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The hypergeometric function, the confluent hypergeometric function and WKB solutions

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Abstract. Relations between the hypergeometric function with a large parameter and Borel sums of WKB solutions of the hypergeometric differential equation with the large parameter are established. The confluent hypergeometric function is also investigated from the viewpoint of exact WKB analysis. As applications, asymptotic expansion formulas for those classical special functions with respect to parameters are obtained.

1. Introduction

In this article, we investigate the classical hypergeometric function ${}_2F_1(a, b, c; x)$ and the confluent hypergeometric function ${}_1F_1(a, c; x)$ from the viewpoint of the exact WKB analysis [2, 13]. It is well known that the hypergeometric function is a solution of the hypergeometric differential equation. There are 24 standard solutions of this equation which are called Kummer's solutions [9]. The hypergeometric function is one of them and others are expressed in terms of it. Linear relations that hold among these solutions are completely known. By using these relations, connection problems for the hypergeometric function can be solved and global properties of the hypergeometric function in the independent variable are obtained. We also know much about the confluent hypergeometric function, which is a standard solution of the confluent hypergeometric differential equation.

Introducing a large parameter in the parameters contained in these equations, we can construct WKB solutions of these equations. It is known that the WKB solutions normalized appropriately are Borel summable [15, 27] and the Borel sums are analytic solutions to the equations. The aim of this article is to give the relations which hold between standard analytic solution bases and the Borel resummed WKB solutions of the equations. This work is based on our previous paper [6] and on the papers [7, 8] by the first and the third authors. Some results given in this article were announced without proofs in [3, 4, 5].

The plan of this paper is as follows. In Section 2, we give a short review of the exact WKB analysis for second-order ordinary differential equations with the large parameter. In Section 3, we recall the definition of WKB solutions normalized at a regular singular point for the equations and the relations which hold between Borel resummed recessive WKB solutions and Frobenius solutions. Section 4 is devoted to investigate the hypergeometric differential equation with a large parameter from the exact WKB theoretic viewpoint. We give the linear relations between the hypergeometric function and the Borel resummed WKB solutions. Similar results can be obtained for the confluent hy-

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pergeometric differential equation. They are given in Section 5. In Section 6, we see that the results obtained in Sections 4 and 5 are consistent with the procedure of confluence and that further confluence to the Bessel equation and the Hermite-Weber equation are also consistent with known results obtained by [25, 26]. In the last section, we give applications of our results to asymptotic expansion formulas of the hypergeometric function and the confluent hypergeometric function with respect to the large parameter.

2. A brief review of exact WKB analysis

We give a brief review of the exact WKB analysis for second-order linear ordinary differential equations with a large parameter η (see [2, 13]). We consider the following linear ordinary differential equation of second order in the complex domain:

$$\left(-\frac{d^2}{dx^2}+\eta^2 Q\right)\psi=0. \tag{2.1}$$

We assume that Q has the form

$$Q=Q(x,\eta)=\sum_{j=0}^{N}\eta^{-j}Q_j(x)$$

for some $N\in\mathbb{N}\cup\{0\}$ and that

(i) Q_j $(j=0,1,\ldots,N)$ are rational functions of x.

(ii) If we write $Q_0(x)=F(x)/G(x)$ with mutually coprime polynomials F and G, then $G(x)Q_j(x)$ $(j=1,2,\ldots,N)$ are polynomials.

A WKB solution ψ of (2.1) is a formal solution of the form

$$\psi=\exp\left(\int S(x,\eta)dx\right).$$

Here $S=\displaystyle\sum_{j=-1}^{\infty}\eta^{-j}S_j$ is a formal solution to the Riccati equation

$$\frac{dS}{dx}+S^2=\eta^2 Q$$

associated with (2.1). The coefficients S_j of η^{-j} of S are determined by the recurrence relations

$$\begin{cases} S_{-1}^2=Q_0, \\ S_{j+1}=-\dfrac{1}{2S_{-1}}\left(\dfrac{dS_j}{dx}+\displaystyle\sum_{k=0}^{j}S_{j-k}S_k-Q_{j+2}\right), \quad j=-1,0,1,2,\ldots. \end{cases}$$

Here we set $Q_j = 0$ for $j > N$. According to the choice of the branch of $S_{-1} = S_{-1}^{(\pm)} = \pm\sqrt{Q_0}$, we have two formal solutions

$$S^{(\pm)} = \sum_{j=-1}^{\infty} \eta^{-j} S_j^{(\pm)}$$

which are defined as far as $Q_0(x) \neq 0$. We set

$$S_{\mathrm{odd}} := \frac{1}{2}(S^{(+)} - S^{(-)}) = \sum_{j=-1}^{\infty} \eta^{-j} S_{\mathrm{odd},\, j},$$
$$S_{\mathrm{even}} := \frac{1}{2}(S^{(+)} + S^{(-)}) = \sum_{j=0}^{\infty} \eta^{-j} S_{\mathrm{even},\, j}.$$

Then we have $S^{(\pm)} = \pm S_{\mathrm{odd}} + S_{\mathrm{even}}$ and

$$S_{\mathrm{even}} = -\frac{1}{2}\frac{d}{dx}\log S_{\mathrm{odd}}.$$

Hence we can take $-1/2 \log S_{\mathrm{odd}}$ as a primitive function of S_{even}. A zero point or a simple pole of Q_0 is called a turning point of (2.1). A turning point τ is said to be simple if it is a simple zero of Q_0. A Stokes curve emanating from a turning point τ is a curve defined by

$$\mathrm{Im} \int_\tau^x \sqrt{Q_0(x)}dx = 0.$$

A Stokes region is, by definition, a connected component of the set obtained by excluding all the Stokes curves, turning points and singular points from the Riemann sphere. Let τ be a simple turning point of (2.1). WKB solutions normalized at the simple turning point τ are defined by

$$\psi_\pm = \frac{1}{\sqrt{S_{\mathrm{odd}}}} \exp\left(\pm \int_\tau^x S_{\mathrm{odd}} dx\right).$$

Here the integration is understood as a half of the integral of $\sqrt{Q_0}$ on a contour starting from x in the second sheet of the Riemann surface of $\sqrt{Q_0}$, going straight to τ (avoiding singular points) and going around it counterclockwise and back to x in the first sheet. This depends on the choice of the contour and we specify it if necessary. There are three Stokes curves emanating from τ. We take a small open disk with the center at τ and consider the intersection of the disk and the complement of the Stokes curves. This set has three connected components. We take two of them and label them as I and II. Let s be a portion of the Stokes curve which is a subset of the boundary of the regions I and II. We may suppose that the region I is on the right side of s when we look at it from the turning point. If all turning points are simple and any Stokes curve flows into some singular point, $\psi_\pm$ are Borel summable in each connected component. This fact is established by Koike and Schäfke [15] (see [27] also). They assume that Q is a polynomial, however, their proofs work in our case with slight modifications. Let $\Psi_\pm^{\mathrm{I}}$ and $\Psi_\pm^{\mathrm{II}}$ denote the Borel sums of $\psi_\pm$ in I and II, respectively. If $\mathrm{Re} \int_\tau^x \sqrt{Q_0(x)}dx > 0$ on s, we say that ψ_+ is dominant and ψ_- recessive on s. In this case, we have the following connection formula of Voros ([1, 13, 30]):

$$\begin{cases} \Psi^{\mathrm{I}}_+ = \Psi^{\mathrm{II}}_+ + i\,\Psi^{\mathrm{II}}_-, \\ \Psi^{\mathrm{I}}_- = \Psi^{\mathrm{II}}_-. \end{cases} \tag{2.2}$$

In the case where $\operatorname{Re}\int_\tau^x \sqrt{Q_0(x)}dx < 0$ on s, we exchange the signs "+" and "−":

$$\begin{cases} \Psi^{\mathrm{I}}_+ = \Psi^{\mathrm{II}}_+, \\ \Psi^{\mathrm{I}}_- = i\Psi^{\mathrm{II}}_+ + \Psi^{\mathrm{II}}_-. \end{cases} \tag{2.3}$$

In this case, we say that ψ_+ is recessive and ψ_- dominant on s.

3. Regular singular points

We keep the notation and assumptions in §2. We further assume that there exists a point $r \in \mathbb{C}$ such that $(x-r)^2Q_0(x)$ is holomorphic at $x = r$ and

$$(x-r)^2Q_0(x)\big|_{x=r} \neq 0.$$

Then $x = r$ is a regular singular point of (2.1).

3.1. WKB solutions normalized at a regular singular point

Let $\rho = \rho_0 + \eta^{-1}\rho_1 + \eta^{-2}\rho_2 + \cdots$ denote the residue of $\sqrt{Q}\,dx$ at $x = r$:

$$\rho = \operatorname*{Res}_{x=r}\sqrt{Q}\,dx$$

and

$$\rho_0 = \operatorname*{Res}_{x=r}\sqrt{Q_0}\,dx.$$

Here the branch of $\sqrt{Q_0}$ is suitably chosen. It follows from [13, Proposition 3.6] that $S_{\mathrm{odd}}dx$ has a simple pole at $x = r$ and the residue is written in the form

$$\operatorname*{Res}_{x=r} S_{\mathrm{odd}}dx = \sigma\eta,$$

where we set

$$\sigma = \rho\sqrt{1 + \frac{1}{4\rho^2\eta^2}}.$$

Definition 3.1. ([4, Definition 1.1]) We set

$$\psi_\pm^{(r)} := \frac{(x-r)^{\pm\sigma\eta}}{\sqrt{S_{\mathrm{odd}}}}\exp\left(\pm\int_r^x\left(S_{\mathrm{odd}} - \frac{\sigma\eta}{x-r}\right)dx\right)$$

and call $\psi_\pm^{(r)}$ the WKB solutions of (2.1) normalized at the regular singular point $x = r$.

Note that dominance of $\psi_\pm^{(r)}$ on the Stokes curves flowing to r can be also specified and it is the same as that of $\psi_\pm$. For example, if $\operatorname{Re}\rho_0 > 0$, then $\psi_-^{(r)}$ is dominant on each Stokes curve flowing into r.

3.2. Recessive WKB solutions

In the sequel, we assume $\operatorname{Re}\rho_0 \neq 0$. We take the branch of $\sqrt{Q_0}$ so that $\operatorname{Re}\rho_0 > 0$ holds. Then $\psi_+^{(r)}$ is recessive on any Stokes curve flowing into r. Hence the connection formulas (2.3) imply that no Stokes phenomenon occurs for $\psi_+^{(r)}$ on any Stokes curve flowing into r. This fact and some observations concerning the radius of convergence of the Borel transform imply

THEOREM 3.2. ([4, Theorem 1.2]) *We set*

$$\tilde{\psi}_+^{(r)} = (x-r)^{-\frac{1}{2}-\sigma\eta}\psi_+^{(r)}.$$

Then there is an open neighborhood U of r such that $\tilde{\psi}_+^{(r)}$ is Borel summable for any $x \in U - \{r\}$ provided η is sufficiently large and $x = r$ is a removable singularity of the Borel sum $\tilde{\Psi}_+^{(r)}$ of $\tilde{\psi}_+^{(r)}$. Moreover, $\tilde{\psi}_+^{(r)}(x,\eta)$ can be evaluated at $x = r$ and

$$\tilde{\Psi}_+^{(r)}(r,\eta) = \tilde{\psi}_+^{(r)}(r,\eta) = (\sigma\eta)^{-\frac{1}{2}}$$

hold.

The Borel sum $\Psi_+^{(r)}$ of $\psi_+^{(r)}$ has the form

$$\Psi_+^{(r)} = (x-r)^{\frac{1}{2}+\sigma\eta}\tilde{\Psi}_+^{(r)}.$$

On the other hand, there is a unique Frobenius solution $f_+(x)$ of (2.1) of the form

$$f_+(x) = (x-r)^{\frac{1}{2}+\sigma\eta}f_{+,0}(x)$$

such that $f_{+,0}$ is holomorphic near $x = r$ and that $f_{+,0}(r) = 1$ holds. Hence we have

THEOREM 3.3. *Under the notation as above, we have*

$$f_+ = (\sigma\eta)^{\frac{1}{2}}\Psi_+^{(r)}.$$

4. The hypergeometric differential equation

In this section, we consider the Gauss hypergeometric equation with parameters $a, b, c \in \mathbb{C}$:

$$x(1-x)\frac{d^2w}{dx^2} + (c-(a+b+1)x)\frac{dw}{dx} - abw = 0. \tag{4.1}$$

If (a,b,c) is generic, (4.1) has standard systems of fundamental solutions (u_1,u_5), (u_2,u_6) and (u_3,u_4) consisting of six of Kummer's 24 solutions ([9, 12]):

$$\begin{aligned}
u_1 &= {}_2F_1(a,b,c;x),\\
u_2 &= {}_2F_1(a,b,a+b+1-c;1-x),\\
u_3 &= (-x)^{-a}\,{}_2F_1(a,a+1-c,a+1-b;\frac{1}{x}),
\end{aligned}$$

$$u_4 = (-x)^{-b}\,{}_2F_1(b, b+1-c, b+1-a; \frac{1}{x}),$$
$$u_5 = x^{1-c}\,{}_2F_1(a+1-c, b+1-c, 2-c; x),$$
$$u_6 = (1-x)^{c-a-b}\,{}_2F_1(c-a, c-b, c-a-b+1; 1-x).$$

Here ${}_2F_1(a, b, c; x)$ denotes the hypergeometric function, which is defined by the hypergeometric series

$$\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)n!} x^n.$$

The linear relations between two of those bases are completely known (see [9], for example).

4.1. The hypergeometric differential equation with a large parameter

To investigate (4.1) from the viewpoint of the exact WKB analysis, we introduce a positive large parameter η by setting

$$a = \alpha_0 + \alpha\eta,$$
$$b = \beta_0 + \beta\eta,$$
$$c = \gamma_0 + \gamma\eta.$$

Here α, β, γ, α_0, β_0, γ_0 are complex numbers. Throughout this paper, we assume that a, b, $c \notin \mathbb{Z}$. The explicit form of (4.1) becomes

$$x(1-x)\frac{d^2w}{dx^2} + (\gamma_0 + \gamma\eta - (\alpha_0 + \beta_0 + 1 + (\alpha+\beta)\eta)x)\frac{dw}{dx} - (\alpha_0 + \alpha\eta)(\beta_0 + \beta\eta)w = 0. \tag{4.2}$$

Since this expression is somewhat lengthy, we keep using a, b and c as far as possible. Introducing a new unknown function ψ by

$$w = x^{-\frac{c}{2}}(1-x)^{-\frac{1}{2}(a+b-c+1)}\psi,$$

we eliminate the first-order term of (4.1) and have an equation of the form of (2.1), namely

$$\left(-\frac{d^2}{dx^2} + \eta^2 Q\right)\psi = 0 \tag{4.3}$$

with

$$Q = \eta^{-2}\frac{((a-b)^2-1)x^2 + 2(2ab - bc - ca + 1)x + c(c-2)}{4x^2(1-x)^2}.$$

This is explicitly written as $Q = Q_0 + \eta^{-1}Q_1 + \eta^{-2}Q_2$ with

$$Q_0 = \frac{(\alpha-\beta)^2x^2 + 2(2\alpha\beta - \alpha\gamma - \beta\gamma)x + \gamma^2}{4x^2(1-x)^2},$$

$$
\begin{aligned}
Q_1 &= \frac{1}{2x^2(1-x)^2}\{(\alpha-\beta)(\alpha_0-\beta_0)x^2 \\
&\qquad + (2(\alpha\beta_0+\alpha_0\beta)-\beta\gamma_0-\beta_0\gamma-\gamma\alpha_0-\gamma_0\alpha+\gamma)x+\gamma(\gamma_0-1)\}, \\
Q_2 &= \frac{1}{4x^2(1-x)^2}\{(\alpha_0-\beta_0+1)(\alpha_0-\beta_0-1)x^2 \\
&\qquad\qquad + 2(2\alpha_0\beta_0-\beta_0\gamma_0-\gamma_0\alpha_0+\gamma_0)x+\gamma_0(\gamma_0-2)\}.
\end{aligned}
$$

Note that Q is invariant under the involutions $\iota_m (m = 0, 1, \ldots, 6)$ on the space of parameters defined by

$$
\begin{aligned}
\iota_0 :&(a,b,c) \mapsto (1-a, 1-b, 2-c), \\
\iota_1 :&(a,b,c) \mapsto (c-b, c-a, c), \\
\iota_2 :&(a,b,c) \mapsto (b, a, c), \\
\iota_3 = \iota_1\iota_2 :&(a,b,c) \mapsto (c-a, c-b, c), \\
\iota_4 = \iota_0\iota_2 :&(a,b,c) \mapsto (1-b, 1-a, 2-c), \\
\iota_5 = \iota_0\iota_1 :&(a,b,c) \mapsto (1+b-c, 1+a-c, 2-c), \\
\iota_6 = \iota_0\iota_1\iota_2 :&(a,b,c) \mapsto (1+a-c, 1+b-c, 2-c).
\end{aligned}
$$

