of S^3 that sends one to the other. A *diagram* of a handlebody-link is a diagram of a spatial trivalent graph whose regular neighborhood is the handlebody-link, where a spatial trivalent graph is a finite trivalent graph embedded in S^3. In this paper, a trivalent graph may contain circle components. Two handlebody-links are equivalent if and only if their diagrams are related by a finite sequence of R1–R6 moves depicted in Figure 1 [3].

Let H be a handlebody-link, which is a regular neighborhood of a spatial trivalent graph K. We give an orientation to the spatial trivalent graph K so that the graph has no source or sink vertices with respect to the orientation. That is, every trivalent vertex of K has one of the orientations given in Figure 2. We call such an orientation a *Y-orientation.* We note that every trivalent graph has a Y-orientation. Let D be a diagram of K. We may represent an orientation of an edge by a normal orientation, which is obtained by rotating a usual orientation counterclockwise by $\pi/2$ on a diagram. We denote by $C(D)$ and $V(D)$ the sets of crossings and vertices of D, respectively. For an arc α incident to a vertex τ, we define $\varepsilon(\tau;\alpha) \in \{1,-1\}$ by

$$\varepsilon(\tau;\alpha) = \begin{cases} 1 & \text{if the orientation of } \alpha \text{ points to } \tau, \\ -1 & \text{otherwise.} \end{cases}$$

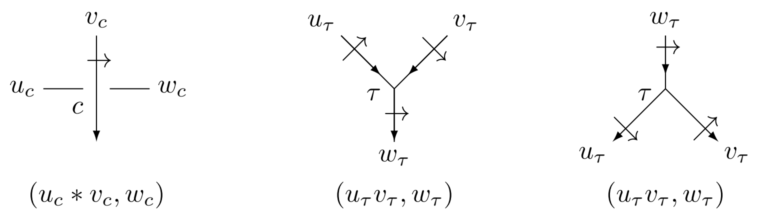


Figure 2: Relators for crossings and vertices

For a crossing $c \in C(D)$, we denote the relator $(u_c * v_c, w_c)$ by r_c, where v_c is the over-arc of c and u_c, w_c are the under-arcs of c such that the normal orientation of v_c points from u_c to w_c. See Figure 2. For a vertex $\tau \in V(D)$, we denote the relator $(u_\tau v_\tau, w_\tau)$ by r_τ, where u_τ, v_τ, w_τ are the arcs incident to a vertex τ such that $\varepsilon(\tau; u_\tau) = \varepsilon(\tau; v_\tau)$ and the normal orientation of w_τ points from u_τ to v_τ. We denote by $\mathcal{A}(D)$ the set of arcs of D. We denote by $\mathcal{A}^{\sqcup}(D)$ the quotient set of $\mathcal{A}(D)$ by the equivalence relation generated by $\bigcup_{\tau \in V(D)} \{u_\tau, v_\tau, w_\tau\}^2$, where we note that the elements of $\mathcal{A}^{\sqcup}(D)$ are disjoint subsets of $\mathcal{A}(D)$. Then we define

$$\mathrm{MCQ}(D) := \langle \mathcal{A}^{\sqcup}(D) \,|\, r_c \ (c \in C(D)), r_\tau \ (\tau \in V(D)) \rangle. \tag{16}$$

In Theorem 6.4, we show that the isomorphism class of $\mathrm{MCQ}(D)$ is an invariant of a handlebody-link H, that is, the isomorphism class does not depend on the choice of the orientation and diagram D of H. We then set $\mathrm{MCQ}(H) := \mathrm{MCQ}(D)$ and call it the *fundamental multiple conjugation quandle* of H. The presentation (16) is called the *Wirtinger presentation* of $\mathrm{MCQ}(H)$ with respect to D.

Example 6.1. In this example, we use the notation introduced in (15). Let D_1 be the diagram of the handlebody-knot 0_1 depicted in Figure 3. We then have

$$\mathrm{MCQ}(D_1) = \langle x_1, x_2, x_3 \,|\, x_2 x_1 = x_3, x_2 x_1 = x_3 \rangle,$$

which can be deformed into the form

$$\langle x_1, x_2 \,|\quad \rangle$$

by using the transformations (T1)–(T3).

