AUSLANDER'S DEFECTS OVER EXTRIANGULATED CATEGORIES: AN APPLICATION FOR THE GENERAL HEART CONSTRUCTION

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Abstract. The notion of extriangulated category was introduced by Nakaoka and Palu giving a simultaneous generalization of exact categories and triangulated categories. Our first aim is to provide an extension to extriangulated categories of Auslander's formula: for some extriangulated category $\mathcal{C}$, there exists a localization sequence $\mathsf{def}\,\mathcal{C} \to \mathsf{mod}\,\mathcal{C} \to \mathsf{lex}\,\mathcal{C}$, where $\mathsf{lex}\,\mathcal{C}$ denotes the full subcategory of finitely presented left exact functors and $\mathsf{def}\,\mathcal{C}$ the full subcategory of Auslander's defects. Moreover we provide a connection between the above localization sequence and the Gabriel-Quillen embedding theorem. As an application, we show that the general heart construction of a cotorsion pair $(\mathcal{U}, \mathcal{V})$ in a triangulated category, which was provided by Abe and Nakaoka, is the same as the construction of a localization sequence $\mathsf{def}\,\mathcal{U} \to \mathsf{mod}\,\mathcal{U} \to \mathsf{lex}\,\mathcal{U}$.

Introduction

Recently, the notion of extriangulated category was introduced in [NP] as a simultaneous generalization of exact categories and triangulated categories. It allows us to unify many results on exact categories and triangulated categories in the same framework [ZZ, INP, LN]. A typical example of extriangulated categories (which are possibly neither exact nor triangulated) is an extension-closed subcategory in a triangulated category. Especially, the cotorsion class of a cotorsion pair in a triangulated category has a natural extriangulated structure.

Our first result is a further investigation of the following Auslander's result. It was proved in [Aus] that, for any abelian category $\mathcal{A}$, the Yoneda embedding $\mathbb{Y} : \mathcal{A} \to \mathsf{mod}\,\mathcal{A}$ has an exact left adjoint Q, where $\mathsf{mod}\,\mathcal{A}$ is the category of finitely presented functors from $\mathcal{A}$ to the category of abelian groups. Moreover the adjoint pair gives rise to a localization sequence

$$\begin{tikzcd} \mathsf{def}\,\mathcal{A} \arrow[r] & \mathsf{mod}\,\mathcal{A} \arrow[r, "Q"] \arrow[l, bend left] & \mathcal{A}. \arrow[l, bend left, "\mathbb{Y}"] \end{tikzcd} \tag{0.1}$$

Following [Len], we call this *Auslander's formula*. Here $\mathsf{def}\,\mathcal{A}$ denotes the full subcategory of Auslander's defects in $\mathsf{mod}\,\mathcal{A}$ (see Definition 2.4). The first aim of this article is to present an extension to extriangulated categories of Auslander's formula: for some

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extriangulated categories $\mathcal{C}$, there exists a localization sequence

$$\begin{tikzcd} \mathsf{def}\,\mathcal{C} \arrow[r] & \mathsf{mod}\,\mathcal{C} \arrow[r, "Q"] \arrow[l, bend left] & \mathsf{lex}\,\mathcal{C} \arrow[l, bend left, "R"] \end{tikzcd} \tag{0.2}$$

where $\mathsf{lex}\,\mathcal{C}$ denotes the full subcategory of left exact functors in $\mathsf{mod}\,\mathcal{C}$ (Theorem 2.9). This localization sequence is closely related to the Gabriel-Quillen embedding theorem of exact categories (see Section 3). Furthermore, using the composed functor $E_{\mathcal{C}} := Q \circ \mathbb{Y} : \mathcal{C} \to \mathsf{mod}\,\mathcal{C} \to \mathsf{lex}\,\mathcal{C}$, we provide characterizations for the given extriangulated category $\mathcal{C}$ to be exact or abelian.

Theorem A (Theorem 2.11)**.** Let $\mathcal{C}$ be an extriangulated category with weak-kernels. Then the following hold.

(1) The functor $E_{\mathcal{C}}$ is exact and fully faithful if and only if $\mathcal{C}$ is an exact category.
(2) The functor $E_{\mathcal{C}}$ is an exact equivalence if and only if $\mathcal{C}$ is an abelian category. If this is the case, we have an equivalence $\mathcal{C} \simeq \mathsf{lex}\,\mathcal{C}$.

Our second result is an application for a cotorsion pair $(\mathcal{U}, \mathcal{V})$ in a triangulated category $\mathcal{T}$. In [Nak, AN], it was proved that there exists an abelian category $\underline{\mathcal{H}}$ associated to the cotorsion pair, called the heart, and a cohomological functor $\mathbb{H} : \mathcal{T} \to \underline{\mathcal{H}}$. This result has been shown for two extremal cases [BBD, KZ], namely, t-structures and 2-cluster tilting subcategories (see [Nak, Proposition 2.6] for details). Since the cotorsion class $\mathcal{U}$ has a natural extriangulated structure, our first result shows the existence of the localization sequence $\mathsf{def}\,\mathcal{U} \to \mathsf{mod}\,\mathcal{U} \to \mathsf{lex}\,\mathcal{U}$. Using this localization, we provide a good understanding for the heart and the cohomological functor.

Theorem B (Theorem 4.7)**.** Let $(\mathcal{U}, \mathcal{V})$ be a cotorsion pair in a triangulated category. Then the heart $\underline{\mathcal{H}}$ of $(\mathcal{U}, \mathcal{V})$ is naturally equivalent to $\mathsf{lex}\,\mathcal{U}$.

The article is organized as follows: In Section 1, we deal with the definitions and basic properties concerning the Serre quotient and the extriangulated category. In Section 2, we construct the localization sequence (0.2) and prove Theorem A. In Section 3, we expand the sequence (0.2) by taking direct colimits and explain how it relates to the Gabriel-Quillen embedding theorem. Section 4 is devoted to a proof of Theorem B.

Notation and convention. All categories and functors appearing in this article are additive, unless otherwise specified. The symbols $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and etc. always denote additive categories, and the set of morphisms $X \to Y$ in $\mathcal{C}$ is denoted by $\mathcal{C}(X, Y)$ or simply denoted by (X, Y) if there is no confusion. If there exists a fully faithful functor $\mathcal{U} \hookrightarrow \mathcal{C}$, we often regard $\mathcal{U}$ as a full subcategory of $\mathcal{C}$. All subcategories in a given additive category are assumed to be full, additive and closed under isomorphisms. We denote by $\mathcal{C}/[\mathcal{U}]$ the ideal quotient category of $\mathcal{C}$ modulo the (two-sided) ideal $[\mathcal{U}]$ in $\mathcal{C}$ consisting of all morphisms having a factorization through an object in $\mathcal{U}$. Consider a functor $F : \mathcal{C} \to \mathcal{B}$. We define the *image* and the *kernel* of F as the full subcategories

$$\operatorname{Im} F := \{Y \in \mathcal{B} \mid {}^{\exists}X \in \mathcal{C},\ F(X) \cong Y\} \quad \text{and} \quad \operatorname{Ker} F := \{X \in \mathcal{C} \mid F(X) = 0\},$$

respectively. For a full subcategory $\mathcal{U}$ in $\mathcal{C}$, the symbol $F|_{\mathcal{U}}$ denotes the restriction of F on $\mathcal{U}$.

Furthermore, we introduce the following notions: For an additive category $\mathcal{C}$, a *(right) $\mathcal{C}$-module* is defined to be a contravariant functor $\mathcal{C} \to \mathsf{Ab}$ and a *morphism* $X \to Y$ between $\mathcal{C}$-modules X and Y is a natural transformation. Thus we define an abelian category $\mathsf{Mod}\,\mathcal{C}$ of $\mathcal{C}$-modules. In the functor category $\mathsf{Mod}\,\mathcal{C}$, the morphism-space $(\mathsf{Mod}\,\mathcal{C})(X,Y)$ is usually denoted by $\mathrm{Hom}_{\mathcal{C}}(X,Y)$. We denote by $\mathsf{mod}\,\mathcal{C}$ the full subcategory of finitely presented $\mathcal{C}$-module in $\mathsf{Mod}\,\mathcal{C}$. Let $\mathcal{U}$ be a full subcategory in $\mathcal{C}$. We denote by $\mathsf{res}_{\mathcal{U}} : \mathsf{Mod}\,\mathcal{C} \to \mathsf{Mod}\,\mathcal{U}$ the restriction functor which sends F to $F|_{\mathcal{U}}$. We call the composed fuctor $\mathsf{res}_{\mathcal{U}} \circ \mathbb{Y} : \mathcal{C} \hookrightarrow \mathsf{Mod}\,\mathcal{C} \to \mathsf{Mod}\,\mathcal{U}$ the *restricted Yoneda functor* of $\mathcal{C}$ relative to $\mathcal{U}$, where $\mathbb{Y}$ is the usual Yoneda embedding. We abbreviate as $\mathbb{Y}_{\mathcal{U}} := \mathsf{res}_{\mathcal{U}} \circ \mathbb{Y}$.

1. Preliminaries

1.1. **The Serre quotient.** We firstly recall the definition and some basic properties of the localization theory of abelian categories. Let $\mathcal{A}$ be an abelian category. We recall that a full subcategory $\mathcal{S}$ in $\mathcal{A}$ is a *Serre subcategory* if, for each exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$, we have $Y \in \mathcal{S}$ if and only if $X, Z \in \mathcal{S}$. In this case, we have a *Serre quotient* $\mathcal{A}/\mathcal{S}$ of $\mathcal{A}$ relative to $\mathcal{S}$ which is known to be abelian. We also recall that the natural quotient functor $Q : \mathcal{A} \to \mathcal{A}/\mathcal{S}$ is exact. We denote this situation by the diagram $\mathcal{S} \to \mathcal{A} \to \mathcal{A}/\mathcal{S}$ of functors.

Proposition 1.1. *Let $\mathcal{A}$ be an abelian category and $\mathcal{S}$ its Serre subcategory. If the quotient functor $Q : \mathcal{A} \to \mathcal{A}/\mathcal{S}$ admits a right adjoint R, then the natural inclusion $\mathcal{S} \hookrightarrow \mathcal{A}$ also admits a right adjoint. This situation will be denoted by the following diagram of functors*

$$\begin{tikzcd} \mathcal{S} \arrow[r] & \mathcal{A} \arrow[l, bend left] \arrow[r, "Q"] & \mathcal{A}/\mathcal{S} \arrow[l, bend left, "R"] \end{tikzcd} \tag{1.1}$$

In this case, the Serre subcategory $\mathcal{S}$ is said to be localizing, *and we call this diagram a* localization sequence *of $\mathcal{A}$ relative to $\mathcal{S}$.*

Dually, we define the *colocalizing subcategory* $\mathcal{S}$ and *colocalization sequence* of $\mathcal{A}$ relative to $\mathcal{S}$. If a given Serre quotient $\mathcal{S} \to \mathcal{A} \to \mathcal{A}/\mathcal{S}$ gives rise to a localizing and colocalizing sequence, we call this a *recollement* of $\mathcal{A}$ relative to $\mathcal{S}$, and denote it as

$$\begin{tikzcd} \mathcal{S} \arrow[r] & \mathcal{A} \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "Q"] & \mathcal{A}/\mathcal{S} \arrow[l, bend right, "L"'] \arrow[l, bend left, "R"] \end{tikzcd} \tag{1.2}$$

The following are standard examples of (co)localization sequences.

Example 1.2. (1) Let $\mathcal{C}$ be an additive category with weak-kernels and $\mathcal{P}$ its contravariantly finite subcategory. Then we have the colocalization sequence below

$$\begin{tikzcd} \mathsf{mod}(\mathcal{C}/[\mathcal{P}]) \arrow[r] & \mathsf{mod}\,\mathcal{C} \arrow[l, bend right] \arrow[r, "\mathsf{res}_{\mathcal{P}}"] & \mathsf{mod}\,\mathcal{P} \arrow[l, bend right, "L"'] \end{tikzcd}$$

(2) Let R be a Noetherian ring with an idempotent e. Then, the sequence of canonical functors $\mathsf{mod}(R/ReR) \to \mathsf{mod}\, R \to \mathsf{mod}\, eRe$ gives rise to a recollement.

We should mention that the converse of Proposition 1.1 does not hold, that is, even if the inclusion $\mathcal{S} \hookrightarrow \mathcal{A}$ admits a right adjoint, the quotient functor $Q : \mathcal{A} \to \mathcal{A}/\mathcal{S}$ does not necessarily admit a right adjoint. The following gives a criterion for a Serre quotient to give rise to a localization sequence, e.g. [Pop, Ch. 4. Thm. 4.5].