Coefficients of ι_m with respect to powers of η induce symmetries of Q_j $(j = 0, 1, 2)$, which are also denoted by ι_m:

$$
\iota_0 : \begin{cases} (\alpha,\beta,\gamma) \mapsto (-\alpha,-\beta,-\gamma), \\ (\alpha_0,\beta_0,\gamma_0) \mapsto (1-\alpha_0, 1-\beta_0, 2-\gamma_0), \end{cases}
$$

$$
\iota_1 : \begin{cases} (\alpha,\beta,\gamma) \mapsto (\gamma-\beta,\gamma-\alpha,\gamma), \\ (\alpha_0,\beta_0,\gamma_0) \mapsto (\gamma_0-\beta_0, \gamma_0-\alpha_0, \gamma_0), \end{cases}
$$

$$
\iota_2 : \begin{cases} (\alpha,\beta,\gamma) \mapsto (\beta,\alpha,\gamma), \\ (\alpha_0,\beta_0,\gamma_0) \mapsto (\beta_0, \alpha_0, \gamma_0), \end{cases}
$$

$$
\iota_3 : \begin{cases} (\alpha,\beta,\gamma) \mapsto (\gamma-\alpha,\gamma-\beta,\gamma), \\ (\alpha_0,\beta_0,\gamma_0) \mapsto (\gamma_0-\alpha_0, \gamma_0-\beta_0, \gamma_0), \end{cases}
$$

$$
\iota_4 : \begin{cases} (\alpha,\beta,\gamma) \mapsto (-\beta,-\alpha,-\gamma), \\ (\alpha_0,\beta_0,\gamma_0) \mapsto (1-\beta_0, 1-\alpha_0, 2-\gamma_0), \end{cases}
$$

$$
\iota_5 : \begin{cases} (\alpha,\beta,\gamma) \mapsto (\beta-\gamma,\alpha-\gamma,-\gamma), \\ (\alpha_0,\beta_0,\gamma_0) \mapsto (1+\beta_0-\gamma_0, 1+\alpha_0-\gamma_0, 2-\gamma_0), \end{cases}
$$

$$\iota_6 : \begin{cases} (\alpha,\beta,\gamma) \mapsto (\alpha-\gamma, \beta-\gamma, -\gamma), \\ (\alpha_0,\beta_0,\gamma_0) \mapsto (1+\alpha_0-\gamma_0, 1+\beta_0-\gamma_0, 2-\gamma_0). \end{cases}$$

We define the sets E_j $(j = 0, 1, 2)$ by

$$\begin{aligned} E_0 &= \{(\alpha,\beta,\gamma) \in \mathbb{C}^3 \mid \alpha\,\beta\,\gamma\,(\alpha-\beta)(\alpha-\gamma)(\beta-\gamma)(\alpha+\beta-\gamma) = 0\}, \\ E_1 &= \{(\alpha,\beta,\gamma) \in \mathbb{C}^3 \mid \operatorname{Re}\alpha \operatorname{Re}\beta \operatorname{Re}(\gamma-\alpha)\operatorname{Re}(\gamma-\beta) = 0\}, \\ E_2 &= \{(\alpha,\beta,\gamma) \in \mathbb{C}^3 \mid \operatorname{Re}(\alpha-\beta)\operatorname{Re}(\alpha+\beta-\gamma)\operatorname{Re}\gamma = 0\}. \end{aligned}$$

If $(\alpha,\beta,\gamma) \notin E_0$, there are two distinct simple turning points τ_0, τ_1 of (4.3) which do not coincide with $0, 1, \infty$. Hereafter we assume that a, b and c satisfy $(\alpha,\beta,\gamma) \notin E_1 \cup E_2$ (hence $(\alpha,\beta,\gamma) \notin E_0$). Then there are no Stokes curves which connect turning point(s). Hence all the Stokes curves flow into some singular point.

To apply the discussion of §3 to (4.3), we define S, S_{odd} and S_{even} for the equation as in §3. We also denote by $\psi_\pm$ and $\psi_\pm^{(0)}$ the WKB solutions of (4.3) normalized at τ_0 and at the origin, respectively. Our assumptions imply that these WKB solutions are Borel summable in any Stokes region. We take the branch of $\sqrt{Q_0}$ as

$$\sqrt{Q_0} \sim \operatorname{sgn}(\operatorname{Re}\gamma)\frac{\gamma}{2x} \tag{4.4}$$

holds near the origin. If $\operatorname{Re}\gamma > 0$, $S^{(\pm)}$ have formal Laurent expansions at the origin of the forms

$$S^{(+)} = \frac{c}{2x} + \frac{2ab - bc - ca + c(c-1)}{2c} + \cdots \tag{4.5}$$

and

$$S^{(-)} = \left(1 - \frac{c}{2}\right)\frac{1}{x} - \frac{2ab - bc - ca + c(c-1)}{2(c-2)} + \ldots, \tag{4.6}$$

respectively. Hence the residue of $S_{\mathrm{odd}}dx$ at the origin can be computed explicitly:

$$\operatorname*{Res}_{x=0} S_{\mathrm{odd}}dx = \frac{c-1}{2}.$$

If $\operatorname{Re}\gamma < 0$, (4.4) is read as $\sqrt{Q_0} \sim -\gamma/(2x)$. Hence we have

$$\operatorname*{Res}_{x=0} S_{\mathrm{odd}}dx = \frac{1-c}{2}.$$

We denote by $S_{\mathrm{odd},j}$ the coefficient of η^{-j} in S_{odd} for $j = -1, 0, 1, 2, \ldots$ and we set

$$S_{\mathrm{odd},\,>0} = \sum_{j>0} \eta^{-j} S_{\mathrm{odd},\,j}, \quad S_{\mathrm{odd},\,\le 0} = \sum_{j\le 0} \eta^{-j} S_{\mathrm{odd},\,j} = \eta S_{-1} + S_{\mathrm{odd},0}.$$

Under the above choice of the branch of $\sqrt{Q_0}$, ψ_+ and $\psi_+^{(0)}$ are recessive WKB solutions on the Stokes curves that flow into the origin. If $\operatorname{Re}\gamma > 0$, we set

$$\tilde{\psi}_+ = x^{-\frac{c}{2}}\psi_+^{(0)}.$$

We can apply Theorem 3.2 and conclude that $\tilde{\psi}_+$ is Borel summable in $U - \{0\}$. The Borel sum $\tilde{\Psi}_+^{(0)}$ of $\tilde{\psi}_+^{(0)}$ has the origin as a removable singularity. Thus

$$w = x^{-\frac{c}{2}}(1-x)^{-\frac{1}{2}(a+b+1-c)}\Psi_+^{(0)}$$

is a holomorphic solution to (4.1) (or (4.2)) near the origin. By Theorem 3.3, we find the relation between u_1 and w. If $\operatorname{Re}\gamma < 0$, we set $\tilde{\psi}_+ = x^{\frac{c}{2}-1}\psi_+^{(0)}$. Then $\tilde{\psi}_+$ is Borel summable in $U - \{0\}$ and the Borel sum $\tilde{\Psi}_+^{(0)}$ of $\tilde{\psi}_+^{(0)}$ has the origin as a removable singularity. Hence $\tilde{w} = x^{c/2-1}(1-x)^{-\frac{1}{2}(a+b+1-c)}\Psi_+^{(0)}$ becomes a holomorphic solution near the origin to the differential equation obtained from (4.1) by the replacement $(a,b,c) \mapsto (a+1-c, b+1-c, 2-c)$ and we find the relation between $x^{c-1}u_5$ and $\tilde{w}$. Hence we obtain the following theorem which was given in [5, Theorem 2.1] (see also [4] for a special case):

Theorem 4.1. *Suppose that the branch of $\sqrt{Q_0}$ is chosen as (4.4). Then $\psi_+^{(0)}$ is Borel summable in a punctured disk with the center at the origin. Let $\Psi_+^{(0)}$ be the Borel sum of $\psi_+^{(0)}$ in the set. If $\operatorname{Re}\gamma > 0$, the hypergeometric function and $\Psi_+^{(0)}$ are related by*

$${}_2F_1(a,b,c;x) = \sqrt{\frac{c-1}{2}}\, x^{-\frac{c}{2}}(1-x)^{-\frac{1}{2}(a+b+1-c)}\Psi_+^{(0)}.$$

If $\operatorname{Re}\gamma < 0$, we have

$$\begin{aligned} &x^{1-c}\,{}_2F_1(a+1-c, b+1-c, 2-c; x) \\ &\qquad = \sqrt{\frac{1-c}{2}}\, x^{-\frac{c}{2}}(1-x)^{-\frac{1}{2}(a+b+1-c)}\Psi_+^{(0)}. \end{aligned}$$

Here the branch of $x^{-\frac{c}{2}}(1-x)^{-\frac{1}{2}(a+b+1-c)}$ is suitably chosen.

4.2. Stokes graphs and residues

Stokes graphs of Eq. (4.3) (or (4.2)) are classified and their types are characterized in terms of (α, β, γ) in [7]. We set

$$\begin{aligned} \omega_1 &= \{(\alpha,\beta,\gamma) \in \mathbb{C}^3 \mid 0 < \operatorname{Re}\alpha < \operatorname{Re}\gamma < \operatorname{Re}\beta\}, \\ \omega_2 &= \{(\alpha,\beta,\gamma) \in \mathbb{C}^3 \mid 0 < \operatorname{Re}\alpha < \operatorname{Re}\beta < \operatorname{Re}\gamma < \operatorname{Re}\alpha + \operatorname{Re}\beta\}, \\ \omega_3 &= \{(\alpha,\beta,\gamma) \in \mathbb{C}^3 \mid 0 < \operatorname{Re}\gamma < \operatorname{Re}\alpha < \operatorname{Re}\beta\}, \\ \omega_4 &= \{(\alpha,\beta,\gamma) \in \mathbb{C}^3 \mid 0 < \operatorname{Re}\gamma < \operatorname{Re}\alpha + \operatorname{Re}\beta < \operatorname{Re}\beta\}. \end{aligned}$$

Theorem 4.2. ([7, Theorem 3.2]) *Let $\mathcal{G}$ denote the group generated by ι_m ($m = 0,1,2$) and Π_j the union of all $g(\omega_j)$ for $g \in \mathcal{G}$ ($j = 1,2,3,4$). Let n_0, n_1 and n_∞ be the numbers of Stokes curves of Eq. (4.3) which flow into 0, 1 and ∞, respectively and set $\boldsymbol{n} = (n_1, n_2, n_3)$. Then we have*

(1) *If $(\alpha,\beta,\gamma) \in \Pi_1$, then* $\boldsymbol{n} = (2,2,2)$.
(2) *If $(\alpha,\beta,\gamma) \in \Pi_2$, then* $\boldsymbol{n} = (4,1,1)$.
(3) *If $(\alpha,\beta,\gamma) \in \Pi_3$, then* $\boldsymbol{n} = (1,4,1)$.
(4) *If $(\alpha,\beta,\gamma) \in \Pi_4$, then* $\boldsymbol{n} = (1,1,4)$.

This theorem is proved as follows. For special three choices of parameters $(\alpha, \beta, \gamma) = (0, 2, 1), (1/2, 1, 1), (1, 2, 1)$, we have double turning points and we can describe the Stokes curves explicitly. Since Q_0 is invariant under the action of $\mathcal{G}$, the Stokes curves are also invariant under the action. We know that no Stokes curve connects turning point(s) if $(\alpha, \beta, \gamma) \notin E_1 \cup E_2$ and that the types, that is $\boldsymbol{n}$'s, are constant on each connected component of Π_k. Hence small perturbation of one of parameters of our special choices yields all the cases.

If the perturbation is sufficiently small, two simple turning points are very close. Hence we can take a segment ℓ connecting two turning points as a branch cut of $\sqrt{Q_0}$. Note that ℓ is contained in a Stokes region. For any $(\alpha, \beta, \gamma) \in \omega_j$, we take a path connecting (α, β, γ) and the above mentioned perturbed parameters in ω_j and take a branch cut of $\sqrt{Q_0}$ which is obtained by a continuous deformation of such an ℓ along the path. Here the deformation is carried out by keeping the cut being contained in the corresponding Stokes region. The branch of $\sqrt{Q_0}$ on the first sheet of the Riemann surface is taken as (4.4). Hence we can specify the dominance of WKB solutions on every Stokes curve on the sheet. For example, if $(\alpha, \beta, \gamma) \in \omega_1$, ψ_- and $\psi_-^{(0)}$ are dominant on each Stokes curve flowing into the origin and ψ_+ and $\psi_+^{(0)}$ are dominant on each Stokes curve flowing into $x = 1$ or to the infinity. The following Figure 4.1 shows a typical configuration of Stokes curves of (4.3) for $(\alpha, \beta, \gamma) \in \omega_1$. Larger dots designate the turning points and the wavy segment connecting them designates the branch cut.

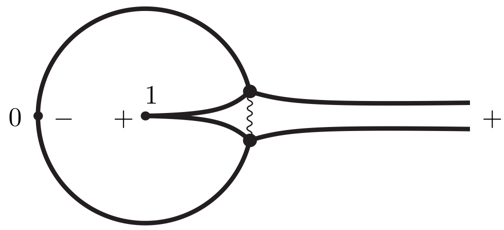

Figure 4.1: Stokes curves for $(\alpha, \beta, \gamma) = (1/2, 1 + 1/300, 1)$

The following table shows the dominance of WKB solutions on the Stokes curves flowing into each singularity. The branch cuts are taken as Figures 4.1, 4.3–4.5.

	0	1	∞
ω_1	$-$	$+$	$+$
ω_2	$-$	$+$	$+$
ω_3	$-$	$+$	$-$
ω_4	$-$	$-$	$+$

Table 4.2: Dominance of WKB solutions

Table 4.2 is read as follows. For example, the second row means that if (α, β, γ) belongs to ω_1, ψ_-, ψ_+ and ψ_+ are dominant on the Stokes curves flowing into 0, 1 and ∞, respectively. Examples of Stokes curves for ω_2, ω_3 and ω_4 are respectively given below:

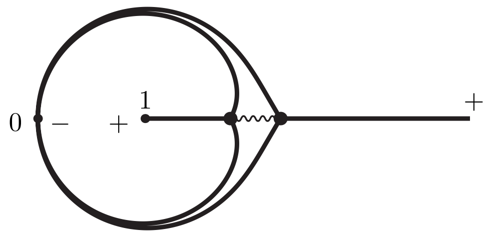

Figure 4.3: $(\alpha, \beta, \gamma) = (1/2, 1-1/300, 1)$

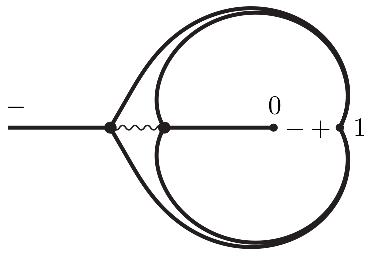

Figure 4.4: $(\alpha, \beta, \gamma) = (1+1/300, 3/2, 1)$

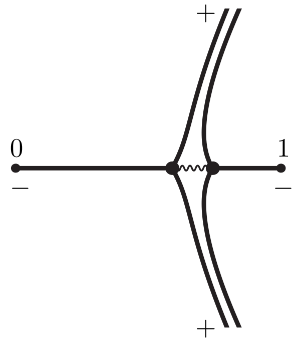

Figure 4.5: $(\alpha, \beta, \gamma) = (-1/100, 3/2, 1)$

Now we can compute the residue of $S_{\mathrm{odd}}dx$ at each singularity by using a method given in [2], [13]. For example, if $(\alpha, \beta, \gamma) \in \omega_1$, we have

$$\operatorname*{Res}_{x=0} S_{\mathrm{odd}}dx = \frac{c-1}{2}, \quad \operatorname*{Res}_{x=1} S_{\mathrm{odd}}dx = \frac{c-a-b}{2} \quad \text{and} \quad \operatorname*{Res}_{x=\infty} S_{\mathrm{odd}}dx = \frac{a-b}{2}.$$

Note that $\operatorname{Re}\gamma > 0$, $\operatorname{Re}(\gamma - \alpha - \beta) < 0$ and $\operatorname{Re}(\alpha - \beta) < 0$. The residues for other cases are given in Table 4.6.

	0	1	∞	sum of residues
ω_1	$\dfrac{c-1}{2}$	$\dfrac{c-a-b}{2}$	$\dfrac{a-b}{2}$	$c-b-\dfrac{1}{2}$
ω_2	$\dfrac{c-1}{2}$	$\dfrac{c-a-b}{2}$	$\dfrac{a-b}{2}$	$c-b-\dfrac{1}{2}$
ω_3	$\dfrac{c-1}{2}$	$\dfrac{c-a-b}{2}$	$\dfrac{b-a}{2}$	$c-a-\dfrac{1}{2}$
ω_4	$\dfrac{c-1}{2}$	$\dfrac{a+b-c}{2}$	$\dfrac{a-b}{2}$	$a-\dfrac{1}{2}$

Table 4.6: Residues of $S_{\mathrm{odd}}dx$ at $x=0,1,\infty$

Since Q is invariant under the action of the involutions ι_m $(m=1,2,\dots,6)$, it is sufficient to consider the case where $(\alpha,\beta,\gamma)\in\omega_k$ for $k=1,2,3,4$.