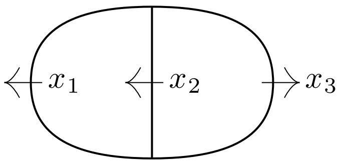
Let D_2 be the diagram of the handlebody-knot 4_1 depicted in Figure 3. We then have

$$\mathrm{MCQ}(D_2) = \left\langle x_1, x_2, x_3; x_4, x_5, x_6; x_7 \,\middle|\, \begin{array}{ll} x_2 x_3 = x_1, & x_6 x_4 = x_5, \\ x_1 * x_4 = x_2, & x_4 * x_2 = x_7, \\ x_3 * x_5 = x_7, & x_5 * x_3 = x_6 \end{array} \right\rangle,$$

which can be deformed into the form

$$\left\langle x_2, x_3; x_5 \,\middle|\, \begin{array}{l} (x_5 * x_3)(x_3 * x_5 * x_2^{-1}) = x_5, \\ (x_2 x_3) * (x_3 * x_5 * x_2^{-1}) = x_2 \end{array} \right\rangle$$

by using the transformations (T1)–(T3).



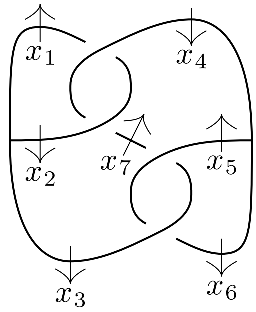


Figure 3: The handlebody-knots 0_1 and 4_1

We denote by O the Y-orientation given to K. We denote the multiple conjugation quandle (16) by $\mathrm{MCQ}(D, O)$. By the following proposition, we see that the isomorphism class of $\mathrm{MCQ}(D)$ does not depend on the choice of a Y-orientation.

Proposition 6.2. *Let D be a diagram of a spatial trivalent graph K. Let O and O' be Y-orientations of K. Then the MCQ homomorphism $f : \mathrm{MCQ}(D, O) \to \mathrm{MCQ}(D, O')$ that sends $[\alpha]$ to $[\alpha]^{\varepsilon(\alpha)}$ for $\alpha \in \mathcal{A}(D)$ is an MCQ isomorphism, where $\varepsilon(\alpha) = 1$ if O and O' coincide on α; otherwise $\varepsilon(\alpha) = -1$. Furthermore, the MCQ isomorphism $f : \mathrm{MCQ}(D, O) \to \mathrm{MCQ}(D, O')$ is realized as a composition of $f_{\mathrm{T1}}, f_{\mathrm{T2}}, f_{\mathrm{T3}}$ and their inverses.*

We remark that, by Proposition 5.5, $f_{\mathrm{T1}}, f_{\mathrm{T2}}, f_{\mathrm{T3}}$ in Proposition 6.2 can be replaced with $f_{\text{T1-1}}, \dots, f_{\text{T1-5}}, f_{\mathrm{T2}}$ and $f_{\text{T3-1}}$.

Proof. The map $f : \mathrm{MCQ}(D, O) \to \mathrm{MCQ}(D, O')$ is a well-defined MCQ isomorphism, since $(x * y, z) \in R$ and $(ab, c) \in R$ imply

$$(x^{-1} * y, z^{-1}), (z * y^{-1}, x), (z^{-1} * y^{-1}, x^{-1}) \in N_{\mathsf{MCQ}}(R) \quad \text{and}$$
$$(b^{-1}a^{-1}, c^{-1}), (a^{-1}c, b), (c^{-1}a, b^{-1}), (cb^{-1}, a), (bc^{-1}, a^{-1}) \in N_{\mathsf{MCQ}}(R)$$

respectively for $R \subset F_{\mathsf{MCQ}}(S_\Lambda) \times F_{\mathsf{MCQ}}(S_\Lambda)$. The MCQ isomorphism f is obtained by repeating the following transformations to the arcs α such that $\varepsilon(\alpha) = -1$:

