FINITE AND SYMMETRIC MORDELL-TORNHEIM MULTIPLE ZETA VALUES

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Abstract. We introduce finite and symmetric Mordell-Tornheim type of multiple zeta values and give a new approach to the Kaneko-Zagier conjecture stating that the finite and symmetric multiple zeta values satisfy the same relations.

1. Introduction and Main results

In [1] the authors described a new approach to study the relationship between finite and symmetric multiple zeta values. This was done by viewing these values as an algebraic and analytic limit of certain q-series evaluated at roots of unity. The purpose of this note is to use this approach to introduce finite and symmetric Mordell-Tornheim multiple zeta values and to give an analogue of the Kaneko-Zagier conjecture, which gives a surprising relationship between these values.

1.1. **Notations.** In this work, we denote by $\mathbb{F}_p$ the finite field with p elements and by $\mathbb{N}$ the set of positive integers. We call a tuple $\mathbf{k} = (k_1, \dots, k_r)$ of positive integers an *index*, $\mathrm{wt}(\mathbf{k}) = k_1 + \cdots + k_r$ the *weight* and $\mathrm{len}(\mathbf{k}) = r$ the *length*. An index $\mathbf{k} = (k_1, \dots, k_r)$ satisfying $k_1 \geq 2$ is said to be *admissible*. We regard the empty index $\emptyset$ as admissible, and let $\mathrm{wt}(\emptyset) = \mathrm{len}(\emptyset) = 0$. We set $F(\emptyset)$ to be a unit element for any function F on indices.

1.2. **A review: the Kaneko-Zagier conjecture.** Kaneko and Zagier [11] introduced the finite multiple zeta value $\zeta_{\mathcal{A}}(\mathbf{k})$ for an index $\mathbf{k} = (k_1, \dots, k_r)$ as an element in the ring $\mathcal{A} = \big(\prod_p \mathbb{F}_p\big)/\big(\bigoplus_p \mathbb{F}_p\big)$ with p running over all primes (see Definition 2.1). Since the ring $\mathcal{A}$ forms a $\mathbb{Q}$-algebra, one can consider the $\mathbb{Q}$-vector subspace $\mathcal{Z}_k^{\mathcal{A}}$ of $\mathcal{A}$ spanned by all finite multiple zeta values of weight k. Zagier numerically observed that $\dim_{\mathbb{Q}} \mathcal{Z}_k^{\mathcal{A}} \overset{?}{=} \dim_{\mathbb{Q}} \mathcal{Z}_k - \dim_{\mathbb{Q}} \mathcal{Z}_{k-2}$, where $\mathcal{Z}_k$ denotes the $\mathbb{Q}$-vector

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space spanned by multiple zeta values of weight k. Here the multiple zeta value is defined for an admissible index $\mathbf{k}=(k_1,\dots,k_r)$ by

$$\zeta(\mathbf{k}) := \sum_{m_1>\cdots>m_r>0} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}}.$$

Inspired by Kontsevich's idea, Kaneko and Zagier are led into a real counterpart of the finite multiple zeta value, called the symmetric multiple zeta value and denoted by $\zeta_{\mathcal{S}}(\mathbf{k})$ (see Definition 2.2), in the quotient $\mathbb{Q}$-algebra $\mathcal{Z}/\zeta(2)\mathcal{Z}$, where $\mathcal{Z}=\sum_{k\geq 0}\mathcal{Z}_k$. They observed that finite and symmetric multiple zeta values satisfy the same relations over $\mathbb{Q}$, and then proposed a one-to-one correspondence between them, which we call the Kaneko-Zagier conjecture (see Conjecture 2.3).

1.3. **Finite Mordell-Tornheim multiple zeta value.** In this paper, we wish to first mimic the above story replacing the multiple harmonic sum $H_n(\mathbf{k})$ defined in (2.1) with the rational number $\omega_n(\mathbf{k})\in\mathbb{Q}$ defined for an index $\mathbf{k}=(k_1,\dots,k_r)$ and $n\in\mathbb{N}$ by

$$\omega_n(\mathbf{k}) := \sum_{\substack{m_1+\cdots+m_r=n\\ m_1,\dots,m_r>0}} \prod_{a=1}^{r} \frac{1}{m_a^{k_a}},$$

and then propose an alternative approach to the Kaneko-Zagier conjecture. Here the notion of $\omega_n(\mathbf{k})$ is originated from a certain truncation of generalized Mordell-Tornheim-Witten sums

$$\omega(\mathbf{k}|\mathbf{l}) = \sum_{\substack{m_1+\cdots+m_r=n_1+\cdots+n_s\\ m_1,\dots,m_r>0\\ n_1,\dots,n_s>0}} \prod_{a=1}^{r}\frac{1}{m_a^{k_a}} \prod_{b=1}^{s}\frac{1}{n_b^{l_b}}$$

studied by [2, 3, 4, 5]. We start by defining a finite analogue of these values.

Definition 1.1. For an index $\mathbf{k}=(k_1,\dots,k_r)$ with $r\geq 2$, we define the *finite multiple omega value* $\omega_{\mathcal{A}}(\mathbf{k})$ by

$$\omega_{\mathcal{A}}(\mathbf{k}) := \left(\omega_p(\mathbf{k}) \mod p\right)_p \in \mathcal{A}.$$

Denote by $\mathcal{Z}_k^{\mathcal{A},\omega}$ the $\mathbb{Q}$-vector space spanned by all finite multiple omega values of weight k. Using PARI-GP [21], we numerically computed the dimension of $\mathcal{Z}_k^{\mathcal{A},\omega}$. For comparison, it is listed together with the conjectural dimension of $\mathcal{Z}_k$ as follows.

k	1	2	3	4	5	6	7	8	9	10	11	12
$\dim \mathcal{Z}_k^{\mathcal{A},\omega}$	0	0	1	0	1	1	1	2	2	3	4	5
$\dim \mathcal{Z}_k$	0	1	1	1	2	2	3	4	5	7	9	12

Similarly to the finite multiple zeta value, the equality $\dim_{\mathbb{Q}} \mathcal{Z}_k^{\mathcal{A},\omega} \overset{?}{=} \dim_{\mathbb{Q}} \mathcal{Z}_k - \dim_{\mathbb{Q}} \mathcal{Z}_{k-2}$ up to $k=12$ is observed. Accordingly, we may expect a real counterpart $\omega_{\mathcal{S}}(\mathbf{k})$ of $\omega_{\mathcal{A}}(\mathbf{k})$ in $\mathcal{Z}/\zeta(2)\mathcal{Z}$.

1.4. **Symmetric Mordell-Tornheim multiple zeta value.** In order to find a proper definition of $\omega_{\mathcal{S}}(\mathbf{k})$, we use the same framework as in our previous work [1]. A crucial feature of this work was that finite and symmetric multiple zeta values are obtained from an algebraic and analytic operation for multiple harmonic q-series $z_n(\mathbf{k};q)$ defined in (4.3), which is a q-analogue of $H_n(\mathbf{k})$, at primitive roots of unity (see Theorem 4.3). The Kaneko-Zagier conjecture then turns out to be the statement of a relationship between 'values' of $z_n(\mathbf{k};q)$ at $q=\zeta_n$ with ζ_n being a primitive n-th root of unity.

Let us begin with the q-analogue of $\omega_n(\mathbf{k})$. For an index $\mathbf{k}=(k_1,\ldots,k_r)$, let

$$\omega_n(\mathbf{k};q) := \sum_{\substack{m_1+\cdots+m_r=n\\ m_1,\ldots,m_r>0}} \prod_{a=1}^{r} \frac{q^{(k_a-1)m_a}}{[m_a]^{k_a}}, \tag{1.1}$$

where $[m]=1+q+\cdots+q^{m-1}=\frac{1-q^m}{1-q}$ denotes the usual q-integer. By definition, it follows that $\omega_n(k_{\sigma(1)},\ldots,k_{\sigma(r)};q)=\omega_n(k_1,\ldots,k_r;q)$ for any permutation $\sigma\in\mathfrak{S}_r$. If $\mathrm{len}(\mathbf{k})\geq 2$, one can consider the value $\omega_n(\mathbf{k};\zeta_n)$ in the cyclotomic field $\mathbb{Q}(\zeta_n)$ at $q=\zeta_n$ a primitive n-th root of unity (note that if ζ_n is not primitive, then $\omega_n(\mathbf{k};\zeta_n)$ is not well-defined). Its connection with the finite multiple omega value is as follows.

Theorem 1.2. *Let ζ_p be a primitive p-th root of unity. Under the identification $\mathbb{Z}[\zeta_p]/(1-\zeta_p)\mathbb{Z}[\zeta_p]=\mathbb{F}_p$ with p being prime, for any index $\mathbf{k}$ with $\mathrm{len}(\mathbf{k})\geq 2$ we have*

$$\left(\omega_p(\mathbf{k};\zeta_p) \mod (1-\zeta_p)\mathbb{Z}[\zeta_p]\right)_p = \omega_{\mathcal{A}}(\mathbf{k}) \in \mathcal{A}.$$

Theorem 1.2 says that the finite multiple omega value is obtained from a certain algebraic substitution $q\to 1$ for $\omega_n(\mathbf{k};q)$. On the other hand, by an analytic limit $q\to 1$ along the unit circle, we are led into the definition of a real counterpart of finite multiple omega value.

Theorem 1.3. *For any index $\mathbf{k}=(k_1,\ldots,k_r)$ with $r\geq 2$, the limit*

$$\Omega(\mathbf{k}) := \lim_{n\to\infty} \omega_n(\mathbf{k};e^{\frac{2\pi i}{n}})$$

exists, and it is given by

$$\Omega(\mathbf{k}) = \sum_{a=1}^{r} (-1)^{k_a} \zeta^{MT}(\underbrace{k_1,\ldots,k_{a-1}}_{a-1},\underbrace{k_{a+1},\ldots,k_r}_{r-a};k_a), \tag{1.2}$$

where $\zeta^{MT}(k_1,\dots,k_r;l)$ *is the Mordell-Tornheim multiple zeta value defined in* (2.4).

Since every Mordell-Tornheim multiple zeta value can be written as $\mathbb{Q}$-linear combinations of multiple zeta values (see [6, Theorem 1.1]), the following definition makes sense.

Definition 1.4. For an index $\mathbf{k}=(k_1,\dots,k_r)$ with $r\geq 2$, we define the *symmetric multiple omega value* $\omega_{\mathcal{S}}(\mathbf{k})$ by

$$\omega_{\mathcal{S}}(\mathbf{k}):=\Omega(\mathbf{k}) \mod \zeta(2)\mathcal{Z}.$$

Using Mathematica [13], we also numerically checked that the identity $\dim_{\mathbb{Q}}\mathcal{Z}_k^{\mathcal{S},\omega}\overset{?}{=}\dim_{\mathbb{Q}}\mathcal{Z}_k-\dim_{\mathbb{Q}}\mathcal{Z}_{k-2}$ holds up to weight 12, where $\mathcal{Z}_k^{\mathcal{S},\omega}$ denotes the $\mathbb{Q}$-vector subspace of $\mathcal{Z}/\zeta(2)\mathcal{Z}$ spanned by all symmetric multiple omega values of weight k. Consequently, for all $k\in\mathbb{N}$, the space $\mathcal{Z}_k^{\mathcal{S},\omega}$ is expected to be isomorphic to $\mathcal{Z}_k^{\mathcal{A},\omega}$ as a $\mathbb{Q}$-vector space.

1.5. **Kaneko-Zagier conjecture revisited.** We compared all $\mathbb{Q}$-linear relations among finite and symmetric multiple omega values up to weight 12, and it is observed that they satisfy the same relations over $\mathbb{Q}$. This further observation will support the following analogous statement to the Kaneko-Zagier conjecture.

Conjecture 1.5. *For any* $k\in\mathbb{N}$*, the* $\mathbb{Q}$*-linear map* $\varphi_k:\mathcal{Z}_k^{\mathcal{A},\omega}\to\mathcal{Z}_k^{\mathcal{S},\omega}$ *given by*

$$\varphi_k(\omega_{\mathcal{A}}(\mathbf{k}))=\omega_{\mathcal{S}}(\mathbf{k})$$

is a well-defined isomorphism.

Another evidence of Conjecture 1.5 will be provided as follows. Let $\mathfrak{h}=\mathbb{Q}\langle x_0,x_1\rangle$ and $\mathfrak{h}^1=\mathbb{Q}+\mathfrak{h}x_1$. We write $y_k=x_0^{k-1}x_1$ for $k\in\mathbb{N}$ and $y_{\mathbf{k}}=y_{k_1}\cdots y_{k_r}$ for an index $\mathbf{k}=(k_1,\dots,k_r)$. Then, the set $\{y_{\mathbf{k}}\mid\mathbf{k}:\text{index}\}$ is a linear basis of $\mathfrak{h}^1$. For $\bullet\in\{\mathcal{A},\mathcal{S}\}$, we write $\zeta_\bullet(y_{\mathbf{k}})=\zeta_\bullet(\mathbf{k})$, and extend these notations to $\mathfrak{h}^1$ by $\mathbb{Q}$-linearly. Denote by $⧢:\mathfrak{h}\times\mathfrak{h}\to\mathfrak{h}$ the standard shuffle product (see (2.2)).