4.3. Voros coefficient at the origin

We relate the WKB solution of (4.3) normalized at a simple turning point and those normalized at the origin. Let τ_0 be one of the simple turning points of (4.3) and let δ denote a path connecting the origin and τ_0. We will specify τ_0 and δ later. Let σ denote the residue of $\eta^{-1}S_{\mathrm{odd}}dx$ at the origin. We set

$$\hat{V}_0=\int_\delta\left(S_{\mathrm{odd}}-\frac{\sigma\eta}{x}\right)dx+\sigma\eta\log\tau_0. \tag{4.7}$$

Here the branch of logarithm is taken suitably. Formally we can relate the WKB solutions $\psi_\pm$ normalized at τ_0 and $\psi_\pm^{(0)}$ normalized at the origin:

$$\psi_\pm^{(0)}=\exp(\pm\hat{V}_0)\psi_\pm. \tag{4.8}$$

We write the formal series $\hat{V}_0$ of η^{-1} in the form

$$\sum_{j=-1}^{\infty}\eta^{-j}V_{0,j}.$$

We decompose this into two parts:

$$\hat{V}_0=V_0+\hat{V}_{0,\le 0},$$

where we set

$$V_0 = \sum_{j=1}^{\infty} \eta^{-j} V_{0,j}, \quad \hat{V}_{0,\le 0} = \eta V_{0,-1} + V_{0,0}.$$

We can rewrite these formal series in the form

$$V_0 = \frac{1}{2} \int_{C_0} S_{\mathrm{odd},>0} dx, \tag{4.9}$$

$$\hat{V}_{0,\le 0} = \lim_{x\to 0} \left(\int_x^{\tau_0} S_{\mathrm{odd},\,\le 0}\, dx + \frac{c-1}{2} \log x \right). \tag{4.10}$$

Here C_0 denotes a contour starting at the origin of the first sheet of the Riemann surface of $\sqrt{Q_0}$, going to τ_0 along δ, detouring τ_0 once counterclockwise and returning to the origin along δ on the second sheet and here the path of integration in (4.10) is taken along δ in the first sheet.

Since the residue of $S_{\mathrm{odd},>0}$ at any regular singular point vanishes, V_0 is independent of the choice of τ_0 and δ. Note that V_0 is invariant under the action of $\iota_m (m = 0, 1, \ldots, 6)$.

Definition 4.3. We call V_0 the Voros coefficient of (4.3) (or of (4.1)) at the origin.

The explicit form of V_0 is given as follows ([5, Theorem 1.1]):

Theorem 4.4. *Suppose that* $\mathrm{Re}\gamma$ *is positive. Then we have*

$$V_0 = \frac{1}{2} \sum_{n=2}^{\infty} \frac{(-1)^{n-1}\eta^{1-n}}{n(n-1)} \left(\frac{B_n(\alpha_0)}{\alpha^{n-1}} + \frac{B_n(\beta_0)}{\beta^{n-1}} + \frac{B_n(\gamma_0-\alpha_0)}{(\gamma-\alpha)^{n-1}} + \frac{B_n(\gamma_0-\beta_0)}{(\gamma-\beta)^{n-1}} - \frac{B_n(\gamma_0)+B_n(\gamma_0-1)}{\gamma^{n-1}} \right). \tag{4.11}$$

Here $B_n(x)$ *denotes* n*-th Bernoulli polynomial*:

$$\frac{te^{xt}}{e^t-1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n.$$

If $\mathrm{Re}\gamma$ *is negative, the right-hand side of* (4.11) *gives* $-V_0$.

The explicit form is obtained in [8, Theorem 2.3] for a special case where $\alpha_0 = \beta_0 = 1/2$, $\gamma_0 = 1$. The proof of Theorem 4.4 is essentially the same as the proof of the special case, however, some careful computations are required. Details are given in [6]. Note that another derivation of the explicit form of V_0 is recently obtained by [11] (see also [10]).

To obtain the explicit form of $\hat{V}_{0,\le 0}$, we should be more careful about the choice of τ_0, δ and the branch of logarithms. From now on, we assume that α, β, γ, α_0, β_0 and γ_0 are real. Our method can be applied for the complex case, however, to state the choice of branch of logarithms precisely, there are many cases to be considered. Hence we treat only the real case for the sake of simplicity.

Let $\tau_\pm$ denote zeros of Q_0, that is,

$$\tau_\pm := \frac{\beta\gamma + \gamma\alpha - 2\alpha\beta \pm 2\sqrt{\Delta}}{(\alpha-\beta)^2},$$

where we set $\Delta = \alpha\beta(\alpha-\gamma)(\beta-\gamma)$. We choose τ_0 and τ_1 as follows:

(i) In the case where $(\alpha,\beta,\gamma) \in \omega_1$ or ω_3, we set

$$\tau_0 = \tau_+, \quad \tau_1 = \tau_-. \tag{4.12}$$

Here we take the branch of the square root as $\mathrm{Im}\sqrt{\Delta} > 0$ if $(\alpha,\beta,\gamma) \in \omega_1$ and $\sqrt{\Delta} > 0$ if $(\alpha,\beta,\gamma) \in \omega_3$.

(ii) In the case where $(\alpha,\beta,\gamma) \in \omega_2$ or ω_4, we set

$$\tau_0 = \tau_-, \quad \tau_1 = \tau_+. \tag{4.13}$$

Here we take the branch of the square root as $\sqrt{\Delta} > 0$.

The path δ in (4.7) is chosen as follows. If $(\alpha,\beta,\gamma) \in \omega_2$, there are two Stokes curves emanating from τ_0 and flowing into the origin. One of the initial directions at τ_0 has a positive imaginary part. We take this Stokes curve as δ. For other cases, there is a unique Stokes curve emanating from τ_0 and flowing into the origin. We take this as δ.

Theorem 4.5. (1) *For $(\alpha,\beta,\gamma) \in \omega_1$, we have*

$$\begin{aligned}\hat{V}_{0,\le 0} = {}& \frac{1}{2}\left(\frac{1}{2}-a\right)\log\alpha + \frac{1}{2}\left(\frac{1}{2}-b\right)\log\beta + \frac{1}{2}\left(\frac{1}{2}-c+a\right)\log(\gamma-\alpha) \\ &+ \frac{1}{2}\left(\frac{1}{2}-c+b\right)\log(\beta-\gamma) + (c-1)\log\gamma + \frac{1}{2}\left(a-\frac{1}{2}\right)\pi i.\end{aligned} \tag{4.14}$$

(2) *For $(\alpha,\beta,\gamma) \in \omega_2$, we have*

$$\begin{aligned}\hat{V}_{0,\le 0} = {}& \frac{1}{2}\left(\frac{1}{2}-a\right)\log\alpha + \frac{1}{2}\left(\frac{1}{2}-b\right)\log\beta + \frac{1}{2}\left(\frac{1}{2}-c+a\right)\log(\gamma-\alpha) \\ &+ \frac{1}{2}\left(\frac{1}{2}-c+b\right)\log(\gamma-\beta) + (c-1)\log\gamma + \frac{1}{2}(a+b-c)\pi i.\end{aligned} \tag{4.15}$$

(3) *For $(\alpha,\beta,\gamma) \in \omega_3$, we have*

$$\begin{aligned}\hat{V}_{0,\le 0} = {}& \frac{1}{2}\left(\frac{1}{2}-a\right)\log\alpha + \frac{1}{2}\left(\frac{1}{2}-b\right)\log\beta + \frac{1}{2}\left(\frac{1}{2}-c+a\right)\log(\alpha-\gamma) \\ &+ \frac{1}{2}\left(\frac{1}{2}-c+b\right)\log(\beta-\gamma) + (c-1)\log\gamma + \frac{1}{2}(1-c)\pi i.\end{aligned} \tag{4.16}$$

(4) *For $(\alpha,\beta,\gamma) \in \omega_4$, we have*

$$\begin{aligned}\hat{V}_{0,\le 0} = {}& \frac{1}{2}\left(\frac{1}{2}-a\right)\log(-\alpha) + \frac{1}{2}\left(\frac{1}{2}-b\right)\log\beta + \frac{1}{2}\left(\frac{1}{2}-c+a\right)\log(\gamma-\alpha) \\ &+ \frac{1}{2}\left(\frac{1}{2}-c+b\right)\log(\beta-\gamma) + (c-1)\log\gamma.\end{aligned} \tag{4.17}$$

PROOF. By an elementary calculus, we see that

$$F = \frac{1}{2}(a+b-c)\log f_1 + \frac{1}{2}(1-c)\log f_2 + \frac{1}{2}(b-a)\log f_3$$

is a primitive function of $S_{\mathrm{odd},\le 0}$. Here we set

$$\begin{cases} f_1 = \dfrac{(\alpha^2+\beta^2-\beta\gamma-\gamma\alpha)x + 2\alpha\beta-\beta\gamma-\gamma\alpha+\gamma^2+(\alpha+\beta-\gamma)\sqrt{G}}{1-x}, \\ f_2 = \dfrac{(2\alpha\beta-\beta\gamma-\gamma\alpha)x+\gamma^2+\gamma\sqrt{G}}{x}, \\ f_3 = (\alpha-\beta)^2x + 2\alpha\beta-\beta\gamma-\gamma\alpha+(\alpha-\beta)\sqrt{G} \end{cases}$$

and $G = (\alpha-\beta)^2x^2 + 2(2\alpha\beta-\beta\gamma-\gamma\alpha)x+\gamma^2$. The integral of $S_{\mathrm{odd},\le 0}$ from τ_0 to τ_1 is evaluated by

$$\int_{\tau_0}^{\tau_1} S_{\mathrm{odd},\le 0}dx = F(\tau_1) - F(\tau_0).$$

Here the path of integration is taken as follows. In the case where $(\alpha,\beta,\gamma)\in\omega_1$, there is an arc connecting τ_0 and τ_1 which is close to the branch cut and lies in the left side of the cut. We take this arc as the path. For other cases, there is an arc connecting τ_0 and τ_1 which is close to the branch cut and lies in $\mathrm{Im}\, x \ge 0$. We take this as the path. Each term of F is evaluated as

$$f_1(\tau_\pm) = \pm 2\sqrt{\Delta}, \quad f_2(\tau_\pm) = \mp 2\sqrt{\Delta}, \quad f_3(\tau_\pm) = \pm 2\sqrt{\Delta}.$$

Hence we have

$$\begin{aligned}\int_{\tau_0}^{\tau_1} S_{\mathrm{odd},\le 0}dx = {} & \frac{1}{2}(a+b-c)\left(\log f_1(\tau_1) - \log f_1(\tau_0)\right) \\ & + \frac{1}{2}(1-c)(\log f_2(\tau_1) - \log f_2(\tau_0)) \\ & + \frac{1}{2}(b-a)(\log f_3(\tau_1) - \log f_3(\tau_0)).\end{aligned}$$

For the case where $(\alpha,\beta,\gamma)\in\omega_1$, this is rewritten as

$$\begin{aligned}\int_{\tau_0}^{\tau_1} S_{\mathrm{odd},\le 0}dx = {} & \frac{1}{2}(a+b-c)\left(\log f_1(\tau_-) - \log f_1(\tau_+)\right) \\ & + \frac{1}{2}(1-c)(\log f_2(\tau_-) - \log f_2(\tau_+)) \\ & + \frac{1}{2}(b-a)(\log f_3(\tau_-) - \log f_3(\tau_+)).\end{aligned}$$

On the other hand, this integral should coincide with

$$\frac{1}{2}\oint S_{\mathrm{odd},\le 0}dx = -\pi i \sum_{x=0,1,\infty} \operatorname*{Res}_x S_{\mathrm{odd}}dx = \left(b-c+\frac{1}{2}\right)\pi i.$$

Here the integral is taken over a closed path encircling the branch cut of S_{-1} with a positive orientation. Using this fact, we can determine the arguments of $f_j(\tau_\pm)$ for each j uniquely. Note that $\mathrm{Im}\sqrt{\Delta} > 0$ and we take the arguments of $f_j(\tau_+)$ so that $-\pi < \arg f_j(\tau_+) \le \pi$. Thus we have

$$\arg f_1(\tau_-) = \frac{3}{2}\pi, \quad \arg f_1(\tau_+) = \frac{1}{2}\pi,$$

$$\arg f_2(\tau_-) = \frac{1}{2}\pi, \quad \arg f_2(\tau_+) = -\frac{1}{2}\pi,$$

$$\arg f_3(\tau_-) = \frac{3}{2}\pi, \quad \arg f_3(\tau_+) = \frac{1}{2}\pi.$$

Next we evaluate f_j's at the origin:

$$f_1(0) = 2\alpha\beta, \quad (xf_2)(0) = 2\gamma^2, \quad f_3(0) = 2(\alpha-\gamma)\beta.$$

Hence we have

$$\begin{aligned}
\hat{V}_{0,\le 0} &= \frac{1}{2}(a+b-c)\log f_1(\tau_+) + \frac{1}{2}(1-c)\log f_2(\tau_+) + \frac{1}{2}(b-a)\log f_3(\tau_+) \\
&\quad - \frac{1}{2}(a+b-c)\log 2\alpha\beta - \frac{1}{2}(1-c)\log 2\gamma^2 - \frac{1}{2}(b-a)\log 2(\alpha-\gamma)\beta \\
&= \frac{1}{2}(a+b-c)\left(\log(2\sqrt{|\Delta|}) + \frac{\pi i}{2}\right) + \frac{1}{2}(1-c)\left(\log 2\sqrt{|\Delta|} - \frac{\pi i}{2}\right) \\
&\quad + \frac{1}{2}(b-a)\left(\log 2\sqrt{|\Delta|} + \frac{\pi i}{2}\right) - \frac{1}{2}(a+b-c)\log 2\alpha\beta \\
&\qquad - \frac{1}{2}(1-c)\log 2\gamma^2 - \frac{1}{2}(b-a)\left(\log 2(\gamma-\alpha)\beta + \pi i\right).
\end{aligned}$$

This turns out to be

$$\begin{aligned}
&\frac{1}{2}\left(\frac{1}{2}-a\right)\log\alpha + \frac{1}{2}\left(\frac{1}{2}-b\right)\log\beta + \frac{1}{2}\left(a-c+\frac{1}{2}\right)\log(\gamma-\alpha) \\
&\qquad + \frac{1}{2}\left(b-c+\frac{1}{2}\right)\log(\beta-\gamma) + (c-1)\log\gamma + \frac{1}{2}\left(a-\frac{1}{2}\right)\pi i.
\end{aligned}$$

This proves (4.14). Similarly (4.15)–(4.17) can be proved.

4.4. Borel sums of the Voros coefficient

We show that the Voros coefficient V_0 is Borel summable under suitable conditions. We compute the Borel sums of V_0 explicitly by using its concrete form given in Theorem 4.4. As is shown in the theorem, V_0 consists of a finite sum of divergent formal power series in η^{-1} of the form:

$$U(s, s_0, \eta) = \frac{1}{2}\sum_{n=2}^{\infty} \frac{(-1)^{n-1}B_n(s_0)\eta^{1-n}}{n(n-1)s^{n-1}}.$$

By the definition, the Borel transform U_B of U with respect to η^{-1} is given by

$$U_{\mathrm{B}}(s, s_0, y) = \frac{1}{2}\sum_{n=2}^{\infty} \frac{(-1)^{n-1}B_n(s_0)}{n!s^{n-1}} y^{n-2},$$

where y denotes the variable of the Borel plane. It is easy to see that the right-hand side becomes

$$-\frac{1}{2y}\left(\frac{\exp\left(-\frac{s_0 y}{s}\right)}{\exp\left(\frac{y}{s}\right)-1} - \frac{s}{y} + \left(s_0 - \frac{1}{2}\right) + \exp\left(-\frac{s_0 y}{s}\right)\right).$$

If the Laplace integral

$$\int_0^{\infty} U_{\mathrm{B}}(s, s_0, y) e^{-\eta y} dy \tag{4.18}$$

converges for $\eta >> 0$, we see that U is Borel summable and the Borel sum is given by this integral. We always assume that η is a real positive large number, however, if necessary, we may consider the case where η has a non-zero imaginary part if (4.18) is well defined.

Lemma 4.6. *If* $\mathrm{Re}\, s \neq 0$, *U is Borel summable. The explicit form of the Borel sum depends on the signature of* $\mathrm{Re}\, s$. *Let $U^{\pm}$ denote the Borel sums of U for* $\pm\mathrm{Re}\, s > 0$. *Then*

$$U^{+}(s, s_0, \eta) = \frac{1}{2}\log\frac{(s\eta)^{s_0+s\eta-\frac{1}{2}}\sqrt{2\pi}}{\Gamma(s_0+s\eta)e^{s\eta}},$$
$$U^{-}(s, s_0, \eta) = \frac{1}{2}\log\frac{\Gamma(1-s_0-s\eta)(-s\eta)^{s_0+s\eta-\frac{1}{2}}}{e^{s\eta}\sqrt{2\pi}}$$

hold.

Proof. Using the Binet formula ([9])

$$\log\Gamma(z) = \left(z-\frac{1}{2}\right)\log z - z + \frac{1}{2}\log 2\pi + \int_0^{\infty}\left(\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\right)\frac{e^{-zt}}{t}dt$$

and

$$\int_0^{\infty}\frac{e^{-at}-e^{-bt}}{t}dt = \log b - \log a, \quad (\mathrm{Re}\, a > 0,\ \mathrm{Re}\, b > 0),$$

we can compute (4.18) directly and have Lemma 4.6.