$$\begin{aligned}
\langle S_\Lambda, S_\mu \cup \{\alpha\} \mid R \rangle &\overset{\mathrm{T2}}{\leftrightarrow} \langle S_\Lambda, S_\mu \cup \{\alpha\}, \{\alpha'\} \mid R, \alpha' = \alpha^{-1} \rangle \\
&\overset{\mathrm{T3}}{\leftrightarrow} \langle S_\Lambda, S_\mu \cup \{\alpha, \alpha'\} \mid R, \alpha' = \alpha^{-1} \rangle \\
&\overset{\mathrm{T1}}{\leftrightarrow} \cdots \overset{\mathrm{T1}}{\leftrightarrow} \langle S_\Lambda, S_\mu \cup \{\alpha, \alpha'\} \mid R, R|_{\alpha = \alpha'^{-1}}, \alpha = \alpha'^{-1} \rangle \\
&\overset{\mathrm{T1}}{\leftrightarrow} \cdots \overset{\mathrm{T1}}{\leftrightarrow} \langle S_\Lambda, S_\mu \cup \{\alpha, \alpha'\} \mid R|_{\alpha = \alpha'^{-1}}, \alpha = \alpha'^{-1} \rangle \\
&\overset{\mathrm{T3}}{\leftrightarrow} \langle S_\Lambda, S_\mu \cup \{\alpha'\}, \{\alpha\} \mid R|_{\alpha = \alpha'^{-1}}, \alpha = \alpha'^{-1} \rangle \\
&\overset{\mathrm{T2}}{\leftrightarrow} \langle S_\Lambda, S_\mu \cup \{\alpha'\} \mid R|_{\alpha = \alpha'^{-1}} \rangle.
\end{aligned}$$

Hence the MCQ isomorphism f is realized as a composition of $f_{\mathrm{T1}}, f_{\mathrm{T2}}, f_{\mathrm{T3}}$ and their inverses. □

Y-oriented R1–R6 moves are R1–R6 moves between two diagrams with Y-orientations which are identical except in the disk where the move is applied.

Proposition 6.3. *Let D be a diagram of a Y-oriented spatial trivalent graph. Let D' be a diagram obtained by applying one of Y-oriented R1–R6 moves to the diagram D once. We denote by $\mathcal{A}(D \cap D')$ the set of arcs in the outside of the disk where the move is applied. Then there exists a unique MCQ isomorphism $f : \mathrm{MCQ}(D) \to \mathrm{MCQ}(D')$ that sends $[\alpha]$ to $[\alpha]$ for any arc $\alpha \in \mathcal{A}(D \cap D')$. Furthermore, the MCQ isomorphism $f : \mathrm{MCQ}(D) \to \mathrm{MCQ}(D')$ is realized as a composition of $f_{\mathrm{T1}}, f_{\mathrm{T2}}, f_{\mathrm{T3}}$ and their inverses.*

We remark that, by Proposition 5.5, $f_{\mathrm{T1}}, f_{\mathrm{T2}}, f_{\mathrm{T3}}$ in Proposition 6.3 can be replaced with $f_{\text{T1-1}}, \ldots, f_{\text{T1-5}}, f_{\mathrm{T2}}$ and $f_{\text{T3-1}}$.

Proof. The MCQ isomorphism $f : \mathrm{MCQ}(D) \to \mathrm{MCQ}(D')$ is unique if it exists, since $\{[\alpha] \,|\, \alpha \in \mathcal{A}(D \cap D')\}$ is a generating set of $\mathrm{MCQ}(D)$ and $\mathrm{MCQ}(D')$. Hence it is sufficient to show that $\mathrm{MCQ}(D)$ and $\mathrm{MCQ}(D')$ are related by a finite sequence of the transformations (T1)–(T3) preserving $\mathcal{A}(D \cap D')$. We demonstrate this sequence for a Y-oriented R5 move: In Figure 4, ε_i indicates the downward orientation of the edge to which ε_i is assigned if $\varepsilon_i = 1$; otherwise the upward orientation. For the left diagram D of the first Y-oriented R5 move in Figure 4, we have