Theorem 1.6. *For any index* $\mathbf{k}=(k_1,\dots,k_r)$ *with* $r\geq 2$*, we have*

$$\omega_{\mathcal{A}}(\mathbf{k})=(-1)^{k_r}\zeta_{\mathcal{A}}(x_0^{k_r}(y_{k_1}⧢\cdots⧢y_{k_{r-1}})), \tag{1.3}$$

$$\omega_{\mathcal{S}}(\mathbf{k})=(-1)^{k_r}\zeta_{\mathcal{S}}(x_0^{k_r}(y_{k_1}⧢\cdots⧢y_{k_{r-1}})). \tag{1.4}$$

By Theorem 1.6, we see that the Kaneko-Zagier conjecture, which insists the equality $\ker\zeta_{\mathcal{A}}\overset{?}{=}\ker\zeta_{\mathcal{S}}$, implies Conjecture 1.5. Numerically, we also observed that all finite and symmetric multiple zeta values can be written as $\mathbb{Q}$-linear combinations of finite and symmetric multiple omega values, respectively. If this is true, then Conjecture 1.5 would imply the map φ_{KZ} defined in Conjecture 2.3 (the Kaneko-Zagier conjecture) induces a well-defined isomorphism of $\mathbb{Q}$-vector spaces. Thus, Conjecture 1.5 could be a new approach to the Kaneko-Zagier conjecture. An advantage of this point of view is that we do not need to regularize our values, but a weak point is that the algebraic structure of the $\mathbb{Q}$-vector space $\mathcal{Z}^{\bullet,\omega}=\sum_{k\geq 0}\mathcal{Z}_k^{\bullet,\omega}$ is not immediately understood (conjecturally, $\mathcal{Z}^{\bullet,\omega}$ forms a $\mathbb{Q}$-algebra).

We remark that Kamano [9] introduces the finite Mordell-Tornheim multiple zeta value $\zeta_{\mathcal{A}}^{MT}(\mathbf{k})$ for each index $\mathbf{k}$. By definition, it follows that $\omega_{\mathcal{A}}(\mathbf{k})=(-1)^{k_r}\zeta_{\mathcal{A}}^{MT}(\mathbf{k})$ for any index $\mathbf{k}$. Then, Theorem 1.2 of [9] is equivalent to (1.3), but our proof is completely different (see Remark 4.5).

1.6. **Contents.** The organization of this paper is as follows. In Section 2, we recall some basics on the Kaneko-Zagier conjecture and Mordell-Tornheim multiple zeta values, thereby also fixing some of our notation. Section 3 is devoted to proving Theorems 1.2 and 1.3 and Section 4 gives a proof of Theorem 1.6. Our proof of (1.4) provides another expression of the limit value $\Omega(\mathbf{k})$ in terms of $\zeta_{\mathcal{S}}^{\sqcup\!\sqcup}(\mathbf{k})$'s, which is mentioned in the end of Section 4. Sections 5 considers relations among the values $\omega_n(\mathbf{k};\zeta_n)$, which is applied to the study of relations of finite and symmetric multiple omega values together with special values of them.

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2. Preliminaries

2.1. **Statement of the Kaneko-Zagier conjecture.** We briefly review the Kaneko-Zagier conjecture.

We define the finite multiple zeta value. It is an element in the ring $\mathcal{A}$, which is introduced by Kontsevich [12, §2.2], defined by

$$\mathcal{A}=\left(\prod_{p:\text{prime}}\mathbb{F}_p\right)\Big/\left(\bigoplus_{p:\text{prime}}\mathbb{F}_p\right).$$

Its element is denoted by $(a_p)_p$, where p runs over all primes and $a_p \in \mathbb{F}_p$. Two elements $(a_p)_p$ and $(b_p)_p$ are identified if and only if $a_p = b_p$ for all but finitely many primes p. Rational numbers $\mathbb{Q}$ can be embedded into $\mathcal{A}$ as follows. For $a \in \mathbb{Q}$, set $a_p = 0$ if p divides the denominator of a and $a_p = a \in \mathbb{F}_p$ otherwise. Then $(a_p)_p \in \mathcal{A}$. In this way, the ring $\mathcal{A}$ forms a commutative algebra over $\mathbb{Q}$.

Let us define the multiple harmonic sum $H_n(\mathbf{k}) \in \mathbb{Q}$ for an index $\mathbf{k} = (k_1, \ldots, k_r)$ and $n \in \mathbb{N}$ by

$$H_n(\mathbf{k}) = \sum_{n>m_1>\cdots>m_r>0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}}. \tag{2.1}$$

Definition 2.1. For an index $\mathbf{k} = (k_1, \ldots, k_r)$, we define the finite multiple zeta value $\zeta_{\mathcal{A}}(\mathbf{k})$ by

$$\zeta_{\mathcal{A}}(\mathbf{k}) = \left(H_p(\mathbf{k}) \mod p\right)_p \in \mathcal{A}.$$

Denote by $\mathcal{Z}_k^{\mathcal{A}}$ the $\mathbb{Q}$-vector subspace of $\mathcal{A}$ spanned by all finite multiple zeta values of weight k and set $\mathcal{Z}^{\mathcal{A}} = \sum_{k\geq 0} \mathcal{Z}_k^{\mathcal{A}}$, which forms a $\mathbb{Q}$-algebra (see [10, 11]).

Let us turn to the symmetric multiple zeta values. We first recall the algebraic setup of multiple zeta values by Hoffman [7]. Let

$$\mathfrak{h} = \mathbb{Q}\langle x_0, x_1 \rangle$$

be the non-commutative polynomial ring with indeterminates x_0 and x_1, and set $y_k = x_0^{k-1} x_1$ $(k \geq 1)$. For a non-empty index $\mathbf{k} = (k_1, \ldots, k_r)$ we set $y_{\mathbf{k}} = y_{k_1} \cdots y_{k_r}$. We define $y_\emptyset = 1$ for the empty index. We also let

$$\mathfrak{h}^1 = \mathbb{Q} + \mathfrak{h} x_1, \quad \mathfrak{h}^0 = \mathbb{Q} + x_0 \mathfrak{h} x_1.$$

The $\mathbb{Q}$-vector subspace $\mathfrak{h}^1$ is the $\mathbb{Q}$-subalgebra freely generated by $\{y_k\}_{k\geq 1}$, and the set of monomials $y_{\mathbf{k}}$ with admissible index $\mathbf{k}$ is a linear basis of $\mathfrak{h}^0$.

Using the iterated integral expression of the multiple zeta value due to Kontsevich, one can prove that the map $\zeta : \mathfrak{h}^0 \to \mathbb{R}$, defined by $\zeta(y_{\mathbf{k}}) = \zeta(\mathbf{k})$ for an admissible index $\mathbf{k}$, is an algebra homomorphism with respect to the shuffle product $ш : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{h}$ given inductively by

$$uw \,ш\, vw' = u(w \,ш\, vw') + v(uw \,ш\, w') \tag{2.2}$$

for $w, w' \in \mathfrak{h}$ and $u, v \in \{x_0, x_1\}$, with the initial condition $w \,ш\, 1 = w = 1 \,ш\, w$. Namely, $\zeta(w_1)\zeta(w_2) = \zeta(w_1 \,ш\, w_2)$ holds for any $w_1, w_2 \in \mathfrak{h}^0$. Equipped with the shuffle product, the vector space $\mathfrak{h}$ forms a commutative $\mathbb{Q}$-algebra and $\mathfrak{h}^1, \mathfrak{h}^0$ are

$\mathbb{Q}$-subalgebras. We write $\mathfrak{h}_{ш}, \mathfrak{h}^1_{ш}, \mathfrak{h}^0_{ш}$ for commutative $\mathbb{Q}$-algebras with the shuffle product. It follows that $\zeta(\mathfrak{h}^0) = \mathcal{Z}$, which is hence a $\mathbb{Q}$-subalgebra of $\mathbb{R}$.

Let us define the shuffle regularized multiple zeta values, following [8]. Recall that the algebra $\mathfrak{h}^1_{ш}$ is freely generated by x_1 over $\mathfrak{h}^0_{ш}$ (see [22]):

$$\mathfrak{h}^1_{ш} \cong \mathfrak{h}^0_{ш}[x_1].$$

Namely, for any word $w \in \mathfrak{h}^1$, there exist $w_i \in \mathfrak{h}^0$ such that

$$w = w_0 + w_1 ш x_1 + w_2 ш x_1^{ш 2} + \cdots + w_n ш x_1^{ш n}.$$

Hence, there is a unique algebra homomorphism $\zeta^{ш} : \mathfrak{h}^1_{ш} \to \mathbb{R}[T]$ such that the map $\zeta^{ш}$ extend $\zeta : \mathfrak{h}^0_{ш} \to \mathbb{R}$ and send $\zeta^{ш}(x_1) = T$. Applying $\zeta^{ш}$ to the above word, we get

$$\zeta^{ш}(w) = \sum_{a=0}^{n} \zeta(w_a) T^a,$$

so by definition it follows that $\zeta^{ш}(\mathfrak{h}^1) = \mathcal{Z}[T]$. For an index $\mathbf{k} = (k_1, \ldots, k_r)$ we write $\zeta^{ш}(y_{\mathbf{k}})\big|_{T=0} = \zeta^{ш}(\mathbf{k}) \in \mathcal{Z}$, which we call the shuffle regularized multiple zeta value. To define the symmetric multiple zeta value, for an index $\mathbf{k} = (k_1, \ldots, k_r)$, let

$$\zeta^{ш}_{\mathcal{S}}(\mathbf{k}) = \sum_{a=0}^{r} (-1)^{k_1+\cdots+k_a} \zeta^{ш}(k_a, k_{a-1}, \ldots, k_1) \zeta^{ш}(k_{a+1}, k_{a+2}, \ldots, k_r). \tag{2.3}$$

Definition 2.2. For an index $\mathbf{k} = (k_1, \ldots, k_r)$, we define the symmetric multiple zeta value $\zeta_{\mathcal{S}}(\mathbf{k})$ by

$$\zeta_{\mathcal{S}}(\mathbf{k}) := \zeta^{ш}_{\mathcal{S}}(\mathbf{k}) \mod \zeta(2)\mathcal{Z},$$

which is an element in the quotient $\mathbb{Q}$-algebra $\mathcal{Z}/\zeta(2)\mathcal{Z}$.

We remark that Yasuda [31, Theorem 6.1] proved that the symmetric multiple zeta value $\zeta_{\mathcal{S}}(\mathbf{k})$ spans the whole space $\mathcal{Z}/\zeta(2)\mathcal{Z}$. Remark that there is another variant of $\zeta^{ш}_{\mathcal{S}}(\mathbf{k})$ replacing $\zeta^{ш}$ with ζ^* the harmonic regularized multiple zeta value in the definition, which is however known to be congruent to $\zeta_{\mathcal{S}}(\mathbf{k})$ modulo $\zeta(2)$ (see [10, Proposition 9.1] and [11]).

The Kaneko-Zagier conjecture is stated as follows.

Conjecture 2.3. *The $\mathbb{Q}$-linear map $\varphi_{KZ} : \mathcal{Z}^{\mathcal{A}} \to \mathcal{Z}/\zeta(2)\mathcal{Z}$ given by*

$$\varphi_{KZ}(\zeta_{\mathcal{A}}(\mathbf{k})) = \zeta_{\mathcal{S}}(\mathbf{k})$$

is a well-defined isomorphism of $\mathbb{Q}$-algebras.

2.2. **Mordell-Tornheim multiple zeta value.** In this subsection, we recall the Mordell-Tornheim type of multiple zeta values.

The Mordell-Tornheim multiple zeta value is defined for an index $\mathbf{k} = (k_1, \ldots, k_r)$ and $l \in \mathbb{N}$ by

$$\zeta^{MT}(\mathbf{k}; l) = \sum_{m_1,\ldots,m_r>0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r} (m_1 + \cdots + m_r)^l}. \tag{2.4}$$

This type of sum for the case $r = 2$ was first studied by Tornheim [25] in 1950, and independently by Mordell [16] in 1958 with $k_1 = k_2 = l$, and then, rediscovered by Witten [29] in 1991 in his volume formula for certain moduli spaces related to theoretical physics (see also Zagier's number theoretical treatment [32]). The case $r \geq 3$ was first introduced by Matsumoto [14] in 2002 as a function of several complex variables, and then investigated by many authors from many point of views (see e.g. [15, 18, 20, 26, 27, 28]).

As a study of special values, it was shown by Bradley-Zhou [6, Theorem 1.1] in 2010 that every Mordell-Tornheim multiple zeta value can be written as a $\mathbb{Q}$-linear combination of multiple zeta values. Moreover, for an index $\mathbf{k} = (k_1, \ldots, k_r)$ Yamamoto [30] proved the following formula:

$$\zeta^{MT}(k_1, \ldots, k_{r-1}; k_r) = \zeta(x_0^{k_r}(y_{k_1} ⧢ \cdots ⧢ y_{k_{r-1}})). \tag{2.5}$$

It is worth mentioning that there seems no study on the opposite implication; "Every multiple zeta value can be written as a $\mathbb{Q}$-linear combination of Mordell-Tornheim multiple zeta values". We numerically checked this implication up to weight 12 using Mathematica:

Conjecture 2.4. *The space $\mathcal{Z}$ is generated by Mordell-Tornheim multiple zeta values over $\mathbb{Q}$.*

Remark that to check Conjecture 2.4, it is necessary to use relations among multiple zeta values. In other words, one can check that there exist a monomial $y_{\mathbf{k}}$ such that it cannot be written as $\mathbb{Q}$-linear combinations of $x_0^{k_r}(y_{k_1} ⧢ \cdots ⧢ y_{k_{r-1}})$'s.