Proposition 1.3. *Consider a Serre quotient $\mathcal{S} \to \mathcal{A} \to \mathcal{A}/\mathcal{S}$. Then the following are equivalent:*

(i) *Both the inclusion $\mathcal{S} \hookrightarrow \mathcal{A}$ and the quotient $\mathcal{A} \to \mathcal{A}/\mathcal{S}$ admit right adjoints;*
(ii) *For each $X \in \mathcal{A}$, there exists an exact sequence $S \to X \to Y$ in $\mathcal{A}$ satisfying $S \in \mathcal{S}$ and $\mathcal{A}(S', Y) = 0 = \mathrm{Ext}^1_{\mathcal{A}}(S', Y)$ for any $S' \in \mathcal{S}$.*

In the case that a given category $\mathcal{A}$ is Grothendieck, we have another criterion, e.g. [Pop, Ch. 4. Prop. 6.3].

Proposition 1.4. *Assume that $\mathcal{A}$ is a Grothendieck category. For a Serre quotient $\mathcal{S} \to \mathcal{A} \to \mathcal{A}/\mathcal{S}$, the following are equivalent:*

(i) *Both the inclusion $\mathcal{S} \hookrightarrow \mathcal{A}$ and the quotient $\mathcal{A} \to \mathcal{A}/\mathcal{S}$ admit right adjoints;*
(ii) *The Serre subcategory $\mathcal{S}$ is closed under coproducts.*

The following subcategories associated to a given Serre subcategory $\mathcal{S}$ play important roles to understand the Serre quotient.

Definition 1.5. Let $\mathcal{A}$ be an abelian category and $\mathcal{S}$ its full subcategory. We define the following full subcategories in $\mathcal{A}$.

(1) Denote by $\mathcal{S}^{\perp_0}$ the full subcategory of objects X satisfying $\mathcal{A}(S, X) = 0$ for any $S \in \mathcal{S}$.
(2) Denote by $\mathcal{S}^{\perp_1}$ the full subcategory of objects X satisfying $\mathrm{Ext}^1(S, X) = 0$ for any $S \in \mathcal{S}$.
(3) We put $\mathcal{S}^{\perp} := \mathcal{S}^{\perp_0} \cap \mathcal{S}^{\perp_1}$ which is called a *perpendicular category of* $\mathcal{S}$ in [GL].

The next lemma is a basic property of a localization sequence which will be freely used in many places.

Lemma 1.6. *Let $\mathcal{S} \to \mathcal{A} \xrightarrow{Q} \mathcal{A}/\mathcal{S}$ be a Serre quotient.*

(1) *The quotient functor Q induces an isomorphism $\mathcal{A}(X, Y) \xrightarrow{\sim} (\mathcal{A}/\mathcal{S})(QX, QY)$ for any $X \in \mathcal{A}$ and $Y \in \mathcal{S}^{\perp}$.*
(2) *The quotient functor Q restricts a fully faithful functor $Q|_{\mathcal{S}^{\perp}} : \mathcal{S}^{\perp} \hookrightarrow \mathcal{A}/\mathcal{S}$. Moreover, if Q has a right adjoint, then it is an equivalence.*
(3) *If Q admits a right adjoint R, then R is fully faithful and* $\mathrm{Im}\, R = \mathcal{S}^{\perp}$ *holds.*

The following lemma says that a special adjoint pair gives rise to a localization sequence.

Proposition 1.7. [Pop, Ch. 4. Thm. 4.9] *Let $R : \mathcal{C} \to \mathcal{A}$ be a fully faithful functor between abelian categories. If R admits an exact left adjoint Q, then a sequence of canonical functors* $\mathrm{Ker}\, Q \to \mathcal{A} \to \mathcal{C}$ *gives rise to a localization sequence.*

1.2. **Extriangulated categories.** We recall the definition and some needed properties of extriangulated categories from [NP]. Throughout $\mathcal{C}$ denotes an additive category. The symbol $\mathcal{C}^{\mathsf{op}}$ denotes the opposite category of $\mathcal{C}$.

Definition 1.8. Consider a biadditive functor $\mathbb{E} : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathsf{Ab}$. For any objects $X, Z \in \mathcal{C}$, an element $\delta \in \mathbb{E}(X, Z)$ is called an $\mathbb{E}$-*extension*. Let δ' be an element in $\mathbb{E}(X', Z')$. A *morphism* $(z, x) : \delta \to \delta'$ of $\mathbb{E}$-extensions is a pair of morphisms $x \in \mathcal{C}(X, X')$ and $z \in \mathcal{C}(Z, Z')$ with $z_*\delta = x^*\delta'$, where we set $z_*\delta := \mathbb{E}(X, z)(\delta)$ and $x^*\delta' := \mathbb{E}(x, Z)(\delta')$.

Definition 1.9. Let δ and δ' be the above $\mathbb{E}$-extensions. By the biadditivity of $\mathbb{E}$, we have a natural isomorphism

$$\mathbb{E}(X \oplus X', Z \oplus Z') \cong \mathbb{E}(X, Z) \oplus \mathbb{E}(X, Z') \oplus \mathbb{E}(X', Z) \oplus \mathbb{E}(X', Z').$$

We denote by $\delta \oplus \delta'$ the element in $\mathbb{E}(X \oplus X', Z \oplus Z')$ corresponding to $(\delta, 0, 0, \delta')$ via this isomorphism.

Definition 1.10. Let X and Z be objects in $\mathcal{C}$. Two sequences of the form $Z \xrightarrow{g} Y \xrightarrow{f} X$ and $Z \xrightarrow{g'} Y' \xrightarrow{f'} X$ in $\mathcal{C}$ are said to be *equivalent* if there exists an isomorphism $y : Y \to Y'$ which makes the following diagram commutative.

$$\begin{tikzcd} & Y \arrow[dr, "f"] \arrow[dd, "y", "\cong"'] & \\ Z \arrow[ur, "g"] \arrow[dr, "g'"'] & & X \\ & Y' \arrow[ur, "f'"'] & \end{tikzcd}$$

We denote the equivalence class of $Z \xrightarrow{g} Y \xrightarrow{f} X$ by $[Z \xrightarrow{g} Y \xrightarrow{f} X]$.

Definition 1.11. Let $\mathcal{C}$ and $\mathbb{E}$ be as above.

(1) For any $X, Z \in \mathcal{C}$, we denote as $0 = [Z \xrightarrow{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} Z \oplus X \xrightarrow{[0\ 1]} X]$.
(2) For any two classes $[Z \xrightarrow{g} Y \xrightarrow{f} X]$ and $[Z' \xrightarrow{g'} Y' \xrightarrow{f'} X']$, we denote by $[Z \xrightarrow{g} Y \xrightarrow{f} X] \oplus [Z' \xrightarrow{g'} Y' \xrightarrow{f'} X']$ the class $[Z \oplus Z' \xrightarrow{g \oplus g'} Y \oplus Y' \xrightarrow{f \oplus f'} X \oplus X']$.

Definition 1.12. Let $\mathfrak{s}$ be a correspondence which associates an equivalence class $\mathfrak{s}(\delta) = [Z \xrightarrow{g} Y \xrightarrow{f} X]$ to any $\mathbb{E}$-extension $\delta \in \mathbb{E}(X, Z)$. This $\mathfrak{s}$ is called a *realization of* $\mathbb{E}$, if it satisfies the following condition

(∗) Let $\delta \in \mathbb{E}(X, Z)$ and $\delta' \in \mathbb{E}(X', Z')$ be $\mathbb{E}$-extensions with

$$\mathfrak{s}(\delta) = [Z \xrightarrow{g} Y \xrightarrow{f} X], \quad \mathfrak{s}(\delta') = [Z' \xrightarrow{g'} Y' \xrightarrow{f'} X'].$$

Then, for any morphism $(z, x) : \delta \to \delta'$, there exists $y \in \mathcal{C}(Y, Y')$ which makes the following diagram commutes:

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "z"'] & Y \arrow[r, "f"] \arrow[d, "y"] & X \arrow[d, "x"] \\ Z' \arrow[r, "g'"] & Y' \arrow[r, "f'"] & X' \end{tikzcd}$$

Under the condition $(*)$, we say that the sequence $Z \xrightarrow{g} Y \xrightarrow{f} X$ *realizes* δ and the triple (z, y, x) *relaizes* (z, x).

Definition 1.13. A realization $\mathfrak{s}$ of $\mathbb{E}$ is said to be *additive*, if it satisfies the following conditions:

(i) For any $X, Z \in \mathcal{C}$, the $\mathbb{E}$-extension $0 \in \mathbb{E}(X, Z)$ satisfies $\mathfrak{s}(0) = 0$;
(ii) For any pair of $\mathbb{E}$-extensions δ and δ', $\mathfrak{s}(\delta \oplus \delta') = \mathfrak{s}(\delta) \oplus \mathfrak{s}(\delta')$ holds.

Definition 1.14. The triple $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is called an *extriangulated category* if the following conditions are satisfied:

(ET1) $\mathbb{E} : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathsf{Ab}$ is an additive bifunctor;
(ET2) $\mathfrak{s}$ is an additive realization of $\mathbb{E}$;
(ET3) Let $\delta \in \mathbb{E}(X, Z)$ and $\delta' \in \mathbb{E}(X', Z')$ be $\mathbb{E}$-extensions realized as

$$\mathfrak{s}(\delta) = [Z \xrightarrow{g} Y \xrightarrow{f} X], \quad \mathfrak{s}(\delta') = [Z' \xrightarrow{g'} Y' \xrightarrow{f'} X'].$$

For any commutative square

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "z"'] & Y \arrow[r, "f"] \arrow[d, "y"] & X \\ Z' \arrow[r, "g'"] & Y' \arrow[r, "f'"] & X' \end{tikzcd}$$

in $\mathcal{C}$, there exists a morphism $(z, x) : \delta \to \delta'$ satisfying $xf = f'y$.
(ET3)$^{\mathsf{op}}$ Dual of (ET3).
(ET4) Let $\delta \in \mathbb{E}(X, Z)$ and $\delta' \in \mathbb{E}(A, Y)$ be $\mathbb{E}$-extensions realized by

$$\mathfrak{s}(\delta) = [Z \xrightarrow{g} Y \xrightarrow{f} X], \quad \mathfrak{s}(\delta') = [Y \xrightarrow{b} B \xrightarrow{a} A].$$

Then there exist an object $C \in \mathcal{C}$, a commutative diagram

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, equal] & Y \arrow[r, "f"] \arrow[d, "b"] & X \arrow[d, "b'"] \\ Z \arrow[r, "g'"] & B \arrow[r, "f'"] \arrow[d, "a"] & C \arrow[d, "a'"] \\ & A \arrow[r, equal] & A \end{tikzcd}$$

in $\mathcal{C}$, and an $\mathbb{E}$-extension $\delta'' \in \mathbb{E}(C, Z)$ realized by $Z \xrightarrow{g'} B \xrightarrow{f'} C$, which satisfy the following compatibilities:

(1) $X \xrightarrow{b'} C \xrightarrow{a'} A$ realizes $f_*\delta'$;
(2) $b'^*\delta'' = \delta$;
(3) $g_*\delta'' = a'^*\delta'$.

(ET4)$^{\mathsf{op}}$ Dual of (ET4).

In the rest, the symbol $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ denotes an extriangulated category. We also write $\mathcal{C} := (\mathcal{C}, \mathbb{E}, \mathfrak{s})$, if there is no confusion.

Definition 1.15. Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category.

(1) A sequence $Z \xrightarrow{g} Y \xrightarrow{f} X$ is called a *conflation* if it realizes some $\mathbb{E}$-extension $\delta \in \mathbb{E}(X, Z)$. In this case, we denote the pair $(Z \xrightarrow{g} Y \xrightarrow{f} X, \delta)$ by $Z \xrightarrow{g} Y \xrightarrow{f} X \overset{\delta}{\dashrightarrow}$, which is called an $\mathbb{E}$-*triangle*.
(2) A morphism $g \in \mathcal{C}(Z, Y)$ is called an *inflation* if it admits some conflation $Z \xrightarrow{g} Y \to X$.
(3) A morphism $f \in \mathcal{C}(Y, X)$ is called an *deflation* if it admits some conflation $Z \to Y \xrightarrow{f} X$.

Like the case of exact categories, the following hold for any extriangulated category: The inflations and deflations are closed under composition by (ET4) and (ET4)$^{\mathsf{op}}$, respectively; The finite coproduct of conflations is also a conflation by the additivity of $\mathbb{E}$ and $\mathfrak{s}$.