Theorem 4.7. *The Voros coefficient V_0 is Borel summable in each connected component of Π_k $(k = 1, 2, 3, 4)$. The explicit form of the Borel sum of V_0 in the component can be computed. For example, if we denote by V_0^k the Borel sum of V_0 for $(\alpha, \beta, \gamma) \in \omega_k$ and we set $\hat{V}_0^k = V_0^k + \hat{V}_{0,\leq 0}$, we have the following explicit forms*:

$$\hat{V_0}^1 = \frac{1}{2}\log\frac{\Gamma(1+b-c)\Gamma(c)\Gamma(c-1)}{\Gamma(a)\Gamma(b)\Gamma(c-a)}e^{\pi i\left(a-\frac{1}{2}\right)}, \tag{4.19}$$

$$\hat{V_0}^2 = \frac{1}{2}\log\frac{2\pi\Gamma(c)\Gamma(c-1)}{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c-b)}e^{\pi i(a+b-c)}, \tag{4.20}$$

$$\hat{V_0}^3 = \frac{1}{2}\log\frac{\Gamma(1+a-c)\Gamma(1+b-c)\Gamma(c)\Gamma(c-1)}{2\pi\Gamma(a)\Gamma(b)}e^{\pi i(1-c)}, \tag{4.21}$$

$$\hat{V_0}^4 = \frac{1}{2}\log\frac{\Gamma(1-a)\Gamma(1+b-c)\Gamma(c)\Gamma(c-1)}{2\pi\Gamma(b)\Gamma(c-a)}. \tag{4.22}$$

Proof. For the case where $(\alpha,\beta,\gamma)\in\omega_1$, we have, by the definition, $\alpha>0$, $\beta>0$, $\gamma-\alpha>0$, $\gamma-\beta<0$ and $\gamma>0$. Note that we have assumed α, β and γ being real. Hence it follows from Lemma 4.6 that the Borel sum of V_0 in ω_1 becomes

$$\frac{1}{2}\log\frac{\Gamma(1+b-c)\Gamma(c)\Gamma(c-1)}{\Gamma(a)\Gamma(b)\Gamma(c-a)}\alpha^{a-\frac{1}{2}}\beta^{b-\frac{1}{2}}(\gamma-\alpha)^{c-a-\frac{1}{2}}(\beta-\gamma)^{c-b-\frac{1}{2}}\gamma^{-2(c-1)}.$$

Combining this with (4.14), we have (4.19). Others can be proved similarly.

4.5. Relation between WKB solutions and the hypergeometric function

In this subsection, we give the relation between the basis (u_1,u_5) of the solution space of (4.1) and the Borel sums of $\psi_\pm$ of the WKB solution of (4.3) normalized at a turning point.

Firstly we consider the case where $(\alpha,\beta,\gamma)\in\omega_1$, i.e., $0<\alpha<\gamma<\beta$. A typical configuration of Stokes curves is as follows:

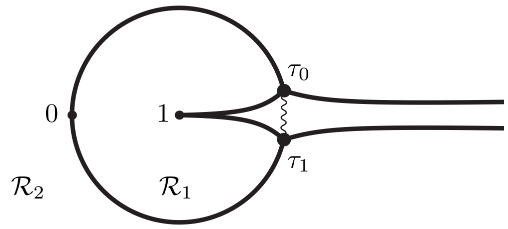

Figure 4.7. An example of Stokes geometry of (4.3) for ω_1

We take τ_0, τ_1 as (4.12). The wavy segment in Figure 4.7 designates the branch cut of S_{-1}. There are 6 Stokes curves and they are labeled s_{jk} ($j=0,1; k=0,1,2$) so that s_{jk} emanates from τ_j and flows into the singular point r_k ($j=0,1; k=0,1;\ r_0=0$, $r_1=1$). Hence s_{j2} flows into the infinity. Let $\mathcal{R}_k$ be the Stokes region surrounded by s_{00}, s_{0k}, s_{1k} and s_{10} ($k=1,2$). Recall that $\psi_\pm$ and $\psi_\pm^{(0)}$ denote the WKB solutions of (4.3) normalized at τ_0 and at the origin, respectively, and the branch of $\sqrt{Q_0}$ is chosen as (4.4) near the origin. These WKB solutions are Borel summable in $\mathcal{R}_k$ ([15, 27]). We denote by $\Psi_\pm^k$ and $\Psi_\pm^{(0),k}$ the Borel sums of $\psi_\pm$ and $\psi_\pm^{(0)}$ in $\mathcal{R}_k$, respectively ($k=1,2$). Since $\operatorname{Re}\gamma>0$, ψ_+ is recessive on s_{00} and on s_{10}. Therefore $\Psi_+^1=\Psi_+^2$ and $\Psi_+^{(0),1}=\Psi_+^{(0),2}$ hold as holomorphic functions. Taking the Borel sum of (4.8), we have

$$\Psi_+^{(0),k} = \exp(\hat{V}_0^1)\Psi_+^k \quad (k=1,2).$$

Thus Theorem 4.1 yields

$$
{}_2F_1(a,b,c;x) = \sqrt{\frac{c-1}{2}}\, x^{-\frac{c}{2}}(1-x)^{-\frac{1}{2}(a+b+1-c)} \exp(\hat{V}_0^1)\Psi_+^k \quad (k=1,2), \tag{4.23}
$$

where the branch of $\sqrt{c-1}$ is fixed by $\sqrt{\gamma} > 0$ and that of $x^{-\frac{c}{2}}(1-x)^{-\frac{1}{2}(a+b+1-c)}$ is chosen so that it is positive for $0 < x < 1$. More explicitly, we have

$$
\begin{aligned}
{}_2F_1(a,b,c;x) = & x^{-\frac{c}{2}}(1-x)^{-\frac{1}{2}(a+b+1-c)} \\
& \times \left(\frac{\Gamma(1+b-c)}{2\Gamma(a)\Gamma(b)\Gamma(c-a)}\right)^{\frac{1}{2}} \Gamma(c) e^{\frac{\pi i}{2}\left(a-\frac{1}{2}\right)} \Psi_+^k \ (k=1,2).
\end{aligned} \tag{4.24}
$$

To find the relation between Ψ_-^1 and (u_1, u_5), we need the monodromy matrix around the origin. Let x_0 be a point in $\mathcal{R}_1$ near τ_0 and let Γ_0 be a closed path starting at x_0, going around the origin once counterclockwise and returning to x_0 so that it does not cross s_{01} and s_{11}. We may assume that Γ_0 crosses s_{00} firstly and next it crosses s_{10}, and it crosses each Stokes curve only once. Then the monodromy matrix M_0 of Γ_0 of the basis (Ψ_+^1, Ψ_-^1) is computed by using the standard theory of the exact WKB analysis ([13, §3.2]):

$$
M_0 = \begin{pmatrix} e^{\pi i c} & i(e^{2\pi i(a-c)} - 1)e^{\pi i c} \\ 0 & e^{-\pi i c} \end{pmatrix}. \tag{4.25}
$$

Since we already have (4.23), we can assume that the relation of the following form holds:

$$
(u_1, u_5) = p(x)(\Psi_+^1, \Psi_-^1) \begin{pmatrix} c_{11} & c_{12} \\ 0 & c_{22} \end{pmatrix}. \tag{4.26}
$$

Here we set

$$
p(x) = x^{-\frac{c}{2}}(1-x)^{-\frac{1}{2}(a+b-c+1)},
$$

$$
c_{11} = \sqrt{\frac{c-1}{2}} \exp \hat{V}_0^1,
$$

and $c_{12},\ c_{22}$ are constants. Taking analytic continuation of (4.26) along Γ_0, we have

$$
(u_1, u_5) \begin{pmatrix} 1 & 0 \\ 0 & e^{-2\pi i c} \end{pmatrix} = p(x)(\Psi_+^1, \Psi_-^1) \begin{pmatrix} e^{-\pi i c} & 0 \\ 0 & e^{-\pi i c} \end{pmatrix} M_0 \begin{pmatrix} c_{11} & c_{12} \\ 0 & c_{22} \end{pmatrix},
$$

or

$$
(u_1, u_5) = p(x)(\Psi_+^1, \Psi_-^1) \begin{pmatrix} c_{11} & e^{2\pi i c}(c_{12} + i(e^{2\pi i(a-c)} - 1)c_{22}) \\ 0 & c_{22} \end{pmatrix}.
$$

If we compare this with (4.26), we obtain

$$
c_{12} = -i \frac{e^{2\pi i a} - e^{2\pi i c}}{e^{2\pi i c} - 1} c_{22}.
$$

To evaluate c_{22}, we use the following

Lemma 4.8. *Set $s(x) = \int_{\tau_0}^{x} S_{-1}dx$. Under the assumptions, notation as above and a suitable choice of $d_0 > 0$, we have*

$$\lim_{\substack{x\to 0,\, x\in\mathcal{R}_1 \\ |\mathrm{Im}\, s(x)|=d_0}} x^{c-1}p(x)\Psi_-^1 = \sqrt{\frac{2}{c-1}}\exp \hat{V}_0^1.$$

The proof of Lemma 4.8 will be given later. Relation (4.26) yields

$$x^{c-1}c_{22}\, p(x)\Psi_-^1 = x^{c-1}u_5 - c_{12}\, {c_{11}}^{-1}x^{c-1}u_1. \tag{4.27}$$

By the definition of u_1 and u_5 and the assumption $\gamma > 0$, we have

$$\lim_{x\to 0} x^{c-1}u_1 = 0, \quad \lim_{x\to 0} x^{c-1}u_5 = 1.$$

Hence (4.27) yields

$$c_{22}\sqrt{\frac{2}{c-1}}\exp \hat{V}_0^1 = 1.$$

Thus we have

Theorem 4.9. *Suppose that $(\alpha,\beta,\gamma)\in\omega_1$. Then the Borel sum (Ψ_+^1,Ψ_-^1) on $\mathcal{R}_1$ of the WKB solution basis (ψ_+,ψ_-) of (4.3) and (u_1,u_5) are related by*

$$(u_1,u_5) = p(x)(\Psi_+^1,\Psi_-^1)\begin{pmatrix} c_{11} & c_{12} \\ 0 & c_{22}\end{pmatrix}. \tag{4.28}$$

Here we set $u_1 = {}_2F_1(a,b,c;x)$, $u_5 = x^{1-c}{}_2F_1(a+1-c,b+1-c,2-c;x)$ and

$$\begin{aligned} c_{11} &= \sqrt{\frac{c-1}{2}}e^{\hat{V}_0^1}, \\ c_{12} &= -i\frac{e^{2\pi i a}-e^{2\pi i c}}{e^{2\pi i c}-1}\sqrt{\frac{c-1}{2}}e^{-\hat{V}_0^1}, \\ c_{22} &= \sqrt{\frac{c-1}{2}}e^{-\hat{V}_0^1}. \end{aligned}$$

Proof of Lemma 4.8. Let us recall the definition of the dominant WKB solution normalized at the origin:

$$\psi_-^{(0)} = \frac{x^{-\frac{c-1}{2}}}{\sqrt{S_{\mathrm{odd}}}}\exp\left(-\int_0^x\left(S_{\mathrm{odd}} - \frac{c-1}{2x}\right)dx\right).$$

We separate the linear part of η in the exponent of the right-hand side and rewrite this as

$$\psi_-^{(0)} = x^{1-\frac{c}{2}}\exp(-\eta t(x))\varphi = x^{1-\frac{c}{2}}\phi,$$

where we set

$$t(x) = \int_0^x \left(S_{-1} - \frac{\gamma}{2x} \right) dx,$$

$$\varphi = \frac{\eta^{-\frac{1}{2}}}{\sqrt{\eta^{-1} x S_{\mathrm{odd}}}} \exp \left(- \int_0^x \left(S_{\mathrm{odd},\geq 0} - \frac{\gamma_0 - 1}{2x} \right) dx \right)$$

with $S_{\mathrm{odd},\geq 0} = S_{\mathrm{odd},0} + S_{\mathrm{odd},>0}$ and

$$\phi = \exp(-\eta\, t(x))\varphi.$$

Note that φ does not have any exponential term of η and it has an expansion of the form

$$\varphi = \sum_{j=0}^{\infty} \phi_j(x) \eta^{-\frac{1}{2}-j}.$$

We can take termwise evaluation at $x = 0$ and have

$$\varphi(0,\eta) = \sum_{j=0}^{\infty} \phi_j(0) \eta^{-\frac{1}{2}-j} = \sqrt{\frac{2}{c-1}} \quad \left(= \lim_{x \to 0} \frac{1}{\sqrt{x S_{\mathrm{odd}}}} \right). \tag{4.29}$$

Let us take the Borel transform of ϕ:

$$\phi_{\mathrm{B}}(x,y) = \sum_{j=0}^{\infty} \frac{\phi_j(x)}{\Gamma\left(j+\frac{1}{2}\right)} (y - t(x))^{j-\frac{1}{2}}. \tag{4.30}$$

The Borel transform of $\psi_-^{(0)}$ is expressed in terms of ϕ_{B}:

$$\psi_{-,\mathrm{B}}^{(0)}(x,y) = x^{1-\frac{\gamma_0}{2}} \sum_{j=0}^{\infty} \frac{\phi_j(x)}{\Gamma\left(j+\frac{1}{2}\right)} (y - \tilde{s}(x))^{j-\frac{1}{2}} \tag{4.31}$$

$$= x^{1-\frac{\gamma_0}{2}} \phi_{\mathrm{B}} \left(x, y - \frac{\gamma}{2} \log x \right). \tag{4.32}$$

Here we set $\tilde{s}(x) = t(x) + \gamma \log x/2$, which is the sum of $s(x)$ and the leading term $V_{0,-1}$ of $\hat{V}_0$:

$$\tilde{s}(x) = s(x) + V_{0,-1}. \tag{4.33}$$

If we rewrite ψ_- in the form

$$\psi_- = \exp(-\eta s(x)) \sum_{j=0}^{\infty} \psi_{-,j}(x) \eta^{-\frac{1}{2}-j},$$

the Borel transform becomes

$$\psi_{-,\mathrm{B}}(x,y) = \sum_{j=0}^{\infty} \frac{\psi_{-,j}(x)}{\Gamma(j+\frac{1}{2})} (y - s(x))^{j-\frac{1}{2}}. \tag{4.34}$$

Eqs. (4.5) and (4.6) imply that $S_j^{(\pm)}(x)$ are holomorphic at the origin for $j > 0$. Hence we can take $d > 0$ uniformly in a neighborhood of the origin so that the series (4.30), (4.31) and (4.34) respectively converge if $0 < |y - t(x)| \le d$, $0 < |y - \tilde{s}(x)| \le d$ and $0 < |y - s(x)| < d$ (cf [1, 13]). Since ψ_- is Borel summable in $\mathcal{R}_1$, $\psi_-^{(0)}$ and ϕ are also Borel summable there. Hence if x is contained in the neighborhood, $\phi_{\mathrm{B}}(x,y)$, $\psi_{-,\mathrm{B}}^{(0)}(x,y)$ and $\psi_{-,\mathrm{B}}(x,y)$ can be analytically continued in y to the sets

$$\{y \mid 0 < |y - t(x)| < d/2\} \cup \{y \mid \mathrm{Re}\,(y - t(x)) > 0,\ |\mathrm{Im}\ (y - t(x))| < d/2\},$$

$$\{y \mid 0 < |y - \tilde{s}(x)| < d/2\} \cup \{y \mid \mathrm{Re}\,(y - \tilde{s}(x)) > 0,\ |\mathrm{Im}\ (y - \tilde{s}(x))| < d/2\}$$

and

$$\{y \mid 0 < |y - s(x)| < d/2\} \cup \{y \mid \mathrm{Re}\,(y - s(x)) > 0,\ |\mathrm{Im}\ (y - s(x))| < d/2\},$$

respectively and have exponential growth as $y \to \infty$. Since $t(x) \to 0$ as $x \to 0$, $\phi_{\mathrm{B}}(x,y)$ is holomorphic in

$$\{y \mid 0 < |y| < d/3\} \cup \{y \mid \mathrm{Re}\,y > 0,\ |\mathrm{Im}\ y| < d/3\}$$

in y if $|x|$ is sufficiently small. Taking the Borel sum of (4.8), we have

$$\Psi_-^1(x,\eta) = \exp(\hat{V}_0^1) \int_{\tilde{s}(x)}^{\infty} \psi_{-,\mathrm{B}}^{(0)}(x,y) \exp(-y\eta) dy.$$

Eq. (4.32) implies the right-hand side can be written as

$$\begin{aligned} x^{1-\frac{c}{2}} \exp(\hat{V}_0^1) &\int_{t(x)}^{\infty} \phi_B(x,y) \exp(-y\eta) dy \\ &= x^{1-\frac{c}{2}} \exp(\hat{V}_0^1) \int_{t(x)}^{\infty} \varphi_B(x, y - t(x)) \exp(-y\eta) dy. \end{aligned}$$

We choose d_0 as $0 < d_0 < d/4$. Then these integrals are well-defined when $x \to 0$ in $\mathcal{R}_1$ satisfying $|\mathrm{Im}\, s(x)| = d_0$. Therefore we have

$$\lim_{\substack{x \to 0,\, x \in \mathcal{R}_1 \\ |\mathrm{Im}\, s(x)| = d_0}} x^{c-1} p(x) \Psi_-^1 = \exp(\hat{V}_0^1) \int_0^{\infty} \varphi_{\mathrm{B}}(0,y) \exp(-y\eta) dy. \tag{4.35}$$

On the other hand, (4.30) implies

$$\phi_{\mathrm{B}}(0,y) = \sum_{j=0}^{\infty} \frac{\phi_j(0)}{\Gamma(j + \frac{1}{2})} y^{j - \frac{1}{2}}.$$

Hence the right-hand side of (4.35) can be evaluated by using term-by-term integration and (4.29):

$$\exp(\hat{V}_0^1) \sum_{j=0}^{\infty} \frac{\phi_j(0)}{\Gamma(j + \frac{1}{2})} \int_0^{\infty} y^{j - \frac{1}{2}} \exp(-y\eta) dt = \sqrt{\frac{2}{c-1}} \exp \hat{V}_0^1.$$

This completes the proof. □

Next we consider the case where $(\alpha, \beta, \gamma) \in \omega_2, \omega_3$ or ω_4. Since $\operatorname{Re}\gamma > 0$ for these cases, we may assume the relations of the form

$$(u_1, u_5) = p(x)(\Psi_+^{1,k}, \Psi_-^{1,k}) \begin{pmatrix} c_{11}^k & c_{12}^k \\ 0 & c_{22}^k \end{pmatrix}$$

for $k = 2, 3, 4$ (cf. (4.26)). Here $\Psi_\pm^{1,k}$ denotes the Borel sum of $\psi_\pm$ in $\mathcal{R}_1$ for the case $(\alpha, \beta, \gamma) \in \omega_k$. We can evaluate c_{11}^k and c_{22}^k similarly to the case of ω_1 and have

$$c_{11}^k = \sqrt{\frac{c-1}{2}} \exp \hat{V}_0^k, \quad c_{22}^k = \sqrt{\frac{c-1}{2}} \exp(-\hat{V}_0^k)$$

for $k = 2, 3, 4$. The relation between c_{12}^k and c_{22}^k depends on k.