3. Proofs of Theorems 1.2 and 1.3

3.1. **Proof of Theorem 1.2.** We prove Theorem 1.2.

Proof of Theorem 1.2. We first notice that the value $[m]\big|_{q=\zeta_p} = (1-\zeta_p^m)/(1-\zeta_p)$ for any prime p is a cyclotomic unit. Hence, the value $\omega_p(\mathbf{k}; \zeta_p)$ lies in $\mathbb{Z}[\zeta_p]$, and taking modulo the ideal $(1-\zeta_p)\mathbb{Z}[\zeta_p]$ generated by $1-\zeta_p$ in $\mathbb{Z}[\zeta_p]$ makes sense. Since $(1-\zeta_p)\mathbb{Z}[\zeta_p]$ is a prime ideal, its residue field $\mathbb{Z}[\zeta_p]/(1-\zeta_p)\mathbb{Z}[\zeta_p]$ is $\mathbb{F}_p$. It follows that $q^{(k-1)m}/[m]^k|_{q=\zeta_p} \equiv m^{-k} \mod (1-\zeta_p)\mathbb{Z}[\zeta_p]$ for all $k \in \mathbb{N}$, and hence

$$\omega_p(\mathbf{k};\zeta_p) \equiv \omega_p(\mathbf{k}) \mod (1-\zeta_p)\mathbb{Z}[\zeta_p].$$

Then, the result follows from the definition of the finite multiple omega value. □

3.2. **Proof of Theorem 1.3.** We now prove Theorem 1.3.

Proof of Theorem 1.3. We see that

$$\left.\frac{1}{[m]}\right|_{q=e^{2\pi i/n}} = e^{-\frac{\pi i}{n}(m-1)}\frac{\sin\frac{\pi}{n}}{\sin\frac{m\pi}{n}} \qquad (n > m > 0).$$

Therefore it holds that

$$\omega_n(\mathbf{k};e^{2\pi i/n}) = \left(e^{\frac{\pi i}{n}}\frac{n}{\pi}\sin\frac{\pi}{n}\right)^{\mathrm{wt}(\mathbf{k})} G_n(\mathbf{k}),$$

where the function $G_n(\mathbf{k})$ is defined by

$$G_n(\mathbf{k}) = \frac{1}{n^{\mathrm{wt}(\mathbf{k})}} \sum_{\substack{m_1,\dots,m_r>0\\ m_1+\cdots+m_r=n}} \prod_{a=1}^{r} g_{k_a}\left(\frac{m_a}{n}\right), \qquad g_k(x) = \left(\frac{\pi}{\sin\pi x}\right)^k e^{(k-2)\pi i x} \ (k \in \mathbb{N}).$$

It suffices to show that $G_n(\mathbf{k})$ converges to the right side of (1.2) in the limit $n \to \infty$. We decompose the range of the sum $I = \{(m_1,\dots,m_r) \in \mathbb{N}^r \mid \sum_{a=1}^{r} m_a = n\}$ into the subsets

$$I_0 = \{(m_1,\dots,m_r) \in I \mid 0 < m_a \le n/2 \ (1 \le \forall a \le r)\},$$
$$I_j = \{(m_1,\dots,m_r) \in I \mid m_j > n/2,\ 0 < m_a \le n/2 \ (\forall a \ne j)\}, \quad (1 \le j \le r).$$

Note that $I = \bigsqcup_{j=0}^{r} I_j$. Now we set

$$A_n(l_1,\dots,l_r) = \frac{1}{n^{\sum_{a=1}^{r} l_a}} \sum_{\substack{n/2 \ge m_1,\dots,m_r>0\\ m_1+\cdots+m_r=n}} \prod_{a=1}^{r} g_{l_a}\left(\frac{m_a}{n}\right),$$

$$B_n(l_1,\dots,l_{r-1};l_r) = \frac{1}{n^{\sum_{a=1}^{r} l_a}} \sum_{\substack{m_1,\dots,m_{r-1}>0\\ m_1+\cdots+m_{r-1}<n/2}} \prod_{a=1}^{r-1} g_{l_a}\left(\frac{m_a}{n}\right) h_{l_r}\left(\frac{m_1+\cdots+m_{r-1}}{n}\right),$$

where

$$h_l(x) = (-1)^l g_l(1-x) = \left(\frac{\pi}{\sin\pi x}\right)^l e^{-(l-2)\pi i x}.$$

Then it holds that

$$\frac{1}{n^{\mathrm{wt}(\mathbf{k})}} \sum_{(m_1,\ldots,m_r)\in I_0} \prod_{a=1}^{r} g_{k_a}\left(\frac{m_a}{n}\right) = A_n(k_1,\ldots,k_r),$$

$$\frac{1}{n^{\mathrm{wt}(\mathbf{k})}} \sum_{(m_1,\ldots,m_r)\in I_j} \prod_{a=1}^{r} g_{k_a}\left(\frac{m_a}{n}\right) = (-1)^{k_j} B_n(k_1,\ldots,k_{j-1},k_{j+1},\ldots,k_r;k_j).$$

Therefore it is enough to show that, for any index $\mathbf{k}$ and $l \in \mathbb{N}$,

$$A_n(\mathbf{k}) \to 0, \qquad B_n(\mathbf{k};l) \to \zeta^{MT}(\mathbf{k};l)$$

as $n \to \infty$.

First, we calculate the limit of $A_n(\mathbf{k})$. Set

$$\tilde{A}_n(\mathbf{k}) = \sum_{\substack{n/2\geq m_1,\ldots,m_r>0\\ m_1+\cdots+m_r=n}} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}}.$$

It holds that

$$|g_k(x)| \leq \left(\frac{\pi}{2x}\right)^k \qquad (0<x\leq \frac{1}{2}). \tag{3.1}$$

Hence $|A_n(\mathbf{k})| \leq C_{\mathbf{k}}\tilde{A}_n(\mathbf{k})$ for a constant $C_{\mathbf{k}}$ which does not depend on n. We prove that the limit of $\tilde{A}_n(\mathbf{k})$ as $n\to\infty$ is equal to zero. For $r \geq 1$ and $1 \leq l \leq n$, set

$$S_{r,l,n} = \sum_{\substack{n/2\geq m_1,\ldots,m_r>0\\ m_1+\cdots+m_r=l}} \frac{1}{m_1\cdots m_r}.$$

We set, by definition, $S_{r,l,n}=0$ unless $r\leq l\leq rn/2$. Since $0\leq \tilde{A}_n(\mathbf{k}) \leq S_{r,n,n}$, it suffices to show

$$S_{r,l,n} \leq \frac{(2(1+\log l))^{r-1}}{l}. \tag{3.2}$$

We prove (3.2) by induction on r. When $r=1$, it is trivial. For $r\geq 2$, setting $k=l-m_r$, we see that

$$S_{r,l,n} = \sum_{k=1}^{l-1} \frac{1}{l-k} S_{r-1,k,n}.$$

By the induction hypothesis, it is estimated as

$$\sum_{k=1}^{l-1}\frac{1}{l-k}S_{r-1,k,n} \leq \sum_{k=1}^{l-1}\frac{1}{l-k}\frac{(2(1+\log k))^{r-2}}{k} \leq (2(1+\log l))^{r-2}\sum_{k=1}^{l-1}\frac{1}{(l-k)k}$$

$$= \frac{(2(1+\log l))^{r-2}}{l} \sum_{k=1}^{l-1} \left(\frac{1}{l-k} + \frac{1}{k} \right)$$

$$= \frac{2(2(1+\log l))^{r-2}}{l} \sum_{k=1}^{l-1} \frac{1}{k} \leq \frac{(2(1+\log l))^{r-1}}{l}.$$

Thus we obtain (3.2) and $A_n(\mathbf{k}) \to 0$ as $n \to \infty$.

Next we calculate the limit of $B_n(\mathbf{k}; l)$ for an index $\mathbf{k} = (k_1, \ldots, k_{r-1})$ and a positive integer l. Set

$$\tilde{B}_n(\mathbf{k}; l) = \sum_{\substack{m_1, \ldots, m_{r-1} > 0 \\ m_1 + \cdots + m_{r-1} < n/2}} \frac{1}{m_1^{k_1} \cdots m_{r-1}^{k_{r-1}} (m_1 + \cdots + m_{r-1})^l}.$$

Since the function $f(z) = z^{k-1}(g_k(z) - z^{-k})$ is regular in a neighborhood of the interval $[0, 1/2]$, there exists a positive constant C such that

$$\left| g_k(x) - \frac{1}{x^k} \right| \leq \frac{C}{x^{k-1}} \qquad (0 < x \leq 1/2). \tag{3.3}$$

Using (3.1), (3.3) and the polynomial identity

$$\prod_{p=1}^{r} X_p - \prod_{p=1}^{r} Y_p = \sum_{j=1}^{r} (\prod_{p=1}^{j-1} X_p)(X_j - Y_j)(\prod_{p=j+1}^{r} Y_p),$$

we obtain the following estimation:

$$\begin{aligned}
&\left| \frac{1}{n^{\sum_{a=1}^{r-1} k_a + l}} \prod_{a=1}^{r-1} g_{k_a}\left(\frac{m_a}{n}\right) h_l\left(\frac{m_1 + \cdots + m_{r-1}}{n}\right) - \prod_{a=1}^{r-1} \frac{1}{m_a^{k_a}} \frac{1}{(m_1 + \cdots + m_{r-1})^l} \right| \\
&\leq \sum_{j=1}^{r-1} \prod_{a=1}^{j-1} \left| \frac{1}{n^{k_a}} g_{k_a}\left(\frac{m_a}{n}\right) \right| \left| \frac{1}{n^{k_j}} g_{k_j}\left(\frac{m_j}{n}\right) - \frac{1}{m_j^{k_j}} \right| \prod_{a=j+1}^{r-1} \frac{1}{m_a^{k_a}} \frac{1}{(m_1 + \cdots + m_{r-1})^l} \\
&\quad + \prod_{a=1}^{r-1} \left| \frac{1}{n^{k_a}} g_{k_a}\left(\frac{m_a}{n}\right) \right| \left| \frac{1}{n^l} h_l\left(\frac{m_1 + \cdots + m_{r-1}}{n}\right) - \frac{1}{(m_1 + \cdots + m_{r-1})^l} \right| \\
&\leq \frac{C'_{\mathbf{k},l}}{n} \sum_{j=1}^{r-1} \left\{ \frac{1}{m_1^{k_1} \cdots m_j^{k_j - 1} \cdots m_{r-1}^{k_{r-1}}} \frac{1}{(m_1 + \cdots + m_{r-1})^l} \right. \\
&\qquad\qquad \left. + \frac{1}{m_1^{k_1} \cdots m_{r-1}^{k_{r-1}}} \frac{1}{(m_1 + \cdots + m_{r-1})^{l-1}} \right\} \\
&\leq \frac{C'_{\mathbf{k},l}}{n} \frac{r}{m_1 \cdots m_{r-1}}
\end{aligned}$$

for some constant $C'_{\mathbf{k},l}$ which does not depend on n. Hence

$$\left|B_n(\mathbf{k}) - \tilde{B}_n(\mathbf{k})\right| \leq \frac{C'_{\mathbf{k},l}}{n} \sum_{\substack{m_1,\dots,m_{r-1}>0 \\ m_1+\cdots+m_{r-1}<n/2}} \frac{r}{m_1\cdots m_{r-1}} \leq C'_{\mathbf{k},l}\,\frac{r}{n}\left(\sum_{n/2>m>0}\frac{1}{m}\right)^{r-1}$$
$$\leq C'_{\mathbf{k},l}\,\frac{r}{n}\left(1+\log\frac{n+1}{2}\right)^{r-1} \to 0$$

as $n\to\infty$. Therefore it holds that

$$\lim_{n\to\infty} B_n(\mathbf{k};l) = \lim_{n\to\infty}\tilde{B}_n(\mathbf{k};l) = \zeta^{MT}(\mathbf{k};l).$$

This completes the proof. □

4. Proof of Theorem 1.6

4.1. **q-multiple polylogarithm.** In this subsection, we briefly recall the q-multiple polylogarithm, introduced by Zhao [33], of one variable $L_{\mathbf{k}}(t)$ and its shuffle relations.

In what follows, we fix a complex parameter q such that $|q|<1$. Define $F_k(m)$ for positive integers k and m by

$$F_k(m) = \frac{q^{(k-1)m}}{[m]^k} \in \mathbb{C}.$$

We introduce the letter $\hat{1}$ and set $\widehat{\mathbb{N}} = \mathbb{N}\sqcup\{\hat{1}\}$. Hereafter we call a tuple of the elements of $\widehat{\mathbb{N}}$ an extended index. In order to give a closed formula for the shuffle product for the q-multiple polylogarithm, we also need the term

$$F_{\hat{1}}(m) = \frac{q^m}{[m]} \in \mathbb{C}$$

for $m\in\mathbb{N}$. For a non-empty extended index $\mathbf{k}=(k_1,\dots,k_r)\in\widehat{\mathbb{N}}^r$ we define the q-multiple polylogarithm of one variable $L_{\mathbf{k}}(q;t)$ by

$$L_{\mathbf{k}}(t) = \sum_{m_1>\cdots>m_r>0} t^{m_1}\prod_{a=1}^{r} F_{k_a}(m_a)$$

as a formal power series in $\mathbb{C}[[t]]$. We set $L_{\emptyset}(t)=1$.