Next, we define the pullback (resp. pushout) as follows: Let δ be an $\mathbb{E}$-extension with a realization $\mathfrak{s}(\delta) = [Z \xrightarrow{g} Y \xrightarrow{f} X]$ in $\mathcal{C}$. For each morphism $x : X' \to X$, we get an $\mathbb{E}$-extension $x^*\delta$ with a realization $\mathfrak{s}(x^*\delta) = [Z \to E \xrightarrow{f'} X']$. Then we have a morphism (id_Z, x) of $\mathbb{E}$-extensions. Since $\mathfrak{s}$ is additive, there exists a commutative diagram

$$\begin{tikzcd} Z \arrow[r] \arrow[d, equal] & E \arrow[r, "f'"] \arrow[d, "y"'] \arrow[dr, phantom, "\text{(Pb)}"] & X' \arrow[d, "x"] \\ Z \arrow[r] & Y \arrow[r, "f"] & X \end{tikzcd}$$

which realizes (id_Z, x). The commutative square (Pb) is called a *pullback* of a deflation f along x. Dually, a morphism $z : Z \to Z'$ induces a commutative diagram:

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "z"'] \arrow[dr, phantom, "\text{(Po)}"] & Y \arrow[r] \arrow[d, "y'"] & X \arrow[d, equal] \\ Z' \arrow[r, "g'"] & E' \arrow[r] & X \end{tikzcd}$$

The commutative square (Po) is called a *pushout* of a inflation g along z.

Lemma 1.16. [NP, Cor. 3.16] *For the above pullback* (Pb) *and pushout* (Po)*, we have* $\mathbb{E}$-*triangles*

$$E \xrightarrow{\begin{bmatrix} f' \\ y \end{bmatrix}} X' \oplus Y \xrightarrow{[x\ -f]} X \dashrightarrow \quad \text{and} \quad Z \xrightarrow{\begin{bmatrix} g \\ z \end{bmatrix}} Y \oplus Z' \xrightarrow{[y'\ -g']} E' \dashrightarrow,$$

respectively.

Via the Yoneda lemma, any $\mathbb{E}$-extension $\delta \in \mathbb{E}(X, Z)$ corresponds to a morphism $\delta_\sharp : \mathcal{C}(, X) \to \mathbb{E}(-, Z)$.

Lemma 1.17. [NP, Cor. 3.12] *Let* $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ *be an extriangulated category. For any* $\mathbb{E}$-*triangle* $Z \xrightarrow{g} Y \xrightarrow{f} X \overset{\delta}{\dashrightarrow}$*, we have an exact sequence*

$$\mathcal{C}(-, Z) \xrightarrow{g\circ -} \mathcal{C}(-, Y) \xrightarrow{f\circ -} \mathcal{C}(-, X) \xrightarrow{\delta_\sharp} \mathbb{E}(-, Z) \xrightarrow{g\circ -} \mathbb{E}(-, Y) \xrightarrow{f\circ -} \mathbb{E}(-, X)$$

in $\mathsf{Mod}\,\mathcal{C}$.

The extriangulated category is a simultaneous generalization of exact categories and triangulated categories. We shall use the following terminology.

Example 1.18. An exact (resp. triangulated) structure in an additive category $\mathcal{C}$ can give rise to an extriangulated structure. In this case, we say that the extriangulated category is *exact* (resp. *triangulated*) (see [NP, Prop. 3.22, Example 2.13] for details).

We end this section by recalling the following fact for later use.

Proposition 1.19. [NP, Cor. 3.18] *Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category, in which any inflation is a monomorphism and any deflation is an epimorphism. If we let $\mathbb{F}$ be the class of conflations given by the $\mathbb{E}$-triangles, then $(\mathcal{C}, \mathbb{F})$ is an exact category.*

2. Auslander's defects over extriangulated categories

2.1. **The Serre subcategory of defects.** We firstly recall that, for an additive category $\mathcal{C}$, although the subcategory $\mathsf{mod}\,\mathcal{C}$ is closed under cokernels and extensions in $\mathsf{Mod}\,\mathcal{C}$, it is not always abelian since it is not necessarily closed under kernels. In fact, the following lemma is well-known.

Lemma 2.1. [Fre, Thm. 1.4] *The following are equivalent for an additive category $\mathcal{C}$:*

(i) *The category $\mathcal{C}$ admits weak-kernels;*
(ii) *The full subcategory $\mathsf{mod}\,\mathcal{C}$ is an exact abelian subcategory in $\mathsf{Mod}\,\mathcal{C}$, that is, it is abelian and the canonical inclusion $\mathsf{mod}\,\mathcal{C} \hookrightarrow \mathsf{Mod}\,\mathcal{C}$ is exact.*

This subsection is devoted to studying basic properties of *effaceable* functors and Auslander's *defects* in $\mathsf{mod}\,\mathcal{C}$. So, we assume that *an extriangulated category $\mathcal{C} := (\mathcal{C}, \mathbb{E}, \mathfrak{s})$ has weak-kernels.* We shall prove that effaceable functors are nothing other than Auslander's defects and they form a Serre subcategory in $\mathsf{mod}\,\mathcal{C}$. To begin, we study the subcategory $\mathsf{eff}\,\mathcal{C}$ of effaceable functors in $\mathsf{mod}\,\mathcal{C}$ defined as below (see [Kel, p. 30], [Gr, p. 141]):

Definition 2.2. An object $F \in \mathsf{mod}\,\mathcal{C}$ is said to be *effaceable* if it satisfies the following condition:

$$\begin{cases} \text{For any } X \in \mathcal{C} \text{ and any } x \in F(X), \\ \text{there exists a deflation } \alpha : Y \twoheadrightarrow X \text{ such that } F(\alpha)(x) = 0. \end{cases} \tag{2.1}$$

We denote by $\mathsf{eff}\,\mathcal{C}$ the full subcategory of all effaceable functors in $\mathsf{mod}\,\mathcal{C}$.

Lemma 2.3. *Let $\mathcal{C}$ be an extriangulated category with weak-kernels. Then the subcategory $\mathsf{eff}\,\mathcal{C}$ is a Serre subcategory in $\mathsf{mod}\,\mathcal{C}$.*

Proof. To show that $\mathsf{eff}\,\mathcal{C}$ is closed under extensions, let $0 \to S_2 \xrightarrow{f} F \xrightarrow{g} S_1 \to 0$ be an exact sequence in $\mathsf{mod}\,\mathcal{C}$ with $S_1, S_2 \in \mathsf{eff}\,\mathcal{C}$. Consider $X \in \mathcal{C}$ and $x \in F(X)$. Since $S_1 \in \mathsf{eff}\,\mathcal{C}$, for $x_1 := (g_X)(x)$, there exists a deflation $\alpha_1 : Y_1 \to X$ with $S_1(\alpha_1)(x_1) = 0$. It forces that $x' := F(\alpha_1)(x)$ satisfies $(g_{Y_1})(x') = 0$. Hence x' belongs to $S_2(Y_1)$. Since $S_2 \in \mathsf{eff}\,\mathcal{C}$, for $x' \in S_2(Y_1)$, there exists a deflation $\alpha_2 : Y_2 \to Y_1$ with $S_2(\alpha_2)(x') = 0$. It is easily checked that $F(\alpha_2\alpha_1)(x) = 0$. The observation here can be understood by chasing the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & S_2(X) \arrow[r] \arrow[d] & F(X) \arrow[r, "g_X"] \arrow[d, "F(\alpha_1)"'] & S_1(X) \arrow[r] \arrow[d, "S_1(\alpha_1)"] & 0 \\
0 \arrow[r] & S_2(Y_1) \arrow[r] \arrow[d, "S_2(\alpha_2)"'] & F(Y_1) \arrow[r, "g_{Y_1}"] \arrow[d] & S_1(Y_1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S_2(Y_2) \arrow[r] & F(Y_2) \arrow[r] & S_1(Y_2) \arrow[r] & 0
\end{tikzcd}$$

Since $\alpha_2\alpha_1$ is a deflation and $F(\alpha_2\alpha_1)(x) = 0$, we have $F \in \mathsf{eff}\,\mathcal{C}$.

By a similar argument, it is easily checked that $\mathsf{eff}\,\mathcal{C}$ is closed under taking factors and subobjects. $\square$

Next we define defects over extriangulated categories and provide a precise connection to effaceable functors.

Definition 2.4. Let $Z \longrightarrow Y \longrightarrow X \overset{\delta}{\dashrightarrow}$ be an $\mathbb{E}$-triangle in an extriangulated category $\mathcal{C}$. Then we have an exact sequence $(-, Z) \to (-, Y) \to (-, X) \to \tilde{\delta} \to 0$ in $\mathsf{mod}\,\mathcal{C}$. The functor $\tilde{\delta}$ is called a *defect of* δ. We denote by $\mathsf{def}\,\mathcal{C}$ the full subcategory in $\mathsf{mod}\,\mathcal{C}$ consisting of all functors isomorphic to defects.

Note that, by Lemma 1.17, the defect $\tilde{\delta}$ is isomorphic to $\operatorname{Im} \delta_\sharp$. The following proposition shows that defects are nothing other than effaceable functors.

Proposition 2.5. *Let $\mathcal{C}$ be an extriangulated category with weak-kernels. Then we have an equality $\mathsf{eff}\,\mathcal{C} = \mathsf{def}\,\mathcal{C}$. In particular, $\mathsf{def}\,\mathcal{C}$ is a Serre subcategory in $\mathsf{mod}\,\mathcal{C}$.*

To prove this proposition, we shall use the following lemma.

Lemma 2.6. *The subcategory $\mathsf{def}\,\mathcal{C}$ is closed under taking kernels and cokernels in $\mathsf{mod}\,\mathcal{C}$.*

Proof. Let δ and δ' be $\mathbb{E}$-extensions with realizations

$$\mathfrak{s}(\delta) = [Z \xrightarrow{g} Y \xrightarrow{f} X] \quad \text{and} \quad \mathfrak{s}(\delta') = [Z' \xrightarrow{g'} Y' \xrightarrow{f'} X'].$$

Consider a morphism $\alpha : \tilde{\delta}' \to \tilde{\delta}$ in $\mathsf{def}\,\mathcal{C}$. We shall show that $\operatorname{Cok} \alpha$ still belongs to $\mathsf{def}\,\mathcal{C}$. By the Yoneda lemma, the morphism α induces a morphism between presentations of $\tilde{\delta}$ and $\tilde{\delta}'$, and hence we have the following commutative diagram

$$\begin{tikzcd}
Z' \arrow[r, "g'"] & Y' \arrow[r, "f'"] \arrow[d, "y"] & X' \arrow[d, "x"] \\
Z \arrow[r, "g"] & Y \arrow[r, "f"] & X
\end{tikzcd}$$

in $\mathcal{C}$. By taking the pullback (Pb) of the deflation f along x, due to Lemma 1.16, we get a conflation $\delta'' : E \xrightarrow{\left[\begin{smallmatrix} -y_1 \\ a \end{smallmatrix}\right]} Y \oplus X' \xrightarrow{[f\ x]} X$. Hence, we have a morphism $y_2 : Y' \to E$ which makes the following diagram commutative:

$$\begin{tikzcd}
Y' \arrow[rrd, bend left, "f'"] \arrow[rd, dotted, "y_2"] \arrow[rdd, bend right, "y"'] & & \\
& E \arrow[r, "a"] \arrow[d, "y_1"'] \arrow[rd, phantom, "(\mathrm{Pb})"] & X' \arrow[d, "x"] \\
& Y \arrow[r, "f"] & X
\end{tikzcd}$$

Moreover, we construct the following commutative diagram

$$\begin{tikzcd}
Z' \arrow[r, "g'"] & Y' \arrow[r, "f'"] \arrow[d, "y_2"'] & X' \arrow[d, equal] \\
Z \arrow[r, "b"] \arrow[d, equal] & E \arrow[r, "a"] \arrow[d, "y_1"'] \arrow[rd, phantom, "(\mathrm{Pb})"] & X' \arrow[d, "x"] \\
Z \arrow[r, "g"] \arrow[d, "-b"'] & Y \arrow[r, "f"] \arrow[d, "\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]"] & X \arrow[d, equal] \\
E \arrow[r, "\left[\begin{smallmatrix}-y_1\\a\end{smallmatrix}\right]"'] & Y \oplus X' \arrow[r, "{[f\ x]}"'] & X
\end{tikzcd}$$

with the rows being conflations, which induces the following commutative diagram

$$\begin{tikzcd}
(-,Z') \arrow[r, "(-,g')"] & (-,Y') \arrow[r, "(-,f')"] \arrow[d, "(-,y)"'] & (-,X') \arrow[r, "p"] \arrow[d, "(-,x)"] & \tilde{\delta}' \arrow[r] \arrow[d, "\alpha"] & 0 \\
(-,Z) \arrow[r, "(-,g)"] \arrow[d, "(-,-b)"'] & (-,Y) \arrow[r, "(-,f)"] \arrow[d, "(-,\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right])"'] & (-,X) \arrow[r] \arrow[d, equal] & \tilde{\delta} \arrow[r] \arrow[d] & 0 \\
(-,E) \arrow[r] & (-,Y \oplus X') \arrow[r] & (-,X) \arrow[r] & \tilde{\delta}'' \arrow[r] \arrow[d] & 0 \\
& & & 0 &
\end{tikzcd}$$

in $\mathsf{mod}\,\mathcal{C}$ with exact rows. As applications of the snake lemma, we obtain an exact sequence $(-,X') \xrightarrow{\alpha p} \tilde{\delta} \to \tilde{\delta}'' \to 0$. Since p is an epimorphism, we have $\operatorname{Cok}\alpha \cong \tilde{\delta}''$. This shows that $\mathsf{def}\,\mathcal{C}$ is closed under taking cokernels. Dually, it can be checked that $\mathsf{def}\,\mathcal{C}$ is closed under taking kernels. $\square$

We remark that a similar proof appears in [Eno19, Thm. A.2] for exact categories. Now we are in position to prove Proposition 2.5.

