On the Positivity of the Dimension of the Global Sections of Adjoint Bundle for Quasi-Polarized Manifold with Numerically Trivial Canonical Bundle *†‡

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Abstract

Let (X, L) denote a quasi-polarized manifold of dimension $n \geq 5$ defined over the field of complex numbers such that the canonical line bundle K_X of X is numerically equivalent to zero. In this paper, we consider the dimension of the global sections of $K_X + mL$ in this case, and we prove that $h^0(K_X + mL) > 0$ for every positive integer m with $m \geq n - 3$. In particular, a Beltrametti-Sommese conjecture is true for quasi-polarized manifolds with numerically trivial canonical divisors.

1 Introduction

Let X be a smooth projective variety of dimension n defined over the field of complex numbers, and let L be an ample (resp. nef and big) line bundle on X. Then, (X, L) is called a *polarized* (resp. *quasi-polarized*) *manifold*. Recently, the positivity of the dimension $h^0(K_X + mL)$ has been discussed, where K_X and m denote the canonical line bundle of X and a natural number, respectively. For $m = n - 1$, Beltrametti and Sommese proposed the following conjecture ([3, Conjecture 7.2.7]):

Conjecture 1 (Beltrametti–Sommese) *Let* (X, L) *be a polarized manifold with* $\dim X = n \geq 3$. *Assume that* $K_X + (n - 1)L$ *is nef. Then* $h^0(K_X + (n - 1)L) > 0$.

For this conjecture, the following partial results have been obtained:

- In [8, Theorem 2.4] and [11, Theorem 3.1], the author proved that this conjecture is true if $n \leq 4$. (See also [5] and [6].) Besides, we also note that Andreatta and Fontanari [1] improved the result in [11].
- In [15, 1.2 Theorem], Höring proved that this conjecture is true if $h^0(L) > 0$.

Moreover, the author has classified (X, L) for the following types in the previously conducted studies:

- Polarized 3-fold (X, L) with $h^0(K_X + 2L) \leq 2$ ([8], [10]).
- Polarized 4-fold (X, L) with $h^0(K_X + 3L) \leq 1$ ([11], [12]).

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More generally, Ionescu proposed the following conjecture (see [17, Open problems, P.321]).

Conjecture 2 (Ionescu) *Let (X, L) be a polarized manifold. Assume that $K_X + L$ is nef. Then, $h^0(K_X + L) > 0$.*

It is known that this conjecture is true if $n \leq 3$ (see [9, Theorem 2.8], [15, 1.5 Theorem]).

Additionally, the author also considered the case where $m \geq n + 1$. In [13, Conjecture 2], we proposed the following conjecture.

Conjecture 3 *Let (X, L) be an n-dimensional polarized manifold with $n \geq 3$. Then, $h^0(K_X + mL) \geq \binom{m-1}{n}$ holds for every integer $m \geq n + 1$. If equality holds for some $m \geq n + 1$, then $(X, L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.*

It has been proved that Conjecture 3 is true only in the following cases:

- The case where $n \leq 4$ ([8, Theorem 2.5] and [13, Theorem 3.1]). (See also [1, Theorem 8] for results concerning with Conjecture 3.)
- The case where $n \geq 5$ and $\dim \mathrm{Bs}|L| \leq 1$ ([13, Theorem 3.2 (i)]).
- The case where $n = 5$ and $h^0(L) > 0$ ([14, Theorem 3.1]).

In this study, we consider the positivity of $h^0(K_X + mL)$ when the canonical bundle K_X is numerically equivalent to zero and L is nef and big. In this case, Cao and Jiang proved that $h^0(K_X + L) > 0$ if $n \leq 4$ (see [7, Remark 5.3]). In this paper, we prove that $h^0(K_X + mL) > 0$ for every integer $m \geq n - 3$ for $n \geq 5$ (Corollary 2.1). In particular, combining this result and the above mentioned result of Cao and Jiang, we deduce that Conjecture 1 is true for quasi-polarized manifolds with numerically trivial canonical divisors (Corollary 2.2).