Recall the algebraic setup (see [24, §2.2] and [34]). Let $\hbar$ be a formal variable and set $\mathcal{C}=\mathbb{Q}[\hbar,\hbar^{-1}]$. We denote the unital non-commutative polynomial ring over $\mathcal{C}$

with two indeterminates a and b by

$$\mathfrak{H} = \mathcal{C}\langle a, b\rangle.$$

For $k \in \widehat{\mathbb{N}}$ we define $e_k \in \mathfrak{H}$ by

$$e_{\hat{1}} = ab, \qquad e_k = a^{k-1}(a+\hbar)b \quad (k \in \mathbb{N}).$$

Let

$$\widehat{\mathfrak{H}}^1 = \mathcal{C}\langle e_k \mid k \in \widehat{\mathbb{N}}\rangle$$

be the subalgebra freely generated by the set $\{e_k\}_{k\in\widehat{\mathbb{N}}}$. For a non-empty extended index $\mathbf{k} = (k_1, \dots, k_r) \in \widehat{\mathbb{N}}^r$ we set $e_{\mathbf{k}} = e_{k_1} \cdots e_{k_r}$. For the empty index we set $e_\emptyset = 1$. Then, the set $\{e_{\mathbf{k}} \mid \mathbf{k} : \text{extended index}\}$ is a linear basis of $\widehat{\mathfrak{H}}^1$.

We define the $\mathbb{Q}$-linear action of $\mathcal{C}$ on $\mathbb{C}[[t]]$ by $(\hbar f)(t) = (1-q)f(t)$ for $f \in \mathbb{C}[[t]]$ and then the $\mathcal{C}$-module homomorphism

$$Ł : \widehat{\mathfrak{H}}^1 \to \mathbb{C}[[t]],\ e_{\mathbf{k}} \mapsto L_{\mathbf{k}}(t).$$

We recall that the map $Ł : \widehat{\mathfrak{H}}^1 \to \mathbb{C}[[t]]$ is viewed as an evaluation map. While the map $\zeta : \mathfrak{h}^0 \to \mathbb{R}$ is given by an iterated integral of the integrands $\frac{dt}{t}, \frac{dt}{1-t}$ corresponding to the indeterminates $x_0, x_1 \in \mathfrak{h}$, we view the indeterminates $a, b, \hbar$ as operations on the formal power series ring $\mathbb{C}[[t]]$ (or the ring of holomorphic functions on $D = \{t \in \mathbb{C} : |t| < 1\}$). They are defined for $f(t) = \sum_{n\geq 0} c_n t^n \in \mathbb{C}[[t]]$ by

$$(af)(t) = (1-q)\sum_{j\geq 1} f(q^j t) = \sum_{n\geq 0} c_n t^n \frac{q^n}{[n]},$$
$$(bf)(t) = \frac{t}{1-t} f(t), \quad (\hbar f)(t) = (1-q)f(t),$$

where a is only defined for f being $f(0) = 0$. By definition, $\hbar$ commutes with others. With this, for any extended index $\mathbf{k} \in \widehat{\mathbb{N}}^r$ it can be shown that

$$L_{\mathbf{k}}(t) = e_{\mathbf{k}}(1).$$

For example, one can compute

$$L_{\hat{1},1}(t) = ab(a+\hbar)b(1) = ab(a+\hbar)\left(\sum_{n>0} t^n\right) = ab\left(\sum_{n>0} t^n \frac{1}{[n]}\right)$$
$$= a\left(\sum_{m>n>0} t^m \frac{1}{[n]}\right) = \sum_{m>n>0} t^m \frac{q^m}{[m][n]},$$

where for the third equality we have used

$$\frac{q^n}{[n]} + (1-q) = \frac{q^n(1-q) + (1-q)(1-q^n)}{1-q^n} = \frac{1}{[n]}.$$

Thus, the map $Ł : \widehat{\mathfrak{H}}^1 \to \mathbb{C}[[t]]$ is alternatively defined by $Ł(e_{\mathbf{k}}) = e_{\mathbf{k}}(1)$.

We define the shuffle product $\sqcup\!\sqcup_\hbar$ as the $\mathcal{C}$-bilinear map $\sqcup\!\sqcup_\hbar : \mathfrak{H} \times \mathfrak{H} \to \mathfrak{H}$ by

$$\begin{aligned} aw \sqcup\!\sqcup_\hbar aw' &= a(aw \sqcup\!\sqcup_\hbar w' + w \sqcup\!\sqcup_\hbar aw' + \hbar\, w \sqcup\!\sqcup_\hbar w'), \\ bw \sqcup\!\sqcup_\hbar w' &= w \sqcup\!\sqcup_\hbar bw' = b(w \sqcup\!\sqcup_\hbar w') \end{aligned}$$

for $w, w' \in \mathfrak{H}$, with the initial condition $1 \sqcup\!\sqcup_\hbar w = w \sqcup\!\sqcup_\hbar 1 = w$. For example, $e_1 \sqcup\!\sqcup_\hbar e_1 = 2abab + ab\hbar b + 2\hbar bab + \hbar b\hbar b = e_1e_1 + e_1e_{\hat{1}}$. The element $w \sqcup\!\sqcup_\hbar w'$ for $w, w' \in \widehat{\mathfrak{H}}^1$ lies in $\widehat{\mathfrak{H}}$, but, as we see in the example, it turns out to be a $\mathbb{Q}$-linear combination of words in $\widehat{\mathfrak{H}}^1$:

Lemma 4.1. *For $w, w' \in \widehat{\mathfrak{H}}^1$ we have $w \sqcup\!\sqcup_\hbar w' \in \widehat{\mathfrak{H}}^1$.*

Proof. For the proof, see Proposition 12.2.20 in [34]. □

With Lemma 4.1, we can state the shuffle relation for q-multiple polylogarithms.

Proposition 4.2. *For $w, w' \in \widehat{\mathfrak{H}}^1$ it holds that $Ł(w \sqcup\!\sqcup_\hbar w') = Ł(w)Ł(w')$.*

Proof. By Lemma 4.1, the expression $Ł(w \sqcup\!\sqcup_\hbar w')$ is well-defined. Then the desired result follows from easily checked identities $(af)\cdot(ag) = a(f\cdot ag) + a(af\cdot g) + \hbar a(f\cdot g)$ for $f, g \in \mathbb{C}[[t]]$ with $f(0) = g(0) = 0$ and $(bf)\cdot g = b(f\cdot g) = f\cdot(bg)$ for $f, g \in \mathbb{C}[[t]]$, where $f \cdot g$ means the product on $\mathbb{C}[[t]]$. □

4.2. **Multiple harmonic q-series at primitive roots of unity.** In this subsection, we recall the main results of our previous work [1].

For a word $w \in \widehat{\mathfrak{H}}^1$, we write $u_m(w)$ the coefficient of t^m in $Ł(w) \in \mathbb{C}[[t]]$

$$\sum_{n \geq 0} u_m(w) t^n := Ł(w) \tag{4.1}$$

and define

$$z_n(w) := \sum_{m=1}^{n-1} u_m(w). \tag{4.2}$$

These are viewed as $\mathcal{C}$-module homomorphisms from $\widehat{\mathfrak{H}}^1$ to $\mathbb{C}$ with the $\mathcal{C}$-linear action defined by $\hbar\alpha = (1-q)\alpha$ for $\alpha \in \mathbb{C}$. Namely, we have $z_n(\hbar e_{\mathbf{k}}) = (1-q) z_n(e_{\mathbf{k}})$ for an extended index $\mathbf{k}=(k_1,\dots,k_r)\in\widehat{\mathbb{N}}^r$, where by definition

$$z_n(e_{\mathbf{k}})=\sum_{n>m_1>\cdots>m_r>0}\prod_{a=1}^r F_{k_a}(m_a).$$

In [1], we studied the value $z_n(e_{\mathbf{k}})$ for an index $\mathbf{k}$ (not extended index) at $q=\zeta_n$ a primitive n-th root of unity. Note that the substitution $q=\zeta_n$ does not make sense if ζ_n is not primitive. For an index $\mathbf{k}=(k_1,\dots,k_r)$, we write

$$z_n(\mathbf{k};q):=z_n(e_{\mathbf{k}})=\sum_{n>m_1>\cdots>m_r>0}\prod_{a=1}^r\frac{q^{(k_a-1)m_a}}{[m_a]^{k_a}}. \tag{4.3}$$

Theorem 4.3. [1, Theorems 1.1 and 1.2]

(i) *For any index $\mathbf{k}\in\mathbb{N}^r$ we have*

$$\left(z_p(\mathbf{k};\zeta_p)\mod(1-\zeta_p)\mathbb{Z}[\zeta_p]\right)_p=\zeta_{\mathcal{A}}(\mathbf{k})\in\mathcal{A}.$$

(ii) *For any index $\mathbf{k}=(k_1,\dots,k_r)\in\mathbb{N}^r$ the limit*

$$\xi(\mathbf{k})=\lim_{n\to\infty}z_n(\mathbf{k};e^{2\pi i/n})$$

exists, and it holds that $\operatorname{Im}\xi(\mathbf{k})\in\pi\mathcal{Z}$ *and*

$$\operatorname{Re}\xi(\mathbf{k})\equiv\zeta_{\mathcal{S}}(\mathbf{k})\mod\zeta(2)\mathcal{Z}.$$

4.3. **Proof of** (1.3)**.** We first prove the following theorem and then (1.3).

Theorem 4.4. *For any index $\mathbf{k}=(k_1,\dots,k_r)\in\mathbb{N}^r$ with $r\geq 2$ and primitive n-th root of unity ζ_n, it holds that*

$$\omega_n(\mathbf{k};\zeta_n)=(-1)^{k_r}\sum_{j=1}^{k_r}\binom{k_r-1}{j-1}(1-q)^{k_r-j}z_n(a^j(e_{k_1}\sqcup\!\sqcup_\hbar\cdots\sqcup\!\sqcup_\hbar e_{k_{r-1}}))\Big|_{q=\zeta_n}.$$

Proof. It is easily seen that the value $\omega_n(\mathbf{k};q)$ is equal to the coefficient of t^n in the product $\prod_{a=1}^r L_{k_a}(t)$ of q-single polylogarithms. From Proposition 4.2 it holds that

$$\prod_{a=1}^r L_{k_a}(t)=\text{Ł}(e_{k_r})\text{Ł}(e_{k_1}\sqcup\!\sqcup_\hbar\cdots\sqcup\!\sqcup_\hbar e_{k_{r-1}}).$$

Hence by definition (4.1) we find that

$$\omega_n(\mathbf{k};q)=\sum_{m=1}^{n-1}u_{n-m}(e_{k_r})u_m(e_{k_1}\sqcup\!\sqcup_\hbar\cdots\sqcup\!\sqcup_\hbar e_{k_{r-1}}).$$

For $1 \le m < n$ and $k \in \mathbb{N}$, we compute

$$\begin{aligned} u_{n-m}(e_k)\big|_{q=\zeta_n} = F_k(n-m)\big|_{q=\zeta_n} &= (-1)^k \frac{q^m}{[m]^k}\bigg|_{q=\zeta_n} \\ &= (-1)^k \sum_{j=1}^{k} \binom{k-1}{j-1} (1-q)^{k-j} \frac{q^{jm}}{[m]^j}\bigg|_{q=\zeta_n}. \end{aligned}$$

Note that the element $e_{k_1} \sqcup\!\sqcup_\hbar \cdots \sqcup\!\sqcup_\hbar e_{k_{r-1}}$ belongs to the $\mathcal{C}$-submodule $\sum_{l\in\mathbb{N}} e_l \widehat{\mathfrak{H}}^1$. Using the equality

$$\frac{q^{jm}}{[m]^j} F_l(m)\big|_{q=\zeta_n} = F_{l+j}(m)\big|_{q=\zeta_n}$$

and $a^j e_l = e_{l+j}$ $(j, l \in \mathbb{N})$, we see that

$$\frac{q^{jm}}{[m]^j} u_m(e_{k_1} \sqcup\!\sqcup_\hbar \cdots \sqcup\!\sqcup_\hbar e_{k_{r-1}})\big|_{q=\zeta_n} = u_m(a^j(e_{k_1} \sqcup\!\sqcup_\hbar \cdots \sqcup\!\sqcup_\hbar e_{k_{r-1}}))\big|_{q=\zeta_n}.$$

Therefore by definition (4.2) we obtain

$$\begin{aligned} \omega_n(\mathbf{k};\zeta_n) &= (-1)^{k_r} \sum_{j=1}^{k_r} \binom{k_r-1}{j-1} (1-q)^{k_r-j} \sum_{m=1}^{n-1} u_m(a^j(e_{k_1} \sqcup\!\sqcup_\hbar \cdots \sqcup\!\sqcup_\hbar e_{k_{r-1}}))\big|_{q=\zeta_n} \\ &= (-1)^{k_r} \sum_{j=1}^{k_r} \binom{k_r-1}{j-1} (1-q)^{k_r-j} z_n(a^j(e_{k_1} \sqcup\!\sqcup_\hbar \cdots \sqcup\!\sqcup_\hbar e_{k_{r-1}}))\big|_{q=\zeta_n}. \end{aligned}$$

This completes the proof. □

We are now in a position to prove (1.3).