If $(\alpha, \beta, \gamma) \in \omega_2$, the type of the Stokes geometry is $(4, 1, 1)$. A typical Stokes curves of (4.3) is given in Figure 4.8:

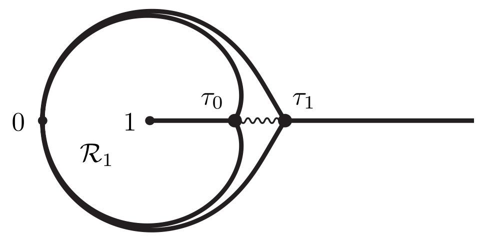

Figure 4.8. An example of Stokes geometry of (4.3) for ω_2.

We take τ_0 and τ_1 as in (4.13). There is a unique Stokes curve emanating from τ_0 and flowing into $x = 1$. We label this s_{01}. Other two Stokes curves emanating from τ_0 flow into the origin. These curves are labeled s_{00+} and s_{00-} so that they are contained in $\operatorname{Im} x \geq 0$ and in $\operatorname{Im} x \leq 0$, respectively. Let $\mathcal{R}_1$ denote the region which is surrounded by s_{01}, s_{00+} and s_{00-}. Let x_0 be a point in $\mathcal{R}_1$ which is close to τ_0 and $\operatorname{Im} x_0 > 0$. Let Γ_0 be a closed path starting from and returning to x_0 which detours the origin once counterclockwise. We assume that Γ_0 does not cross s_{01}. Then the monodromy matrix M_0 of Γ_0 of the basis $(\Psi_+^{1,2}, \Psi_-^{1,2})$ is computed as

$$M_0 = \begin{pmatrix} e^{\pi ic} & -ie^{\pi ic}(e^{2\pi i(b-c)} - 1)(e^{2\pi i(a-c)} - 1) \\ 0 & e^{-\pi ic} \end{pmatrix}.$$

Using the relation obtained by the analytic continuation of (4.26) along Γ_0, we have

$$c_{12}^2 = i\frac{(e^{2\pi i(b-c)} - 1)(e^{2\pi i(a-c)} - 1)e^{2\pi ic}}{e^{2\pi ic} - 1} c_{22}^2.$$

Other cases can be managed similarly. What we need is to specify the region $\mathcal{R}_1$ of taking the Borel sums. If $(\alpha, \beta, \gamma) \in \omega_3$, the type of the Stokes geometry is $(1, 4, 1)$. We take τ_0 and τ_1 as in (4.12). There is a unique Stokes curve emanating from τ_0 and flowing into the origin. We label this curve s_{00}. Other two Stokes curves emanating from τ_0 flow into $x = 1$. We label these curves s_{01+} and s_{01-}. We take $\mathcal{R}_1$ as the region surrounded by s_{00}, s_{01+} and s_{01-}. Let x_0 be a point in $\mathcal{R}_1$ close to τ_0 and $\operatorname{Im} x_0 > 0$ and let Γ_0 be a simple closed path in $\mathcal{R}_1 \cup s_{00}$ with the base point x_0 detouring the origin counterclockwise. We have the monodromy matrix M_0 of Γ_0 with respect to $(\Psi_+^{1,3}, \Psi_-^{1,3})$ of the form

$$M_0 = \begin{pmatrix} e^{\pi i c} & -ie^{\pi i c} \\ 0 & e^{-\pi i c} \end{pmatrix} \tag{4.36}$$

and hence the relation

$$c_{12}^3 = i\frac{e^{2\pi i c}}{e^{2\pi i c} - 1} c_{22}^3.$$

If $(\alpha, \beta, \gamma) \in \omega_4$, the type of the Stokes geometry is $(1, 1, 4)$. We take τ_0 and τ_1 as in (4.13). One of the Stokes curves emanating from τ_0 flows into the origin, which is denoted by s_{00}, and other two go to the infinity. Let $\mathcal{R}_1$ be the region surrounded by these Stokes curves. We take a base point x_0 in $\mathcal{R}_1$ close τ_0, $\operatorname{Im} x_0 > 0$ and a simple closed path Γ_0 in $\mathcal{R}_1 \cup s_{00}$ with the base point x_0 detouring the origin counterclockwise. The monodromy matrix of Γ_0 of $(\Psi_+^{1,4}, \Psi_-^{1,4})$ is computed in the form

$$M_0 = \begin{pmatrix} e^{\pi i c} & -ie^{-\pi i c} \\ 0 & e^{-\pi i c} \end{pmatrix} \tag{4.37}$$

and hence we have

$$c_{12}^4 = \frac{i}{e^{2\pi i c} - 1} c_{22}^4.$$

Summing up and restating Theorem 4.9 again as the first case, we have

Theorem 4.10. *Suppose that $(\alpha, \beta, \gamma) \in \omega_k$ $(k = 1, 2, 3, 4)$. Then the Borel sum $(\Psi_+^{1,k}, \Psi_-^{1,k})$ of the WKB solution basis (ψ_+, ψ_-) of (4.3) on the region $\mathcal{R}_1$ specified as above and (u_1, u_5) are related by*

$$(u_1, u_5) = p(x)(\Psi_+^{1,k}, \Psi_-^{1,k}) \begin{pmatrix} c_{11}^k & c_{12}^k \\ 0 & c_{22}^k \end{pmatrix}. \tag{4.38}$$

Here we set $u_1 = {}_2F_1(a, b, c; x)$, $u_5 = x^{1-c}{}_2F_1(a + 1 - c, b + 1 - c, 2 - c; x)$,

$$c_{11}^k = \sqrt{\frac{c-1}{2}} e^{\hat{V}_0^k}, \quad c_{22}^k = \sqrt{\frac{c-1}{2}} e^{-\hat{V}_0^k}, \tag{4.39}$$

$$c_{12}^1 = -i\frac{e^{2\pi i a} - e^{2\pi i c}}{e^{2\pi i c} - 1} \sqrt{\frac{c-1}{2}} e^{-\hat{V}_0^1}, \tag{4.40}$$

$$c_{12}^2 = i\frac{(e^{2\pi i(a-c)} - 1)(e^{2\pi i(b-c)} - 1)e^{2\pi i c}}{e^{2\pi i c} - 1} \sqrt{\frac{c-1}{2}} e^{-\hat{V}_0^2}, \tag{4.41}$$

$$c_{12}^3 = i\frac{e^{2\pi ic}}{e^{2\pi ic}-1}\sqrt{\frac{c-1}{2}}e^{-\hat{V}_0^3}, \tag{4.42}$$

and

$$c_{12}^4 = \frac{i}{e^{2\pi ic}-1}\sqrt{\frac{c-1}{2}}e^{-\hat{V}_0^4}. \tag{4.43}$$

Explicit forms of c_{ij}^k are as follows:

$$c_{11}^1 = \left(\frac{\Gamma(1+b-c)}{2\Gamma(a)\Gamma(b)\Gamma(c-a)}\right)^{\frac{1}{2}}\Gamma(c)e^{\frac{\pi i}{2}\left(a-\frac{1}{2}\right)}, \tag{4.44}$$

$$c_{12}^1 = -i\frac{e^{2\pi ia}-e^{2\pi ic}}{e^{2\pi ic}-1}\left(\frac{\Gamma(a)\Gamma(b)\Gamma(c-a)}{2\Gamma(1+b-c)}\right)^{\frac{1}{2}}\frac{e^{-\frac{\pi i}{2}\left(a-\frac{1}{2}\right)}}{\Gamma(c-1)}, \tag{4.45}$$

$$c_{22}^1 = \left(\frac{\Gamma(a)\Gamma(b)\Gamma(c-a)}{2\Gamma(1+b-c)}\right)^{\frac{1}{2}}\frac{e^{-\frac{\pi i}{2}\left(a-\frac{1}{2}\right)}}{\Gamma(c-1)}, \tag{4.46}$$

$$c_{11}^2 = \frac{\sqrt{\pi}\,\Gamma(c)e^{\frac{\pi i}{2}(a+b-c)}}{(\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c-b))^{\frac{1}{2}}}, \tag{4.47}$$

$$c_{12}^2 = i\frac{(e^{2\pi i(a-c)}-1)(e^{2\pi i(b-c)}-1)e^{2\pi ic}(\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c-b))^{\frac{1}{2}}}{2\sqrt{\pi}(e^{2\pi ic}-1)\Gamma(c-1)e^{\frac{\pi i}{2}(a+b-c)}}, \tag{4.48}$$

$$c_{22}^2 = \frac{(\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c-b))^{\frac{1}{2}}}{2\sqrt{\pi}\Gamma(c-1)e^{\frac{\pi i}{2}(a+b-c)}}, \tag{4.49}$$

$$c_{11}^3 = \left(\frac{\Gamma(1+a-c)\Gamma(1+b-c)}{\pi\Gamma(a)\Gamma(b)}\right)^{\frac{1}{2}}\frac{\Gamma(c)}{2}e^{\frac{\pi i}{2}(1-c)}, \tag{4.50}$$

$$c_{12}^3 = i\frac{e^{2\pi ic}}{e^{2\pi ic}-1}\left(\frac{\Gamma(a)\Gamma(b)}{\Gamma(1+a-c)\Gamma(1+b-c)}\right)^{\frac{1}{2}}\frac{\sqrt{\pi}}{\Gamma(c-1)}e^{\frac{\pi i}{2}(c-1)}, \tag{4.51}$$

$$c_{22}^3 = \left(\frac{\Gamma(a)\Gamma(b)}{\Gamma(1+a-c)\Gamma(1+b-c)}\right)^{\frac{1}{2}}\frac{\sqrt{\pi}}{\Gamma(c-1)}e^{\frac{\pi i}{2}(c-1)}, \tag{4.52}$$

$$c_{11}^4 = \left(\frac{\Gamma(1-a)\Gamma(1+b-c)}{\Gamma(b)\Gamma(c-a)}\right)^{\frac{1}{2}}\frac{\Gamma(c)}{2\sqrt{\pi}}, \tag{4.53}$$

$$c_{12}^4 = \frac{i}{e^{2\pi ic}-1}\left(\frac{\Gamma(b)\Gamma(c-a)}{\Gamma(1-a)\Gamma(1+b-c)}\right)^{\frac{1}{2}}\frac{\sqrt{\pi}}{\Gamma(c-1)}, \tag{4.54}$$

$$c_{22}^4 = \left(\frac{\Gamma(b)\Gamma(c-a)}{\Gamma(1-a)\Gamma(1+b-c)}\right)^{\frac{1}{2}}\frac{\sqrt{\pi}}{\Gamma(c-1)}. \tag{4.55}$$

REMARK 4.11. We can compute the connection coefficients for each of other connected components of Π_k in the same way as given above if $\operatorname{Re}\gamma > 0$. On the other hand, for the case where $\operatorname{Re}\gamma < 0$, the Borel sums of the recessive WKB solutions are multiples of u_5. Thus the connection formula has the form

$$(u_5, u_1) = (p\Psi_+, p\Psi_-)\begin{pmatrix} c_{11} & c_{12} \\ 0 & c_{22} \end{pmatrix}.$$

Our method can be also applied to this case and we can compute the connection matrices.

Remark 4.12. Theorem 4.10 gives the relations between (u_1, u_5) and the Borel sums of WKB solution basis in the region $\mathcal{R}_1$, however, computing these relations in other Stokes regions is easy because we have connection formulas for WKB solutions. To get global information concerning WKB solutions, we need to use WKB solutions $\tilde{\psi}_\pm$ normalized at another turning point τ_1. The relation between $\tilde{\psi}_\pm$ and $\psi_\pm$ is given as

$$\psi_\pm = \exp\left(\pm \int_{\tau_0}^{\tau_1} S_{\mathrm{odd}} dx\right)\tilde{\psi}_\pm.$$

Here the path of integration is chosen suitably. As is discussed in §4.2 and in §4.3, the integral can be replaced by a half of contour integral of S_{odd} along a closed path encircling τ_0 and τ_1 counterclockwise and we have

$$\begin{aligned} \int_{\tau_0}^{\tau_1} S_{\mathrm{odd}} dx &= \frac{1}{2}\oint S_{\mathrm{odd}} dx \\ &= -\pi i \sum_{b\in\{0,1,\infty\}} \operatorname*{Res}_{x=b} S_{\mathrm{odd}} dx. \end{aligned}$$

Under the choice of the branch cut of $\sqrt{Q_0}$ in § 4.2, the sums of the residues of $S_{\mathrm{odd}}dx$ are given in Table 4.6. For other choices of the cut, the possible values of the integral are as follows:

$$\left(\pm\left(a - c + \frac{1}{2}\right) - 1\right)\pi i, \quad \left(\pm\left(b - c + \frac{1}{2}\right) - 1\right)\pi i,$$
$$\left(\pm\left(a - \frac{1}{2}\right) - 1\right)\pi i, \quad \left(\pm\left(b - \frac{1}{2}\right) - 1\right)\pi i.$$

We can choose one by using the information of the branch of S_{-1}. Thus starting from (4.28) or (4.38) and using the connection formulas for WKB solutions $\psi_\pm$, $\tilde{\psi}_\pm$, we find the relation between (u_1, u_5) and the Borel sum of WKB solution-basis (ψ_+, ψ_-) in any Stokes region. Moreover, combining such a relation and the connection formulas for Kummer's solutions, we can express any Kummer's solution in terms of the Borel sums of $\psi_\pm$ in any Stokes region.

Remark 4.13. Combining the formulas given in Theorem 4.10, one can obtain formulas describing the parametric Stokes phenomena for WKB solutions. For example, Eq. (4.38) for $k = 1, 2$ give the relation

$$\begin{aligned} \Psi_+^{1,1} =& c_{11}^2 \left(c_{11}^1\right)^{-1} \Psi_+^{1,2} \\ =& (1 - e^{2\pi i(b-c)})^{\frac{1}{2}} \Psi_+^{1,2}. \end{aligned}$$

This kind of formulas are given in [8, Theorem 4.4] for the case $\alpha_0 = \beta_0 = 1/2$, $\gamma_0 = 1$.

We have to note that in [8], the choice of the branch of WKB solutions is not specified accurately. Hence the formulas given there should be read under suitable choice of the branch of WKB solutions.

5. The confluent hypergeometric differential equation

In this section, we consider the Kummer confluent hypergeometric differential equation with parameters a, $c \in \mathbb{C}$:

$$x\frac{d^2w}{dx^2} + (c-x)\frac{dw}{dx} - aw = 0. \tag{5.1}$$

This equation has four standard solutions [9]:

$$\begin{aligned}
y_1 &= {}_1F_1(a, c; x), \\
y_2 &= x^{1-c}{}_1F_1(a-c+1, 2-c; x), \\
y_3 &= e^x{}_1F_1(c-a, c; x), \\
y_4 &= x^{1-c}e^x{}_1F_1(1-a, 2-c; -x),
\end{aligned}$$

where

$${}_1F_1(a, c; x) = \frac{\Gamma(c)}{\Gamma(a)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)}{\Gamma(c+n)n!}x^n$$

is Kummer's series, which defines an entire function of x.