Throughout this paper, we work over the field of complex numbers, and we use the customary notations in algebraic geometry.

2 Main Result

First, we note that we can prove the following two propositions by the same method as in the proof of [7, Theorems 5.1 and 5.2].

Proposition 2.1 *Let X be a smooth projective variety of dimension $2k + 1$ with $k \geq 1$, and let L be a nef and big divisor on X. Assume that K_X is numerically equivalent to zero. Then, the following holds: For every positive integers $\alpha_1, \ldots, \alpha_k$ with $1 \leq \alpha_1 < \ldots < \alpha_k$, there exists i such that $h^0(K_X + \alpha_i L) > 0$.*

Proposition 2.2 *Let X be a smooth projective variety of dimension $4k + 2$ or $4k + 4$ with $k \geq 0$, and let L be a nef and big divisor on X. Assume that K_X is numerically equivalent to zero. Then, the following holds: For every positive integers $\alpha_1, \ldots, \alpha_{2k+1}$ with $1 \leq \alpha_1 < \ldots < \alpha_{2k+1}$, there exists i such that $h^0(K_X + \alpha_i L) > 0$.*

Remark 2.1 Let X be a smooth projective variety of dimension n. Assume that the canonical bundle K_X of X is numerically equivalent to zero. Then, for any line bundle L on X, we have $\chi(tL) = (-1)^n \chi(-tL + K_X) = (-1)^n \chi(-tL)$. In particular, $\chi(tL)$ is an even (resp. odd) function of integers if n is even (resp. odd), and we also note that $\chi(\mathcal{O}_X) = 0$ if n is odd. Furthermore, we see that if $\chi(aL) = 0$ for some $a \in \mathbb{N}$, then $\chi(-aL) = 0$.

Theorem 2.1 *Let X be a smooth projective variety of dimension n, and let L be a nef and big divisor on X. Assume that K_X is numerically equivalent to zero. Then the following hold.*

(a) *If n is odd with $n \geq 5$, then $h^0(K_X + mL) > 0$ for every integer m with $m \geq n-3$.*

(b) *If n is odd with $n \geq 7$, then $h^0(K_X + (n-4)L) > 0$.*

(c) *If $n = 4k+4$ with $1 \leq k \in \mathbb{Z}$, then $h^0(K_X + mL) > 0$ for every integer m with $m \geq n-5$.*

(d) *If $n = 4k+2$ with $1 \leq k \in \mathbb{Z}$, then $h^0(K_X + mL) > 0$ for every integer m with $m \geq n-3$.*

Proof. First of all, we note that we have $h^i(K_X+mL) = h^i(mL) = 0$ for every positive integers i and m by the Kawamata-Viehweg vanishing theorem because K_X is numerically equivalent to zero. Hence $h^0(K_X + mL) = \chi(K_X + mL) = \chi(mL) = h^0(mL)$ for every positive integer m.

(I) First, we study the case (a) in Theorem 2.1. Here we set $n = 2k+1$, where $2 \leq k \in \mathbb{Z}$.

(I.1) If $k = 2$, then we consider the pair $(h^0(L), h^0(mL))$. We remark that $m \geq n-3 = 2$. Then we see from Proposition 2.1 that $h^0(L) > 0$ or $h^0(mL) > 0$. If $h^0(mL) > 0$, then we are done. If $h^0(L) > 0$, then it follows $h^0(mL) > 0$.

(I.2) We assume that $k \geq 3$ (i.e. $n \geq 7$). Then, we take the following string of $k-2$ pairs

(1) $\quad (h^0(L), h^0((m-1)L)), (h^0(2L), h^0((m-2)L)), \ldots, (h^0((k-2)L), h^0((m-k+2)L)).$

Here we note that there are no overlaps among iL's in the string (1) if and only if $m-k+2 > k-2$, that is, $m \geq 2k-3 = n-4$. We also remark the following.