Proof of (1.3). When p is prime and $q = \zeta_p$, for $m < p$ it holds that $F_{\hat{1}}(m) \equiv m^{-1} \mod (1-\zeta_p)\mathbb{Z}[\zeta_p]$ and $F_k(m) \equiv m^{-k} \mod (1-\zeta_p)\mathbb{Z}[\zeta_p]$ for any $k \in \mathbb{N}$. Hence, under the identification $\mathbb{Z}[\zeta_p]/(1-\zeta_p)\mathbb{Z}[\zeta_p] = \mathbb{F}_p$, for any $w \in \widehat{\mathfrak{H}}^1$ it holds that

$$(z_p(w)\big|_{q=\zeta_p} \mod (1-\zeta_p)\mathbb{Z}[\zeta_p])_p = \zeta_{\mathcal{A}}(\rho(w)),$$

where $\rho : \widehat{\mathfrak{H}}^1 \to \mathfrak{h}^1$ is the $\mathbb{Q}$-algebra homomorphism given by

$$\rho(\hbar) = 0, \quad \rho(e_{\hat{1}}) = y_1, \quad \rho(e_k) = y_k \ (k \in \mathbb{N}).$$

Note that $\widehat{\mathfrak{H}}^1$ (resp. $\mathfrak{h}^1$) is invariant under the left multiplication with a (resp. x_0) because $ae_{\hat{1}} = e_2 - \hbar e_{\hat{1}}$, $ae_k = e_{k+1}$ and $x_0 y_k = y_{k+1}$ for $k \in \mathbb{N}$. These relations imply that

$$\rho(ae_{\mathbf{k}}) = x_0 \rho(e_{\mathbf{k}}) \tag{4.4}$$

for any non-empty extended index $\mathbf{k}\in\widehat{\mathbb{N}}^r$. Hence, by Theorem 4.4, for any index $\mathbf{k}=(k_1,\dots,k_r)\in\mathbb{N}^r$ we obtain

$$(\omega(\mathbf{k};\zeta_p)\mod (1-\zeta_p)\mathbb{Z}[\zeta_p])_p=(-1)^{k_r}\zeta_{\mathcal{A}}(x_0^{k_r}\rho(e_{k_1}\sqcup\!\sqcup_\hbar\cdots\sqcup\!\sqcup_\hbar e_{k_{r-1}})).$$

Because of Theorem 4.3 (i), it suffices to show that

$$\rho(w_1\sqcup\!\sqcup_\hbar w_2)=\rho(w_1)\sqcup\!\sqcup\rho(w_2)$$

for any $w_1,w_2\in\widehat{\mathfrak{H}}^1$. This is done by induction on the sum of weights (degrees) of w_1 and w_2. We may assume that w_1 and w_2 are monomials in $\{e_k\}_{k\in\widehat{\mathbb{N}}}$. If $w_1=1$ or $w_2=1$, it is trivial. Hence we consider the case where $w_j=e_{k_j}w_j'$ $(j=1,2)$ for some $k_j\in\widehat{\mathbb{N}}$.

First, we consider the case $k_1=k_2=\hat{1}$. Then

$$\begin{aligned}w_1\sqcup\!\sqcup_\hbar w_2&=abw_1'\sqcup\!\sqcup_\hbar abw_2'=a(bw_1'\sqcup\!\sqcup_\hbar abw_2'+abw_1'\sqcup\!\sqcup_\hbar bw_2'+\hbar bw_1'\sqcup\!\sqcup_\hbar bw_2')\\&=e_{\hat{1}}(w_1'\sqcup\!\sqcup_\hbar e_{\hat{1}}w_2'+e_{\hat{1}}w_1'\sqcup\!\sqcup_\hbar w_2')+e_{\hat{1}}(e_1-e_{\hat{1}})(w_1'\sqcup\!\sqcup_\hbar w_2').\end{aligned}$$

Using the induction hypothesis and $\rho(e_{\hat{1}})=\rho(e_1)=y_1=x_1$, we see that

$$\begin{aligned}\rho(w_1\sqcup\!\sqcup_\hbar w_2)&=x_1(\rho(w_1')\sqcup\!\sqcup x_1\rho(w_2')+x_1\rho(w_1')\sqcup\!\sqcup\rho(w_2'))\\&=x_1\rho(w_1')\sqcup\!\sqcup x_1\rho(w_2')=\rho(e_{\hat{1}}w_1')\sqcup\!\sqcup\rho(e_{\hat{1}}w_2')=\rho(w_1)\sqcup\!\sqcup\rho(w_2).\end{aligned}$$

For the case $(k_1,k_2)=(1,\hat{1})$ or $(1,1)$, we obtain the desired equality in a similar way by using

$$\begin{aligned}(e_1w_1')\sqcup\!\sqcup_\hbar(e_1w_2')&=e_1(w_1'\sqcup\!\sqcup_\hbar e_{\hat{1}}w_2'+e_{\hat{1}}w_1'\sqcup\!\sqcup_\hbar w_2')+e_1(e_1-e_{\hat{1}})(w_1'\sqcup\!\sqcup_\hbar w_2'),\\(e_1w_1')\sqcup\!\sqcup_\hbar(e_{\hat{1}}w_2')&=e_1(w_1'\sqcup\!\sqcup_\hbar e_{\hat{1}}w_2')+e_{\hat{1}}(e_{\hat{1}}w_1'\sqcup\!\sqcup_\hbar w_2')+e_{\hat{1}}(e_1-e_{\hat{1}})(w_1'\sqcup\!\sqcup_\hbar w_2').\end{aligned}$$

Next, we consider the case where $k_1=\hat{1}$ and $k_2\geq 2$. In this case, one can write $w_2=aw_2''$ with $w_2''\in\widehat{\mathfrak{H}}^1$. Hence we see that

$$w_1\sqcup\!\sqcup_\hbar w_2=abw_1'\sqcup\!\sqcup_\hbar aw_2''=e_{\hat{1}}(w_1'\sqcup\!\sqcup_\hbar w_2)+a(w_1\sqcup\!\sqcup_\hbar w_2'')+\hbar e_{\hat{1}}(w_1'\sqcup\!\sqcup_\hbar w_2'').$$

Hence the induction hypothesis and (4.4) imply that

$$\begin{aligned}\rho(w_1\sqcup\!\sqcup_\hbar w_2)&=x_1(\rho(w_1')\sqcup\!\sqcup\rho(w_2))+x_0(\rho(w_1)\sqcup\!\sqcup\rho(w_2''))\\&=x_1(\rho(w_1')\sqcup\!\sqcup x_0\rho(w_2''))+x_0(x_1\rho(w_1')\sqcup\!\sqcup\rho(w_2''))\\&=x_1\rho(w_1')\sqcup\!\sqcup x_0\rho(w_2'')=\rho(w_1)\sqcup\!\sqcup\rho(w_2).\end{aligned}$$

The case $k_1=1$ and $k_2\geq 2$ is a consequence of the above case, since

$$e_1w_1'\sqcup\!\sqcup_\hbar w_2=e_{\hat{1}}w_1'\sqcup\!\sqcup_\hbar w_2+\hbar bw_1'\sqcup\!\sqcup_\hbar w_2$$

$$= e_{\hat{1}} w_1' ш_\hbar w_2 + \hbar b(w_1' ш_\hbar w_2)$$
$$= e_{\hat{1}} w_1' ш_\hbar w_2 + (e_1 - e_{\hat{1}})(w_1' ш_\hbar w_2)$$

and $\rho(e_1(w_1' ш_\hbar w_2)) = \rho(e_{\hat{1}}(w_1' ш_\hbar w_2))$, we get

$$\rho(e_1 w_1' ш_\hbar w_2) = \rho(e_{\hat{1}} w_1' ш_\hbar w_2).$$

Finally, we consider the case where $k_j \geq 2\,(j = 1, 2)$. We write $w_j = a w_j''$ with $w_j'' \in \widehat{\mathfrak{H}}^1$. Then it holds that

$$w_1 ш_\hbar w_2 = a(w_1'' ш_\hbar w_2 + w_1 ш_\hbar w_2'' + \hbar\, w_1'' ш_\hbar w_2'').$$

Using the induction hypothesis and (4.4), we see that

$$\rho(w_1 ш_\hbar w_2) = x_0(\rho(w_1'') ш x_0\rho(w_2'') + x\rho(w_1'') ш \rho(w_2'')) = x_0\rho(w_1'') ш x_0\rho(w_2'')$$
$$= \rho(w_1) ш \rho(w_2).$$

This completes the proof. □

Remark 4.5. In [9], Kamano introduced the finite Mordell-Tornheim multiple zeta value for an index $\mathbf{k} = (k_1, \ldots, k_r) \in \mathbb{N}^r$ by

$$\zeta_{\mathcal{A}}^{MT}(\mathbf{k}) = \left(\sum_{\substack{m_1,\ldots,m_{r-1}\geq 1 \\ m_1+\cdots+m_{r-1}<p}} \frac{1}{m_1^{k_1}\cdots m_{r-1}^{k_{r-1}}(m_1+\cdots+m_{r-1})^{k_r}} \mod p \right)_p \in \mathcal{A}$$

and, as a finite analogue of (2.5), proved that

$$\zeta_{\mathcal{A}}^{MT}(\mathbf{k}) = \zeta_{\mathcal{A}}(x_0^{k_r}(y_{k_1} ш \cdots ш y_{k_{r-1}})). \tag{4.5}$$

Since $\omega_{\mathcal{A}}(\mathbf{k}) = (-1)^{k_r}\zeta_{\mathcal{A}}^{MT}(\mathbf{k})$ for each index $\mathbf{k} = (k_1, \ldots, k_r) \in \mathbb{N}^r$, Theorem 4.4 can be viewed as a lift of Kamano's result (4.5) to the values $\omega_n(\mathbf{k}; \zeta_n)$.

4.4. **Proof of** (1.4)**.** The proof of (1.4) is based on identities of words. Let $\phi : \mathfrak{h}^1 \to \mathbb{R}$ be the $\mathbb{Q}$-linear map given by $\phi(1) = 1$ and

$$\phi(y_{k_1}\cdots y_{k_r}) = \sum_{a=0}^{r} (-1)^{k_1+\cdots+k_a}(y_{k_a}y_{k_{a-1}}\cdots y_{k_1}) ш (y_{k_{a+1}}y_{k_{a+2}}\cdots y_{k_r}). \tag{4.6}$$

Theorem 4.6. *For any index $(k_1, \ldots, k_r) \in \mathbb{N}^r$ with $r \geq 2$, we have*

$$\phi((-1)^{k_r}x_0^{k_r}(y_{k_1} ш \cdots ш y_{k_{r-1}}))$$
$$= \sum_{a=1}^{r} (-1)^{k_a} x_0^{k_a}(y_{k_1} ш \cdots ш y_{k_{a-1}} ш y_{k_{a+1}} ш \cdots ш y_{k_r}).$$

First, we give a proof of (1.4), and then prove Theorem 4.6 below.

Proof of (1.4). Since $\zeta^{ш} : \mathfrak{h}^1_{ш} \to \mathbb{R}$ is a homomorphism, it follows from the definition (2.3) that $\zeta^{ш}_{\mathcal{S}}(w) = \zeta^{ш}(\phi(w))$ for any $w \in \mathfrak{h}^1$. By Theorem 4.6 and (2.5), we see that

$$\begin{aligned}
&(-1)^{k_r}\zeta^{ш}_{\mathcal{S}}\left(x_0^{k_r}(y_{k_1} ш \cdots ш y_{k_{r-1}})\right)\\
&= \zeta^{ш}\left(\phi\left((-1)^{k_r}x_0^{k_r}(y_{k_1} ш \cdots ш y_{k_{r-1}})\right)\right)\\
&= \sum_{a=1}^{r}(-1)^{k_a}\zeta^{ш}\left(x_0^{k_a}(y_{k_1} ш \cdots ш y_{k_{a-1}} ш y_{k_{a+1}} ш \cdots ш y_{k_r})\right)\\
&= \sum_{a=1}^{r}(-1)^{k_a}\zeta^{MT}(k_1,\dots,k_{a-1},k_{a+1},\dots,k_r;k_a),
\end{aligned}$$

which by Theorem 1.3 is equal to $\Omega(k_1,\dots,k_r)$. Taking modulo $\zeta(2)$, we get the desired result. □

In what follows, we will prove Theorem 4.6 by calculating the generating series of both sides. For that purpose we introduce the element

$$y(t) = \sum_{k=1}^{\infty} y_k t^{k-1} = \frac{1}{1-x_0t}\,x_1 \in \mathfrak{h}^1[[t]]$$

with the indeterminate t. We begin with the following lemma, which is used to prove Propositions 4.8, 4.9 and 4.10 below, and then, give a proof of Theorem 4.6.