Proof of Proposition 2.5. To show $\mathsf{eff}\,\mathcal{C} \subseteq \mathsf{def}\,\mathcal{C}$, let S be an object in $\mathsf{eff}\,\mathcal{C}$. Since $S \in \mathsf{mod}\,\mathcal{C}$, there exists an epimorphism $(-,X) \xrightarrow{x} S \to 0$. Thanks to the Yoneda Lemma, we regard x as an element in $S(X)$. Since S satisfies the condition (2.1), we get a conflation $Z \to Y \xrightarrow{f} X$ such that $S(f)(x) = 0$. Let δ be an $\mathbb{E}$-extension with a realization $\mathfrak{s}(\delta) = [Z \to Y \to X]$. We consider the defect of δ, namely, there exists an exact sequence $(-,Z) \to (-,Y) \xrightarrow{(-,f)} (-,X) \to \tilde{\delta} \to 0$ in $\mathsf{mod}\,\mathcal{C}$. Since $S(f)(x) = 0$ is equivalent to that the composed morphism $(-,Y) \xrightarrow{(-,f)} (-,X) \xrightarrow{x} S$ is zero, we have thus obtained the following commutative diagram

$$\begin{tikzcd}
(-,Z) \arrow[r] & (-,Y) \arrow[r, "(-,f)"] \arrow[dr, "0"'] & (-,X) \arrow[r, two heads] \arrow[d, "x"] & \tilde{\delta} \arrow[r] \arrow[dl, dotted, two heads] & 0 \\
 & & S & &
\end{tikzcd}$$

By the commutativity, the above dotted arrow from $\tilde{\delta}$ to S is an epimorphism. Thus we have an exact sequence $0 \to K \to \tilde{\delta} \to S \to 0$ in $\mathsf{mod}\,\mathcal{C}$. Since $\mathsf{eff}\,\mathcal{C}$ is a Serre subcategory, we get $K \in \mathsf{eff}\,\mathcal{C}$. By a similar argument to the above, there exists a defect $\tilde{\delta}'$ and an epimorphism $\tilde{\delta}' \to K \to 0$. This induces an exact sequence $\tilde{\delta}' \to \tilde{\delta} \to S \to 0$. Thanks to Lemma 2.6, we conclude $S \in \mathsf{def}\,\mathcal{C}$.

To show the converse $\mathsf{eff}\,\mathcal{C} \supseteq \mathsf{def}\,\mathcal{C}$, let δ be an $\mathbb{E}$-extension with a realization $\mathfrak{s}(\delta) = [X'' \xrightarrow{g} X' \xrightarrow{f} X]$ and consider its defect $\tilde{\delta}$, that is, there exists an exact sequence

$$(-,X'') \to (-,X') \to (-,X) \xrightarrow{p} \tilde{\delta} \to 0$$

in $\mathsf{mod}\,\mathcal{C}$. To show $\tilde{\delta} \in \mathsf{eff}\,\mathcal{C}$, fix an object $Y \in \mathcal{C}$ and an element $y \in \tilde{\delta}(Y)$. Since $p_Y : (Y,X) \to \tilde{\delta}(Y)$ is an epimorphism, we get $h \in (Y,X)$ such that $p_Y(h) = y$. By taking the pullback of f along h, we have the following commutative diagram with the rows being conflations:

$$\begin{tikzcd}
X'' \arrow[r, "g"] \arrow[d, equal] & X' \arrow[r, "f"] \arrow[dr, phantom, "\text{(Pb)}"] & X \\
X'' \arrow[r] & Y' \arrow[r, "\alpha"] \arrow[u] & Y \arrow[u, "h"']
\end{tikzcd}$$

By using the Yoneda lemma, we can check that $\tilde{\delta}(\alpha)(y) = 0$, which says $\tilde{\delta} \in \mathsf{eff}\,\mathcal{C}$. We have thus obtained $\mathsf{eff}\,\mathcal{C} = \mathsf{def}\,\mathcal{C}$. □

By the discussion so far, we get a Serre subcategory $\mathsf{eff}\,\mathcal{C} = \mathsf{def}\,\mathcal{C}$ in $\mathsf{mod}\,\mathcal{C}$. Thus we have a Serre quotient of $\mathsf{mod}\,\mathcal{C}$ relative to $\mathsf{def}\,\mathcal{C}$. Since it is basic to study the perpendicular category $(\mathsf{def}\,\mathcal{C})^{\perp}$ to understand the Serre quotient, we shall show that it coincides with the subcategory of left exact functors in $\mathsf{mod}\,\mathcal{C}$.

Definition 2.7. Let $\mathcal{A}$ and $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an abelian category and an extriangulated category, respectively. A contravariant functor $F : \mathcal{C} \to \mathcal{A}$ is said to be

(1) *half-exact*, if F sends a conflation $X \to Y \to Z$ to an exact sequence $FZ \to FY \to FX$;
(2) *left exact*, if F is half-exact and sends a deflation $Y \to Z$ to a monomorphism $FZ \to FY$;
(3) *right exact*, if F is half-exact and sends an inflation $X \to Y$ to an epimorphism $FY \to FX$;
(4) *exact*, if F is left exact and right exact.

We denote by $\mathsf{lex}\,\mathcal{C}$ (resp. $\mathsf{Lex}\,\mathcal{C}$) the full subcategory of all left exact functors in $\mathsf{mod}\,\mathcal{C}$ (resp. $\mathsf{Mod}\,\mathcal{C}$).

Let us remark that, if $\mathcal{C}$ is a triangulated category, the left exact functors should be zero. In fact, any morphism in $\mathcal{C}$ is an inflation as well as a deflation. So, a left exact functor sends any morphism to an isomorphism, in particular, it sends each triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \to X[1]$ to an exact sequence $0 \to F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z)$ with $F(f), F(g)$ being isomorphisms. The exactness forces $F(X) = F(Y) = F(Z) = 0$.

The following shows that the perpendicular category of $\mathsf{def}\,\mathcal{C}$ coincides with $\mathsf{lex}\,\mathcal{C}$.

Proposition 2.8. *Let $\mathcal{C}$ be an extriangulated category with weak-kernels. The following assertions hold.*

(1) *Let $F \in \mathsf{mod}\,\mathcal{C}$. Then, F sends deflations to monomorphisms if and only if $F \in (\mathsf{def}\,\mathcal{C})^{\perp_0}$.*

(2) *We have an equality $(\mathsf{def}\,\mathcal{C})^{\perp} = \mathsf{lex}\,\mathcal{C}$.*

Proof. Let $Z \xrightarrow{g} Y \xrightarrow{f} X \overset{\delta}{\dashrightarrow}$ be an $\mathbb{E}$-triangle in $\mathcal{C}$ and consider an associated exact sequence

$$(-, Z) \to (-, Y) \to (-, X) \to \tilde{\delta} \to 0. \tag{2.2}$$

(1) We assume that F sends deflations to monomorphisms, that is, we have a monomorphism $0 \to F(X) \xrightarrow{F(f)} F(Y)$. We shall show that $(\tilde{\delta}, F) = 0$. Taking a morphism $\alpha \in (\tilde{\delta}, F)$, we get the following commutative diagram:

$$\begin{tikzcd}
(-,Z) \arrow[r] & (-,Y) \arrow[r, "{(-,f)}"] \arrow[drr, "0"'] & (-,X) \arrow[r, two heads] \arrow[dr, "\beta"] & \tilde{\delta} \arrow[r] \arrow[d, "\alpha"] & 0 \\
 & & & F &
\end{tikzcd}$$

By the Yoneda Lemma, we have $F(f)(\beta) = 0$. The injectivity of $F(f)$ forces $\beta = 0$. Hence we have $\alpha = 0$, showing $F \in (\mathsf{def}\,\mathcal{C})^{\perp_0}$.

Assume $F \in (\mathsf{def}\,\mathcal{C})^{\perp_0}$. To show the injectivity of $F(f)$, consider $x \in F(X)$ satisfying $F(f)(x) = 0$. Since the composed morphism $(-, Y) \xrightarrow{(-,f)} (-, X) \xrightarrow{x} F$ is zero, x factors through $\tilde{\delta} \in \mathsf{def}\,\mathcal{C}$. Since $(F, \tilde{\delta}) = 0$, we have $x = 0$.

(2) Consider an object $F \in \mathsf{lex}\,\mathcal{C}$, i.e., we have an exact sequence $0 \to F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z)$. We shall show $\mathrm{Ext}^1(\tilde{\delta}, F) = 0$. By applying $(-, F)$ to an exact sequence $0 \to G \to (-, X) \to \tilde{\delta} \to 0$ obtained from (2.2), we get an exact sequence

$$0 \to (\tilde{\delta}, F) \to F(X) \xrightarrow{\varphi} (G, F) \to \mathrm{Ext}^1(\tilde{\delta}, F) \to 0$$

Thus, it is enough to show that the morphism φ is surjective. For any $h \in (G, F)$, we have the following commutative diagram

$$\begin{tikzcd}
(-,Z) \arrow[rr, "{(-,g)}"] \arrow[ddrrr, "0"'] & & (-,Y) \arrow[rr, "{(-,f)}"] \arrow[dr, "p"'] & & (-,X) \arrow[ddl, dotted, "h'"] \\
 & & & G \arrow[ur] \arrow[d, "h"'] & \\
 & & & F &
\end{tikzcd}$$

In fact, regarding hp as an element in $F(Y)$, we get $F(g)(hp) = 0$. Since $F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z)$ is exact, we have an element $h' \in F(X)$ which satisfies $F(f)(h') = hp$ and corresponds to the dotted arrow in the above diagram. Hence we conclude that φ is surjective. Thus $\mathrm{Ext}^1(\tilde{\delta}, F) = 0$. Hence we have $F \in (\mathsf{def}\,\mathcal{C})^{\perp}$.

Fix $F \in (\mathsf{def}\,\mathcal{C})^{\perp}$. To show that $F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z)$ is exact, we consider an element $y \in F(Y)$ with $F(g)(y) = 0$. We extract an exact sequence $(-,Z) \xrightarrow{(-,g)} (-,Y) \xrightarrow{p} G \to 0$ from (2.2). The assumption implies that the composed morphism $y \circ (-,g)$ is zero. Thus we get a morphism $x : G \to F$ such that $xp = y$, which are depicted as follows:

$$\begin{tikzcd}
(-,Z) \arrow[r, "(-,g)"] \arrow[dr, "0"'] & (-,Y) \arrow[r, "p"] \arrow[d, "y"] & G \arrow[r] \arrow[dl, dotted, "x"] & 0 \\
& F & &
\end{tikzcd}$$

Applying the functor $(-,F)$ to the exact sequence $0 \to G \xrightarrow{\iota} (-,X) \to \tilde{\delta} \to 0$, we have the following exact sequence

$$0 \to (\tilde{\delta}, F) \to F(X) \xrightarrow{\psi} (G,F) \to \mathrm{Ext}^1(\tilde{\delta}, F).$$

Since $\tilde{\delta} \in \mathsf{def}\,\mathcal{C}$ and $F \in (\mathsf{def}\,\mathcal{C})^{\perp}$, the morphism $\psi : F(X) \xrightarrow{\sim} (G,F)$ is an isomorphism. We get $x' := \psi^{-1}(x) \in F(X)$. Regarding x' as a morphism $(-,X) \to F$, we have $y = x \circ p = (x' \circ \iota) \circ p = x' \circ (-,f)$. This means $F(f)(x') = y$, showing the exactness at $F(Y)$. Combining (1), we conclude that F is left exact. $\square$

The following is our first result which is a generalization of Auslander's formula ([Aus], [ARS, IV. 4]).