5.1. The confluent hypergeometric differential equation with a large parameter

We introduce a large parameter η in (5.1) by setting

$$\begin{aligned}
a =& \alpha_0 + \alpha\eta, \\
c =& \gamma_0 + \gamma\eta
\end{aligned}$$

with α, α_0, γ, $\gamma_0 \in \mathbb{C}$ and substituting x by ηx. Then we have

$$x\frac{d^2w}{dx^2} + (c-\eta x)\frac{dw}{dx} - a\eta w = 0 \tag{5.2}$$

or

$$x\frac{d^2w}{dx^2} + (\gamma_0 + \gamma\eta - \eta x)\frac{dw}{dx} - (\alpha_0 + \alpha\eta)\eta w = 0. \tag{5.3}$$

The guage transformation

$$w = \exp\left(\frac{\eta x}{2}\right)x^{-\frac{c}{2}}\psi$$

eliminates the first order term of (5.3) and yields Whittaker's equation:

$$\left(-\frac{d^2}{dx} + \eta^2 Q(x,\eta)\right)\psi = 0, \tag{5.4}$$

where

$$Q = \frac{x^2 + 2(2a-c)\eta^{-1}x + c(c-2)\eta^{-2}}{4x^2}.$$

This can be explicitly written as $Q(x,\eta) = Q_0(x) + \eta^{-1}Q_1(x) + \eta^{-2}Q_2(x)$ with

$$\begin{aligned} Q_0(x) &= \frac{x^2 + 2(2\alpha-\gamma)x + \gamma^2}{4x^2}, \\ Q_1(x) &= \frac{(2\alpha_0-\gamma_0)x + \gamma(\gamma_0-1)}{2x^2}, \\ Q_2(x) &= \frac{\gamma_0(\gamma_0-2)}{4x^2}. \end{aligned}$$

Further transformation $S = \psi'/\psi$ gives us a Riccati equation:

$$\frac{dS}{dx} + S^2 = \eta^2 Q.$$

Note that Q is invariant under the involution ι on parameters defined by

$$\iota(a,c) = (1+a-c, 2-c).$$

The leading term and the subleading term of ι are also denoted by ι:

$$\iota : \begin{cases} (\alpha,\gamma) \mapsto (\alpha-\gamma, -\gamma), \\ (\alpha_0,\gamma_0) \mapsto (1+\alpha_0-\gamma_0, 2-\gamma_0), \end{cases}$$

which preserve Q_j $(j=0,1,2)$.

We set

$$\begin{aligned} E_0 &= \{(\alpha,\gamma) \in \mathbb{C}^2 \,|\, \alpha\gamma(\alpha-\gamma) = 0\}, \\ E_1 &= \{(\alpha,\gamma) \in \mathbb{C}^2 \,|\, \operatorname{Re}\alpha \operatorname{Re}\gamma \operatorname{Re}(\alpha-\gamma) = 0\}. \end{aligned}$$

The condition $(\alpha,\gamma) \notin E_0$ allows us to confirm that there are two distinct simple turning points τ_i $(i=0,1)$ of (5.4) and each of which does not coincide with the singular points of (5.4), namely, the origin and the infinity. Hereafter we assume that $(\alpha,\gamma) \notin E_1$ (hence $(\alpha,\gamma) \notin E_0$). Then there are no Stokes curves connecting turning point(s).

We can construct a formal solution S of (5.1), WKB solutions $\psi_\pm$ and $\psi_\pm^{(0)}$ of (5.4) normalized at τ_0 and at the origin, respectively, as in §2. The odd part S_{odd} of S is also defined and the explicit forms of the leading term S_{-1} and the subleading term $S_{\mathrm{odd},0}$ of S_{odd} have the following forms:

$$S_{-1} = S_{\mathrm{odd},-1} = \frac{\sqrt{x^2 + 2(2\alpha-\gamma)x + \gamma^2}}{2x},$$

$$S_{\mathrm{odd},0} = \frac{(2\alpha_0-\gamma_0)x + \gamma(\gamma_0-1)}{2x\sqrt{x^2 + 2(2\alpha-\gamma)x + \gamma^2}}.$$

We fix the branch of $\sqrt{Q_0}$ near the origin as

$$\sqrt{Q_0} \sim \operatorname{sgn}(\operatorname{Re}\gamma)\frac{\gamma}{2x} \tag{5.5}$$

holds. Then we have

$$\operatorname*{Res}_{x=0} S_{\mathrm{odd}} dx = \pm\frac{c-1}{2}$$

according to the signature of $\operatorname{Re}\gamma$. As in §4, ψ_+ is recessive at the origin and hence THEOREM 3.3 yields the following

THEOREM 5.1. *Suppose that the branch of $\sqrt{Q_0}$ is chosen as* (5.5)*. Then $\psi_+^{(0)}$ is Borel summable in a punctured disk with the center at the origin. Let $\Psi_+^{(0)}$ be the Borel sum of $\psi_+^{(0)}$ in the set. The confluent hypergeometric function and $\Psi_+^{(0)}$ are related by*

$$ {}_1F_1(a,c;\eta x) = \sqrt{\frac{c-1}{2}} e^{\frac{\eta x}{2}} x^{-\frac{c}{2}} \Psi_+^{(0)} \quad \text{if } \operatorname{Re}\gamma > 0$$

and

$$x^{1-c}{}_1F_1(a-c+1, 2-c;\eta x) = \sqrt{\frac{1-c}{2}} e^{\frac{\eta x}{2}} x^{-\frac{c}{2}} \Psi_+^{(0)} \quad \text{if } \operatorname{Re}\gamma < 0.$$

Here the branches of $x^{-\frac{c}{2}}$ and of x^{1-c} are chosen appropriately.

5.2. Stokes graphs and residues

Stokes curves of (5.4) (or of (5.3)) are classified topologically and their types are characterized in terms of (α, γ). Let ω_j $(j = 1, 3, 4)$ be regions in the space of parameters $(\alpha, \gamma) \in \mathbb{C}^2$ defined by

$$\begin{aligned}
\omega_1 &= \{(\alpha,\gamma) \in \mathbb{C}^2 | 0 < \mathrm{Re}\ \alpha < \mathrm{Re}\ \gamma\},\\
\omega_3 &= \{(\alpha,\gamma) \in \mathbb{C}^2 | 0 < \mathrm{Re}\ \gamma < \mathrm{Re}\ \alpha\},\\
\omega_4 &= \{(\alpha,\gamma) \in \mathbb{C}^2 | \mathrm{Re}\ \alpha < 0 < \mathrm{Re}\ \gamma\}.
\end{aligned}$$

Further, we set

$$\Pi_j = \omega_j \cup \iota(\omega_j) \quad (j = 1, 3, 4).$$

Note that

$$\Pi_1 \cup \Pi_3 \cup \Pi_4 = \{(\alpha,\gamma) \in \mathbb{C}^2 | \operatorname{Re}\alpha \operatorname{Re}\gamma \operatorname{Re}(\gamma-\alpha) \neq 0\}.$$

Configuration of ω_j and $\iota(\omega_j)$ $(j = 1, 3, 4)$ in the $\operatorname{Re}\alpha$-$\operatorname{Re}\gamma$ plane is shown in Fig. 5.1.

THEOREM 5.2. ([24]) *For a given pair of (α,γ) which does not belong to E_1, there are three possible destinations of a Stokes curve*: (i) *The origin.* (ii) $+\infty$. (iii) $-\infty$*. Let n_0, n_1 and n_2 denote the number of Stokes curves of* (5.4) *which flow into the origin, the positive infinity and the negative infinity, respectively and let $\boldsymbol{n}$ denote (n_0, n_1, n_2). Then we have*

(1) *If* $(\alpha, \gamma) \in \Pi_1$*, then* $\boldsymbol{n} = (2, 2, 2)$.
(2) *If* $(\alpha, \gamma) \in \Pi_3$*, then* $\boldsymbol{n} = (1, 4, 1)$.
(3) *If* $(\alpha, \gamma) \in \Pi_4$*, then* $\boldsymbol{n} = (1, 1, 4)$.

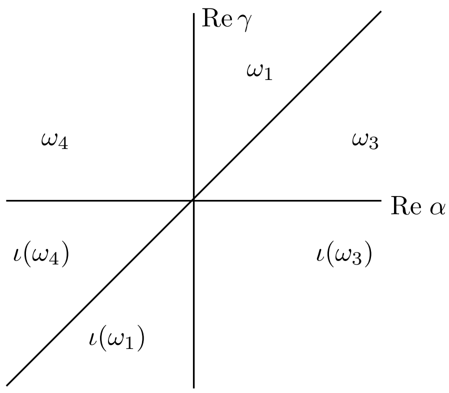

Figure 5.1.

Examples of Stokes curves of (5.4) for the types in the above theorem are given in Figures 5.2–5.4.

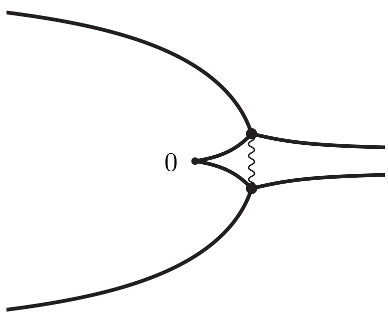

Figure 5.2: type $(2, 2, 2)$

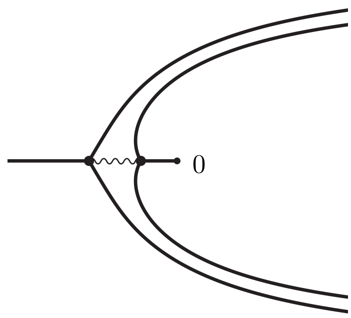

Figure 5.3: type $(1, 4, 1)$

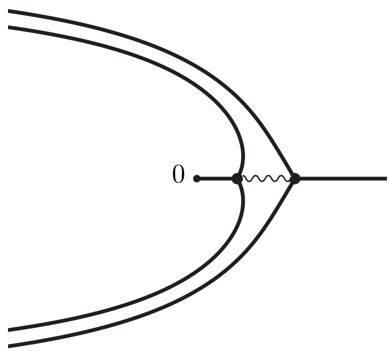

Figure 5.4: type $(1, 1, 4)$

Here the small dots in the figures designate the origin and larger ones turning points. For each cases, we can take the segment connecting two turning points as the branch cut for $\sqrt{Q_0}$. The following table shows the dominance of WKB solutions on the Stokes curves under the choice of the branch as (5.5).

	0	$-\infty$	$+\infty$
ω_1	$-$	$+$	$-$
ω_3	$-$	$-$	$+$
ω_4	$-$	$+$	$-$

Table 5.5 : Dominance of WKB solutions

The residue of $S_{\mathrm{odd}}dx$ at each singularity can be computed and given in the following table.

	0	∞	sum of residues
ω_1	$\dfrac{c-1}{2}$	$\dfrac{2a-c}{2}$	$a-\dfrac{1}{2}$
ω_3	$\dfrac{c-1}{2}$	$\dfrac{c-2a}{2}$	$c-a-\dfrac{1}{2}$
ω_4	$\dfrac{c-1}{2}$	$\dfrac{2a-c}{2}$	$a-\dfrac{1}{2}$

Table 5.6: Residues of $S_{\mathrm{odd}}dx$ at $x = 0, \infty$

5.3. Voros coefficients for the confluent hypergeometric differential equation

The Voros coefficient of the origin of (5.3) or, equivalently, of (5.4), is defined in the same say as is given in §4. Let δ be a path connecting the origin and τ_0. We will specify τ_0 and δ later. Let σ denote the residue of $\eta^{-1}S_{\mathrm{odd}}dx$ at the origin and we set

$$\hat{V}_0 = \int_\delta \left(S_{\mathrm{odd}} - \frac{\sigma\eta}{x} \right) dx + \sigma\eta \log \tau_0. \tag{5.6}$$

We can repeat the formal discussion given in § 4.3. We relate the WKB solutions $\psi_\pm$ of (5.4) normalized at τ_0 and $\psi_\pm^{(0)}$ normalized at the origin:

$$\psi_\pm^{(0)} = \exp(\pm\hat{V}_0)\psi_\pm.$$

We write the formal series $\hat{V}_0$ of η^{-1} in the form

$$\sum_{j=-1}^{\infty} \eta^{-j} V_{0,j}.$$

We decompose this into two parts:

$$\hat{V}_0 = V_0 + \hat{V}_{0,\le 0},$$

where we set

$$V_0 = \sum_{j=1}^{\infty} \eta^{-j} V_{0,j}, \quad \hat{V}_{0,\le 0} = \eta V_{0,-1} + V_{0,0}.$$

Definition 5.3. We call V_0 the Voros coefficient of (5.4) (or of (5.3)) at the origin.

The explicit form of V_0 is obtained by [3] for a special case where $\alpha_0 = 1/2, \gamma_0 = 1$ and by [24] in general.

Theorem 5.4. *Suppose that* $\operatorname{Re}\gamma$ *is positive. Then we have*

$$V_0 = \frac{1}{2}\sum_{n=2}^{\infty} \frac{(-1)^{n-1}\eta^{1-n}}{n(n-1)} \left(\frac{B_n(\alpha_0)}{\alpha^{n-1}} + \frac{B_n(\gamma_0-\alpha_0)}{(\gamma-\alpha)^{n-1}} - \frac{B_n(\gamma_0)+B_n(\gamma_0-1)}{\gamma^{n-1}} \right), \tag{5.7}$$

Here $B_n(x)$ *denotes the* n*-th Bernoulli polynomial.*

Remark 5.5. (i) This is also obtained by formal computation of taking the term-by-term limit $\beta \to \infty$ in (4.11).
(ii) If $\operatorname{Re}\gamma < 0$, we have to multiply the right-hand sides of (5.7) by (-1).
(iii) Since the residues of $S_{\mathrm{odd},>0}dx$ at $x = 0, \infty$ vanish, V_0 does not depend on the choices of the turning point and of the path of integration.
(iv) The Voros coefficient of the irregular singular point $x = \infty$ is also defined as was given in [3] for the special case. If we take the branch of $\sqrt{Q_0}$ as $\sqrt{Q_0} \sim -1/2$ when $x \to +\infty$, the principal part of $S_{\mathrm{odd}}dx$ at the infinity has the form

$$-\frac{\eta}{2} - \left(a - \frac{c}{2}\right)\frac{1}{x}.$$

Hence we define the WKB solutions $\psi_\pm^{(\infty)}$ normalized at the irregular singular point $x = \infty$ by

$$\psi_\pm^{(\infty)} = \frac{x^{\mp\left(a-\frac{c}{2}\right)} \exp\left(\mp\dfrac{\eta x}{2}\right)}{\sqrt{S_{\mathrm{odd}}}} \exp\left(\pm \int_\infty^x \left(S_{\mathrm{odd}} + \left(\frac{\eta}{2} + \left(a - \frac{c}{2}\right)\frac{1}{x}\right)\right) dx \right)$$

and $\hat{V}_\infty$ of (5.4) by

$$\hat{V}_\infty = \int_\infty^{\tau_0} \left(S_{\mathrm{odd}} - \left(\frac{\eta}{2} + \left(a - \frac{c}{2}\right)\frac{1}{x}\right)\right) dx + \frac{\eta\tau_0}{2} + \left(a - \frac{c}{2}\right)\log\tau_0.$$

It is expressed in the form $\hat{V}_\infty = V_\infty + \hat{V}_{\infty,\le 0}$ as above and V_∞ is called the Voros coefficient of (5.4) at the infinity. The explicit form was also given in [24] as follows:

$$V_\infty = -\frac{1}{2}\sum_{n=2}^{\infty}\frac{(-1)^{n-1}\eta^{1-n}}{n(n-1)}\left(\frac{B_n(\alpha_0)}{\alpha^{n-1}} - \frac{B_n(\gamma_0-\alpha_0)}{(\gamma-\alpha)^{n-1}}\right). \tag{5.8}$$

(v) The Voros coefficient for a degenerate case of (5.4) was obtained by Koike and Takei [16]. Our assumption does not cover this case, however, it can be obtained by taking the limit $\gamma \to 0$ and suitable specialization for α, α_0 and γ_0 in (5.8).

From now on, we assume that α, γ, α_0 and γ_0 to be real. We set

$$\tau_\pm = \gamma - 2\alpha \pm 2\sqrt{\alpha(\alpha-\gamma)}.$$

Here the branch of the square root is chosen suitably. Of course $\tau_\pm$ are the roots of the equation $Q_0(x) = 0$. We choose τ_0 and τ_1 as follows:

(i) In the case where $(\alpha,\gamma) \in \omega_1$ or ω_3, we set

$$\tau_0 = \tau_+, \quad \tau_1 = \tau_-. \tag{5.9}$$

Here we take the branch of the square root as $\operatorname{Im}\sqrt{\alpha(\alpha-\gamma)} > 0$ if $(\alpha,\gamma) \in \omega_1$ and $\sqrt{\alpha(\alpha-\gamma)} > 0$ if $(\alpha,\gamma) \in \omega_3$.

(ii) In the case where $(\alpha,\gamma) \in \omega_4$, we set

$$\tau_0 = \tau_-, \quad \tau_1 = \tau_+. \tag{5.10}$$

Here we take the branch of the square root as $\sqrt{\alpha(\alpha-\gamma)} > 0$.

We can choose the path δ of integration in (5.6) in a similar manner to § 4.3. The explicit forms of $\hat{V}_{0,\le 0}$ are obtained also by similar computations to the proof of Theorem 4.5.

Theorem 5.6. (1) *For $(\alpha,\gamma) \in \omega_1$, we have*

$$\hat{V}_{0,\le 0} = \frac{1}{2}\left(\frac{1}{2}-a\right)\log\alpha + \frac{1}{2}\left(\frac{1}{2}-c+a\right)\log(\gamma-\alpha) + (c-1)\log\gamma - \frac{\gamma\eta}{2} + \frac{1}{2}\left(a-\frac{1}{2}\right)\pi i. \tag{5.11}$$

(2) *For $(\alpha,\gamma) \in \omega_3$, we have*

$$\hat{V}_{0,\le 0} = \frac{1}{2}\left(\frac{1}{2}-a\right)\log\alpha + \frac{1}{2}\left(\frac{1}{2}-c+a\right)\log(\alpha-\gamma) + (c-1)\log\gamma - \frac{\eta\gamma}{2} + \frac{1}{2}(1-c)\pi i. \tag{5.12}$$

(3) *For $(\alpha,\gamma) \in \omega_4$, we have*

$$\hat{V}_{0,\le 0} = \frac{1}{2}\left(\frac{1}{2}-a\right)\log(-\alpha) + \frac{1}{2}\left(\frac{1}{2}-c+a\right)\log(\gamma-\alpha) + (c-1)\log\gamma - \frac{\eta\gamma}{2}. \tag{5.13}$$

We can prove this theorem in a similar manner to that of Theorem 4.5 and we omit the proof.