Lemma 4.7. *For $w, w' \in \mathfrak{h}^1$ it holds that*

$$y(t_1)w ш y(t_2)w' = y(t_1+t_2)\left(w ш y(t_2)w' + y(t_1)w ш w'\right).$$

Proof. We denote the left side by I. Noting $y(t) = x_1 + tx_0y(t)$, one computes

$$\begin{aligned}
I &= (x_1 + t_1x_0y(t_1))\,w ш (x_1 + t_2x_0y(t_2))\,w'\\
&= x_1w ш x_1w' + t_1x_0y(t_1)w ш x_1w' + x_1w ш t_2x_0y(t_2)w' + t_1x_0y(t_1)w ш t_2x_0y(t_2)w'\\
&= x_1(w ш x_1w') + x_1(x_1w ш w')\\
&+ t_1x_0(y(t_1)w ш x_1w') + t_1x_1(x_0y(t_1)w ш w')\\
&+ t_2x_1(w ш x_0y(t_2)w') + t_2x_0(x_1w ш y(t_2)w')\\
&+ t_1t_2x_0(y(t_1)w ш x_0y(t_2)w') + t_1t_2x_0(x_0y(t_1)w ш y(t_2)w')\\
&= x_0\left\{t_1y(t_1)w ш x_1w' + x_1w ш t_2y(t_2)w' + t_1y(t_1)w ш x_0t_2y(t_2)w' + t_1x_0y(t_1)w ш t_2y(t_2)w'\right\}\\
&+ x_1\left\{w ш x_1w' + x_1w ш w' + x_0t_1y(t_1)w ш w' + w ш x_0t_2y(t_2)w'\right\}
\end{aligned}$$

$$
\begin{aligned}
&= x_0\left(t_1 y(t_1)w \sqcup y(t_2)w' + t_2 y(t_1)w \sqcup y(t_2)w'\right)\\
&+ x_1\left(w \sqcup y(t_2)w' + y(t_1)w \sqcup w'\right)\\
&= (t_1+t_2)x_0 I + x_1\left(w \sqcup y(t_2)w' + y(t_1)w \sqcup w'\right).
\end{aligned}
$$

Hence, $(1-(t_1+t_2)x_0)I = x_1\left(w \sqcup y(t_2)w' + y(t_1)w \sqcup w'\right)$, which implies the desired formula. □

From Lemma 4.7 one can compute

$$
y(t_1) \sqcup y(t_2) = y(t_1+t_2)\left(y(t_2)+y(t_1)\right)
$$

and

$$
\begin{aligned}
(y(t_1) \sqcup y(t_2)) \sqcup y(t_3) &= y(t_1+t_2)\left(y(t_2)+y(t_1)\right) \sqcup y(t_3)\\
&= y(t_1+t_2)y(t_1) \sqcup y(t_3) + y(t_1+t_2)y(t_2) \sqcup y(t_3)\\
&= y(t_1+t_2+t_3)(y(t_1) \sqcup y(t_3) + y(t_1+t_2)y(t_1))\\
&+ y(t_1+t_2+t_3)(y(t_2) \sqcup y(t_3) + y(t_1+t_2)y(t_2))\\
&= y(t_1+t_2+t_3)\left(y(t_1+t_3)(y(t_1)+y(t_3)) + y(t_1+t_2)y(t_1)\right)\\
&+ y(t_1+t_2+t_3)\left(y(t_2+t_3)(y(t_2)+y(t_3)) + y(t_1+t_2)y(t_2)\right)\\
&= y(t_1+t_2+t_3)y(t_1+t_2)y(t_1) + y(t_1+t_2+t_3)y(t_1+t_3)y(t_1)\\
&+ y(t_1+t_2+t_3)y(t_1+t_2)y(t_2) + y(t_1+t_2+t_3)y(t_2+t_3)y(t_2)\\
&+ y(t_1+t_2+t_3)y(t_2+t_3)y(t_3) + y(t_1+t_2+t_3)y(t_1+t_3)y(t_3).
\end{aligned}
$$

Using Lemma 4.7 repeatedly, we obtain the following formula for the shuffle product of $y(t)$:

Proposition 4.8. *It holds that*

$$
y(t_1) \sqcup \cdots \sqcup y(t_r) = \sum_{\sigma\in\mathfrak{S}_r} \prod_{1\le j\le r}^{\curvearrowleft} y\left(\textstyle\sum_{i=1}^{j} t_{\sigma(i)}\right),
$$

where $\prod_{m\le j\le n}^{\curvearrowleft} X_j$ *denotes the ordered product* $X_n X_{n-1}\cdots X_m$ *for* $m\le n$.

We denote by $[r]$ the set $\{1,2,\dots,r\}$ for $r\ge 1$. For a subset $I=\{p_1,\dots,p_s\}$ of $[r]$ with $|I|=s$, we define

$$
\begin{aligned}
S_I(t_1,\dots,t_r) &= y(t_{p_1}) \sqcup \cdots \sqcup y(t_{p_s})\\
&= \sum_{\sigma\in\mathfrak{S}_s} \prod_{1\le j\le s}^{\curvearrowleft} y\left(\textstyle\sum_{i=1}^{j} t_{p_{\sigma(i)}}\right).
\end{aligned}
$$

Proposition 4.9. *For any subset $I=\{p_1,\dots,p_s\}$ of $[r]$ with $|I|=s$, it holds that*

$$\phi(y(u)S_I(t_1,\dots,t_r)) = y(u)S_I(t_1,\dots,t_r) - y(-u)\sqcup S_I(t_1,\dots,t_r) + \sum_{b\in I} y(-t_b)\left(y(-u)\sqcup S_{I\setminus\{b\}}(t_1,\dots,t_r)\right).$$

Proof. From the definition (4.6) of the map ϕ, we see that

$$\begin{aligned}
\phi(y(s_1)\cdots y(s_r)) &= \sum_{k_1,\dots,k_r\geq 1}\phi(y_{k_1}\cdots y_{k_r})s_1^{k_1-1}\cdots s_r^{k_r-1}\\
&= \sum_{a=0}^{r}(-1)^{k_1+\cdots+k_a}\sum_{k_1,\dots,k_r\geq 1} y_{k_a}\cdots y_{k_1}\sqcup y_{k_{a+1}}\cdots y_{k_r}s_1^{k_1-1}\cdots s_r^{k_r-1}\\
&= \sum_{a=0}^{r}(-1)^a \underbrace{y(-s_a)\cdots y(-s_1)}_{a}\sqcup\underbrace{y(s_{a+1})\cdots y(s_r)}_{r-a}\\
&= \sum_{a=0}^{r}(-1)^a\left(\overset{\curvearrowleft}{\prod_{1\leq j\leq a}} y(-s_j)\right)\sqcup\left(\overset{\curvearrowright}{\prod_{a+1\leq j\leq r}} y(s_j)\right).
\end{aligned}$$

For simplicity we set $\alpha_j^\sigma=\sum_{i=1}^{j} t_{p_{\sigma(i)}}$ for $1\leq j\leq s$ and $\sigma\in\mathfrak{S}_s$. Proposition 4.8 and the above formula imply that

$$\begin{aligned}
&\phi(y(u)S_I(t_1,\dots,t_r))\\
&= \sum_{\sigma\in\mathfrak{S}_s}\phi(y(u)y(\alpha_s^\sigma)\cdots y(\alpha_1^\sigma))\\
&= \sum_{\sigma\in\mathfrak{S}_s} y(u)y(\alpha_s^\sigma)\cdots y(\alpha_1^\sigma)\\
&+ (-1)^1\sum_{\sigma\in\mathfrak{S}_s} y(-u)\sqcup y(\alpha_s^\sigma)\cdots y(\alpha_1^\sigma)\\
&+ \sum_{a=2}^{s+1}(-1)^a\sum_{\sigma\in\mathfrak{S}_s}\left(\left\{\overset{\curvearrowright}{\prod_{s-a+2\leq j\leq s}} y(-\alpha_j^\sigma)\right\}y(-u)\right)\sqcup\left(\overset{\curvearrowleft}{\prod_{1\leq j\leq s-a+1}} y(\alpha_j^\sigma)\right).
\end{aligned} \tag{4.7}$$

Applying Lemma 4.7 except for the first two terms and for the last term with $a=s+1$, we obtain

$$\begin{aligned}
&= y(u)S_I(t_1,\dots,t_r) - y(-u)\sqcup S_I(t_1,\dots,t_r)\\
&+ \sum_{\sigma\in\mathfrak{S}_s}\sum_{a=2}^{s}(-1)^a y(-t_{p_{\sigma(s-a+2)}})\left(\left\{\overset{\curvearrowright}{\prod_{s-a+2\leq j\leq s}} y(-\alpha_j^\sigma)\right\}y(-u)\right)\sqcup\left(\overset{\curvearrowleft}{\prod_{1\leq j\leq s-a}} y(\alpha_j^\sigma)\right)
\end{aligned}$$

$$+\sum_{\sigma\in\mathfrak{S}_s}\sum_{a=2}^{s}(-1)^a y(-t_{p_{\sigma(s-a+2)}})\left(\left\{\overset{\curvearrowright}{\prod_{s-a+3\le j\le s}} y(-\alpha_j^\sigma)\right\} y(-u)\right)ш\left(\overset{\curvearrowleft}{\prod_{1\le j\le s-a+1}} y(\alpha_j^\sigma)\right)$$
$$+(-1)^{s+1}\sum_{\sigma\in\mathfrak{S}_s} y(-\alpha_1^\sigma)\cdots y(-\alpha_s^\sigma)y(-u)\,.$$

In the third term with $3\le a\le s$, change $a\to a+1$ and $\sigma\to\sigma'=\sigma(s-a+2,s-a+1)$, where $(s-a+2,s-a+1)\in\mathfrak{S}_s$ is the transposition. Then, since $\alpha_j^{\sigma'}=\alpha_j^\sigma$ $(j\neq s-a+1)$, it cancels with the second term except for the term with $a=s$. We see that the second term with $a=s$ also cancels with the last term by changing $\sigma\to\sigma(1,2)$. For the third term with $a=2$, decomposing the range $\mathfrak{S}_s$ as $\bigsqcup_{l=1}^s \mathfrak{S}_s^l$, where $\mathfrak{S}_s^l=\{\sigma\in\mathfrak{S}_s\,|\,\sigma(s)=l\}$, we get

$$\begin{aligned}&=y(u)S_I(t_1,\dots,t_r)-y(-u)ш S_I(t_1,\dots,t_r)\\&+\sum_{l=1}^{s} y(-t_{p_l})(y(-u)ш\sum_{\sigma\in\mathfrak{S}_s^l} y(\alpha_{s-1}^\sigma)\cdots y(\alpha_1^\sigma))\\&=y(u)S_I(t_1,\dots,t_r)-y(-u)ш S_I(t_1,\dots,t_r)\\&+\sum_{b\in I} y(-t_b)(y(-u)ш S_{I\setminus\{b\}}(t_1,\dots,t_r)),\end{aligned}$$

which completes the proof. □

In the same way as above, we obtain the following formula.

Proposition 4.10. *For any subset I of $[r]$, it holds that*

$$\phi(S_I(t_1,\dots,t_r))=S_I(t_1,\dots,t_r)-\sum_{b\in I} y(-t_b)S_{I\setminus\{b\}}(t_1,\dots,t_r).$$

Proof. We use the same notation with the proof of Proposition 4.9. Replacing $y(u)$ with 1 in (4.7) and doing the same game, we get

$$\begin{aligned}&\phi(S_I(t_1,\dots,t_r))\\&=\sum_{\sigma\in\mathfrak{S}_s}\sum_{a=0}^{s}(-1)^a\left(\overset{\curvearrowright}{\prod_{s-a+1\le j\le s}} y(-\alpha_j^\sigma)\right)ш\left(\overset{\curvearrowleft}{\prod_{1\le j\le s-a}} y(\alpha_j^\sigma)\right)\\&=\sum_{\sigma\in\mathfrak{S}_s}\overset{\curvearrowleft}{\prod_{1\le j\le s}} y(\alpha_j^\sigma)\\&+\sum_{\sigma\in\mathfrak{S}_s}\sum_{a=1}^{s-1}(-1)^a y(-t_{p_{\sigma(s-a+1)}})\left(\overset{\curvearrowright}{\prod_{s-a+2\le j\le s}} y(-\alpha_j^\sigma)\right)ш\left(\overset{\curvearrowleft}{\prod_{1\le j\le s-a}} y(\alpha_j^\sigma)\right)\end{aligned}$$

$$+\sum_{\sigma\in\mathfrak{S}_s}\sum_{a=1}^{s-1}(-1)^a y(-t_{p_{\sigma(s-a+1)}})\left(\overset{\curvearrowright}{\prod_{s-a+1\le j\le s}} y(-\alpha_j^\sigma)\right)\sqcup\left(\overset{\curvearrowleft}{\prod_{1\le j\le s-a-1}} y(\alpha_j^\sigma)\right)$$
$$+\sum_{\sigma\in\mathfrak{S}_s}(-1)^s \overset{\curvearrowright}{\prod_{1\le j\le s}} y(-\alpha_j^\sigma)$$
$$=\sum_{\sigma\in\mathfrak{S}_s}\overset{\curvearrowleft}{\prod_{1\le j\le s}} y(\alpha_j^\sigma)-\sum_{\sigma\in\mathfrak{S}_s} y(-t_{p_{\sigma(s)}})\overset{\curvearrowleft}{\prod_{1\le j\le s-1}} y(\alpha_j^\sigma)$$
$$=S_I(t_1,\dots,t_r)-\sum_{b\in I}y(-t_b)S_{I\setminus\{b\}}(t_1,\dots,t_r).$$

We complete the proof. ∎

We are ready to prove Theorem 4.6.