Theorem 2.9. *Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category with weak-kernels. Then, we have a Serre quotient*

$$\mathsf{def}\,\mathcal{C} \longrightarrow \mathsf{mod}\,\mathcal{C} \xrightarrow{Q} \frac{\mathsf{mod}\,\mathcal{C}}{\mathsf{def}\,\mathcal{C}}. \tag{2.3}$$

Moreover, if the quotient functor Q has a right adjoint, then we have a localization sequence

$$\begin{tikzcd}
\mathsf{def}\,\mathcal{C} \arrow[r] & \mathsf{mod}\,\mathcal{C} \arrow[r, "Q"] \arrow[l, bend left] & \mathsf{lex}\,\mathcal{C} \arrow[l, bend left, "R"]
\end{tikzcd} \tag{2.4}$$

where R deotes the canonical inclusion.

Proof. Lemma 2.3 and Proposition 2.5 show that the subcategory $\mathsf{def}\,\mathcal{C}$ is a Serre subcategory in $\mathsf{mod}\,\mathcal{C}$. Thus we obtain the Serre quotient (2.3). It remains to show that there is an equivalence $\frac{\mathsf{mod}\,\mathcal{C}}{\mathsf{def}\,\mathcal{C}} \simeq \mathsf{lex}\,\mathcal{C}$. This follows from Lemma 1.6 and Proposition 2.8. $\square$

Auslander's formula (0.1) shows that, if $\mathcal{C}$ is abelian, the quotient functor always has a right adjoint. However, even if a given category $\mathcal{C}$ is exact, the quotient functor Q does not necessarily have a right adjoint. The author is grateful to Haruhisa Enomoto for providing to him an example which shows the last fact.

Example 2.10. Let $R := k[\![x, y]\!]$ be a ring of formal power series over a commutative field k. We consider

- the category $\mathsf{mod}\, R$ of finitely generated modules;
- the full subcategory $\mathsf{proj}\, R$ of all finitely generated projectives;
- the Serre subcategory $\mathsf{fl}\, R$ of modules of finite length.

Then we have the quotient functor $Q : \mathsf{mod}\, R \to \frac{\mathsf{mod}\, R}{\mathsf{fl}\, R}$ which does not have a right adjoint. In fact, if Q has a right adjoint R, we have an equivalence $\frac{\mathsf{mod}\, R}{\mathsf{fl}\, R} \simeq (\mathsf{fl}\, R)^{\perp}$ by Lemma 1.6. Furthermore, it is basic that, for each R-module X, it belongs to $(\mathsf{fl}\, R)^{\perp}$ if and only if it satisfies $\mathrm{Hom}_R(k, X) = 0 = \mathrm{Ext}^1_R(k, X)$ if and only if $\mathsf{depth} X \geq 2$. Since the Krull dimension of R is two, we have $(\mathsf{fl}\, R)^{\perp} = \mathsf{proj}\, R$. This contradicts to the fact that $\mathsf{proj}\, R$ is not abelian.

Thanks to [Eno18, Exa. 3.4, Thm. 3.7], there exists an exact structure on $\mathsf{proj}\, R$ which induces an equivalence $\mathsf{fl}\, R \simeq \mathsf{def}(\mathsf{proj}\, R)$. Using an equivalence $\mathsf{mod}\, R \simeq \mathsf{mod}(\mathsf{proj}\, R)$, we have a Serre quotient $\mathsf{mod}(\mathsf{proj}\, R) \to \frac{\mathsf{mod}(\mathsf{proj}\, R)}{\mathsf{def}(\mathsf{proj}\, R)}$ which does not have a right adjoint.

Define the composed functor $E_{\mathcal{C}} := Q\mathbb{Y} : \mathcal{C} \to \frac{\mathsf{mod}\, \mathcal{C}}{\mathsf{def}\, \mathcal{C}}$. The following gives characterizations for $\mathcal{C}$ to be exact or abelian via this functor.

Theorem 2.11. *Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category with weak-kernels. Then the following hold.*

(1) *The functor $E_{\mathcal{C}}$ is exact and fully faithful if and only if $\mathcal{C}$ is an exact category.*
(2) *The functor $E_{\mathcal{C}}$ is an exact equivalence if and only if $\mathcal{C}$ is an abelian category. If this is the case, we have an equivalence $\mathcal{C} \simeq \mathsf{lex}\, \mathcal{C}$.*

In the rest of this subsection, we shall prove Theorem 2.11.

Lemma 2.12. *The functor $E_{\mathcal{C}}$ is right exact.*

Proof. Let $X \longrightarrow Y \longrightarrow Z \overset{\delta}{\dashrightarrow}$ be an $\mathbb{E}$-triangle in $\mathcal{C}$ which defines a defect $\tilde{\delta}$ by the exactness of a sequence $(-, X) \to (-, Y) \to (-, Z) \to \tilde{\delta} \to 0$ in $\mathsf{mod}\, \mathcal{C}$. Since $\tilde{\delta} \in \mathcal{S}$ and the quotient functor Q is exact, the assertion follows. □

Lemma 2.13. *Suppose that $\mathcal{C}$ is an exact category with weak-kernels. Then, the functor $E_{\mathcal{C}} : \mathcal{C} \to \frac{\mathsf{mod}\, \mathcal{C}}{\mathsf{def}\, \mathcal{C}}$ is exact and fully faithful.*

Proof. We shall show the exactness of $E_{\mathcal{C}}$. Note that a conflation $\delta : X \to Y \to Z$ in $\mathcal{C}$ gives rise to a projective resolution of the defect $\tilde{\delta}$. By Lemma 2.12, applying Q to the projective resolution yields an exact sequence $0 \to Q(-, X) \to Q(-, Y) \to Q(-, Z) \to 0$ in $\frac{\mathsf{mod}\, \mathcal{C}}{\mathsf{def}\, \mathcal{C}}$.

We shall show that $E_{\mathcal{C}}$ is fully faithful. Since every representable functor in $\mathsf{mod}\, \mathcal{C}$ is left exact, it belongs to $\mathsf{lex}\, \mathcal{C} = (\mathsf{def}\, \mathcal{C})^{\perp}$. Due to Lemma 1.6, the fully faithfulness of $E_{\mathcal{C}}$ follows. □

We recall the following result from [Aus, p. 205].

Lemma 2.14 (Auslander's formula)**.** *Suppose that $\mathcal{C}$ is abelian. Then, the Yoneda embedding $\mathbb{Y} : \mathcal{C} \hookrightarrow \mathsf{mod}\,\mathcal{C}$ admits an exact left adjoint Q. Moreover, we have a localization sequence:*

$$\begin{tikzcd} \mathsf{def}\,\mathcal{C} \arrow[r] & \mathsf{mod}\,\mathcal{C} \arrow[r, "Q"] \arrow[l, bend left] & \mathcal{C}. \arrow[l, bend left, "\mathbb{Y}"] \end{tikzcd} \tag{2.5}$$

Now we are ready to prove Theorem 2.11.

Proof of Theorem 2.11. (1) The 'if' part follows from Lemma 2.13. To show the 'only if' part, we suppose that $E_{\mathcal{C}}$ is exact and fully faithful. Thanks to Proposition 1.19, we have only to show that, for each conflation $X \xrightarrow{f} Y \xrightarrow{g} Z$, the morphism f is a monomorphism and the morphism g is an epimorphism. Since $E_{\mathcal{C}}$ is exact, we get an exact sequence $0 \to Q(-,X) \to Q(-,Y) \to Q(-,Z) \to 0$ in $\frac{\mathsf{mod}\,\mathcal{C}}{\mathsf{def}\,\mathcal{C}}$. Since $E_{\mathcal{C}}$ is fully faithful, the assertion is obvious.

(2) If $\mathcal{C}$ is abelian, by comparing Auslander's defect formula (2.5) with the Serre quotient $\mathsf{def}\,\mathcal{C} \to \mathsf{mod}\,\mathcal{C} \to \frac{\mathsf{mod}\,\mathcal{C}}{\mathsf{def}\,\mathcal{C}}$, we get an equality $\mathcal{C} = \frac{\mathsf{mod}\,\mathcal{C}}{\mathsf{def}\,\mathcal{C}} = \mathsf{lex}\,\mathcal{C}$ as subcategories in $\mathsf{mod}\,\mathcal{C}$. The 'only if' part is obvious because $\frac{\mathsf{mod}\,\mathcal{C}}{\mathsf{def}\,\mathcal{C}}$ is abelian. □

2.2. **The case of enough projectives.** We study the case that an extriangulated category $\mathcal{C}$ has enough projectives.

Definition 2.15. Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category. We say that $\mathcal{C}$ *has enough projectives* if there exists a full subcategory $\mathcal{P}$ in $\mathcal{C}$ with $\mathbb{E}(\mathcal{P}, \mathcal{C}) = 0$ and, for every $C \in \mathcal{C}$, there exists a conflation $C' \to P \to C$ with $P \in \mathcal{P}$.

In this case, we have nicer forms of the quotient functor $Q : \mathsf{mod}\,\mathcal{C} \to \frac{\mathsf{mod}\,\mathcal{C}}{\mathsf{def}\,\mathcal{C}}$ and the functor $E_{\mathcal{C}} : \mathcal{C} \to \frac{\mathsf{mod}\,\mathcal{C}}{\mathsf{def}\,\mathcal{C}}$.

Proposition 2.16. *Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category with weak-kernels which has enough projectives. Let $\mathcal{P}$ be the subcategory of projectives in $\mathcal{C}$ and consider the restriction functor $\mathsf{res}_{\mathcal{P}} : \mathsf{mod}\,\mathcal{C} \to \mathsf{mod}\,\mathcal{P}$. Then the following hold.*

1. *There exists an equivalence $Q' : \frac{\mathsf{mod}\,\mathcal{C}}{\mathsf{def}\,\mathcal{C}} \simeq \mathsf{mod}\,\mathcal{P}$ with $\mathsf{res}_{\mathcal{P}} \cong Q' \circ Q$.*
2. *The functor $E_{\mathcal{C}}$ is isomorphic to the restricted Yoneda functor $\mathbb{Y}_{\mathcal{P}} : \mathcal{C} \to \mathsf{mod}\,\mathcal{P}$.*
3. *An equality $\mathsf{def}\,\mathcal{C} = \mathsf{mod}(\mathcal{C}/[\mathcal{P}])$ holds in $\mathsf{mod}\,\mathcal{C}$.*

Proof. (1) Let $\tilde{\delta}$ be a defect corresponding to an $\mathbb{E}$-triangle $Z \longrightarrow Y \longrightarrow X \overset{\delta}{\dashrightarrow}$. Since $\tilde{\delta}$ is a subobject of $\mathbb{E}(-,Z)$, it vanishes on $\mathcal{P}$. Thus, the restriction functor $\mathsf{res}_{\mathcal{P}} : \mathsf{mod}\,\mathcal{C} \to \mathsf{mod}\,\mathcal{P}$ vanishes on $\mathsf{def}\,\mathcal{C}$. Since $\mathsf{res}_{\mathcal{P}}$ is exact, by the universality, there uniquely exists an exact functor $Q' : \frac{\mathsf{mod}\,\mathcal{C}}{\mathsf{def}\,\mathcal{C}} \to \mathsf{mod}\,\mathcal{P}$ such that $\mathsf{res}_{\mathcal{P}} \cong Q' \circ Q$. We shall show that Q' is an equivalence.

It is obvious that Q' is full and dense. To show the faithfulness of Q', we consider a morphism $\alpha : F \to G$ in $\mathsf{mod}\,\mathcal{C}$ such that $\mathsf{res}_{\mathcal{P}}(\alpha) = 0$. Note that there exists an epimorphism $(-,C) \to F \to 0$. Since $\mathcal{C}$ has enough projectives, we have an $\mathbb{E}$-triangle

$C' \longrightarrow P \longrightarrow C \overset{\delta}{\dashrightarrow}$ with $P \in \mathcal{P}$. The following commutative diagram shows that the functor $\operatorname{Im}\alpha$ belongs to $\mathsf{def}\,\mathcal{C}$.

$$\begin{tikzcd}
(-,C') \arrow[r] & (-,P) \arrow[r] \arrow[ddr, "0"'] & (-,C) \arrow[r, two heads] \arrow[d, two heads] & \tilde{\delta} \arrow[r] \arrow[ddl] & 0 \\
 & & F \arrow[d, two heads, "\alpha"'] & & \\
 & & \operatorname{Im}\alpha & &
\end{tikzcd}$$

Thus we get $\operatorname{Im}\alpha \in \mathsf{def}\,\mathcal{C}$, which shows that the morphisms α factors through an object in $\mathsf{def}\,\mathcal{C}$. This shows the faithfulness of Q'.