($*$) If there exists an integer i with $1 \leq i \leq k-2$ such that $h^0(iL) > 0$ and $h^0((m-i)L) > 0$, then we have $h^0(mL) \geq h^0(iL) + h^0((m-i)L) - 1 > 0$ by [16, 15.6.2 Lemma]. So we may assume that $h^0(iL) = 0$ or $h^0((m-i)L) = 0$ for every $i = 1, 2, \ldots, k-2$.

So we may assume that we can pick up $k-2$ integers $1 \leq \beta_1 < \ldots < \beta_{k-2} \leq m-1$ such that $h^0(\beta_i L) = 0$ for every i with $1 \leq i \leq k-2$.

Now we consider the set

$$\mathcal{A} = \{t \in \mathbb{Z} \mid k-1 \leq t \leq m-k+1\}.$$

We note that $h^0(tL)$ is not contained in the string (1) for every $t \in \mathcal{A}$. Then, there is at most one $p \in \mathcal{A}$ such that $h^0(pL) = 0$ by Proposition 2.1 since we have already $k-2$ zeros of $h^0(tL)$ in the string (1).

(I.2.1) Assume that $m \geq n-2 = 2k-1$. Here we note that $k-1 \neq m-k+1$. We consider the pair $(h^0((k-1)L), h^0((m-k+1)L))$. This is not contained in the string (1). So by Propossition 2.1, at least one of them is positive. If $h^0((k-1)L) > 0$ and $h^0((m-k+1)L) > 0$, then we get $h^0(mL) > 0$ by the same argument as ($*$) above. If $h^0((k-1)L) = 0$ (resp. $h^0((m-k+1)L) = 0$), then by applying Proposition 2.1 to $\{\beta_1, \ldots, \beta_{k-2}, k-1, m\}$ (resp. $\{\beta_1, \ldots, \beta_{k-2}, m-k+1, m\}$), we also get $h^0(mL) > 0$.

(I.2.2) Assume that $m = n-3 = 2k-2$. In this case, $\mathcal{A} = \{k-1\}$. So by applying Proposition 2.1 to $\{\beta_1, \ldots, \beta_{k-2}, k-1, m\}$, we see that $h^0((k-1)L) > 0$ or $h^0(mL) > 0$. If $h^0(mL) > 0$, then we are done. So we may assume that $h^0((k-1)L) > 0$. But then we get $h^0(mL) > 0$ because $mL = 2(k-1)L$.

(II) Next, we study the case (b) in Theorem 2.1. We set $n = 2k+1$, where $3 \leq k \in \mathbb{Z}$. We remark that $n-4=2k-3$ in this case. Then, we take $k-2$ pairs $(h^0(L), h^0((2k-4)L))$, ... , $(h^0((k-2)L), h^0((k-1)L))$. By the same argument as $(*)$ in (I.2), we may assume that $h^0(iL)=0$ or $h^0((2k-3-i)L)=0$ for every $i=1,2,\ldots,k-2$. Let $a_1,\ldots,a_{k-2}$ be integers such that $1\le a_1<\ldots<a_{k-2}\le 2k-4$ and $h^0(a_iL)=0$ for every i with $1\le i\le k-2$.

If $h^0(L)>0$, then we have $0<h^0((2k-3)L)=h^0((n-4)L)$ and we are done. So we may assume that $h^0(L)=0$. Then, we note that we can take $a_1=1$. If $h^0((2k-4)L)=0$, then by applying Proposition 2.1 to $\{a_1,\ldots,a_{k-2},2k-4,2k-3\}$, we have $h^0((2k-3)L)>0$ holds. Hence we may also assume that $h^0((2k-4)L)>0$, that is, $a_{k-2}\le 2k-5=n-6$.