5.4. Relation between WKB solutions and the confluent hypergeometric function

We can apply a similar discussion given in §§4.4 and 4.5 to the Voros coefficient of (5.4) and to the WKB solutions of it. Hence we give our results without proof.

Theorem 5.7. *The Voros coefficient* V_0 *of the origin of* (5.4) *is Borel summable in each connected component of* Π_k*, namely,* ω_k *or* $\iota(\omega_k)$ $(k = 1, 3, 4)$*. Let* V_0^k *denote the Borel sums of* V_0 *for* $(\alpha, \gamma) \in \omega_k$*. We set* $\hat{V}_0^k = V_0^k + \hat{V}_{0,\leq 0}$*. Then we have*

$$\hat{V}_0^1 = \frac{1}{2} \log \frac{\Gamma(c)\Gamma(c-1)e^{\pi i(a-\frac{1}{2})}\eta^{1-c}}{\Gamma(a)\Gamma(c-a)}, \tag{5.14}$$

$$\hat{V}_0^3 = \frac{1}{2} \log \frac{\Gamma(1+a-c)\Gamma(c)\Gamma(c-1)\eta^{1-c}e^{\pi i(1-c)}}{2\pi\Gamma(a)}, \tag{5.15}$$

$$\hat{V}_0^4 = \frac{1}{2} \log \frac{\Gamma(1-a)\Gamma(c)\Gamma(c-1)\eta^{1-c}}{2\pi\Gamma(c-a)}. \tag{5.16}$$

To state the relations between the WKB solutions of (5.4) and the confluent hypergeometric function, we have to specify the Stokes regions $\mathcal{R}_k$ where we take the Borel sum of the WKB solutions. Recall that $\psi_\pm$ denotes the WKB solutions of (5.4) normalized at a simple turning point τ_0.

For the case $(\alpha, \gamma) \in \omega_1$, we take τ_0, τ_1 as (5.9). Since the type of the Stokes geometry is $(2, 2, 2)$, there are six Stokes curves s_{jk} $(j = 0, 1; k = 0, 1, 2)$ such that s_{jk} emanate from τ_j, s_{j0} flow into the origin, s_{j1} into $+\infty$ and s_{j2} into $-\infty$. Let $\mathcal{R}_k$ be the Stokes region surrounded by s_{00}, s_{0k}, s_{1k} and s_{10} $(k = 1, 2)$. We denote by $\Psi_\pm^{k,1}$ and $\Psi_\pm^{(0),k,1}$ be the Borel sums in $\mathcal{R}_k$ of $\psi_\pm$ and $\psi_\pm^{(0)}$, respectively. We take a point x_0 in $\mathcal{R}_1$ which is close to the origin. Then our situation is completely similar to the case $(\alpha, \beta, \gamma) \in \omega_1$ in §4.5. Hence we can take a contour Γ_0 as in the case and compute the monodromy matrix M_0 of Γ_0 of $(\Psi_+^{1,1}, \Psi_-^{1,1})$. The explicit form of M_0 is the same as (4.25). Thus the discussion given in §4.5 works in the present situation.

For the case $(\alpha, \gamma) \in \omega_3$, we take τ_0, τ_1 as (5.9). The type of the Stokes geometry is $(1, 4, 1)$. There is a unique Stokes curve emanating from τ_0 and flowing into the origin. We label this curve s_{00}. Other two Stokes curves emanating from τ_0 flow into $+\infty$. We label these curves s_{01+} and s_{01-}. We take $\mathcal{R}_1$ as the Stokes region surrounded by s_{00}, s_{01+} and s_{01-}. We denote by $(\Psi_+^{1,3}, \Psi_-^{1,3})$ the Borel sum of (ψ_+, ψ_-) in $\mathcal{R}_1$. Let x_0 be a point in $\mathcal{R}_1$ close to τ_0 and $\operatorname{Im} x_0 > 0$ and let Γ_0 be a simple closed path in $\mathcal{R}_1 \cup s_{00}$ with the base point x_0 detouring the origin counterclockwise.

For the case $(\alpha, \gamma) \in \omega_4$, we take τ_0, τ_1 as (5.10). The type of the Stokes geometry is $(1, 1, 4)$. One of the Stokes curves emanating from τ_0 flows into the origin, which is denoted by s_{00}, and other two go to $-\infty$. Let $\mathcal{R}_1$ be the Stokes region surrounded by these Stokes curves. We denote by $(\Psi_+^{1,4}, \Psi_-^{1,4})$ the Borel sum of (ψ_+, ψ_-) in $\mathcal{R}_1$. We take a base point x_0 in $\mathcal{R}_1$ close τ_0, $\mathrm{Im} x_0 > 0$ and a simple closed path Γ_0 in $\mathcal{R}_1 \cup s_{00}$ with the base point x_0 detouring the origin counterclockwise.

The monodromy matrix M_0 of Γ_0 with respect to $(\Psi_+^{1,k}, \Psi_-^{1,k})$ $(k = 3, 4)$ has the same form as is given by (4.36) if $(\alpha, \gamma) \in \omega_3$ and by (4.37) if $(\alpha, \gamma) \in \omega_4$, respectively. Note that the parameter b of (4.1) does not appear in those expressions.

Therefore the discussion given in § 4.5 works in these cases and we have the following theorem.

THEOREM 5.8. *Suppose that $(\alpha, \gamma) \in \omega_k$ ($k = 1, 3$ or 4). Let $\mathcal{R}_1$ be a Stokes region specified as above. Set $v_1 = {}_1F_1(a, c; \eta x)$, $v_2 = x^{1-c}{}_1F_1(a - c + 1, 2 - c : \eta x)$ and $p(x) = e^{\frac{\eta x}{2}} x^{-\frac{c}{2}}$. The Borel sum $(\Psi_+^{1,k}, \Psi_-^{1,k})$ on $\mathcal{R}_1$ of the WKB solution basis (ψ_+, ψ_-) and (v_1, v_2) are related by*

$$(v_1, v_2) = p(x)(\Psi_+^{1,k}, \Psi_-^{1,k}) \begin{pmatrix} c_{11}^k & c_{12}^k \\ 0 & c_{22}^k \end{pmatrix}. \tag{5.17}$$

Here we set

$$c_{11}^k = \sqrt{\frac{c-1}{2}} e^{\hat{V}_0^k}, \quad c_{22}^k = \sqrt{\frac{c-1}{2}} e^{-\hat{V}_0^k} \tag{5.18}$$

with V_0^k given by (5.14), (5.15), (5.16) for $k = 1, 3, 4$ and

$$c_{12}^1 = -i \frac{e^{2\pi i a} - e^{2\pi i c}}{e^{2\pi i c} - 1} \sqrt{\frac{c-1}{2}} e^{-\hat{V}_0^1}, \tag{5.19}$$

$$c_{12}^3 = i \frac{e^{2\pi i c}}{e^{2\pi i c} - 1} \sqrt{\frac{c-1}{2}} e^{-\hat{V}_0^3}, \tag{5.20}$$

and

$$c_{12}^4 = \frac{i}{e^{2\pi i c} - 1} \sqrt{\frac{c-1}{2}} e^{-\hat{V}_0^4}. \tag{5.21}$$

Explicit forms of c_{ij}^k are as follows:

$$c_{11}^1 = \frac{\Gamma(c) e^{\frac{\pi i}{2}(a - \frac{1}{2})}}{(2\Gamma(a)\Gamma(c-a))^{\frac{1}{2}}} \eta^{\frac{1-c}{2}}, \tag{5.22}$$

$$c_{12}^1 = -i \frac{e^{2\pi i a} - e^{2\pi i c}}{e^{2\pi i c} - 1} \frac{(\Gamma(a)\Gamma(c-a))^{\frac{1}{2}}}{\sqrt{2}\Gamma(c-1)} e^{-\frac{\pi i}{2}(a - \frac{1}{2})} \eta^{\frac{c-1}{2}}, \tag{5.23}$$

$$c_{22}^1 = \frac{(\Gamma(a)\Gamma(c-a))^{\frac{1}{2}}}{\sqrt{2}\Gamma(c-1)} e^{-\frac{\pi i}{2}(a - \frac{1}{2})} \eta^{\frac{c-1}{2}}, \tag{5.24}$$

$$c_{11}^3 = \left(\frac{\Gamma(1 + a - c)}{\Gamma(a)} \right)^{\frac{1}{2}} \frac{\Gamma(c)}{2\sqrt{\pi}} e^{\frac{1-c}{2}\pi i} \eta^{\frac{1-c}{2}}, \tag{5.25}$$

$$c_{12}^3 = i \frac{e^{2\pi i c}}{e^{2\pi i c} - 1} \left(\frac{\Gamma(a)}{\Gamma(1 + a - c)} \right)^{\frac{1}{2}} \frac{\sqrt{\pi}}{\Gamma(c-1)} e^{\frac{c-1}{2}\pi i} \eta^{\frac{c-1}{2}}, \tag{5.26}$$

$$c_{22}^3 = \left(\frac{\Gamma(a)}{\Gamma(1 + a - c)} \right)^{\frac{1}{2}} \frac{\sqrt{\pi}}{\Gamma(c-1)} e^{\frac{c-1}{2}\pi i} \eta^{\frac{c-1}{2}}, \tag{5.27}$$

$$c_{11}^4 = \left(\frac{\Gamma(1-a)}{\Gamma(c-a)}\right)^{\frac{1}{2}} \frac{\Gamma(c)}{2\sqrt{\pi}}\eta^{\frac{1-c}{2}}, \tag{5.28}$$

$$c_{12}^4 = \frac{i}{e^{2\pi ic}-1}\left(\frac{\Gamma(c-a)}{\Gamma(1-a)}\right)^{\frac{1}{2}} \frac{\sqrt{\pi}}{\Gamma(c-1)}\eta^{\frac{c-1}{2}}, \tag{5.29}$$

$$c_{22}^4 = \left(\frac{\Gamma(c-a)}{\Gamma(1-a)}\right)^{\frac{1}{2}} \frac{\sqrt{\pi}}{\Gamma(c-1)}\eta^{\frac{c-1}{2}}. \tag{5.30}$$

6. Confluence

In this section, we will see that our results given respectively in the preceding two sections are consistent with the procedure of confluence of the singularities $x = 1$ and $x = \infty$ of the Gauss hypergeometric differential equation. In addition, we can see this observation works for further confluence.

Throughout this section, we assume $\operatorname{Re}\gamma > 0$ unless otherwise stated.

6.1. From Gauss to Kummer

As is well known, replacing x by x/b in (4.1) and taking the limit $b \to \infty$, we obtain (5.1). Hence we have (5.4) if we substitute x by x/β in (4.3) and take the limit $\beta \to \infty$. To distinguish the notation given in §4 and in §5, we use the superscripts G and K, respectively. That is, Q, $\psi_\pm$, S, S_{odd}, p, τ_0, V_0 and c_{ij}^k in §4 (resp. in §5) will be denoted by Q^{G}, $\psi_\pm^{\mathrm{G}}$, S^{G}, $S_{\mathrm{odd}}^{\mathrm{G}}$, p^{G}, τ_0^{G}, V_0^{G} and $c_{ij}^{\mathrm{G},k}$ (resp. Q^{K}, $\psi_\pm^{\mathrm{K}}$, S^{K}, $S_{\mathrm{odd}}^{\mathrm{K}}$, p^{K}, τ_0^{K}, V_0^{K} and $c_{ij}^{\mathrm{K},k}$). Then we can see immediately that

$$\lim_{\beta\to\infty} \frac{1}{\beta^2} Q_j^{\mathrm{G}}\left(\frac{x}{\beta}\right) = Q_j^{\mathrm{K}}(x)$$

holds for $j = 0, 1, 2$. Hence we have the following relations of term-by-term limits:

$$\lim_{\beta\to+\infty} \frac{1}{\beta} S^{\mathrm{G}}\left(\frac{x}{\beta}, \eta\right) = S^{\mathrm{K}}(x,\eta),$$

$$\lim_{\beta\to+\infty} \frac{1}{\beta} S_{\mathrm{odd}}^{\mathrm{G}}\left(\frac{x}{\beta}, \eta\right) = S_{\mathrm{odd}}^{\mathrm{K}}(x,\eta),$$

$$\lim_{\beta\to+\infty} \beta^{\frac{1}{2}} \psi_\pm^{\mathrm{G}}\left(\frac{x}{\beta}; \eta\right) = \psi_\pm^{\mathrm{K}}(x,\eta),$$

$$\lim_{\beta\to+\infty} \beta^{-\frac{c}{2}} p^{\mathrm{G}}\left(\frac{x}{\beta}\right) = p^{\mathrm{K}}(x).$$

We observe that if we replace x by x/β in (4.3), $V_{0,\le 0}^{\mathrm{G}}$ turns out to be

$$V_{0,\le 0}^{\mathrm{G}} + (c-1)/2 \log\beta.$$

On the other hand, the Voros coefficient V_0^{G} is invariant under this replacement since it is defined by using $S^{\mathrm{G}}_{\mathrm{odd},>0}dx$. Hence, as we noted in REMARK 5.5 (i),

$$\lim_{\beta\to\infty} V_0^{\mathrm{G}} = V_0^{\mathrm{K}}$$

and we have

$$\lim_{\beta\to+\infty}\left(\hat{V}_0^{\mathrm{G}} + \frac{c-1}{2}\log\beta\right) = \hat{V}_0^{\mathrm{K}}.$$

For the Borel sums of $\hat{V}_0^{\mathrm{G}}$, we also have

$$\lim_{\beta\to+\infty}\left(\hat{V}_0^{\mathrm{G},k} + \frac{c-1}{2}\log\beta\right) = \hat{V}_0^{\mathrm{K},k}$$

for $k = 1, 3, 4$. Similarly, we have

$$\lim_{\beta\to+\infty} \beta^{\frac{c-1}{2}} c_{11}^{\mathrm{G},k} = c_{11}^{\mathrm{K},k},$$

$$\lim_{\beta\to+\infty} \beta^{\frac{1-c}{2}} c_{12}^{\mathrm{G},k} = c_{12}^{\mathrm{K},k}$$

and

$$\lim_{\beta\to+\infty} \beta^{\frac{1-c}{2}} c_{22}^{\mathrm{G},k} = c_{22}^{\mathrm{K},k}$$

for $k = 1, 3, 4$. Since

$$\lim_{\beta\to+\infty} u_1\left(\frac{x}{\beta}\right) = v_1(x),$$

and

$$\lim_{\beta\to+\infty} \beta^{1-c} u_5\left(\frac{x}{\beta}\right) = v_2(x),$$

we have

$$\lim_{\beta\to+\infty} \beta^{\frac{1}{2}}\Psi_+^{\mathrm{G},k}\left(\frac{x}{\beta},\eta\right) = \Psi_+^{\mathrm{K},k}(x,\eta),$$

$$\lim_{\beta\to+\infty} \beta^{\frac{1}{2}}\Psi_-^{\mathrm{G},k}\left(\frac{x}{\beta},\eta\right) = \Psi_-^{\mathrm{K},k}(x,\eta)$$

for $k = 1, 3, 4$ and we see that (5.17) and (4.38) are consistent with the confluence. We note that such a consistency holds for the Voros coefficients V_∞^{G} and V_∞^{K} (see (5.8)) at the infinity. We do not define the Voros coefficient V_∞^{G} at the infinity for (4.1) (or (4.2), (2.1)) in this paper, but it can be defined and computed explicitly in a similar manner. See [3, 8] for a special case.