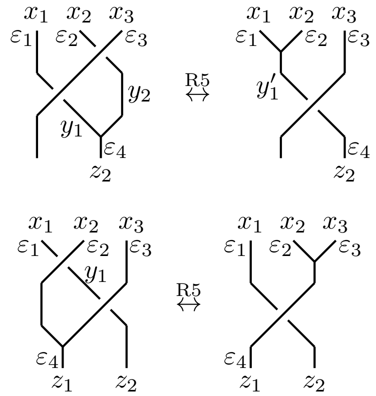
$$\begin{aligned}
\mathrm{MCQ}(D) &\overset{\mathrm{T1}}{\leftrightarrow} \langle S_\Lambda, S_{\mu_1} \cup \{x_1\}, S_{\mu_2} \cup \{x_2\}, S_{\mu_3} \cup \{x_3\}, S_{\mu_4} \cup \{y_1, y_2, z_2\} \,| \\
&\qquad \boldsymbol{r}, y_1 = x_1 * x_3^{\varepsilon_3}, y_2 = x_2 * x_3^{\varepsilon_3}, z_2 = (y_1^{\varepsilon_1} y_2^{\varepsilon_2})^{\varepsilon_4} \rangle \\
&\overset{\mathrm{T1}}{\leftrightarrow} \langle S_\Lambda, S_{\mu_1} \cup \{x_1\}, S_{\mu_2} \cup \{x_2\}, S_{\mu_3} \cup \{x_3\}, S_{\mu_4} \cup \{y_1, y_2, z_2\} \,| \\
&\qquad \boldsymbol{r}, y_1 = x_1 * x_3^{\varepsilon_3}, y_2 = x_2 * x_3^{\varepsilon_3}, z_2 = ((x_1 * x_3^{\varepsilon_3})^{\varepsilon_1} (x_2 * x_3^{\varepsilon_3})^{\varepsilon_2})^{\varepsilon_4} \rangle \\
&\overset{\mathrm{T3}}{\leftrightarrow} \langle S_\Lambda, S_{\mu_1} \cup S_{\mu_2} \cup \{x_1, x_2\}, S_{\mu_3} \cup \{x_3\}, S_{\mu_4} \cup \{z_2\}, \{y_1\}, \{y_2\} \,| \\
&\qquad \boldsymbol{r}, y_1 = x_1 * x_3^{\varepsilon_3}, y_2 = x_2 * x_3^{\varepsilon_3}, z_2 = ((x_1 * x_3^{\varepsilon_3})^{\varepsilon_1} (x_2 * x_3^{\varepsilon_3})^{\varepsilon_2})^{\varepsilon_4} \rangle \\
&\overset{\mathrm{T2}}{\leftrightarrow} \langle S_\Lambda, S_{\mu_1} \cup S_{\mu_2} \cup \{x_1, x_2\}, S_{\mu_3} \cup \{x_3\}, S_{\mu_4} \cup \{z_2\} \,| \\
&\qquad \boldsymbol{r}, z_2 = ((x_1 * x_3^{\varepsilon_3})^{\varepsilon_1} (x_2 * x_3^{\varepsilon_3})^{\varepsilon_2})^{\varepsilon_4} \rangle.
\end{aligned}$$

For the right diagram D' of the first Y-oriented R5 move in Figure 4, we have

$$\begin{aligned}
\mathrm{MCQ}(D') &\overset{\mathrm{T1}}{\leftrightarrow} \langle S_\Lambda, S_{\mu_1} \cup S_{\mu_2} \cup \{x_1, x_2, y_1'\}, S_{\mu_3} \cup \{x_3\}, S_{\mu_4} \cup \{z_2\} \,| \\
&\qquad \boldsymbol{r}, y_1' = (x_1^{\varepsilon_1} x_2^{\varepsilon_2})^{\varepsilon_4}, z_2 = y_1' * x_3^{\varepsilon_3} \rangle \\
&\overset{\mathrm{T1}}{\leftrightarrow} \langle S_\Lambda, S_{\mu_1} \cup S_{\mu_2} \cup \{x_1, x_2, y_1'\}, S_{\mu_3} \cup \{x_3\}, S_{\mu_4} \cup \{z_2\} \,| \\
&\qquad \boldsymbol{r}, y_1' = (x_1^{\varepsilon_1} x_2^{\varepsilon_2})^{\varepsilon_4}, z_2 = (x_1^{\varepsilon_1} x_2^{\varepsilon_2})^{\varepsilon_4} * x_3^{\varepsilon_3} \rangle \\
&\overset{\mathrm{T3}}{\leftrightarrow} \langle S_\Lambda, S_{\mu_1} \cup S_{\mu_2} \cup \{x_1, x_2\}, S_{\mu_3} \cup \{x_3\}, S_{\mu_4} \cup \{z_2\}, \{y_1'\} \,| \\
&\qquad \boldsymbol{r}, y_1' = (x_1^{\varepsilon_1} x_2^{\varepsilon_2})^{\varepsilon_4}, z_2 = (x_1^{\varepsilon_1} x_2^{\varepsilon_2})^{\varepsilon_4} * x_3^{\varepsilon_3} \rangle \\
&\overset{\mathrm{T2}}{\leftrightarrow} \langle S_\Lambda, S_{\mu_1} \cup S_{\mu_2} \cup \{x_1, x_2\}, S_{\mu_3} \cup \{x_3\}, S_{\mu_4} \cup \{z_2\} \,| \\
&\qquad \boldsymbol{r}, z_2 = (x_1^{\varepsilon_1} x_2^{\varepsilon_2})^{\varepsilon_4} * x_3^{\varepsilon_3} \rangle.
\end{aligned}$$