Here we assume by contradiction that $h^0((n-4)L)=h^0((2k-3)L)=0$. Since n is odd, we can describe $\chi(tL)$ as follows by Remark 2.1.

$$\chi(tL)=\alpha(t^2-a_1^2)\cdots(t^2-a_{k-2}^2)(t^2-(2k-3)^2)(t^2-\beta)t, \tag{2}$$

where $\alpha,\beta\in\mathbb{R}$. We note that α is positive, and $a_i\le 2k-5$ for every i with $1\le i\le k-2$. Moreover, $(2k-4)^2-\beta<0$ holds because $h^0((2k-4)L)>0$, $\alpha>0$, $(2k-4)^2-(2k-3)^2<0$ and $a_i^2<(2k-4)^2$ for every i. Namely, $\beta>(2k-4)^2>0$. Here, we note that the coefficient of t^{2k-1} in the right hand side of (2) is

$$-\alpha\left(\beta+\sum_{i=1}^{k-2}a_i^2+(2k-3)^2\right).$$

On the other hand, by employing the Hirzebruch-Riemann-Roch formula for $\chi(tL)$, we have

$$-\alpha\left(\beta+\sum_{i=1}^{k-2}a_i^2+(2k-3)^2\right)=\frac{L^{n-2}c_2(X)}{12(n-2)!} \tag{3}$$

and the right hand side of (3) is non-negative by [18, Theorem 6.6]. But the left hand side of (3) is negative because $\alpha>0$ and $\beta>0$. Hence, this is a contradiction. Therefore, $h^0((n-4)L)>0$ holds.

(III) Here we consider the cases (c) and (d) in Theorem 2.1. Then, we take the following string of $2k-1$ pairs

$$(h^0(L),h^0((m-1)L)),(h^0(2L),h^0((m-2)L)),\ldots,(h^0((2k-1)L),h^0((m-2k+1)L)). \tag{4}$$

First, we note the following.

$$\begin{cases}\text{In (c), we consider the case that } n=4k+4 \text{ and } m\ge n-5=4k-1.\\ \text{In (d), we consider the case that } n=4k+2 \text{ and } m\ge n-3=4k-1.\end{cases}$$

We also note that there are no overlaps among iL's in the string (4) in these cases. By an argument similar to $(*)$ in (I.2), we obtain that there are at least $2k-1$ zeros of $h^0(tL)$ in the string (4).

Here, we consider the set

$$\mathcal{B}=\{t\in\mathbb{Z}\mid 2k\le t\le m-2k\}.$$

We remark that $h^0(tL)$ is not contained in the string (4) for every $t\in\mathcal{B}$. We see from Proposition 2.2 that there is at most one $p\in\mathcal{B}$ such that $h^0(pL)=0$.

(III.1) We assume that $m\ge 4k$. Then, we remark that $\mathcal{B}$ is nonempty.

(III.1.1) If $m>4k$ (namely, $m\ge n-1$ (resp. $m\ge n-3$) if $n=4k+2$ (resp. $n=4k+4$)), then $2k\ne m-2k$. Moreover, we remark that $2k+(m-2k)=m$. Since $2k$ and $m-2k$ are elements of $\mathcal{B}$, we can conclude that $h^0(mL)>0$ by using Proposition 2.2 in a similar way to the case (I.2.1) above.

(III.1.2) If $m = 4k$ (namely, $m = n-2$ (resp. $m = n-4$) if $n = 4k+2$ (resp. $n = 4k+4$)), then $\mathcal{B} = \{2k\}$. Then we can conclude that $h^0(mL) = h^0(4kL) > 0$ by using Proposition 2.2 in a similar way to the case (I.2.2) above.

(III.2) Finally, we consider the case that $m = 4k-1$.

In the string (4), we consider the pair $(h^0(L), h^0((4k-2)L))$. If $h^0(L) > 0$, then we can see that $h^0(mL) > 0$. So we may assume that $h^0(L) = 0$. If $h^0((4k-2)L) = 0$, then there are at least $2k$ zeros of $h^0(tL)$ in the string (4). Hence, we see from Proposition 2.2 that $h^0(mL) > 0$. Therefore, we may assume that $h^0(L) = 0$ and $h^0((4k-2)L) > 0$.