Proof of Theorem 4.6. Let

$$J(t_1,\dots,t_{r-1},t_r):=\sum_{k_1,\dots,k_r\ge 1}(-1)^{k_r}x_0^{k_r}(y_{k_1}\sqcup\cdots\sqcup y_{k_{r-1}})t_1^{k_1-1}\cdots t_{r-1}^{k_{r-1}-1}t_r^{k_r-1}.$$

Then, we have

$$\sum_{k_1,\dots,k_r\ge 1}\sum_{a=1}^{r}(-1)^{k_a}x_0^{k_a}(y_{k_1}\sqcup\cdots\sqcup y_{k_{a-1}}\sqcup y_{k_{a+1}}\sqcup\cdots\sqcup y_{k_r})t_1^{k_1-1}\cdots t_r^{k_r-1}$$
$$=\sum_{a=1}^{r}J(\underbrace{t_1,\dots,t_{a-1}}_{a-1},\underbrace{t_{a+1},\dots,t_r}_{r-a},t_a)$$

Therefore, for $r\ge 2$, it suffices to show that

$$\phi\big(J(t_1,\dots,t_r)\big)=\sum_{a=1}^{r}J(\underbrace{t_1,\dots,t_{a-1}}_{a-1},\underbrace{t_{a+1},\dots,t_r}_{r-a},t_a). \tag{4.8}$$

For simplicity we set

$$\alpha_j=\sum_{i=1}^{j}t_i\ (1\le j\le r)\text{ and }\alpha_j^\sigma=\sum_{i=1}^{j}t_{\sigma(i)}\ (1\le j\le r-1,\sigma\in\mathfrak{S}_{r-1}).$$

From Proposition 4.8, we obtain

$$J(t_1,\dots,t_r)=-\frac{x_0}{1+x_0t_r}(y(t_1)\sqcup\cdots\sqcup y(t_{r-1}))$$
$$=-\frac{x_0}{1+x_0t_r}\sum_{\sigma\in\mathfrak{S}_{r-1}}y(\alpha_{r-1}^\sigma)\cdots y(\alpha_2^\sigma)y(\alpha_1^\sigma)$$

$$= -\sum_{\sigma\in\mathfrak{S}_{r-1}} \frac{x_0}{1+x_0t_r} y(\alpha_{r-1})y(\alpha_{r-2}^\sigma)\cdots y(\alpha_2^\sigma)y(\alpha_1^\sigma).$$

Since

$$\begin{aligned}\frac{x_0}{1+x_0t_r}y(\alpha_{r-1}) &= -\frac{1}{\alpha_r}\left(\frac{1}{1+x_0t_r}x_1 - \frac{1}{1-x_0\alpha_{r-1}}x_1\right)\\ &= -\frac{1}{\alpha_r}(y(-t_r)-y(\alpha_{r-1})),\end{aligned}$$

again by Proposition 4.8 it can be reduced to

$$\begin{aligned}&= \frac{1}{\alpha_r}\sum_{\sigma\in\mathfrak{S}_{r-1}} y(-t_r)y(\alpha_{r-2}^\sigma)\cdots y(\alpha_2^\sigma)y(\alpha_1^\sigma)\\ &- \frac{1}{\alpha_r}\sum_{\sigma\in\mathfrak{S}_{r-1}} y(\alpha_{r-1}^\sigma)y(\alpha_{r-2}^\sigma)\cdots y(\alpha_2^\sigma)y(\alpha_1^\sigma)\\ &= \frac{1}{\alpha_r}\sum_{a\in[r-1]} y(-t_r)S_{[r-1]\setminus\{a\}}(t_1,\ldots,t_{r-1}) - \frac{1}{\alpha_r}S_{[r-1]}(t_1,\ldots,t_{r-1})\\ &= \frac{1}{\alpha_r}\sum_{a\in[r-1]} y(-t_r)S_{[r]\setminus\{a,r\}}(t_1,\ldots,t_r) - \frac{1}{\alpha_r}S_{[r]\setminus\{r\}}(t_1,\ldots,t_r).\end{aligned}$$

Applying ϕ, Proposition 4.9 for the case $u=-t_r$ and $I=[r]\setminus\{a,r\}$ gives

$$\begin{aligned}&\phi\left(y(-t_r)S_{[r]\setminus\{a,r\}}(t_1,\ldots,t_r)\right)\\ &= y(-t_r)S_{[r]\setminus\{a,r\}}(t_1,\ldots,t_r) - y(t_r)\sqcup S_{[r]\setminus\{a,r\}}(t_1,\ldots,t_r)\\ &+ \sum_{b\in[r-1]\setminus\{a\}} y(-t_b)\left(y(t_r)\sqcup S_{[r]\setminus\{a,b,r\}}(t_1,\ldots,t_r)\right)\\ &= y(-t_r)S_{[r]\setminus\{a,r\}}(t_1,\ldots,t_r) - S_{[r]\setminus\{a\}}(t_1,\ldots,t_r)\\ &+ \sum_{b\in[r-1]\setminus\{a\}} y(-t_b)S_{[r]\setminus\{a,b\}}(t_1,\ldots,t_r)\\ &= \sum_{b\in[r]\setminus\{a\}} y(-t_b)S_{[r]\setminus\{a,b\}}(t_1,\ldots,t_r) - S_{[r]\setminus\{a\}}(t_1,\ldots,t_r).\end{aligned}$$

Likewise, Proposition 4.10 for the case $I=[r]\setminus\{r\}$ shows

$$\phi\left(S_{[r]\setminus\{r\}}(t_1,\ldots,t_r)\right) = S_{[r]\setminus\{r\}}(t_1,\ldots,t_r) - \sum_{b\in[r]\setminus\{r\}} y(-t_b)S_{[r]\setminus\{b,r\}}(t_1,\ldots,t_r).$$

Therefore we have

$$\begin{aligned}
\phi\big(J(t_1,\dots,t_r)\big) &= \frac{1}{\alpha_r}\sum_{a\in[r-1]}\left(\sum_{b\in[r]\setminus\{a\}} y(-t_b)S_{[r]\setminus\{a,b\}}(t_1,\dots,t_r) - S_{[r]\setminus\{a\}}(t_1,\dots,t_r)\right)\\
&\quad - \frac{1}{\alpha_r}\left(S_{[r]\setminus\{r\}}(t_1,\dots,t_r) - \sum_{b\in[r]\setminus\{r\}} y(-t_b)S_{[r]\setminus\{b,r\}}(t_1,\dots,t_r)\right)\\
&= \frac{1}{\alpha_r}\sum_{a\in[r]}\left(\sum_{b\in[r]\setminus\{a\}} y(-t_b)S_{[r]\setminus\{a,b\}}(t_1,\dots,t_r) - S_{[r]\setminus\{a\}}(t_1,\dots,t_r)\right)\\
&= \sum_{a=1}^{r} J(\underbrace{t_1,\dots,t_{a-1}}_{a-1},\underbrace{t_{a+1},\dots,t_r}_{r-a},t_a),
\end{aligned}$$

which proves (4.8). We complete the proof. $\square$

4.5. **Remark on the value $\Omega(\mathbf{k})$.** In the proof of (1.4), we actually proved the equality

$$\Omega(\mathbf{k}) = (-1)^{k_r}\zeta_{\mathcal{S}}^{\sqcup\!\sqcup}(x_0^{k_r}(y_{k_1}\sqcup\!\sqcup\cdots\sqcup\!\sqcup y_{k_{r-1}})) \tag{4.9}$$

for any index $\mathbf{k}=(k_1,\dots,k_r)\in\mathbb{N}^r$. Accordingly, relations of $\Omega(\mathbf{k})$ give rise to relations of $\zeta_{\mathcal{S}}^{\sqcup\!\sqcup}(\mathbf{k})$ (without taking modulo $\zeta(2)$). For example, from (1.2) and [16, (5)], we have

$$\Omega(\{1\}^k) = -k\zeta^{MT}(\{1\}^{k-1};1) = -k!\zeta(k), \tag{4.10}$$

where $\{1\}^k$ means a sequence of 1 repeated k times. On the other hand, it follows that

$$\zeta_{\mathcal{S}}^{\sqcup\!\sqcup}(x_0(\underbrace{y_1\sqcup\!\sqcup\cdots\sqcup\!\sqcup y_1}_{k-1})) = (k-1)!\zeta_{\mathcal{S}}^{\sqcup\!\sqcup}(2,\{1\}^{k-2}).$$

Therefore, we obtain

$$\zeta_{\mathcal{S}}^{\sqcup\!\sqcup}(2,\{1\}^{k-2}) = k\zeta(k).$$

Similarly to Conjecture 2.4, we observed that all multiple zeta value up to weight 12 can be written as $\mathbb{Q}$-linear combinations of $\Omega(\mathbf{k})$'s.

Conjecture 4.11. *The space $\mathcal{Z}$ is generated by the set $\{\Omega(\mathbf{k})\mid \mathbf{k}: index\}$.*

We further remark that Ono, Seki and Yamamoto [19] introduced the 't-adic' symmetric Mordell-Tornheim multiple zeta value as a counterpart of Kamano's finite Mordell-Tornheim multiple zeta value. It is defined for an index $\mathbf{k}=(k_1,\dots,k_r,k_{r+1})$

by

$$\zeta^{MT}_{\widehat{\mathcal{S}}}(\mathbf{k};t)=\lim_{M\to\infty}\sum_{i=1}^{r+1}\zeta^{MT}_{\widehat{\mathcal{S}},M}(i;\mathbf{k};t),$$

where for $1\le i\le r$ we set

$$\zeta^{MT}_{\widehat{\mathcal{S}},M}(i;\mathbf{k};t)=\sum_{\substack{m_1,\dots,m_{i-1}>0\\ m_{i+1},\dots,m_{r+1}>0\\ M_{i-1}+M_{i+1,r+1}<M}}\frac{1}{(t-M_{i-1}-M_{i+1,r+1})^{k_i}(t-m_{r+1})^{k_{r+1}}}\prod_{\substack{j=1\\ j\neq i}}^{r}\frac{1}{m_j^{k_j}}$$

with $M_0=0$, $M_i=m_1+\cdots+m_i$ for $1\le i\le r+1$, $M_{i,j}=m_i+m_{i+1}+\cdots+m_j$ for $i\le j$ and $M_{i,j}=0$ if $i>j$, and also let

$$\zeta^{MT}_{\widehat{\mathcal{S}},M}(r+1;\mathbf{k};t)=\sum_{\substack{m_1,\dots,m_r>0\\ M_r<M}}\frac{1}{m_1^{k_1}\cdots m_r^{k_r}M_r^{k_{r+1}}}.$$

Letting $t\to 0$ (or taking the constant term), we see that

$$\zeta^{MT}_{\widehat{\mathcal{S}}}(\mathbf{k};0)=(-1)^{k_{r+1}}\sum_{a=1}^{r+1}(-1)^{k_a}\zeta^{MT}(k_1,\dots,k_{a-1},k_{a+1},\dots,k_{r+1};k_a).$$

Combining this with (1.2), we obtain another expression of $\Omega(\mathbf{k})$ as follows:

$$\Omega(k_1,\dots,k_r)=\lim_{M\to\infty}\sum_{i=1}^{r}(-1)^{k_r}\zeta^{MT}_{\mathcal{S},M}(i;k_1,\dots,k_r;0).$$

The authors were informed that in [19], they prove a truncated version of the formula

$$\sum_{a=1}^{r}(-1)^{k_a}\zeta^{MT}(\underbrace{k_1,\dots,k_{a-1}}_{a-1},\underbrace{k_{a+1},\dots,k_r}_{r-a};k_a)=(-1)^{k_r}\zeta^{\sqcup\!\sqcup}_{\mathcal{S}}(x_0^{k_r}(y_{k_1}\sqcup\!\sqcup\cdots\sqcup\!\sqcup\, y_{k_{r-1}})),$$

which provides an alternative proof of (4.9).

5. Further remarks

5.1. Relations of $\omega_n(\mathbf{k};\zeta_n)$. In this subsection, we consider relations among the values $\omega_n(\mathbf{k};\zeta_n)$ which are valid for all n, as well as its application to relations among finite and symmetric multiple omega values.

For that purpose we consider the $\mathbb{Q}$-algebra

$$\mathcal{R}=\prod_{n\geq 1}\mathbb{Q}(\zeta_n).$$

In this $\mathbb{Q}$-algebra, we will count the number of linearly independent relations of the form

$$\sum_{\substack{\mathbf{k}:\text{index}\\ m\geq 0}} a_{\mathbf{k}}^{(m)}(1-\zeta_n)^m\omega_n(\mathbf{k};\zeta_n)=0 \quad (a_{\mathbf{k}}^{(m)}\in\mathbb{Q}) \tag{5.1}$$

which holds for all n. It is apparent that the relation (5.1) induces relations

$$\sum_{\mathbf{k}:\text{index}} a_{\mathbf{k}}^{(0)}\omega_{\mathcal{A}}(\mathbf{k})=0 \quad\text{and}\quad \sum_{\mathbf{k}:\text{index}} a_{\mathbf{k}}^{(0)}\Omega(\mathbf{k})=0,$$

by applying Theorems 1.2 and 1.3. Hence, finding relations of the form (5.1) will be a first step toward to capture all relations among finite and symmetric multiple omega values (note that the coefficient $a_{\mathbf{k}}^{(m)}$ could be a polynomial in n over $\mathbb{Q}$, but we get rid of such relations below). For an index $\mathbf{k}$, we then consider a universal object

$$\omega(\mathbf{k})=\left(\omega_n(\mathbf{k};e^{\frac{2\pi i}{n}})\right)_{n\geq 1}\in\mathcal{R}\,,$$

which we call a cyclotomic multiple omega value.