(2) This directly follows from (1). In fact, we have $Q'E_{\mathcal{C}} = Q'Q\mathbb{Y} \cong \mathsf{res}_{\mathcal{P}}\mathbb{Y} = \mathbb{Y}_{\mathcal{P}}$.

(3) Since $\mathcal{P}$ is contravariantly finite in $\mathcal{C}$, by Example 1.2(1), we have a Serre quotient $\mathsf{mod}(\mathcal{C}/[\mathcal{P}]) \to \mathsf{mod}\,\mathcal{C} \to \mathsf{mod}\,\mathcal{P}$. By comparing it with a Serre quotient $\mathsf{def}\,\mathcal{C} \to \mathsf{mod}\,\mathcal{C} \to \frac{\mathsf{mod}\,\mathcal{C}}{\mathsf{def}\,\mathcal{C}}$, we have a desired equality. $\square$

We end this section by showing that, in the case that $\mathcal{C}$ is an exact category having enough projectives, the quotient functor $Q : \mathsf{mod}\,\mathcal{C} \to \frac{\mathsf{mod}\,\mathcal{C}}{\mathsf{def}\,\mathcal{C}} \simeq \mathsf{mod}\,\mathcal{P}$ always admits a right adjoint.

Proposition 2.17. *Let $(\mathcal{C}, \mathbb{E})$ be an exact category with weak-kernels which has enough projectives. Then, the restriction functor $\mathsf{res}_{\mathcal{P}} : \mathsf{mod}\,\mathcal{C} \to \mathsf{mod}\,\mathcal{P}$ admits a right adjoint R. Moreover, it induces a recollement*

$$\begin{tikzcd}
\mathsf{def}\,\mathcal{C} \arrow[r] & \mathsf{mod}\,\mathcal{C} \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "\mathsf{res}_{\mathcal{P}}"] & \mathsf{mod}\,\mathcal{P} \arrow[l, bend right, "L"'] \arrow[l, bend left, "R"]
\end{tikzcd} \tag{2.6}$$

Proof. It is basic that $\mathsf{res}_{\mathcal{P}}$ is full and dense. Clearly, the subcategory $\mathcal{C}$ is contravariantly finite in $\mathsf{mod}\,\mathcal{C}$. By Proposition 2.16, the composition $\mathbb{Y}_{\mathcal{P}} : \mathcal{C} \hookrightarrow \mathsf{mod}\,\mathcal{P}$ is fully faithful. It is straightforward that the subcategory $\mathcal{C}$ is a contravariantly finite in $\mathsf{mod}\,\mathcal{P}$. Therefore, by Example 1.2, we have a colocalization sequence $\mathsf{mod}(\mathsf{mod}\,\mathcal{P}/[\mathcal{C}]) \to \mathsf{mod}(\mathsf{mod}\,\mathcal{P}) \xrightarrow{\mathsf{res}_{\mathcal{C}}} \mathsf{mod}\,\mathcal{C}$, that is, $\mathsf{res}_{\mathcal{C}}$ has a left adjoint L. Let $\mathbb{Y} : \mathsf{mod}\,\mathcal{P} \hookrightarrow \mathsf{mod}(\mathsf{mod}\,\mathcal{P})$ be the Yoneda functor. It is easily checked that the functor $R := \mathsf{res}_{\mathcal{C}} \circ \mathbb{Y} : \mathsf{mod}\,\mathcal{P} \to \mathsf{mod}\,\mathcal{C}$ is a right adjoint of $\mathsf{res}_{\mathcal{P}}$. By Proposition 1.1 and its dual, we have a desired recollement. $\square$

We should mention that there are related results for exact categories with enough projectives [Eno17, Prop. A] and Frobenius exact categories [Che, Thm. 4.2].

3. Direct colimits of Auslander's defects

Let $\mathcal{C}$ be a skeletally small extriangulated category with weak-kernels. We consider an expansion of $\mathsf{eff}\,\mathcal{C}$ by taking direct colimits. To begin, we recall basic propeties of direct colimits in $\mathsf{Mod}\,\mathcal{C}$. A poset $(I, \leq)$ is called a *directed set* if, for any $i, j \in I$ there exists $k \in I$ with $i \leq k, j \leq k$. Regarding a directed set I as a category, for a covariant functor

$F : I \to \mathsf{Mod}\,\mathcal{C}$, the associated colimit $\varinjlim_I F_i$ is called a *direct colimit* of $\{F_i\}_{i\in I}$. For any colimit $\varinjlim_I F_i$, there exists an exact sequence

$$\coprod_{k\in K} F_k \to \coprod_{j\in J} F_j \to \varinjlim_I F_i \to 0$$

for some sets K and J. For any skeletally small additive category $\mathcal{C}$, it is well-known that any object in $\mathsf{Mod}\,\mathcal{C}$ can be obtained as a direct colimit of objects in $\mathsf{mod}\,\mathcal{C}$. We denote by $\mathsf{Def}\,\mathcal{C}$ the full subcategory in $\mathsf{Mod}\,\mathcal{C}$ consisting of direct colimits of objects in $\mathsf{def}\,\mathcal{C}$. Our first aim of this section is to expand Theorem 2.9 by taking direct colimits as follows:

Theorem 3.1. *Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be a skeletally small extriangulated category with weak-kernels. Then, the Serre quotient*

$$\mathsf{def}\,\mathcal{C} \longrightarrow \mathsf{mod}\,\mathcal{C} \xrightarrow{Q} \frac{\mathsf{mod}\,\mathcal{C}}{\mathsf{eff}\,\mathcal{C}}.$$

induces the following localization sequence

$$\begin{tikzcd} \mathsf{Def}\,\mathcal{C} \arrow[r] & \mathsf{Mod}\,\mathcal{C} \arrow[r] \arrow[l, bend left] & \mathsf{Lex}\,\mathcal{C} \arrow[l, bend left, "R"] \end{tikzcd} \tag{3.1}$$

where R denotes the canonical inclusion.

The first Serre quotient is given in Theorem 2.9. We have only to show that the localization sequence (3.1) exists. To show $\mathsf{Def}\,\mathcal{C}$ is a Serre subcategory in $\mathsf{Mod}\,\mathcal{C}$. We consider the effaceable functors in $\mathsf{Mod}\,\mathcal{C}$, that is, we say a functor $F \in \mathsf{Mod}\,\mathcal{C}$ to be *effaceable* if it satisfies the condition (2.1) in Definition 2.2 and denote by $\mathsf{Eff}\,\mathcal{C}$ the full subcategory of effaceable functors. Clearly $\mathsf{eff}\,\mathcal{C} = \mathsf{mod}\,\mathcal{C} \cap \mathsf{Eff}\,\mathcal{C}$ holds. In the following, we shall show that $\mathsf{Eff}\,\mathcal{C}$ coincides with $\mathsf{Def}\,\mathcal{C}$.

Lemma 3.2. *Let $\mathcal{C}$ be a skeletally small extriangulated category with weak-kernels. Then the subcategory $\mathsf{Eff}\,\mathcal{C}$ is localizing in $\mathsf{Mod}\,\mathcal{C}$.*

Proof. By a similar argument given in Lemma 2.3, we can easily check that $\mathsf{Eff}\,\mathcal{C}$ is a Serre subcategory in $\mathsf{Mod}\,\mathcal{C}$.

Thanks to Lemma 1.4, it suffices to show that $\mathsf{Eff}\,\mathcal{C}$ is closed under coproducts. Let $\{S_i\}_{i\in I}$ be a set of objects in $\mathsf{Eff}\,\mathcal{C}$ and put $F := \coprod_{i\in I} S_i$. Consider $X \in \mathcal{C}$ and $x := \{x_i\}_{i\in I} \in F(X)$. There exists a finite subset $J \subseteq I$ such that $x_i = 0$ for $i \in I \setminus J$. For each $j \in J$, since $S_j \in \widetilde{\mathcal{S}}$, we get a deflation $Y_j \xrightarrow{\alpha_j} X$ such that $S_j(\alpha_j)(x_j) = 0$. Thus we have a deflation $\coprod_{j\in J} Y_j \xrightarrow{\alpha} \coprod_{j\in J} X$. Let $\pi : \coprod_{j\in J} X \to X$ be a natural summation morphism. Since π is a splitting epimorphism, we have a deflation $\pi\alpha : \coprod_{j\in J} Y_j \to X$. By a standard argument, we can verify that $F(\pi\alpha)(x) = 0$. This shows $F \in \mathsf{Eff}\,\mathcal{C}$. $\square$

Lemma 3.3. *An equality $\mathsf{Def}\,\mathcal{C} = \mathsf{Eff}\,\mathcal{C}$ holds in $\mathsf{Mod}\,\mathcal{C}$.*

Proof. Let S be an object in $\mathsf{Def}\,\mathcal{C}$. Then there exists an exact sequence

$$\coprod_{i\in I} S_i \to \coprod_{j\in J} S_j \to S \to 0 \tag{3.2}$$

in $\mathsf{Mod}\,\mathcal{C}$ with $S_i, S_j \in \mathsf{def}\,\mathcal{C}$ for any $i \in I, j \in J$. By Lemma 3.2, we get $S \in \mathsf{Eff}\,\mathcal{C}$.

Let S be an object in $\mathsf{Eff}\,\mathcal{C}$ with a set of morphisms $\{f_i : (-, X_i) \to S\}_{i\in I}$ forming an epimorphism $\coprod_{i\in I}(-, X_i) \to S \to 0$. Since $S \in \mathsf{Eff}\,\mathcal{C}$, for each $i \in I$, there exists a deflation $\alpha_i : Y_i \to X_i$ which satisifies that the composition $(-, Y_i) \xrightarrow{(-,\alpha_i)} (-, X_i) \xrightarrow{f_i} S$ is zero. We consider a set of exact sequences $\{(-, Y_i) \xrightarrow{(-,\alpha_i)} (-, X_i) \to S_i \to 0\}_{i\in I}$ and the coproduct of them:

$$\coprod_{i\in I}(-, Y_i) \to \coprod_{i\in I}(-, X_i) \to \coprod_{i\in I} S_i \to 0.$$

Since each S_i belongs to $\mathsf{def}\,\mathcal{C}$, we have $\coprod S_i \in \mathsf{Def}\,\mathcal{C}$. Since S is a factor object of $\coprod S_i$, we conclude that S also belongs to $\mathsf{Def}\,\mathcal{C}$. $\square$

Now we are in position to prove Proposition 3.1.

Proof of Proposition 3.1. By Lemmas 3.2 and 3.3, we have a localizaing subcategory $\mathsf{Def}\,\mathcal{C}$ which gives rise to a localization sequence

$$\begin{tikzcd} \mathsf{Def}\,\mathcal{C} \arrow[r] & \mathsf{Mod}\,\mathcal{C} \arrow[l, bend left] \arrow[r, "Q"] & \frac{\mathsf{Mod}\,\mathcal{C}}{\mathsf{Def}\,\mathcal{C}} \arrow[l, bend left, "R"] \end{tikzcd}.$$

It remains to show an equality $(\mathsf{Def}\,\mathcal{C})^\perp = \mathsf{Lex}\,\mathcal{C}$ in $\mathsf{Mod}\,\mathcal{C}$. Let $F \in \mathsf{Mod}\,\mathcal{C}$. By a similar argument in Proposition 2.8(2), we can easily check $\mathsf{Lex}\,\mathcal{C} \subseteq (\mathsf{def}\,\mathcal{C})^\perp$ and $(\mathsf{Def}\,\mathcal{C})^\perp \subseteq \mathsf{Lex}\,\mathcal{C}$. It remains to check a containment $(\mathsf{def}\,\mathcal{C})^\perp \subseteq (\mathsf{Def}\,\mathcal{C})^\perp$. Applying $(-, F)$ to the sequence (3.2), we have exact sequences

$$0 \to (S, F) \to \prod_{j\in J}(S_j, F) \quad \text{and} \quad 0 \to \mathrm{Ext}^1_{\mathcal{C}}(S, F) \to \prod_{j\in J} \mathrm{Ext}^1_{\mathcal{C}}(S_j, F).$$

Since $(S_j, F) = \mathrm{Ext}^1_{\mathcal{C}}(S_j, F) = 0$ for any $j \in J$, we conclude $F \in (\mathsf{Def}\,\mathcal{C})^\perp$. Thanks to Lemma 1.6(3), we have an equivalence $\frac{\mathsf{Mod}\,\mathcal{C}}{\mathsf{Def}\,\mathcal{C}} \simeq \mathsf{Lex}\,\mathcal{C}$. This finishes the proof. $\square$

We end this section by mentioning that, in the case that $\mathcal{C}$ is exact, the localization sequence (3.1) provides the Gabriel-Quillen embedding functor. The theorem is stated as below. A proof is provided in [Kel, Appendix B] (see also [Buh, Thm. A.1]).