Thus we obtain the desired sequence, since we have

$$((x_1 * x_3^{\varepsilon_3})^{\varepsilon_1} (x_2 * x_3^{\varepsilon_3})^{\varepsilon_2})^{\varepsilon_4} = (x_1^{\varepsilon_1} x_2^{\varepsilon_2})^{\varepsilon_4} * x_3^{\varepsilon_3}$$



$$y_1 = x_1 * x_3^{\varepsilon_3},\quad y_2 = x_2 * x_3^{\varepsilon_3},\quad z_2 = (y_1^{\varepsilon_1} y_2^{\varepsilon_2})^{\varepsilon_4},\quad y'_1 = (x_1^{\varepsilon_1} x_2^{\varepsilon_2})^{\varepsilon_4},\quad z_2 = y'_1 * x_3^{\varepsilon_3}.$$

$$y_1 = x_1 * x_2^{\varepsilon_2},\quad z_2 = y_1 * x_3^{\varepsilon_3},\quad z_1 = (x_2^{\varepsilon_2} x_3^{\varepsilon_3})^{\varepsilon_4},\quad z_1 = (x_2^{\varepsilon_2} x_3^{\varepsilon_3})^{\varepsilon_4},\quad z_2 = x_1 * z_1^{\varepsilon_4}.$$

Figure 4: Y-oriented R5 moves

in $F_{\mathsf{MCQ}}(S_\Lambda, S_{\mu_1} \cup S_{\mu_2} \cup \{x_1, x_2\}, S_{\mu_3} \cup \{x_3\}, S_{\mu_4} \cup \{z_2\})$. The deformation can be summarized as follows: Let $\boldsymbol{x}$ be the sequence of the top arcs $x_1, \ldots, x_l$ of the move in D. Let $\boldsymbol{z}$ be the sequence of the bottom arcs $z_1, \ldots, z_m$ of the move in D. Let $\boldsymbol{y}$ be the sequence of the other arcs $y_1, \ldots, y_n$ of the move in D. We then have

$$\begin{aligned}
\mathrm{MCQ}(D) &\overset{\mathrm{T1}}{\leftrightarrow} \langle \mathcal{A}^{\sqcup}(D) \,|\, \boldsymbol{r}, \boldsymbol{y} = \boldsymbol{w}_{\mathrm{middle}}(\boldsymbol{x}), \boldsymbol{z} = \boldsymbol{w}_{\mathrm{bottom}}(\boldsymbol{x}, \boldsymbol{y}) \rangle \\
&\overset{\mathrm{T1}}{\leftrightarrow} \langle \mathcal{A}^{\sqcup}(D) \,|\, \boldsymbol{r}, \boldsymbol{y} = \boldsymbol{w}_{\mathrm{middle}}(\boldsymbol{x}), \boldsymbol{z} = \boldsymbol{w}_{\mathrm{bottom}}(\boldsymbol{x}, \boldsymbol{w}_{\mathrm{middle}}(\boldsymbol{x})) \rangle \\
&\overset{\mathrm{T3}}{\leftrightarrow} \langle S_\Lambda, \boldsymbol{y} \,|\, \boldsymbol{r}, \boldsymbol{y} = \boldsymbol{w}_{\mathrm{middle}}(\boldsymbol{x}), \boldsymbol{z} = \boldsymbol{w}_{\mathrm{bottom}}(\boldsymbol{x}, \boldsymbol{w}_{\mathrm{middle}}(\boldsymbol{x})) \rangle \\
&\overset{\mathrm{T2}}{\leftrightarrow} \langle S_\Lambda \,|\, \boldsymbol{r}, \boldsymbol{z} = \boldsymbol{w}_{\mathrm{bottom}}(\boldsymbol{x}, \boldsymbol{w}_{\mathrm{middle}}(\boldsymbol{x})) \rangle,
\end{aligned}$$