We give a family of relations among the cyclotomic multiple omega values. Set $\zeta=\left(\zeta_n\right)_{n\geq 1}\in\mathcal{R}$.

Theorem 5.1. *Suppose that $k_1,\dots,k_r\geq 2$. Then it holds that*

$$\sum_{j=1}^{r}\sum_{\substack{l_1,\dots,l_{j-1}\geq 2\\ l_j,\dots,l_r\geq 1}}\left\{\prod_{p=1}^{j-1}\binom{k_p-2}{l_p-2}\right\}\binom{k_j-2}{l_j-1}\left\{\prod_{p=j+1}^{r}\binom{k_p-1}{l_p-1}\right\}$$
$$\times(1-\zeta)^{\sum_{p=1}^{r}(k_p-l_p)-1}\omega(l_1,\dots,l_r)=0.$$

Proof. We show that for all $n\geq 1$ and primitive roots of unity ζ_n we have

$$\sum_{j=1}^{r}\sum_{\substack{l_1,\dots,l_{j-1}\geq 2\\ l_j,\dots,l_r\geq 1}}\left\{\prod_{p=1}^{j-1}\binom{k_p-2}{l_p-2}\right\}\binom{k_j-2}{l_j-1}\left\{\prod_{p=j+1}^{r}\binom{k_p-1}{l_p-1}\right\}$$
$$\times(1-\zeta_n)^{\sum_{p=1}^{r}(k_p-l_p)-1}\omega_n(l_1,\dots,l_r;\zeta_n)=0. \tag{5.2}$$

For $m_1,\dots,m_r\geq 1$ it holds that

$$\sum_{j=1}^{r}q^{m_1+\cdots+m_{j-1}}[m_j]=[m_1+\cdots+m_r].$$

Suppose that $\sum_{j=1}^r m_j = n$. Divide the both sides with $\prod_{p=1}^r [m_p]^{k_p}$ and set $q = \zeta_n$. Then we have

$$\sum_{j=1}^r \prod_{p=1}^{j-1} \frac{q^{m_p}}{[m_p]^{k_p}} \frac{1}{[m_j]^{k_j-1}} \prod_{p=j+1}^r \frac{1}{[m_p]^{k_p}} \bigg|_{q=\zeta_n} = 0.$$

Now the desired equality follows from

$$\frac{q^m}{[m]^k} = \sum_{l\geq 2} \binom{k-2}{l-2} (1-q)^{k-l} F_l(m) \qquad (k \geq 2),$$

$$\frac{1}{[m]^k} = \sum_{l\geq 1} \binom{k-1}{l-1} (1-q)^{k-l} F_l(m) \qquad (k \geq 1).$$

We complete the proof. □

For example, taking $k_1 = k_2 = \cdots = k_r = 2$ we obtain for all $r \geq 2$

$$\sum_{j=1}^r \binom{r}{j} (1-\zeta)^{j-1} \omega(\{2\}^{r-j}, \{1\}^j) = 0\,. \tag{5.3}$$

Using Theorems 1.2 and 1.3, we obtain the following.

Corollary 5.2. *Suppose that $k_1, \ldots, k_r \geq 2$. Then we have*

$$\sum_{j=1}^r \omega_{\mathcal{A}}(k_1, \ldots, k_j - 1, \ldots, k_r) = 0,$$

$$\sum_{j=1}^r \Omega(k_1, \ldots, k_j - 1, \ldots, k_r) = 0. \tag{5.4}$$

Proof. The first equation is an immediate from (5.2) and Theorem 1.2. For the second equation, set $\zeta_n = e^{2\pi i/n}$ in (5.2), and take the limit $n \to 0$. Then the terms which satisfy $l_p = k_p\,(p \neq j)$ and $l_j = k_j - 1$ only remain, and by Theorem 1.3 we get the desired formula. □

Remark that the relation (5.4) can be also deduced from the following relations of the Mordell-Tornheim multiple zeta values. For $k_1, \ldots, k_r, l \geq 2$ we have

$$\begin{aligned} \zeta^{MT}(k_1, \ldots, k_r; l-1) &= \sum_{m_1, \ldots, m_r \geq 1} \frac{m_1 + \cdots + m_r}{m_1^{k_1} \cdots m_r^{k_r} (m_1 + \cdots + m_r)^l} \\ &= \sum_{a=1}^r \zeta^{MT}(k_1, \ldots, k_a - 1, \ldots, k_r; l), \end{aligned}$$

which is found in the literature (see e.g. the proof of [20, Proposition 4.16]).

5.2. **Observation on cyclotomic multiple omega values.** In this subsection, we work on the $\mathbb{Q}$-vector subspace $\mathcal{Z}_k^\omega$ of $\mathcal{R}$ defined for $k \geq 1$ by

$$\mathcal{Z}_k^\omega = \left\langle (1-\zeta)^m \omega(k_1, \dots, k_r) \mid 0 \leq m < k,\ 2 \leq r \leq k,\ k_1 + \cdots + k_r + m = k \right\rangle_{\mathbb{Q}}.$$

In the following we discuss the relations among the generators of $\mathcal{Z}_k^\omega$ and its dimension. From Theorems 1.2 and 1.3, one obtains two well-defined $\mathbb{Q}$-linear maps $\varphi_k^\bullet : \mathcal{Z}_k^\omega \to \mathcal{Z}_k^{\bullet,\omega}$ given for $\bullet \in \{\mathcal{A}, \mathcal{S}\}$ and each generator by

$$\varphi_k^\bullet\big((1-\zeta)^m \omega(\mathbf{k})\big) = \begin{cases} \omega_\bullet(\mathbf{k}) & \text{if } m = 0 \\ 0 & \text{if } m > 0 \end{cases}.$$

By definition these maps are surjective, so we get

$$\begin{array}{ccccc} & & \mathcal{Z}_k^\omega & & \\ & \overset{\varphi_k^{\mathcal{A}}}{\swarrow} & & \overset{\varphi_k^{\mathcal{S}}}{\searrow} & \\ \mathcal{Z}_k^{\mathcal{A},\omega} & & \overset{\varphi_k}{\dashrightarrow} & & \mathcal{Z}_k^{\mathcal{S},\omega} \end{array},$$

where the map $\varphi_k : \mathcal{Z}_k^{\mathcal{A},\omega} \to \mathcal{Z}_k^{\mathcal{S},\omega}$ denotes the conjectured isomorphism of Conjecture 1.5. This picture suggests that $\ker \varphi_k^{\mathcal{A}} \overset{?}{=} \ker \varphi_k^{\mathcal{S}}$. Clearly, we have $(1-\zeta)\mathcal{Z}_{k-1}^\omega \subset \ker \varphi_k^\bullet$ for $\bullet \in \{\mathcal{A}, \mathcal{S}\}$, but, besides these, there are more elements in $\ker \varphi_k^\bullet$, since Theorem 5.1 gives relations among elements in $\mathcal{Z}_k^\omega$. Numerically, we get the following relations.

weight 3:

$$\omega(2,1) = -\frac{1}{2}(1-\zeta)\omega(1,1).$$

weight 4:

$$\begin{aligned} \omega(3,1) &\overset{?}{=} \omega(2,1,1) + \frac{1}{2}(1-\zeta)\omega(1,1,1) + \frac{1}{4}(1-\zeta)^2\omega(1,1), \\ \omega(2,2) &\overset{?}{=} -\omega(2,1,1) - \frac{1}{2}(1-\zeta)\omega(1,1,1) + \frac{1}{4}(1-\zeta)^2\omega(1,1). \end{aligned}$$

weight 5:

$$\begin{aligned} \omega(4,1) &\overset{?}{=} -\frac{3}{2}(1-\zeta)\omega(2,1,1) - \frac{3}{4}(1-\zeta)^2\omega(1,1,1) - \frac{1}{8}(1-\zeta)^3\omega(1,1), \\ \omega(3,2) &\overset{?}{=} \frac{1}{2}(1-\zeta)\omega(2,1,1) + \frac{1}{4}(1-\zeta)^2\omega(1,1,1) - \frac{1}{8}(1-\zeta)^3\omega(1,1), \\ \omega(2,2,1) &= -(1-\zeta)\omega(2,1,1) - \frac{1}{3}(1-\zeta)^2\omega(1,1,1). \end{aligned}$$

Notice that the first and the last relation are consequences of (5.3). Remember that $\omega(k_{\sigma(1)},\dots,k_{\sigma(r)}) = \omega(k_1,\dots,k_r)$ holds for any permutation $\sigma \in \mathfrak{S}_r$. The above relations conjecturally give all relations in each weight. Doing numerical calculation until weight 11, we get the following table:

k	1	2	3	4	5	6	7	8	9	10	11
$\dim \mathcal{Z}_k^\omega \overset{?}{=}$	0	1	2	4	7	12	19	30	45	68	99
$\dim \mathcal{Z}_k^\omega/(1-\zeta)\mathcal{Z}_{k-1}^\omega \overset{?}{=}$	0	1	1	2	3	5	7	11	15	23	31

Notice that the numbers in the second row are given by the number of partitions of $k-2$ until weight $k=9$. Until this weight it seems that the elements of the form $(1-\zeta)^m\omega(k_1,\dots,k_r,1,1)$ with $m \geq 0$, $r \geq 0$, $k_1 \geq \dots \geq k_r \geq 1$ and $k_1+\cdots+k_r+m = k-2$ form a basis. But starting in weight 10 there seem to be elements, which are not linear combinations of these.

In contrast to the $z_n(\mathbf{k};q)$ introduced in [1] (see (4.3)) the algebraic structure of the cyclotomic multiple omega values is not known yet. We are expecting that $\mathcal{Z}_{k_1}^\omega \mathcal{Z}_{k_2}^\omega \subset \mathcal{Z}_{k_1+k_2}^\omega$ holds for $k_1, k_2 \geq 2$. Namely, the space $\mathcal{Z}^\omega = \sum_{k\geq 0} \mathcal{Z}_k^\omega$ forms a $\mathbb{Q}$-subalgebra of $\mathcal{R}$, where $\mathcal{Z}_0^\omega = \mathbb{Q}$. For example, one can check numerically that

$$\begin{aligned} \omega(1,1)^2 &\overset{?}{=} -5\omega(2,1,1) - \frac{5}{2}(1-\zeta)\omega(1,1,1) + \frac{1}{4}(1-\zeta)^2\omega(1,1)\,, \\ \omega(1,1)\omega(1,1,1) &\overset{?}{=} -2\omega(2,1,1,1) - 3\omega(3,1,1) - (1-\zeta)\omega(1,1,1,1) \\ &\quad - 3(1-\zeta)\omega(2,1,1) - \frac{1}{3}(1-\zeta)^2\omega(1,1,1)\,. \end{aligned} \tag{5.5}$$

Since $\varphi^\bullet_{\mathrm{wt}(\mathbf{k})+\mathrm{wt}(\mathbf{l})}(\omega(\mathbf{k})\omega(\mathbf{l})) = \omega_\bullet(\mathbf{k})\omega_\bullet(\mathbf{l})$ and $\omega_\bullet(1,1) = 0$ for $\bullet \in \{\mathcal{A},\mathcal{S}\}$ (see Section 5.3 below), the equations (5.5) would imply the relations

$$\omega_\bullet(2,1,1) \overset{?}{=} 0, \qquad 2\omega_\bullet(2,1,1,1) + 3\omega_\bullet(3,1,1) \overset{?}{=} 0\,.$$

5.3. **Special values.** In this subsection, we make a list of known values of finite and symmetric multiple omega values.

(i) From Theorem 1.6, for $\bullet \in \{\mathcal{A},\mathcal{S}\}$ and $k_1,k_2 \geq 1$ we obtain $\omega_\bullet(k_1,k_2) = (-1)^{k_2}\zeta_\bullet(k_1+k_2)$. Since $\zeta_\bullet(k) = 0$, we see that

$$\omega_\bullet(k_1,k_2) = 0.$$

(ii) Letting $k_1 = \cdots = k_r = k \geq 2$ in Corollary 5.2, we obtain

$$\omega_\bullet(\{k\}^{r-1}, k-1) = 0.$$

(iii) By (4.10), for $k \geq 2$ we see that

$$\omega_{\mathcal{S}}(\{1\}^k) \equiv -k!\zeta(k) \mod \zeta(2)\mathcal{Z}.$$

This also follows from (1.4) and a special case of the sum formula for symmetric multiple zeta values by Murahara [17, Theorem 1.2]. Likewise, from (1.3) and a special case of the sum formula for finite multiple zeta values by Saito and Wakabayashi [23], for $k \geq 2$ we have

$$\omega_{\mathcal{A}}(\{1\}^k) = -k!Z(k),$$

where $Z(k)$ is the 'true' analogue of $\zeta(k)$ in $\mathcal{A}$ defined for $k \geq 1$ by

$$Z(k) = \left(\frac{B_{p-k}}{k} \mod p\right)_p \in \mathcal{A} \qquad (B_{p-k} : \text{Bernoulli number}).$$

So far, we were not able to find explicit formulas for $\omega_\bullet(\{a\}^k)$ (note that $\zeta_\bullet(\{a\}^k) = 0$ for any $a, k \in \mathbb{N}$ and $\bullet \in \{\mathcal{A}, \mathcal{S}\}$).

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