Proposition 3.4. *Let $\mathcal{C}$ be a skeletally small exact category. Then the following hold.*

(1) *There exists a localization sequence*

$$\begin{tikzcd} \mathsf{Eff}\,\mathcal{C} \arrow[r] & \mathsf{Mod}\,\mathcal{C} \arrow[l, bend left] \arrow[r, "Q"] & \mathsf{Lex}\,\mathcal{C} \arrow[l, bend left, "R"] \end{tikzcd}$$

where R denotes a canonical inclusion.

(2) *The composition functor $Q \circ \mathbb{Y} : \mathcal{C} \to \mathsf{Lex}\,\mathcal{C}$ is a fully faithful exact functor which reflects exactness. This functor is known as the* Gabriel-Quillen embedding functor.

Since $\mathsf{Eff}\,\mathcal{C} = \mathsf{Def}\,\mathcal{C}$, if $\mathcal{C}$ is an exact category with weak-kernels, then the above localization sequence coincides with the one (3.1) in Theorem 3.1.

4. General heart construction via Left exact functors

Throughout this section, we fix a triangulated category $\mathcal{T}$ with a translation [1]. A pair $(\mathcal{U}, \mathcal{V})$ of full subcategories in $\mathcal{T}$ is called a *cotorsion pair* if it is closed under direct summands and satisfies the following conditions:

- $\mathcal{T}(U, V[1]) = 0$ for any $U \in \mathcal{U}$ and $V \in \mathcal{V}$;
- For any object X of $\mathcal{T}$, there exists a triangle $U \to X \to V[1] \to U$ with $U \in \mathcal{U}$ and $V \in \mathcal{V}$.

Since $\mathcal{U}$ is extension-closed and contravariantly finite in $\mathcal{T}$, it gives rise to an extriangulated category with weak-kernels by setting $\mathbb{E}(+,-) := \mathcal{U}(+,-[1])$. First we shall show that the quotient functor $Q : \mathsf{mod}\,\mathcal{U} \to \frac{\mathsf{mod}\,\mathcal{U}}{\mathsf{def}\,\mathcal{U}}$ has a right adjoint, and hence we have an equivalence $\frac{\mathsf{mod}\,\mathcal{U}}{\mathsf{def}\,\mathcal{U}} \simeq \mathsf{lex}\,\mathcal{U}$. Let us start with the following easy lemma.

Lemma 4.1. *The following hold.*

(1) *For any $V \in \mathcal{V}$, the functor $(-, V[2])|_{\mathcal{U}}$ belongs to $(\mathsf{def}\,\mathcal{U})^{\perp}$.*
(2) *For any $U \in \mathcal{U}$, the functor $(-, U[1])|_{\mathcal{U}}$ belongs to $\mathsf{def}\,\mathcal{U}$.*

Proof. (1) Since $(\mathcal{U}, V[1]) = 0$, the functor $(-, V[2])|_{\mathcal{U}}$ is left exact.

(2) We shall check that the functor $(-, U[1])|_{\mathcal{U}}$ satisfies the condition (2.1). Consider $X \in \mathcal{U}$ and $x \in (X, U[1])$. Then we get a conflation $U \to U' \xrightarrow{\alpha} X$ which is a part of a triangle $U \to U' \xrightarrow{\alpha} X \xrightarrow{x} U[1]$ in $\mathcal{T}$. Obviously $(X, U[1]) \xrightarrow{-\circ\alpha} (U', U[1])$ sends x to $\alpha \circ x = 0$, which shows $(-, U[1])|_{\mathcal{U}} \in \mathsf{def}\,\mathcal{U}$. $\square$

The next proposition shows the existence of a right adjoint of the quotient functor Q.

Proposition 4.2. *The quotient functor $Q : \mathsf{mod}\,\mathcal{U} \to \frac{\mathsf{mod}\,\mathcal{U}}{\mathsf{def}\,\mathcal{U}}$ has a right adjoint R. Moreover, R induces an equivalence $R : \frac{\mathsf{mod}\,\mathcal{U}}{\mathsf{def}\,\mathcal{U}} \xrightarrow{\sim} \mathsf{lex}\,\mathcal{U}$.*

Proof. Let F be an object in $\mathsf{mod}\,\mathcal{U}$ with a projective presentation $(-, U_1) \xrightarrow{f\circ-} (-, U_0) \to F \to 0$. Thanks to Proposition 1.3, it is enough to show that there exists an exact sequence $S \to F \to G$ with $S \in \mathsf{def}\,\mathcal{U}$ and $G \in (\mathsf{def}\,\mathcal{U})^{\perp}$. The morphism f induces a triangle $K \to U_1 \xrightarrow{f} U_0 \to K[1]$ in $\mathcal{T}$. For the object K, there exists a triangle $U_2 \to K \to V_2[1] \to U_2[1]$ with $U_2 \in \mathcal{U}$ and $V_2 \in \mathcal{V}$ induced by the cotorsion pair $(\mathcal{U}, \mathcal{V})$. We complete the composed morphism $U_2 \to K \to U_1$ to a triangle

$$U_2 \to U_1 \to C \to U_2[1], \tag{4.1}$$

and then we have the commutative diagram below

$$\begin{tikzcd}
 & & V_2[1] \arrow[r, equal] \arrow[d] & V_2[1] \arrow[d] \\
U_2 \arrow[r] \arrow[d] & U_1 \arrow[r] \arrow[d, equal] & C \arrow[r] \arrow[d, "g"] & U_2[1] \arrow[d] \\
K \arrow[r] & U_1 \arrow[r] & U_0 \arrow[r] \arrow[d] & K[1] \arrow[d] \\
 & & V_2[2] \arrow[r, equal] & V_2[2]
\end{tikzcd} \tag{4.2}$$

where all rows and columns are triangles in $\mathcal{T}$. The third column induces an exact sequence

$$0 \to (-, C)|_{\mathcal{U}} \xrightarrow{g\circ -} (-, U_0) \to (-, V_2[2])|_{\mathcal{U}} \tag{4.3}$$

Thus we have a monomorphism $\varphi := (g \circ -) : (-, C)|_{\mathcal{U}} \rightarrowtail (-, U_0)$. The second and third rows in (4.2) induce the following commutative diagram

$$\begin{tikzcd}
(-, U_2) \arrow[r] \arrow[d] & (-, U_1) \arrow[r] \arrow[d, equal] & (-, C)|_{\mathcal{U}} \arrow[r] \arrow[d, tail, "\varphi"'] \arrow[dr, phantom, "(*)"] & S \arrow[r] \arrow[d, tail, "\phi"] & 0 \\
(-, K)|_{\mathcal{U}} \arrow[r] & (-, U_1) \arrow[r, "f\circ -"] & (-, U_0) \arrow[r] & F \arrow[r] & 0
\end{tikzcd}$$

with exact rows. Since the square $(*)$ is a pullback, we have an isomorphism $\operatorname{Cok} \varphi \cong \operatorname{Cok} \phi$. By the sequence (4.3), we have an exact sequence

$$S \xrightarrow{\phi} F \to (-, V_2[2])|_{\mathcal{U}}$$

in $\mathsf{mod}\,\mathcal{U}$. Lemma 4.1 shows $(-, V_2[2])|_{\mathcal{U}} \in (\mathsf{def}\,\mathcal{U})^{\perp}$ and $(-, U_2[1])|_{\mathcal{U}} \in \mathsf{def}\,\mathcal{U}$. Since S is a subobject of $(-, U_2[1])|_{\mathcal{U}}$, we get $S \in \mathsf{def}\,\mathcal{U}$. This finishes the proof. $\square$

By combining Proposition 4.2 and Theorem 3.1, we have the following localization sequence.

Corollary 4.3. *For a cotorsion pair $(\mathcal{U}, \mathcal{V})$ in a triangulated category $\mathcal{T}$, there exists a localization sequence*

$$\begin{tikzcd}
\mathsf{def}\,\mathcal{U} \arrow[r] & \mathsf{mod}\,\mathcal{U} \arrow[r, "Q"] \arrow[l, bend left] & \mathsf{lex}\,\mathcal{U} \arrow[l, bend left, "R"]
\end{tikzcd}$$

where R denotes the canonical inclusion.

Finally we study a connection between $\mathsf{lex}\mathcal{U}$ and the heart of the cotorsion pair $(\mathcal{U}, \mathcal{V})$. Let us introduce the following notion: For two classes $\mathcal{U}$ and $\mathcal{V}$ of objects in $\mathcal{T}$, we denote by $\mathcal{U} * \mathcal{V}$ the class of objects X occurring in a triangle $U \to X \to V \to U[1]$ with $U \in \mathcal{U}$ and $V \in \mathcal{V}$.

Definition 4.4. Let $(\mathcal{U}, \mathcal{V})$ be a cotorsion pair in a triangulated category $\mathcal{T}$. We define the following associated categories:

- Put $\mathcal{W} := \mathcal{U} \cap \mathcal{V}$;
- For a sequence $\mathcal{W} \subseteq \mathcal{S} \subseteq \mathcal{T}$ of subcategories, we put $\underline{\mathcal{S}} := \mathcal{S}/[\mathcal{W}]$ and denote by $\pi : \mathcal{S} \to \underline{\mathcal{S}}$ the canonical ideal quotient functor;
- We put $\mathcal{T}^+ := \mathcal{W} * \mathcal{V}[1], \mathcal{T}^- := \mathcal{U}[-1] * \mathcal{W}$ and $\mathcal{H} := \mathcal{T}^+ \cap \mathcal{T}^-$.

We call the category $\underline{\mathcal{H}}$ the *heart of* $(\mathcal{U}, \mathcal{V})$.

As mentioned in Introduction, the heart $\underline{\mathcal{H}}$ is abelian and there exists a *cohomological* functor $\mathbb{H} : \mathcal{T} \to \underline{\mathcal{H}}$, namely, $\mathbb{H}$ sends any triangle $X \to Y \to Z \to X[1]$ in $\mathcal{T}$ to an exact sequence $\mathbb{H}X \to \mathbb{H}Y \to \mathbb{H}Z \to \mathbb{H}X[1]$ in $\underline{\mathcal{H}}$. We recall a construction of the cohomological functor $\mathbb{H} : \mathcal{T} \to \underline{\mathcal{H}}$. By definition we have the following containments

$$\begin{tikzcd} & \underline{\mathcal{T}}^+ \arrow[dr, hook, "i_+"] & \\ \underline{\mathcal{H}} \arrow[ur, hook] \arrow[dr, hook] \arrow[rr, phantom, "\circlearrowleft"] & & \underline{\mathcal{T}} \\ & \underline{\mathcal{T}}^- \arrow[ur, hook, "i_-"'] & \end{tikzcd}$$

where the arrows denote the canonical inclusions. It is a crucial property of the above inclusions that i_+ and i_- admit a left adjoint and a right adjoint, respectively. First, the following statement says that there exists a nice left $(\mathcal{T}^+)$-approximation (resp. right $(\mathcal{T}^-)$-approximation) for each $X \in \mathcal{T}$.

Proposition 4.5. [Nak, Prop. 4.3] *Let X be an object in $\mathcal{T}$.*

(a) *There exists a diagram of the form*

$$\begin{tikzcd} U'[-1] \arrow[rr] \arrow[dr] & & X \arrow[r, "\alpha^+"] & X^+ \arrow[r] & U' \\ & U_X \arrow[ur] & & & \end{tikzcd} \tag{4.4}$$

with the first row forming a triangle, $U', U_X \in \mathcal{U}$ and $X^+ \in \mathcal{T}^+$. We call this triangle and the morphism α^+ a reflection triangle for X *and a* reflection morphism for X, *respectively.*

(b) *There exists a diagram of the form*

$$\begin{tikzcd} V' \arrow[r] & X^- \arrow[r, "\alpha^-"] & X \arrow[rr] \arrow[dr] & & V'[1] \\ & & & V_X \arrow[ur] & \end{tikzcd} \tag{4.5}$$

with the first row forming a triangle, $V', V_X \in \mathcal{V}$ and $X^- \in \mathcal{T}^-$. We call this triangle and the morphism α^- a coreflection triangle for X *and a* coreflection morphism for X, *respectively.*

In particular, the subcategory $\mathcal{T}^+$ (resp. $\mathcal{T}^-$) is covariantly finite (resp. contravariantly finite) in $\mathcal{T}$.