where $\boldsymbol{y} = \boldsymbol{w}_{\mathrm{middle}}(\boldsymbol{x})$ and $\boldsymbol{z} = \boldsymbol{w}_{\mathrm{bottom}}(\boldsymbol{x}, \boldsymbol{y})$ are respectively the sequences of the relators

$$\begin{aligned} y_1 &= w^1_{\mathrm{middle}}(x_1, \ldots, x_l), \\ &\vdots \\ y_n &= w^n_{\mathrm{middle}}(x_1, \ldots, x_l), \end{aligned} \qquad \text{and} \qquad \begin{aligned} z_1 &= w^1_{\mathrm{bottom}}(x_1, \ldots, x_l, y_1, \ldots, y_n), \\ &\vdots \\ z_m &= w^m_{\mathrm{bottom}}(x_1, \ldots, x_l, y_1, \ldots, y_n) \end{aligned}$$

derived from crossings and vertices of the move in D by using the transformation (T1) if necessary. Here $w^i_{\mathrm{middle}}(x_1, \ldots, x_l)$ and $w^i_{\mathrm{bottom}}(x_1, \ldots, x_l, y_1, \ldots, y_n)$ are elements of $F_{\mathsf{MCQ}}(\mathcal{A}^{\sqcup}(D))$ represented by MCQ words using $x_1, \ldots, x_l$ and $x_1, \ldots, x_l, y_1, \ldots, y_n$, respectively. We denote by

$$z_i = w^i_{\mathrm{bottom}}(\boldsymbol{x}, \boldsymbol{w}_{\mathrm{middle}}(\boldsymbol{x}))$$

the relator obtained from $z_i = w^i_{\mathrm{bottom}}(x_1, \ldots, x_l, y_1, \ldots, y_n)$ by substituting

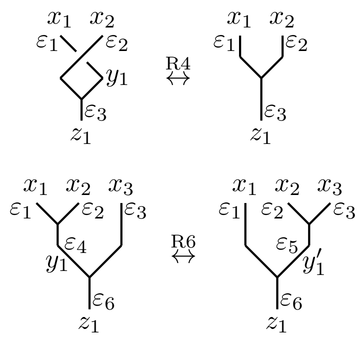
$$w^1_{\mathrm{middle}}(x_1, \ldots, x_l), \ldots, w^n_{\mathrm{middle}}(x_1, \ldots, x_l)$$

for $y_1, \ldots, y_n$. Then

$$\boldsymbol{z} = \boldsymbol{w}_{\mathrm{bottom}}(\boldsymbol{x}, \boldsymbol{w}_{\mathrm{middle}}(\boldsymbol{x}))$$

is the sequence of the relators

$$\begin{aligned} z_1 &= w^1_{\mathrm{bottom}}(\boldsymbol{x}, \boldsymbol{w}_{\mathrm{middle}}(\boldsymbol{x})), \\ &\vdots \\ z_m &= w^m_{\mathrm{bottom}}(\boldsymbol{x}, \boldsymbol{w}_{\mathrm{middle}}(\boldsymbol{x})). \end{aligned}$$



$$y_1 = x_1 * x_2^{\varepsilon_2},\quad z_1 = (x_2^{\varepsilon_2} y_1^{\varepsilon_1})^{\varepsilon_3},\quad z_1 = (x_1^{\varepsilon_1} x_2^{\varepsilon_2})^{\varepsilon_3}.$$

$$y_1 = (x_1^{\varepsilon_1} x_2^{\varepsilon_2})^{\varepsilon_4},\quad z_1 = (y_1^{\varepsilon_4} x_3^{\varepsilon_3})^{\varepsilon_6},\quad y_1' = (x_2^{\varepsilon_2} x_3^{\varepsilon_3})^{\varepsilon_5},\quad z_1 = (x_1^{\varepsilon_1} y_1'^{\varepsilon_5})^{\varepsilon_6}.$$