(III.2.1) We consider the case that $(n, m) = (4k+2, 4k-1)$. Assume by contradiction that $h^0(mL) = 0$. Then, $\chi(tL)$ has $2k-1$ zeros $1 = a_1 < \ldots < a_{2k-1} \leq 4k-3 = m-2$ and another zero $m = 4k-1$. Therefore, we see from Remark 2.1 that $\chi(tL)$ has $4k$ zeros $\pm a_1, \ldots, \pm a_{2k-1}, \pm(4k-1)$ and we may write

$$\chi(tL) = \alpha(t^2 - a_1^2)\cdots(t^2 - a_{2k-1}^2)(t^2 - (4k-1)^2)(t^2 - \beta),$$

where $\alpha, \beta \in \mathbb{R}$. But then, we can get a contradiction by an argument similar to (II) above. Therefore, $0 < h^0(mL) = h^0((n-3)L)$ holds.

(III.2.2) We consider the case that $(n, m) = (4k+4, 4k-1)$. Assume by contradiction that $h^0(mL) = 0$. Then, $\chi(tL) = 0$ has $2k-1$ zeros

$$1 = d_1 < \ldots < d_{2k-1} \leq 4k-3 \tag{5}$$

and another zero $m = 4k-1$. Hence, we see from Remark 2.1 that $\chi(tL)$ has $4k$ zeros $\pm d_1, \ldots, \pm d_{2k-1}, \pm(4k-1)$ and we may write

$$\chi(tL) = \alpha(t^2 - d_1^2)\cdots(t^2 - d_{2k-1}^2)(t^2 - (4k-1)^2)(t^4 - pt^2 + q), \tag{6}$$

where $\alpha, p, q \in \mathbb{R}$. We note that $\alpha > 0$.

Assume that $p \geq 0$. Then, the coefficient of t^{4k+2} in the right hand side of (6) is

$$-\alpha\left(p + \sum_{i=1}^{2k-1} d_i^2 + (4k-1)^2\right)$$

and this is negative. On the other hand, by employing the Hirzebruch-Riemann-Roch formula for $\chi(tL)$, we have

$$-\alpha\left(p + \sum_{i=1}^{2k-1} d_i^2 + (4k-1)^2\right) = \frac{L^{n-2}c_2(X)}{12(n-2)!} \tag{7}$$

and the right hand side of (7) is non-negative by [18, Theorem 6.6]. Hence we get a contradiction and we see that

$$p < 0. \tag{8}$$

By (6), we have

$$\begin{aligned} 0 &< h^0((4k-2)L) = \chi((4k-2)L) \\ &= \alpha((4k-2)^2 - d_1^2)\cdots((4k-2)^2 - d_{2k-1}^2)((4k-2)^2 - (4k-1)^2)((4k-2)^4 - p(4k-2)^2 + q). \end{aligned}$$

Hence, we see from (5) and (8) that $q < 0$ holds, and we get

$$\chi(\mathcal{O}_X) = \alpha(-d_1^2)\cdots(-d_{2k-1}^2)(-(4k-1)^2)q < 0.$$

But, this is impossible because $\chi(\mathcal{O}_X) \geq 0$ by the Beauville-Bogomolov decomposition (see [2], [4], [7, Theorem 2.1]).

Therefore we get $0 < h^0(mL) = h^0((n-5)L)$. □

By Theorem 2.1 and [7, Remark 5.3], we obtain the following corollaries.

Corollary 2.1 *Let X be a smooth projective variety of dimension $n \geq 5$, and let L be a nef and big divisor on X. Assume that K_X is numerically equivalent to zero. Then, $h^0(K_X + mL) > 0$ for every integer $m \geq n-3$.*

Corollary 2.2 *Let (X, L) be a quasi-polarized manifold. Assume that K_X is numerically equivalent to zero. Then, Conjecture 1 is true.*

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