By [Nak, Lem. 4.6], if $X \in \mathcal{T}^-$, then the object X^+ in (4.4) belongs to $\mathcal{H}$. Dually, $X \in \mathcal{T}^+$ forces $X^- \in \mathcal{H}$. Furthermore, by taking the stable categories with respect to $\mathcal{W}$, the assignments $X \mapsto X^+$ and $X \mapsto X^-$ behave very nicely as follows.

Proposition 4.6. [Nak, Cor. 4.4, Thm. 5.7]

(a) *The assignment $X \mapsto X^+$ gives rise to a functor $\tau^+ : \underline{\mathcal{T}} \to \underline{\mathcal{T}}^+$. Moreover, the functor τ^+ is a left adjoint of the inclusion $i_+ : \underline{\mathcal{T}}^+ \hookrightarrow \underline{\mathcal{T}}$.*

(b) *The assignment $X \mapsto X^-$ gives rise to a functor $\tau^- : \underline{\mathcal{T}} \to \underline{\mathcal{T}}^-$. Moreover, the functor τ^- is a right adjoint of the inclusion $i_- : \underline{\mathcal{T}}^- \hookrightarrow \underline{\mathcal{T}}$.*

Furthermore, we have an isomorphism $\tau^- \circ \tau^+ \cong \tau^+ \circ \tau^-$ and the functor $\mathbb{H} := \tau^- \circ \tau^+$ is cohomological.

By Corollary 4.3, we have a localization sequence of $\mathsf{mod}\,\mathcal{U}[-1]$ relative to $\mathsf{def}\,\mathcal{U}[-1]$ and the functor $E_{\mathcal{U}[-1]} : \mathcal{U}[-1] \to \mathsf{lex}\mathcal{U}[-1]$. We consider the following diagram:

$$\begin{tikzcd}
\mathcal{H} \arrow[r, hook] \arrow[d, "\pi"'] & \mathcal{T} \arrow[r, "\mathbb{Y}_{\mathcal{U}[-1]}"] & \mathsf{mod}\,\mathcal{U}[-1] \arrow[r, "Q"] & \mathsf{lex}\mathcal{U}[-1] \\
\underline{\mathcal{H}} \arrow[rrru, dotted, "\Psi"'] & & &
\end{tikzcd} \tag{4.6}$$

There uniquely exists a dotted arrow Ψ which makes the diagram commutative up to isomorphism. In fact, the functor $\mathcal{H} \to \mathsf{lex}\mathcal{U}[-1]$ composed with all horizontal arrows sends any $W \in \mathcal{W}$ to zero, because of $(\mathcal{U}[-1], W) = 0$. Thus the functor $\Psi : \underline{\mathcal{H}} \to \mathsf{lex}\mathcal{U}[-1]$ sends $\pi(H)$ to $(-, H)|_{\mathcal{U}[-1]}$ for any $H \in \mathcal{H}$.

The aim of this section is to prove the following result.

Theorem 4.7. *Let $(\mathcal{U}, \mathcal{V})$ be a cotorsion pair in a triangulated category $\mathcal{T}$. Then the following hold.*

(1) *There exists a natural equivalence $\Psi : \underline{\mathcal{H}} \xrightarrow{\sim} \mathsf{lex}\mathcal{U}[-1]$ which sends $\pi(H)$ to $(-, H)|_{\mathcal{U}[-1]}$ for any $H \in \mathcal{H}$.*

(2) *The cohomological functor $\mathbb{H}$ is isomorphic to the composed functor $\mathcal{T} \to \mathsf{mod}\,\mathcal{U}[-1] \xrightarrow{Q} \mathsf{lex}\mathcal{U}[-1] \xrightarrow{\Psi^{-1}} \underline{\mathcal{H}}$.*

To prove Theorem 4.7, we use the following lemma.

Lemma 4.8. *Let X be an object in $\mathcal{T}$. Then there exist morphisms $X \xrightarrow{\alpha^+} X^+ \xleftarrow{\alpha^-} X^{\pm}$ with $X^+ \in \mathcal{T}^+$ and $X^{\pm} \in \mathcal{H}$ so that they induce isomorphisms*

$$Q(-, X)|_{\mathcal{U}[-1]} \xrightarrow{\sim} Q(-, X^+)|_{\mathcal{U}[-1]} \xleftarrow{\sim} Q(-, X^{\pm})|_{\mathcal{U}[-1]}$$

in $\mathsf{lex}\mathcal{U}[-1]$. Moreover, we have an isomorphism $\mathbb{H}(X) \cong \pi(X^{\pm})$.

Proof. For a given $X \in \mathcal{T}$, we consider a reflection triangle (4.4) for X. Since Lemma 4.1(3) tells $(-, U')|_{\mathcal{U}[-1]}, (-.U_X)|_{\mathcal{U}[-1]} \in \mathsf{def}\,\mathcal{U}[-1]$, the induced morphism $Q(-, \alpha^+)|_{\mathcal{U}[-1]}$ is an isomorphism in $\mathsf{lex}\mathcal{U}[-1]$. Similarly, considering a coreflection triangle for X^+, we have a desired morphism. We skip the details. □

Proof of Theorem 4.7. (1) For readability purposes we divide the proof in some steps.

Claim 1. The functor Ψ is dense.

Proof. Let F be an object in $\mathsf{mod}\,\mathcal{U}[-1]$ with projective presentation $(-, U'[-1]) \xrightarrow{(-,f)} (-, U[-1]) \to F \to 0$. Complete triangles $\eta : U'[-1] \xrightarrow{f} U[-1] \to X \to U$ in $\mathcal{T}$. Since $(-, U)|_{\mathcal{U}[-1]} \in \mathsf{def}\,\mathcal{U}[-1]$ by Lemma 4.1(2), we have the following exact sequence

$$Q(-, U'[-1]) \xrightarrow{Q(-,f)} Q(-, U[-1]) \to Q(-, X)|_{\mathcal{U}[-1]} \to 0$$

in $\mathsf{lex}\mathcal{U}[-1]$. Obviously we get $QF \cong Q(-, X)|_{\mathcal{U}[-1]}$.

By Lemma 4.8, there exist morphisms $X \xrightarrow{\alpha^+} X^+ \xleftarrow{\alpha^-} X^{\pm}$ with $X^{\pm} \in \mathcal{H}$. Applying $\mathbb{Y}_{\mathcal{U}[-1]} \circ Q$ to this diagram, we have isomorphims $Q(-, X)|_{\mathcal{U}[-1]} \xrightarrow{\sim} Q(-, X^+)|_{\mathcal{U}[-1]} \xleftarrow{\sim} Q(-, X^{\pm})|_{\mathcal{U}[-1]}$. Since $X^{\pm} \in \mathcal{H}$, this shows that Ψ is dense. □

Claim 2. The functor $(-,H)|_{\mathcal{U}[-1]}$ is left exact for any $H \in \mathcal{H}$.

Proof. Since $(-,H)|_{\mathcal{U}[-1]}$ is half exact, thanks to Proposition 2.8(1), it suffices to show that $(-,H)|_{\mathcal{U}[-1]}$ sends a deflation to a monomorphism. To this end, let $U' \to U \to U''$ be a conflation in $\mathcal{U}$, equivalently, there exists a triangle $U'[-1] \to U[-1] \to U''[-1] \xrightarrow{a} U'$. Applying the functor $(-,H)$ to the above triangle, we get an exact sequence

$$(U',H) \xrightarrow{-\circ a} (U''[-1],H) \to (U[-1],H) \to (U'[-1],H)$$

To show the morphism $(U',H) \xrightarrow{-\circ a} (U''[-1],H)$ is zero, we take an element $b \in (U',H)$. Since $H \in \mathcal{W} * \mathcal{V}[1]$, we have the following commutative diagram

$$\begin{tikzcd} U'[-1] \arrow[r] & U[-1] \arrow[r] & U''[-1] \arrow[r, "a"] & U' \arrow[d, "b"] \arrow[dll, dotted] \arrow[dr, "0"] & & \\ & & W \arrow[r] & H \arrow[r] & V[1] \arrow[r] & W[1] \end{tikzcd}$$

with $W \in \mathcal{W}$ and $V \in \mathcal{V}$. Thus the morphism $b \circ a : U''[-1] \to H_1$ factors through W, which shows $b \circ a = 0$. We have then concluded $(-,H)|_{\mathcal{U}[-1]} \in \mathsf{lex}\,\mathcal{U}[-1]$. ☐

Claim 3. The functor Ψ is full.

Proof. Let H_1 and H_2 be objects in $\mathcal{H}$. By Claim 2, $(-,H_i)|_{\mathcal{U}[-1]}$ belongs to $\mathsf{Lex}\,\mathcal{U}[-1]$ for $i = 1, 2$. Thus, it suffices to show that, for any morphism $\phi : (-,H_1)|_{\mathcal{U}[-1]} \to (-,H_1)|_{\mathcal{U}[-1]}$, there exists a morphism $c : H_1 \to H_2$ such that $\Psi(c) \cong \phi$. Since $H_i \in \mathcal{U}[-1] * \mathcal{W}$, there exist triangles $\eta_i : W_i[-1] \to U_i[-1] \to H_i \to W_i$ with $U_i \in \mathcal{U}$ and $W_i \in \mathcal{W}$ for each $i = 1, 2$. These triangles induce projective presentations $(-,W_i[-1]) \to (-,U_i[-1]) \to (-,H_i)|_{\mathcal{U}[-1]} \to 0$ in $\mathsf{mod}\,\mathcal{U}[-1]$. Then the morphism ϕ induces a morphism $(a,b,c) : \eta_1 \to \eta_2$ between triangles. Hence we have an isomorphism $\Psi(c) \cong \phi$. ☐

Claim 4. The functor Ψ is faithful.

Proof. Let $h : H_0 \to H_1$ be a morphism in $\mathcal{H}$ such that the induced morphism

$$(-,H_0)|_{\mathcal{U}[-1]} \xrightarrow{h\circ -} (-,H_1)|_{\mathcal{U}[-1]}$$

factors through an object in $\mathsf{def}\,\mathcal{U}[-1]$. Since $(-,H_0)|_{\mathcal{U}[-1]} \in \mathsf{lex}\,\mathcal{U}[-1]$ by Claim 2, this morphism $h\circ -$ should be zero. Since $H_0 \in \mathcal{U}[-1] * \mathcal{W}$, we have the following commutative diagram

$$\begin{tikzcd} U_0[-1] \arrow[r] \arrow[dr, "0"'] & H_0 \arrow[r] \arrow[d, "h"] & W_0 \arrow[r] \arrow[dl, dotted] & U_0 \\ & H_1 & & \end{tikzcd}$$

with $U_0 \in \mathcal{U}$ and $W_0 \in \mathcal{W}$, showing that h factors through W_0. ☐

Combining the above claims, we conclude that Ψ is an equivalence.

(2) Consider a morphism $c : X_1 \to X_2$ in $\mathcal{T}$. By Proposition 4.5, we have the following commutative diagram

$$\begin{tikzcd} X_1 \arrow[r, "\alpha_1^+"] \arrow[d, "c"'] & X_1^+ \arrow[d, "c^+"] & X_1^\pm \arrow[l, "\alpha_1^-"'] \arrow[d, "c^\pm"] \\ X_2 \arrow[r, "\alpha_2^+"] & X_2^+ & X_2^\pm \arrow[l, "\alpha_2^-"'] \end{tikzcd} \tag{4.7}$$

with $X_i^+ \in \mathcal{T}^+$ and $X_i^\pm \in \mathcal{H}$ for $i = 1, 2$. Denote by Φ the composed functor $\mathcal{T} \to \mathsf{mod}\,\mathcal{U}[-1] \xrightarrow{Q} \mathsf{lex}\,\mathcal{U}[-1] \xrightarrow{\Psi^{-1}} \underline{\mathcal{H}}$. Applying Φ to (4.7), by Lemma 4.8, we have an isomorphism $\Phi(c) \cong \Phi(c^\pm)$. By the definition of the cohomological functor, we have $\pi(c^\pm) = \tau^+\tau^-(c) = \mathbb{H}(c)$. By the commutativity of (4.6), we have an isomorphism $\Phi(c^\pm) \cong \pi(c^\pm)$. Hence we have an isomorphism $\Phi \cong \mathbb{H}$. $\square$

By combining Proposition 2.16 and Theorem 4.7, we recover the following result.

Corollary 4.9. [LN, Prop. 4.15][Liu, Thm. 2.10] *Let $(\mathcal{U}, \mathcal{V})$ be a cotorsion pair in a triangulated category $\mathcal{T}$ and $\mathcal{P}$ the full subcategory of projectives in the extrianguated category $\mathcal{U}$. If $\mathcal{U}$ has enough projectives, then we have an equivalence $\underline{\mathcal{H}} \xrightarrow{\sim} \mathsf{mod}\,\mathcal{P}$.*

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