Figure 5: Y-oriented R4 and R6 moves

In the same manner, we have

$$\begin{aligned}
\mathrm{MCQ}(D') &\overset{\mathrm{T1}}{\leftrightarrow} \langle \mathcal{A}^{\sqcup}(D') \,|\, \boldsymbol{r}, \boldsymbol{y}' = \boldsymbol{w}'_{\mathrm{middle}}(\boldsymbol{x}), \boldsymbol{z} = \boldsymbol{w}'_{\mathrm{bottom}}(\boldsymbol{x}, \boldsymbol{y}') \rangle \\
&\overset{\mathrm{T1}}{\leftrightarrow} \langle \mathcal{A}^{\sqcup}(D') \,|\, \boldsymbol{r}, \boldsymbol{y}' = \boldsymbol{w}'_{\mathrm{middle}}(\boldsymbol{x}), \boldsymbol{z} = \boldsymbol{w}'_{\mathrm{bottom}}(\boldsymbol{x}, \boldsymbol{w}'_{\mathrm{middle}}(\boldsymbol{x})) \rangle \\
&\overset{\mathrm{T3}}{\leftrightarrow} \langle S_\Lambda, \boldsymbol{y}' \,|\, \boldsymbol{r}, \boldsymbol{y}' = \boldsymbol{w}'_{\mathrm{middle}}(\boldsymbol{x}), \boldsymbol{z} = \boldsymbol{w}'_{\mathrm{bottom}}(\boldsymbol{x}, \boldsymbol{w}'_{\mathrm{middle}}(\boldsymbol{x})) \rangle \\
&\overset{\mathrm{T2}}{\leftrightarrow} \langle S_\Lambda \,|\, \boldsymbol{r}, \boldsymbol{z} = \boldsymbol{w}'_{\mathrm{bottom}}(\boldsymbol{x}, \boldsymbol{w}'_{\mathrm{middle}}(\boldsymbol{x})) \rangle,
\end{aligned}$$

where $\boldsymbol{w}_{\mathrm{bottom}}(\boldsymbol{x}, \boldsymbol{w}_{\mathrm{middle}}(\boldsymbol{x})) = \boldsymbol{w}'_{\mathrm{bottom}}(\boldsymbol{x}, \boldsymbol{w}'_{\mathrm{middle}}(\boldsymbol{x}))$. This sequence also works for Y-oriented R4–R6 moves (see Figures 4 and 5). The sequences for Y-oriented R1–R3 moves can be obtained in a similar manner. It is left to the reader to construct them. □

The following theorem follows from Propositions 6.2 and 6.3.

Theorem 6.4. *The fundamental multiple conjugation quandle* $\mathrm{MCQ}(H)$ *is an invariant of a handlebody-link* H*. Furthermore, Wirtinger presentations of the fundamental multiple conjugation quandle of a handlebody-link are related by a finite sequence of the transformations (T1)–(T3).*

At the end of this section, we give some properties of the fundamental multiple conjugation quandle: group representations, colorings and mirror images. Let D be a diagram of a handlebody-link H. We give a Y-orientation to D. We denote by $E(H)$ the exterior of H. We remark that we obtain a presentation of the fundamental group $G(H) := \pi_1(E(H))$ by replacing $\mathcal{A}^{\sqcup}(D)$ by $\mathcal{A}(D)$ and replacing r_c by $v_c^{-1} u_c v_c w_c^{-1}$ and r_τ by $u_\tau v_\tau w_\tau^{-1}$ in (16), which is the Wirtinger presentation of $G(H)$ with respect to D. We denote it by $G(D)$. Then, the map $p : \mathrm{MCQ}(D) \to G(D)$ defined by $p([x]) = [x]$ for $x \in \mathcal{A}(D)$ is an MCQ homomorphism, where we regard $G(D)$ as a multiple conjugation quandle with the conjugation operation. For a group representation $\rho : G(D) \to G$, we call the MCQ homomorphism $\rho \circ p : \mathrm{MCQ}(D) \to G$ the *induced MCQ representation* of ρ, where we regard G as a multiple conjugation quandle with the conjugation operation.

Let X be a multiple conjugation quandle. An X*-coloring* of D is a map $C : \mathcal{A}(D) \to X$ satisfying the conditions

$$C(u_c) * C(v_c) = C(w_c) \qquad \text{and} \qquad C(u_\tau) C(v_\tau) = C(w_\tau) \tag{17}$$