6.2. From Kummer to Bessel

Replacing x by x/α in (5.2) and letting $\alpha \to \infty$, we have

$$x\frac{d^2w}{dx^2} + c\frac{dw}{dx} - \eta^2 w = 0, \tag{6.1}$$

which has standard solutions

$$\begin{aligned} w_1 &= {}_0F_1(c;\eta^2 x) = \sum_{n=0}^{\infty} \frac{\Gamma(c)}{\Gamma(c+n)n!}(\eta^2 x)^n, \\ w_2 &= x^{1-c}{}_0F_1(2-c;\eta^2 x). \end{aligned}$$

Schrödinger form of (6.1) is as follows:

$$\left(-\frac{d^2}{dx} + \eta^2 Q(x,\eta)\right)\psi = 0, \tag{6.2}$$

where we set $w = x^{-\frac{c}{2}}\psi$, $Q = Q_0 + \eta^{-1}Q_1 + \eta^{-2}Q_2$ and

$$Q_0 = \frac{4x+\gamma^2}{4x^2}, \quad Q_1 = \frac{\gamma(\gamma_0 - 1)}{2x^2}, \quad Q_2 = \frac{\gamma_0(\gamma_0-2)}{4x^2}.$$

The Voros coefficient V_0^{C} of (6.2) (or of (6.1)) of the origin and $\hat{V}_{0,\le 0}^{\mathrm{C}}$ are defined in a similar way to §§ 4, 5 and we have:

$$V_0^{\mathrm{C}} = -\frac{1}{2}\sum_{n=2}^{\infty} \frac{(-1)^{n-1}\eta^{1-n}}{n(n-1)} \frac{B_n(\gamma_0) + B_n(\gamma_0 - 1)}{\gamma^{n-1}}, \tag{6.3}$$

$$\hat{V}_{0,\le 0}^{\mathrm{C}} = (c-1)\log\gamma - \gamma\eta + \frac{1}{2}(1-c)\pi i. \tag{6.4}$$

These are obtained from (5.7), (5.11) by taking limit:

$$\lim_{\alpha\to\infty} V_0^{\mathrm{K}} = V_0^{\mathrm{C}}, \quad \lim_{\alpha\to+\infty}\left(\hat{V}_{0,\le 0}^{\mathrm{K}} + \frac{c-1}{2}\log\alpha\right) = \hat{V}_{0,\le 0}^{\mathrm{C}}.$$

If $\operatorname{Re}\gamma > 0$, (w_1, w_2) and the Borel sum $(\Psi_+^{\mathrm{C}}, \Psi_-^{\mathrm{C}})$ of the WKB solution basis (ψ_+, ψ_-) of (6.2) near the origin (normalized at a turning point) is related by

$$(w_1, w_2) = (x^{-\frac{c}{2}}\Psi_+^{\mathrm{C}}, x^{-\frac{c}{2}}\Psi_-^{\mathrm{C}}) \begin{pmatrix} \dfrac{\Gamma(c)}{2\sqrt{\pi}}\eta^{1-c}e^{\frac{1}{2}(1-c)\pi i} & i\dfrac{e^{2\pi i c}}{e^{2\pi i c}-1}\dfrac{\sqrt{\pi}}{\Gamma(c-1)}\eta^{c-1}e^{\frac{1}{2}(c-1)\pi i} \\ 0 & \dfrac{\sqrt{\pi}}{\Gamma(c-1)}\eta^{c-1}e^{\frac{1}{2}(c-1)\pi i} \end{pmatrix},$$

where the branches of power functions of x are chosen suitably. This is obtained from (5.17) ($k = 3$) and (5.25)–(5.27) by taking the limit as $\alpha \to \infty$.

It is well known that setting $x = -t^2/4$ and $w = t^{1-c}v$ in (6.1) yields the Bessel equation (with a large parameter):

$$\frac{d^2v}{dt^2} + \frac{1}{t}\frac{dv}{dt} + \left(\eta^2 - \frac{(1-c)^2}{t^2}\right) v = 0. \tag{6.5}$$

The Voros coefficient V_0^{Be} of this equation of the origin is defined and computed in [25]:

$$V_0^{\mathrm{Be}} = \frac{1}{2}\sum_{n=2}^{\infty} \frac{(-1)^n \eta^{1-n}}{n(n-1)} \frac{B_n(\nu_0) + B_n(\nu_0+1)}{\nu^{n-1}} \tag{6.6}$$

under the notation $c = 1 - \nu_0 - \nu\eta$ (hence $\mathrm{Re}\,\gamma < 0$ if $\mathrm{Re}\,\nu > 0$). Setting $\gamma = -\nu$ and $\gamma_0 = 1 - \nu_0$ in (6.3), we reproduce (6.6) up to signature which comes from the choice of the branch of the leading term of S_{-1} in the WKB solutions.

We note that the Voros coefficient of (6.1) (or of (6.2), (6.5)) at the infinity is defined similarly, however, it becomes trivial. This is consistent with the fact that the term-by-term limit of (5.8) as $\alpha \to \infty$ vanishes.

6.3. From Kummer to Hermite-Weber

The Hermite-Weber equation with the large parameter η is obtained from (5.2) by setting $c = 1/2$ and $x = t^2$:

$$\frac{d^2w}{dt^2} - 2\eta t\frac{dw}{dt} - 4a\eta w = 0.$$

Setting $t = z/\sqrt{2}$ and $w = e^{\eta z^2/4}\psi$, we have Weber's equation of the form

$$\left(-\frac{d^2}{dz^2} + \eta^2\left(\frac{z^2}{4} - \left(\nu + \left(\nu_0 + \frac{1}{2}\right)\eta^{-1}\right)\right)\right)\psi = 0 \tag{6.7}$$

with $\nu = -2\alpha$, $\nu_0 = -2\alpha_0$ ($a = \alpha_0 + \alpha\eta$). The Voros coefficient V^{HW} of this equation (at the infinity) can be defined similarly to Remark 5.5, (iv) and the explicit form is obtained in the form

$$V^{\mathrm{HW}} = \frac{1}{2}\sum_{n=2}^{\infty} \frac{(-1)^{n-1}\eta^{1-n}}{n(n-1)} \frac{B_n(\nu_0+1)}{\nu^{n-1}}. \tag{6.8}$$

Setting $\gamma_0 = 1/2$ and taking the limit as $\gamma \to 0$ in (5.8), we obtain (6.8). Takei [26] defined and gave an expression of the Voros coefficient of (6.7) for the case where $\nu = \lambda$, $\nu_0 = -1/2$ (see [23] also). We reproduce the expression, namely Sato's conjecture, as this special case.

We know that the Voros coefficient for the Airy equation is trivial and this corresponds to, roughly speaking, that the term-by-term limit as $\nu \to \infty$ of (6.8) vanishes. Hence this limit is also consistent with the confluent diagram for the hypergeometric differential equations.

7. Applications

7.1. Asymptotic expansion formulas of the hypergeometric function with respect to the large parameter

We employ the same notation as in §4. Combining Theorem 4.1 and Watson's lemma, we obtain an asymptotic expansion formula in terms of the WKB solution $\psi_+^{(0)}$ for ${}_2F_1(a,b,c;x)$ with $(a,b,c) = (\alpha_0 + \alpha\eta, \beta_0 + \beta\eta, \gamma_0 + \gamma\eta)$ as $\eta \to +\infty$ in a neighborhood of the origin. There are many works concerning asymptotic expansions of the hypergeometric function and the confluent hypergeometric functions with respect to parameters ([14, 17, 18, 19, 20, 21, 22, 28, 29, 31] and the references cited in these works). Most of them are interested in the case where $(\alpha,\beta,\gamma) \in E_1 \cup E_2$, while we assume that $(\alpha,\beta,\gamma) \notin E_1 \cup E_2$ and x is not on the Stokes curves. We assume that $\alpha,\beta,\gamma,\alpha_0,\beta_0,\gamma_0$ are real.

Theorem 7.1. *Suppose that γ is positive and $(\alpha,\beta,\gamma) \notin E_1 \cup E_2$. Let U be a small neighborhood of the origin. Then the hypergeometric function has the following asymptotic expansion as $\eta \to \infty$:*

$$ {}_2F_1(a,b,c;x) \sim \sqrt{\frac{c-1}{2}}\, x^{-\frac{c}{2}}(1-x)^{-\frac{1}{2}(a+b+1-c)}\psi_+^{(0)} \quad (x \in U). $$

The origin is a removable singularity of the right-hand side. This asymptotic expansion formula is valid for x in the union of Stokes regions that intersect U and the Stokes curve(s) flowing into the origin.

Asymptotic expansion formulas of the standard basis (u_1,u_5) of the hypergeometric differential equation in terms of (ψ_+,ψ_-) follow from Theorem 4.10.

Theorem 7.2. *Let k denote one of $1,2,3,4$. Suppose that $(\alpha,\beta,\gamma) \in \omega_k$. Let $\mathcal{R}_1$ be the Stokes region specified in §4. Then (u_1,u_5) has the following asymptotic expansion for $x \in \mathcal{R}_1$ as $\eta \to \infty$:*

$$ (u_1,u_5) \sim p(x)(\psi_+,\psi_-)\begin{pmatrix} \hat{c}_{11}^k & \hat{c}_{12}^k \\ 0 & \hat{c}_{22}^k \end{pmatrix} \tag{7.1} $$

with

$$ \hat{c}_{11}^k = \sqrt{\frac{c-1}{2}}e^{\hat{V}_0}, \quad \hat{c}_{22}^k = \sqrt{\frac{c-1}{2}}e^{-\hat{V}_0}, \tag{7.2} $$

$$ \hat{c}_{12}^1 = -i\frac{e^{2\pi ia} - e^{2\pi ic}}{e^{2\pi ic} - 1}\sqrt{\frac{c-1}{2}}e^{-\hat{V}_0}, \tag{7.3} $$

$$ \hat{c}_{12}^2 = i\frac{(e^{2\pi i(a-c)} - 1)(e^{2\pi i(b-c)} - 1)e^{2\pi ic}}{e^{2\pi ic} - 1}\sqrt{\frac{c-1}{2}}e^{-\hat{V}_0}, \tag{7.4} $$

$$ \hat{c}_{12}^3 = i\frac{e^{2\pi ic}}{e^{2\pi ic} - 1}\sqrt{\frac{c-1}{2}}e^{-\hat{V}_0}, \tag{7.5} $$

and

$$\hat{c}_{12}^4 = \frac{i}{e^{2\pi i c} - 1}\sqrt{\frac{c-1}{2}}e^{-\hat{V}_0}. \tag{7.6}$$

Here $\hat{V}_0 = V_0 + \hat{V}_{0,\leq 0}$ *and the explicit form of* $\hat{V}_{0,\leq 0}$ *is given in* THEOREM 4.5*, which depends on* k.

To get an asymptotic expansion formula of ${}_2F_1(a,b,c;x)$ for x away from $\mathcal{R}_1$, we need to use connection formulas of WKB solutions (cf. REMARK 4.12). For example, we consider the case where $(\alpha,\beta,\gamma) \in \omega_4$. Recall that an example of the configuration of the Stokes curves is shown in Figure 4.5 and that the Stokes region $\mathcal{R}_1$ is specified in § 4.5 for ω_4. The turning points τ_0 and τ_1 are taken as (4.13). The Stokes curves emanating from τ_0 are labeled s_{01}, s_{02}, s_{03} counterclockwise so that s_{01} flows into the origin. Similarly, the Stokes curves emanating from τ_1 are labeled s_{11}, s_{12}, s_{13} counterclockwise so that s_{11} flows into $x=1$. Let $\mathcal{R}_2$ be the Stokes region surrounded s_{02}, s_{13}, s_{12} and s_{03}. Let $\mathcal{R}_3$ be the Stokes region surrounded by s_{11}, s_{12} and s_{13}. We consider the intersection of $\mathcal{R}_1$ (resp. $\mathcal{R}_3$) and a small open disc with the center at τ_0 (resp. τ_1). This set has two connected components. Let D_1 (resp. D_3) denote one of them having a subset of $s_{01} \cup s_{03} \cup \{\tau_0\}$ (resp. $s_{11} \cup s_{12} \cup \{\tau_1\}$) as a part of its boundary. We denote by $\psi_\pm$ (reps. $\tilde{\psi}_\pm$) the WKB solutions normalized at τ_0 (resp. τ_1). Let $\Psi_\pm^k$ and $\tilde{\Psi}_\pm^k$ $(k=1,3)$ be the Borel sums of $\psi_\pm$ and $\tilde{\psi}_\pm$ in D_k, respectively. Let $\Psi_\pm^2$ and $\tilde{\Psi}_\pm^2$ be the Borel sums of $\psi_\pm$ and $\tilde{\psi}_\pm$ in $\mathcal{R}_2$, respectively. Since $(\alpha,\beta,\gamma) \in \omega_4$, we have $\gamma > 0$, $\alpha+\beta-\gamma > 0$ and $\alpha - \beta < 0$. Hence our choice of the branch cut and the branch of $\sqrt{Q_0}$ yield

$$\int_{\tau_0}^{\tau_1} S_{\mathrm{odd}} dx = \left(a - \frac{1}{2}\right)\pi i.$$

Here the path of integration is taken so that it is included in the closure of $\mathcal{R}_2$. Hence we have

$$\psi_\pm = \tilde{\psi}_\pm \exp\left(\pm\left(a-\frac{1}{2}\right)\pi\right),$$

$$\Psi_\pm^2 = \tilde{\Psi}_\pm^2 \exp\left(\pm\left(a-\frac{1}{2}\right)\pi\right),$$

$$\Psi_\pm^3 = \tilde{\Psi}_\pm^3 \exp\left(\pm\left(a-\frac{1}{2}\right)\pi\right).$$

Applying the Voros connection formulas ([13], [30]), we obtain the following relations:

$$(\Psi_+^1, \Psi_-^1) = (\Psi_+^2, \Psi_-^2)\begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix},$$

$$(\tilde{\Psi}_+^2, \tilde{\Psi}_-^2) = (\tilde{\Psi}_+^3, \tilde{\Psi}_-^3)\begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix}.$$

On the other hand, we know the relation (Theorem 4.10, (4.38), (4.53))

$$(u_1, u_5) = p(x)(\Psi^1_+, \Psi^1_-) \begin{pmatrix} c^4_{11} & c^4_{12} \\ 0 & c^4_{22} \end{pmatrix}.$$

Here the superscript $k = 4$ in $\Psi_\pm$ is omitted. Thus we have

$$(u_1, u_5) = p\,(\tilde{\Psi}^3_+, \tilde{\Psi}^3_-) \begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix} \begin{pmatrix} -ie^{a\pi i} & 0 \\ 0 & ie^{-a\pi i} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix} \begin{pmatrix} c^4_{11} & c^4_{12} \\ 0 & c^4_{22} \end{pmatrix},$$

or, in terms of (Ψ^3_+, Ψ^3_-),

$$(u_1, u_5) = p\,(\Psi^3_+, \Psi^3_-) \begin{pmatrix} 1 & 0 \\ i(e^{2a\pi i} - 1) & 1 \end{pmatrix} \begin{pmatrix} c^4_{11} & c^4_{12} \\ 0 & c^4_{22} \end{pmatrix}.$$

Taking analytic continuation, we see that these relations are valid for all $x \in \mathcal{R}_3$. If we replace (Ψ^3_+, Ψ^3_-) by the WKB solutions (ψ_+, ψ_-) and c^k_{ij} by their formal expressions, we have the following asymptotic expansion formula for $x \in \mathcal{R}_3$ as $\eta \to \infty$:

$$(u_1, u_5) \sim \sqrt{\frac{c-1}{2}}\, p(x)(\psi_+, \psi_-) \begin{pmatrix} 1 & 0 \\ i(e^{2a\pi i} - 1) & 1 \end{pmatrix} \begin{pmatrix} e^{\hat{V}_0} & \dfrac{ie^{-\hat{V}_0}}{e^{2c\pi i} - 1} \\ 0 & e^{-\hat{V}_0} \end{pmatrix}. \tag{7.7}$$

7.2. Asymptotic expansion formulas of the confluent hypergeometric function with respect to the large parameter

We use the same notation as in § 5. Combining Theorem 5.4 and Watson's lemma, we obtain an asymptotic expansion formula in terms of the WKB solution $\psi^{(0)}_+$ for ${}_1F_1(a, c; \eta x)$ with $(a, c) = (\alpha_0 + \alpha\eta, \gamma_0 + \gamma\eta)$ as $\eta \to +\infty$ in a neighborhood of the origin. We assume that $\alpha, \gamma, \alpha_0, \gamma_0$ are real.

Theorem 7.3. *Suppose that γ is positive and $(\alpha, \beta, \gamma) \notin E_1$. Let U be a small neighborhood of the origin. Then the confluent hypergeometric function has the following asymptotic expansion as $\eta \to \infty$:*

$${}_1F_1(a, c; \eta x) \sim \sqrt{\frac{c-1}{2}}\, e^{\frac{\eta x}{2}} x^{-\frac{c}{2}} \psi^{(0)}_+ \quad (x \in U).$$

The origin is a removable singularity of the right-hand side. This asymptotic expansion formula is valid for x in the union of Stokes regions that intersect U and the Stokes curve(s) flowing into the origin.

Asymptotic expansion formula of the standard basis (v_1, v_2) of the confluent hypergeometric differential equation follows from Theorem 5.8.

Theorem 7.4. *Let k denote one of 1, 3 or 4. Suppose that $(\alpha, \gamma) \in \omega_k$. Let $\mathcal{R}_1$ be the Stokes region specified in §5. Then (v_1, v_2) has the following asymptotic expansion for $\mathcal{R}_1$ as $\eta \to +\infty$:*

$$(v_1, v_2) \sim p(x)(\psi_+, \psi_-) \begin{pmatrix} \hat{c}^k_{11} & \hat{c}^k_{12} \\ 0 & \hat{c}^k_{22} \end{pmatrix} \tag{7.8}$$

with

$$\hat{c}^k_{11} = \sqrt{\frac{c-1}{2}}e^{\hat{V}_0}, \quad \hat{c}^k_{22} = \sqrt{\frac{c-1}{2}}e^{-\hat{V}_0}, \tag{7.9}$$

$$\hat{c}^1_{12} = -i\frac{e^{2\pi i a} - e^{2\pi i c}}{e^{2\pi i c} - 1}\sqrt{\frac{c-1}{2}}e^{-\hat{V}_0}, \tag{7.10}$$

$$\hat{c}^3_{12} = i\frac{e^{2\pi i c}}{e^{2\pi i c} - 1}\sqrt{\frac{c-1}{2}}e^{-\hat{V}_0}, \tag{7.11}$$

and

$$\hat{c}^4_{12} = \frac{i}{e^{2\pi i c} - 1}\sqrt{\frac{c-1}{2}}e^{-\hat{V}_0}. \tag{7.12}$$

Here $\hat{V}_0 = V_0 + \hat{V}_{0,\leq 0}$ *and the explicit form of* $\hat{V}_{0,\leq 0}$ *is given in* THEOREM 5.6*, which depends on* k*.*

As in § 6, we may obtain asymptotic expansion formula for the confluent hypergeometric function for x away from the above set by using the Voros connections formulas.

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