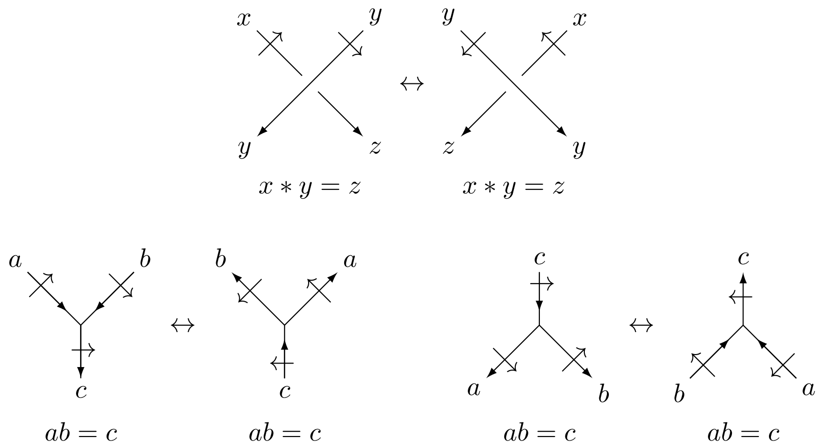


Figure 6: Mirror images

for each crossing $c \in C(D)$ and vertex $\tau \in V(D)$, where u_c, v_c, w_c, u_τ, v_τ and w_τ are the arcs described in Figure 2. We denote by $\mathrm{Col}_X(D)$ the set of X-colorings of D. By Proposition 5.4, an X-coloring of D can be regarded as an MCQ homomorphism from $\mathrm{MCQ}(D)$ to X, which we call an *MCQ representation* of $\mathrm{MCQ}(D)$ to X. Under this identification, we have $\mathrm{Col}_X(D) = \mathrm{Hom}(\mathrm{MCQ}(D), X)$. When X is a group G with the conjugation operation, the conditions (17) turn into

$$C(v_c)^{-1}C(u_c)C(v_c) = C(w_c) \qquad \text{and} \qquad C(u_\tau)C(v_\tau) = C(w_\tau),$$

which are the group coloring conditions. It is well-known that a group coloring can be regarded as a group homomorphism. Under the identification, we have $\mathrm{Col}_G(D) = \mathrm{Hom}_{\mathsf{Grp}}(G(D), G)$, where $\mathrm{Hom}_{\mathsf{Grp}}(G(D), G)$ is the set of group homomorphisms from $G(D)$ to G. The two identifications give a bijection between $\mathrm{Hom}_{\mathsf{Grp}}(G(D), G)$ and $\mathrm{Hom}(\mathrm{MCQ}(D), G)$. More precisely, the bijection from $\mathrm{Hom}_{\mathsf{Grp}}(G(D), G)$ to $\mathrm{Hom}(\mathrm{MCQ}(D), G)$ is the map that sends a group representation into its induced MCQ representation. Let D_1, D_2 be the diagrams of the handlebody-knots depicted in Figure 3. From the bijection, we have

$$\begin{aligned} |\mathrm{Hom}(\mathrm{MCQ}(D_1), SL(2, \mathbb{Z}/2\mathbb{Z}))| &= |\mathrm{Hom}_{\mathsf{Grp}}(G(D_1), SL(2, \mathbb{Z}/2\mathbb{Z}))| = 11, \\ |\mathrm{Hom}(\mathrm{MCQ}(D_2), SL(2, \mathbb{Z}/2\mathbb{Z}))| &= |\mathrm{Hom}_{\mathsf{Grp}}(G(D_2), SL(2, \mathbb{Z}/2\mathbb{Z}))| = 14, \end{aligned}$$

where the last equalities follow from [5].

Proposition 6.5. *Let H be a handlebody-link, and H^* the mirror image of H. Then, we have* $\mathrm{MCQ}(H) \cong \mathrm{MCQ}(H^*)$.

Proof. Let D be a diagram of a handlebody-link H. We give a Y-orientation O to D. Let D^* be the mirror image of D, which is a diagram of H^*. We denote by O^* the Y-orientation induced from O by the reflection. Let $-O^*$ be the Y-orientation obtained from O^* by reversing the orientations of all edges of D^*. See Figure 6. We have $\mathrm{MCQ}(H) \cong \mathrm{MCQ}(D, O) = \mathrm{MCQ}(D^*, -O^*) \cong \mathrm{MCQ}(H^*)$. □

In a manner similar to the proof of Proposition 6.5, we have $\mathrm{MCQ}(H_1 \sqcup H_2) \cong \mathrm{MCQ}(H_1 \sqcup H_2^*)$, where $\sqcup$ stands for the split union of two handlebody-links.

Problem 6.6. Is the fundamental multiple conjugation quandle $\mathrm{MCQ}(H)$ a complete invariant for handlebody-knots H up to mirror images?

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